Amplification of gravitationally induced entanglement

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Observation of gravitationally induced entanglement between two massive particles can be viewed as implying the existence of the nonclassical nature of gravity. However, weak interaction in the gravitational field is extremely small so that gravitationally induced entanglement is exceptionally challenging to test in practice. For addressing this key challenge, here we propose a criterion based on the logical contradictions of weak entanglement, which may boost the sensitivity of the signal due to the gravitationally induced entanglement. Specifically, we make use of the weak-value scenario and Einstein-Podolsky-Rosen steering. We prove that it is impossible for a classical mediator to act on two local quantum objects to simulate amplified-weak-value phenomenon in two-setting Einstein-Podolsky-Rosen steering. Our approach can amplify the signal of gravitationally induced entanglement that were previously impossible to observe by any desired factor that depends on the magnitude of the weak value. Our results not only open up the possibility of exploring nonclassical nature of gravity in the near future, but they also pave the way for weak entanglement criterion of a more general nature.

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I. INTRODUCTION

Quantum theory and general relativity, the two backbones of modern physics, have been verified with very high precision in their respective fields. Yet, it is hard to unify them into a unique corpus of laws. One possible route to that general theory is the quantization of gravity, with the same spirit as other field theories. However, there is a long-standing debate whether gravity should be quantized [1–5]. Traditionally, it is believed that the effects of quantum gravity should occur at high energy scales or in the short length regime which are beyond the reach of current technology. Recently, there has been a revival of the idea of a tabletop probe, which highlights the interaction of the probe mass with the gravitational field generated by another mass [6–10]. Especially, two gravity-induced-entanglement tests, sometimes called the Bose-Marletto-Vedral (BMV) experiments [11,12], have been proposed, which may be use to expose the quantum nature of gravity. BMV's protocol aims to provide a firm evidence on whether the gravitational field is mediated by the transfer of quantum information. Bose *et al.* suggest that if we admit the central principle of quantum information theory—entanglement between two systems cannot be created by local operations and classical communication (LOCC)—then gravitationally induced entanglement indicates that gravity must be quantum [12]. On the other hand, Marletto and Vedral argue for this view through a more general information-theoretic argument [11,13], which is based on constructor theory [14]. Specifically, one does not need to assume any specific dynamics law of mediators (in this case the gravitational field) to justify the conclusions of creating entanglement in the experimental proposal [11,13]. In this article, we will focus on the quantum formalism. Till now, a variety of advanced theoretical and experimental proposals have been suggested to investigate the gravitationally induced entanglement and nonclassicality [15–40].

Entanglement witnesses are a suitable method for measuring the gravitationally induced entanglement [41,42]. Unfortunately, due to the extremely weak strength of gravity, a "strong" and detectable entanglement signal might require a longer interaction time of massive particles in a superposition of two locations (matter-wave-like interferometer), which poses a serious challenge to current experimental techniques. As we all know, a general rule of thumb is that the larger and heavier a particle is, the shorter its coherence time. In particular, the experiments must be implemented within the coherent time otherwise the loss of entanglement due to decoherence would prevent us from concluding anything about the quantum nature of gravity. Is it possible to detect weakly entangled signals with limited coupling time for a given mass of particles and a finite resolution or sensitivity of the measurement devices? Could we amplify the signals of these nonclassical correlations?

This is an issue that has not been mainly considered in previous studies [15–40] and is also the main motivation for our present paper.

There is a famous parametric amplification approach in quantum information field, called weak-value amplification [43–45]. Weak values have their root in quantum weak measurement, which describes a weak coupled measurements, proposed by Aharonov, Albert, and Vaidman [43]. Weak-value amplification exploits the fact that the post-selection of the weak measurement of a pointer can yield an amplified shift that is exceptionally sensitive to small changes in an interaction parameter. This has been successfully applied to the estimation of a range of small physical parameters [45], including beam deflection [46,47], frequency shifts [48], phase shifts [49], and so on.

In this article, we propose a criterion for determining weak, gravitationally induced entanglement, which makes use of a weak-value scenario and Einstein-Podolsky-Rosen (EPR) steering [50–52]. Specifically, we unify the weak measurements (weak value amplification scenarios) in the framework of EPR steering. Similar to the Bell test [53,54], we consider two sets of measurement bases that can be randomly selected, one of which is the normal measurement basis (e.g., the computational basis) and the other one corresponding to weak value amplification. We present a comparison of two predictions of the quantum and classical mediator, the measurement probability distribution and the measurement visibility. We show that in the case of weak entanglement, the classical mediator (in this case, the gravitational field) cannot simulate the results related to the measurement visibility of weak-value basis, thus ruling out the separable model. Concretely, our approach can amplify the signal of gravitationally induced entanglement by any desired factor that depends on the magnitude of the weak value. Compared to the previous protocols, our approach allows us to observe entangled signals that were previously impossible to observe. Besides, our criterion is not limited to the detection of weak entanglement in gravity. It is applicable to more general case of weak entanglement, including potentially macroscopic entanglement.

II. QUANTUM FORMALISM OF BMV EXPERIMENTS

Here we focus on the quantum formalism of BMV experiments. As shown in Fig. 1, the BMV proposal is presented. Two quantum mass $\mathcal{Q}_{\mathcal{A}}$ and $\mathcal{Q}_{\mathcal{B}}$ are initially at a distance from each other. Each mass individually undergoes Mach-Zehnder-type interference in parallel, and thus interacts with the other mass via the gravitational field, which plays the role of the mediator $\mathcal{M}$. Under the assumption of locality, observation of gravitationally induced entanglement between $\mathcal{Q}_{\mathcal{A}} \oplus \mathcal{Q}_{\mathcal{B}}$ is the indirect evidence of nonclassicality (quantumness) of the mediator $\mathcal{M}$ [11–13]. Specifically, The initial state of system $\mathcal{Q}_{\mathcal{A}}$ and system $\mathcal{Q}_{\mathcal{B}}$ is a separable state (by the first beam splitter), donated as

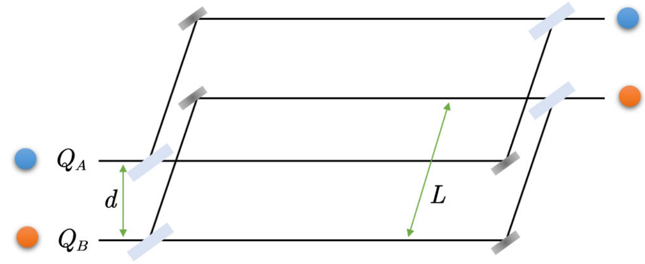

FIG. 1. Symmetric Bose-Marletto-Vedral experiment for testing gravitationally induced entanglement. There are tow mass $\mathcal{Q}_{\mathcal{A}}$ and $\mathcal{Q}_{\mathcal{B}}$. Each mass individually undergoes Mach-Zehnder-type interference in parallel and interacts with the other mass via gravity.

$\varrho_A \otimes \varrho_B = |+\rangle_A\langle +| \otimes |+\rangle_B\langle +|$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle$. Since the masses on different paths interact via the gravitational field, the state of the composite system becomes, before they enter their respective final beam splitters, $\varrho_{AB} = \Lambda(\varrho_A \otimes \varrho_B)$, where $\Lambda(\cdot)$ is the map of channel (operation) acting on quantum systems $\mathcal{Q}_{\mathcal{A}}$ and $\mathcal{Q}_{\mathcal{B}}$ induced by the mediator $\mathcal{M}$. If the quantum state ϱ_{AB} cannot be written as $\sum_i p_i \varrho_A^i \otimes \varrho_B^i$, then ϱ_{AB} is an entangled state, which indicates that the action $\Lambda(\cdot)$ is an entanglement operation. These results may be the evidence of quantumness for the mediator $\mathcal{M}$ [11,13,39]. On the contrary, a classical mediator can only produce unentangled quantum state for quantum systems $\mathcal{Q}_{\mathcal{A}}$ and $\mathcal{Q}_{\mathcal{B}}$, that is $\varrho_{AB}^C = \Lambda^C(\varrho_A \otimes \varrho_B) = \sum_i p_i \varrho_A^i \otimes \varrho_B^i$, where $\Lambda^C(\cdot)$ denotes the effective channel induced by a classical mediator.

A. Weak-value scenario of BMV experiments

Under the time evolution of the joint state of the two masses is purely due to their mutual gravitational interaction [11,13]. Thus, the channel $\Lambda(\cdot)$ of quantum mediator $\mathcal{M}$ mentioned above (in Fig. 1) is an unitary operation. Specifically, the unitary is given as $U = \exp(-i\frac{H\tau}{\hbar}) = \cos(\frac{\Delta\phi\tau}{2\hbar})I \otimes I + i\sin(\frac{\Delta\phi\tau}{2\hbar})Z \otimes Z$, where $H = -\frac{\Delta\phi}{2}Z \otimes Z$ is the Hamiltonian of the two mass is [23,24] in which $\Delta\phi = Gm_1m_2(\frac{1}{d} - \frac{1}{\sqrt{d^2+L^2}})$. After the unitary evolution, the joint quantum state of $\mathcal{Q}_{\mathcal{A}}$ and $\mathcal{Q}_{\mathcal{B}}$ becomes

$$|\Psi\rangle = U|+\rangle_A \otimes |+\rangle_B, \\ = \cos\left(\frac{\Delta\phi\tau}{2\hbar}\right)|+\rangle_A|+\rangle_B + i\sin\left(\frac{\Delta\phi\tau}{2\hbar}\right)|-\rangle_A|-\rangle_B, \quad (1)$$

which is a two-qubit entangled state. Without loss generality, $|\Psi\rangle$ can be rewritten in other basis. For the qubit case, the identity can be expressed as $I = |\epsilon\rangle\langle\epsilon| + |\epsilon^\perp\rangle\langle\epsilon^\perp|$, satisfying $\langle\epsilon^\perp|\epsilon\rangle = 0$. Here we define $|\epsilon\rangle = \epsilon|0\rangle - \sqrt{1-\epsilon^2}|1\rangle$ and $|\epsilon^\perp\rangle = \sqrt{1-\epsilon^2}|0\rangle + \epsilon|1\rangle$, where ϵ is a real positive number. Now the composite state becomes

$$|\Psi\rangle = (|\epsilon\rangle\langle\epsilon| + |\epsilon^{\perp}\rangle\langle\epsilon^{\perp}|) \otimes I|\Psi\rangle,$$
$$= \alpha|\epsilon\rangle_A \otimes |\tilde{\chi}_{\epsilon}\rangle_B + \beta|\epsilon^{\perp}\rangle_A \otimes |\tilde{\chi}_{\epsilon^{\perp}}\rangle_B, \qquad (2)$$

where $\alpha = \langle\epsilon|+\rangle$, $\beta = \langle\epsilon^{\perp}|+\rangle$, $|\tilde{\chi}_{\epsilon}\rangle = \cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle_B + i\sin(\frac{\Delta\phi\tau}{2\hbar})A_w^{\epsilon}|-\rangle$, and $|\tilde{\chi}_{\epsilon^{\perp}}\rangle = \cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle + i\sin(\frac{\Delta\phi\tau}{2\hbar}) A_w^{\epsilon^{\perp}}|-\rangle$. As can be seen, the weak values are embedded in the quantum states $|\tilde{\chi}_{\epsilon}\rangle$ and $|\tilde{\chi}_{\epsilon^{\perp}}\rangle$, which are given as $A_w^{\epsilon} = \frac{\langle\epsilon|Z|+\rangle}{\langle\epsilon|+\rangle} = \frac{\epsilon+\sqrt{1-\epsilon^2}}{\epsilon-\sqrt{1-\epsilon^2}}$ and $A_w^{\epsilon^{\perp}} = \frac{\langle\epsilon^{\perp}|Z|+\rangle}{\langle\epsilon^{\perp}|+\rangle} = \frac{-\epsilon+\sqrt{1-\epsilon^2}}{\epsilon+\sqrt{1-\epsilon^2}} = -\frac{1}{A_w^{\epsilon}}$, respectively. From the nature of $|\Psi\rangle$, it follows that if the quantum system $\mathcal{Q}_{\mathcal{A}}$ is projected into $|\epsilon\rangle$ ($|\epsilon^{\perp}\rangle$), then the quantum state of system $\mathcal{Q}_{\mathcal{B}}$ will collapse to (unnormalized) state $|\tilde{\chi}_{\epsilon}\rangle$ ($|\tilde{\chi}_{\epsilon^{\perp}}\rangle$), and vice versa. The amplified weak-value A_w^{ϵ} can be achieved if the result of the collapse of quantum system $\mathcal{Q}_{\mathcal{A}}$ to $|\epsilon\rangle$ when $\alpha = \langle\epsilon|+\rangle$ is very small. From this perspective, the generation of weak value can be explained as it originated from EPR steering [50–52], which is determine by the measurements of one of the parties [55].

As we have shown above, the weak value A_w^{ϵ} ($A_w^{\epsilon^{\perp}}$) determine the form of the quantum state $|\tilde{\chi}_{\epsilon}\rangle$ ($|\tilde{\chi}_{\epsilon^{\perp}}\rangle$). The larger the weak value A_w^{ϵ}, the bigger (smaller) the component $|-\rangle$ of the quantum state $|\tilde{\chi}_{\epsilon}\rangle$ ($|\tilde{\chi}_{\epsilon^{\perp}}\rangle$). According to the theory of weak-value amplification, the quantum state $|\tilde{\chi}_{\epsilon}\rangle$ is more likely to be accurately measured with a big weak value A_w^{ϵ} when the phase parameter $\frac{\Delta\phi\tau}{2\hbar}$ is extremely small. So one may use such amplification phenomenon to enhance sensitivity of signal of gravitationalliy induced entanglement. Unfortunately, the weak-value amplification approach is specific to parametric amplification, and it cannot be used directly to rule out the possibility of classical models. Therefore, one needs to find an entanglement criterion with weak-value amplification to exclude the model of classical mediator.

III. EPR STEERING AND WEAK-VALUE SCENARIO

Here we focus on how to construct an entanglement criterion with the weak-value amplification. As we have shown above, the weak-value scenario is a special case of EPR steering, which corresponds to a one-measurement setting. It is known that the experimental results of a one-measurement setting in EPR steering can be easily simulated by a local model. Therefore, from this point of view, the amplified weak value in BMV experiments may be simulated by a classical mediator. That is, one cannot determine the quantumness of gravity directly with a weak value amplification scenario. In general, EPR steering scenario needs at least two different measurement bases (two-measurement setting) to determine whether the joint quantum state is steerable (entangled) or not. Hence, one may consider exploiting the EPR-steering scenario to determine entanglement while keeping the measurement basis corresponding to the weak value amplification as one of the two measurement bases for EPR steering.

Nowadays, EPR steering has been heavily studied, including the detection of various linear and nonlinear inequalities (see review [56]). There is also some quantum steering paradox based on logical contradictions [57–59]. However, we will show that none of these can be directly used for the verification of weakly amplified versions of quantum steering. The reason is that all of these depend on the expectation value, which is related to the probability (the probability of weak amplification is very low). This could lead to experimental errors masking the true entangled signal. In the following, we consider not only the probability distribution of the steered quantum states, but we also introduce a physical quantity: the visibility of the measurement (Π) of one of subsystem, i.e., $\mathcal{Q}_{\mathcal{B}}$. We express this quantity in terms of $V = \mathrm{Tr}(\varrho_B\Pi)$, where ϱ_B is the steered normalized density matrix of $\mathcal{Q}_{\mathcal{B}}$. We will show that the genuine entanglement signal is hidden in the visibility. Satisfying all conditions of probability distribution and visibility allows us to exclude any separable state model.

IV. WEAK ENTANGLEMENT CRITERION

In general EPR steering scenario, there are two parities, one of which is trusted and the other is untrusted. In that case, local hidden state model is considered [52] to simulate the predictions of genuine EPR steering. Fortunately, in the following, we do not need to make use of local hidden state model (a separable model is considered) to analyze the steering since two parities are trusted (controlled by ourselves) and the system $\mathcal{Q}_{\mathcal{A}}$ and $\mathcal{Q}_{\mathcal{B}}$ are genuine quantum states. As mentioned above, the quantum states generated by quantum mediator is $|\Psi\rangle = \cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle_A|+\rangle_B + i\sin(\frac{\Delta\phi\tau}{2\hbar})|-\rangle_A|-\rangle_B$ [60]. Here we set two measurement bases for $\mathcal{Q}_{\mathcal{A}}$ as $\{|0\rangle, |1\rangle\}$ and $\{|\epsilon\rangle, |\epsilon^{\perp}\rangle\}$, respectively, and we have four steered but not normalized quantum states

$$\tilde{\rho}_B^{\langle 0|_A}(Q) = \frac{1}{2}|\phi_+\rangle\langle\phi_+|_B,$$
$$\tilde{\rho}_B^{\langle 1|_A}(Q) = \frac{1}{2}|\phi_-\rangle\langle\phi_-|_B,$$
$$\tilde{\varrho}_B^{\langle\epsilon|_A}(Q) = |\alpha|^2\mathrm{Tr}(|\tilde{\chi}_{\epsilon}\rangle\langle\tilde{\chi}_{\epsilon}|)|\chi_{\epsilon}\rangle\langle\chi_{\epsilon}|_B,$$
$$\tilde{\varrho}_B^{\langle\epsilon^{\perp}|_A}(Q) = |\beta|^2\mathrm{Tr}(|\tilde{\chi}_{\epsilon^{\perp}}\rangle\langle\tilde{\chi}_{\epsilon^{\perp}}|)|\chi_{\epsilon^{\perp}}\rangle\langle\chi_{\epsilon^{\perp}}|_B, \qquad (3)$$

where $\alpha = \langle\epsilon|+\rangle$, $\beta = \langle\epsilon^{\perp}|+\rangle$, $|\phi_{\pm}\rangle = \cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle \pm i\sin(\frac{\Delta\phi\tau}{2\hbar})|-\rangle$, $|\chi_{\epsilon}\rangle\langle\chi_{\epsilon}| = \frac{|\tilde{\chi}_{\epsilon}\rangle\langle\tilde{\chi}_{\epsilon}|}{\mathrm{Tr}(|\tilde{\chi}_{\epsilon}\rangle\langle\tilde{\chi}_{\epsilon}|)}$, $|\chi_{\epsilon^{\perp}}\rangle\langle\chi_{\epsilon^{\perp}}| = \frac{|\tilde{\chi}_{\epsilon^{\perp}}\rangle\langle\tilde{\chi}_{\epsilon^{\perp}}|}{\mathrm{Tr}(|\tilde{\chi}_{\epsilon^{\perp}}\rangle\langle\tilde{\chi}_{\epsilon^{\perp}}|)}$, and $\tilde{\varrho}_B$ represent the unnormalized quantum states [(Q) represents the quantum mediator]. Equation (3) indicates that when we project the quantum states of $\mathcal{Q}_{\mathcal{A}}$ to $|0\rangle, |1\rangle, |\epsilon\rangle, |\epsilon^{\perp}\rangle\}$, we get the quantum states of $\mathcal{Q}_{\mathcal{B}}$ are $|\phi_+\rangle, |\phi_-\rangle, |\chi_{\epsilon}\rangle, |\chi_{\epsilon^{\perp}}\rangle$ with probabilities

$$\{p^{(0,\phi_+)}, p^{(1,\phi_-)}, p^{(\epsilon,\chi_\epsilon)}, p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}\} = \left\{\frac{1}{2}, \frac{1}{2}, |\alpha|^2\mathrm{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|), |\beta|^2\mathrm{Tr}(|\tilde{\chi}_{\epsilon^\perp}\rangle\langle\tilde{\chi}_{\epsilon^\perp}|)\right\}, \quad (4)$$

respectively. If there exists a separable model (classical mediator) that can fake the results of Eq. (4), then one is not convinced that $\mathcal{Q}_\mathcal{A}$ can steer $\mathcal{Q}_\mathcal{B}$'s quantum state (namely, $\mathcal{Q}_\mathcal{A}$ and $\mathcal{Q}_\mathcal{B}$ are unentangled). Otherwise the separable model contradicts with the quantum predictions. However, since the precision of the measurement devices is limited [61], we may not be able to measure the signal of weak entanglement. One can verify that when the entanglement is extremely weak (i.e., $\frac{\Delta\phi\tau}{2\hbar}$ is very small), the probability $p^{(\epsilon,\chi_\epsilon)} = |\alpha|^2\mathrm{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|) = |\alpha|^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]$ will be very small so that the measurement device with resolution $\gamma \geq p^{(\epsilon,\chi_\epsilon)}/(p^{(\epsilon,\chi_\epsilon)} + p^{(\epsilon^\perp,\chi_{\epsilon^\perp})})$ cannot distinguish whether $\mathcal{Q}_\mathcal{A}$ and $\mathcal{Q}_\mathcal{B}$ are entangled or separable or not (See Appendix A). Therefore, we cannot directly determine weak entanglement in this way. Similarly, entanglement witness and other inequality methods to calculate the expectation value will also fail in this case.

Here we note that the Eqs. (3) and (4) may cover the predictions of visibility of the measurement of $\mathcal{Q}_\mathcal{B}$ (steered state) when the heralded probability is very small, i.e., $p^{(\epsilon,\chi_\epsilon)} \to 0$. Here we show that the visibility of measurement of system $\mathcal{Q}_\mathcal{B}$ is more robust and powerful to detect weak entanglement. Without loss generality, we define the visibility of projective measurement Π_i is

$$V_{\Pi_i}(\varrho_B) = \mathrm{Tr}(\Pi_i\varrho_B), \quad (5)$$

One can see that, for a pure qubit, the maximal visibility is 1 while for a mixed state, it is impossible to obtain the visibility equals to 1. Let us set that $\{\Pi_0, \Pi_1, \Pi_2, \Pi_3\}$ are $\{|\phi_+\rangle\langle\phi_+|, |\phi_-\rangle\langle\phi_-|, |\chi_\epsilon\rangle\langle\chi_\epsilon|, |\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|\}$, respectively. Now we have four sets of measurement visibility

$$\begin{aligned}
V_{\Pi_0}(\varrho_B^{\langle 0|_A}) &= \mathrm{Tr}(|\phi_+\rangle\langle\phi_+|\varrho_B^{\langle 0|_A}),\\
V_{\Pi_1}(\varrho_B^{\langle 1|_A}) &= \mathrm{Tr}(|\phi_-\rangle\langle\phi_-|\varrho_B^{\langle 1_A}),\\
V_{\Pi_2}(\varrho_B^{\langle \epsilon|_A}) &= \mathrm{Tr}(|\chi_\epsilon\rangle\langle\chi_\epsilon|\varrho_B^{\langle \epsilon|_A}),\\
V_{\Pi_3}(\varrho_B^{\langle \epsilon^\perp|_A}) &= \mathrm{Tr}(|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|\varrho_B^{\langle \epsilon^\perp|_A}),
\end{aligned} \quad (6)$$

where $\varrho_B^{\langle 0|_A}, \varrho_B^{\langle 1|_A}, \varrho_B^{\langle \epsilon|_A}, \varrho_B^{\langle \epsilon^\perp|_A}$ are the steered and normalized quantum state of $\mathcal{Q}_\mathcal{B}$. Obviously, the steered states of Eq. (3) satisfy that all above V are equal to 1. That is

$$V_{\Pi_0}^Q(\varrho_B^{\langle 0|_A}) = V_{\Pi_1}^Q(\varrho_B^{\langle 1|_A}) = V_{\Pi_2}^Q(\varrho_B^{\langle \epsilon|_A}) = V_{\Pi_3}^Q(\varrho_B^{\langle \epsilon^\perp|_A}) = 1, \quad (7)$$

in which label Q indicates quantum prediction. As we analyzed before, in the case of extremely weak entanglement, the results of Eq. (4) (probability distribution) can be simulated by a separable state ϱ_{AB}^C (See Appendix A). Naturally, one may wonder whether separable states ϱ_{AB}^C can also emulate the measurement visibility. Our finding is that the visibility of measurement setting $V_{\Pi_2}(\varrho_B^{\langle \epsilon|_A})$ corresponding to the weak value amplification cannot be simulated. Specifically, the classical visibility of measurement Π_2 is given as (See Appendix A)

$$V_{\Pi_2}^C(\varrho_B^{\langle \epsilon|_A}) = \mathrm{Tr}(|\chi_\epsilon\rangle\langle\chi_\epsilon|\varrho_B^{\langle \epsilon|_A}) = \frac{\frac{1}{2}\epsilon^2 p^{(0,\phi_+)}\langle\chi_\epsilon|\phi_+\rangle\langle\phi_+|\chi_\epsilon\rangle + \frac{1}{2}(1-\epsilon^2)p^{(1,\phi_-)}\langle\chi_\epsilon|\phi_-\rangle\langle\phi_-|\chi_\epsilon\rangle + \frac{1}{2}p^{(\epsilon,\chi_\epsilon)}}{\frac{1}{2}\epsilon^2 p^{(0,\phi_+)} + \frac{1}{2}(1-\epsilon^2)p^{(1,\phi_-)} + \frac{1}{2}p^{(\epsilon,\chi_\epsilon)}}. \quad (8)$$

When the entanglement is extremely weak, without loss generality, we set $\cos(\frac{\Delta\phi\tau}{2\hbar}) \approx 1$, $\sin(\frac{\Delta\phi\tau}{2\hbar}) = \frac{\Delta\phi\tau}{2\hbar}$. The measurement basis $\{|\epsilon\rangle, |\epsilon^\perp\rangle\}$ is chosen to realize weak-value amplification (i.e., $A_w^\epsilon = k\frac{1}{\frac{\Delta\phi\tau}{2\hbar}}$ and $A_w^{\epsilon^\perp} = \frac{1}{A_w^\epsilon} \approx \frac{\Delta\phi\tau}{2k\hbar} \ll 1$, where k is a coefficient) when $\epsilon \to \frac{1}{\sqrt{2}}$ and we have $\alpha = \langle\epsilon|+\rangle \approx 0$, $\beta = \langle\epsilon^\perp|+\rangle \approx 1$, and $\alpha k \ll 1$. Upon substituting these approximations into Eq. (8) (discard the second-order small quantity $|\alpha|^2$, $(\frac{\Delta\phi\tau}{2\hbar})^2$ and set $1 \pm (\frac{\Delta\phi\tau}{2\hbar})^2 \approx 1$), we obtain (See Appendix A)

$$V_{\Pi_2}^C(\varrho_B^{\langle \epsilon|_A}) \approx \frac{1}{1+k^2}, \quad (9)$$

while $V_{\Pi_0}^C \approx V_{\Pi_1}^C \approx V_{\Pi_3}^C \approx 1$. One can see that this result is contradictory to the results in Eq. (7). The measured visibility of weak entanglement is all equal to 1, however, the separable model has a $\frac{1}{1+k^2}$. If $k=1$, then we have $V_{\Pi_2}^C(\varrho_B^{\langle \epsilon|_A}) \approx \frac{1}{2}$. This is a logical contradiction of weak entanglement. It is clear that the distinguishability of measurement visibility is much greater than the probability distribution of measurement. Therefore, the signal of weak entanglement is amplified. Another implication of amplifying entanglement seems to be that we can reduce the experimental requirement in tests of gravitationally induced entanglement. Given the sensitivity of the measurement device, our scheme can achieved as $X = A_w^\epsilon$, saving for coupling strength of gravity. For example, if $A_w^\epsilon = 10^4$, then we can reduce the mass of two systems by 10 times, and shorten the coupling time by 100 times (See Appendix B). Our methodology does not depend on a specific physical system. Hence, different physical systems may realize amplification of gravitationally induced entanglement by this approaches. Besides, our weak entanglement criterion remains valid at a certain degree of decoherence and the limited precision of measurement device (See Appendix C and D).

V. TESTS OF GRAVITATIONALLY INDUCED ENTANGLEMENT

As a result of technological advance in quantum manipulation of matter at larger mass scales [62–64] and in gravitational measurements at smaller mesoscopic mass scales [65], probing nonclassical nature of gravity becomes possible. In Refs. [12,40], the spin degrees of freedom of the particles are used to construct the Stern-Gerlach interferometry to test quantum gravity. Remarkably, there are many other physical realizations apart from spin degrees of freedom on probing gravitationally induced entanglement [15–38], such as neutrinolike oscillations [15], optomechanics [18], and atomic interferometers [27] ana in Appendix.

Very recently, there is a promising experimental proposal that uses two-level systems coupled to a massive resonator (a harmonic oscillator) to probe gravitationally induced entanglement [66]. Unlike Ref. [27], it enhances the gravitational interaction of two 2-level systems by a massive particle (as a mediator), where the effective gravity-induced coupling strength is increased by a factor of $\frac{g_b}{w}$ [66]. Our criterion can also be applied to this scenario to achieve additional amplification.

VI. DISCUSSION AND CONCLUSION

Historically, there is a one measurement steering protocol the same as the amplification by LOCC. In particular, in Gisin's paper in 1995, it was called "Hidden quantum nonlocality revealed by local filters" [67]. However, this local filter is essentially a positive operator valued measures and needs to be performed with the help of an additional Hilbert space, such as an additional ancillary qubit. The positive operator valued measures measurement will increase the experimental difficulty because it requires additional coupling to a new quantum system and measuring it. Certainly, if one does not consider the difficulty of measurement, then one may perform two types of entanglement amplification. The first type of amplification can be achieved by using the local filter method, and the second type of amplification is achieved by the way we propose in the paper.

From a fundamental perspective, our work combines weak value theory and quantum correlation theory for the first time. We show that weak-valued amplification in the two-setting protocol is impossible to be simulated classically. Our results also support the fact that weak values are quantum, whereas in the past it was controversial whether weak values were quantum or not [68–72].

Our scheme is applicable to any weakly entangled pure state, while allowing for the presence of partial decoherence and noise. It be expected to significantly reduce the requirements for experiments, allowing for the test of gravitationally induced entanglement in the near future. From a more general point of view, our result is a general weak entanglement criterion. We reveal how the hidden weakly entangled information is again presented as it is. Compared to the previous protocols, our approach allows us to observe entangled signals that were previously impossible to observe. As an outlook, we expect that the criterion can be extended to the more general mixed states, which may make it more possible to detect the entanglement of macroscopic objects.

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APPENDIX A: SEPARABLE STATE MODEL FOR CLASSICAL MEDIATOR

1. Separable state model

Here we analyze the model of separable states for $\mathcal{Q}_{\mathcal{A}}$ and $\mathcal{Q}_{\mathcal{B}}$ (induced by a classical mediator) that can simulate the results of quantum entanglement. As we mentioned before, the quantum states generated by classical mediator is $\varrho_{AB} = \Lambda^C(\varrho_A \otimes \varrho_B) = \sum_i p_i \varrho_A^i \otimes \varrho_B^i$, we have

$$\tilde{\varrho}_B^{\langle 0|_A}(C) = \sum_i p_i \mathrm{Tr}(|0\rangle\langle 0|\varrho_A^i)\varrho_B^i,$$
$$\tilde{\varrho}_B^{\langle 1|_A}(C) = \sum_i p_i \mathrm{Tr}(|1\rangle\langle 1|\varrho_A^i)\varrho_B^i,$$
$$\tilde{\varrho}_B^{\langle \epsilon|_A}(C) = \sum_i p_i \mathrm{Tr}(|\epsilon\rangle\langle \epsilon|\varrho_A^i)\varrho_B^i,$$
$$\tilde{\varrho}_B^{\langle \epsilon^\perp|_A}(C) = \sum_i p_i \mathrm{Tr}(|\epsilon^\perp\rangle\langle \epsilon^\perp|\varrho_A^i)\varrho_B^i, \tag{A1}$$

where $\tilde{\varrho}_B(C)$ is the unnormalized quantum state with the classical mediator. If a classical mediator can simulate all the results of a quantum mediator, then it must satisfy $\tilde{\varrho}_B^{\langle 0|}(Q) = \tilde{\varrho}_B^{\langle 0|}(C)$, $\tilde{\varrho}_B^{\langle 1|}(Q) = \tilde{\varrho}_B^{\langle 1|}(C)$, $\tilde{\varrho}_B^{\langle \epsilon|}(Q) = \tilde{\varrho}_B^{\langle \epsilon|}(C)$, and $\tilde{\varrho}_B^{\langle \epsilon^\perp|}(Q) = \tilde{\varrho}_B^{\langle \epsilon^\perp|}(C)$. So we have

$$
\begin{aligned}
&p^{(0,\chi_{\phi_+})}|0\rangle\langle 0|_A\otimes|\phi_+\rangle\langle\phi_+|_B\\
&=\sum_i p_i\mathrm{Tr}(|0\rangle\langle 0|\varrho_A^i)\mathrm{Tr}(|\phi_+\rangle\langle\phi_+|\varrho_B^i)\varrho_A^i\otimes\varrho_B^i,\\
&p^{(1,\chi_{\phi_-})}|1\rangle\langle 1|_A\otimes|\phi_-\rangle\langle\phi_-|_B\\
&=\sum_i p_i\mathrm{Tr}(|1\rangle\langle 1|\varrho_A^i)\mathrm{Tr}(|\phi_-\rangle\langle\phi_-|\varrho_B^i)\varrho_A^i\otimes\varrho_B^i,\\
&p^{(\epsilon,\chi_\epsilon)}|\epsilon\rangle\langle\epsilon|_A\otimes|\chi_\epsilon\rangle\langle\chi_\epsilon|_B\\
&=\sum_i p_i\mathrm{Tr}(|\epsilon\rangle\langle\epsilon|\varrho_A^i)\mathrm{Tr}(|\chi_\epsilon\rangle\langle\chi_\epsilon|\varrho_B^i)\varrho_A^i\otimes\varrho_B^i,\\
&p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}|\epsilon^\perp\rangle\langle\epsilon^\perp|_A\otimes|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|_B\\
&=\sum_i p_i\mathrm{Tr}(|\epsilon^\perp\rangle\langle\epsilon^\perp|\varrho_A^i)\mathrm{Tr}(|\chi_\epsilon\rangle\langle\chi_{\epsilon^\perp}|\varrho_B^i)\varrho_A^i\otimes\varrho_B^i, \qquad \text{(A2)}
\end{aligned}
$$

where

$$
\begin{aligned}
&\{p^{(0,\phi_+)},p^{(1,\phi_-)},p^{(\epsilon,\chi_\epsilon)},p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}\}\\
&=\left\{\frac{1}{2},\frac{1}{2},\alpha^2\mathrm{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|),\beta^2\mathrm{Tr}(|\tilde{\chi}_{\epsilon^\perp}\rangle\langle\tilde{\chi}_{\epsilon^\perp}|)\right\}. \qquad \text{(A3)}
\end{aligned}
$$

It is well known that a pure state cannot be obtained by a convex sum of other different states, namely, a density matrix of pure state can only be expanded by itself. Let us look at Eq. (A2), because the left-hand side is proportional to a pure state; without loss of generality, one has

$$
\begin{aligned}
p^{(0,\phi_+)}&=p_j\mathrm{Tr}(|0\rangle\langle 0|\varrho_A^j)\mathrm{Tr}(|\phi_+\rangle\langle\phi_+|\varrho_B^j)\quad\text{and}\quad\varrho_A^j\otimes\varrho_B^j=|0\rangle\langle 0|_A\otimes|\phi_+\rangle\langle\phi_+|_B,\\
p^{(1,\phi_-)}&=p_k\mathrm{Tr}(|1\rangle\langle 1|\varrho_A^k)\mathrm{Tr}(|\phi_-\rangle\langle\phi_-|\varrho_B^k)\quad\text{and}\quad\varrho_A^k\otimes\varrho_B^k=|1\rangle\langle 1|_A\otimes|\phi_-\rangle\langle\phi_-|_B,\\
p^{(\epsilon,\chi_\epsilon)}&=p_m\mathrm{Tr}(|\epsilon\rangle\langle\epsilon|\varrho_A^m)\mathrm{Tr}(|\chi_\epsilon\rangle\langle\chi_\epsilon|\varrho_B^m)\quad\text{and}\quad\varrho_A^m\otimes\varrho_B^m=|\epsilon\rangle\langle\epsilon|_A\otimes|\chi_\epsilon\rangle\langle\chi_\epsilon|_B,\\
p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}&=p_n\mathrm{Tr}(|\epsilon^\perp\rangle\langle\epsilon^\perp|\varrho_A^n)\mathrm{Tr}(|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|\varrho_B^n)\quad\text{and}\quad\varrho_A^n\otimes\varrho_B^n=|\epsilon^\perp\rangle\langle\epsilon^\perp|_A\otimes|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|_B. \qquad \text{(A4)}
\end{aligned}
$$

Naturally, one may consider mixing these four pure states with corresponding probabilities to construct a separable model to simulate the prediction of quantum mediator. Since two bases $a=\{0,1\}$ and $b=\{\epsilon,\epsilon^\perp\}$ are randomly selected (with probability $\frac{1}{2}$). Therefore, the separable state induced by classical mediator can be written as

$$
\begin{aligned}
\varrho_{AB}^C=\Lambda^C(\varrho_A\otimes\varrho_B)&=\frac{1}{2}[p^{(0,\phi_+)}|0\rangle\langle 0|_A\otimes|\phi_+\rangle\langle\phi_+|_B+p^{(1,\phi_-)}|1\rangle\langle 1|_A\otimes|\phi_-\rangle\langle\phi_-|_B]\\
&+\frac{1}{2}[p^{(\epsilon,\chi_\epsilon)}|\epsilon\rangle\langle\epsilon|_A\otimes|\chi_\epsilon\rangle\langle\chi_\epsilon|_B+p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}|\epsilon^\perp\rangle\langle\epsilon^\perp|_A\otimes|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|_B]. \qquad \text{(A5)}
\end{aligned}
$$

It should be noted that if our test only uses a single basis $b=\{\epsilon,\epsilon^\perp\}$, then it is easy to find a separable state to simulate the results of quantum mediator (one can verify it). Similar to quantum steering scenario, two or more than two basis are considered, in theory, there is no classical quantum mediator can simulate it (probability distribution). However, ideal projective measurements cannot be implemented in experiments since they need infinite resource costs [61]. That is, the measurement device has limited resolution. One can verify that when the entanglement is extremely weak (i.e., $\frac{\Delta\phi\tau}{2\hbar}$ is very small), the probability $p^{(\epsilon,\chi_\epsilon)}=\alpha^2\mathrm{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|)=\alpha^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]$ will be very small so that the measurement device with resolution $\eta\geq p^{(\epsilon,\chi_\epsilon)}/(p^{(\epsilon,\chi_\epsilon)}+p^{(\epsilon^\perp,\chi_{\epsilon^\perp})})$ cannot distinguish whether $\mathcal{Q}_\mathcal{A}$ and $\mathcal{Q}_\mathcal{B}$ are entangled or separable or not.

Example.—Suppose $\frac{\Delta\phi\tau}{2\hbar}$ is very small and A_w^ϵ is very large, we approximate $\cos(\frac{\Delta\phi\tau}{2\hbar})\approx 1$ and $\sin(\frac{\Delta\phi\tau}{2\hbar})\approx\frac{\Delta\phi\tau}{2\hbar}$. So we have $|\tilde{\chi}_{\epsilon^\perp}\rangle\approx|+\rangle$ and $|\tilde{\chi}_\epsilon\rangle=|+\rangle+i\frac{\Delta\phi\tau}{2\hbar}A_w^\epsilon|-\rangle$. Here we set $\frac{\Delta\phi\tau}{2\hbar}A_w^\epsilon=1$, the quantum state $|\tilde{\chi}_\epsilon\rangle$ becomes $|+\rangle+i|-\rangle$. So we have

$$\{p^{(0,\phi_+)},p^{(1,\phi_-)},p^{(\epsilon,\chi_\epsilon)},p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}\}=\left\{\frac{1}{2},\frac{1}{2},2\alpha^2,\beta^2\right\}. \qquad \text{(A6)}$$

Recall that $\alpha=\langle\epsilon|+\rangle$, $\beta=\langle\epsilon^\perp|+\rangle$, $|\epsilon\rangle=\epsilon|0\rangle-\sqrt{1-\epsilon^2}|1\rangle$, and $|\epsilon^\perp\rangle=\sqrt{1-\epsilon^2}|0\rangle+\epsilon|1\rangle$. If we want to get a big weak value $A_w^\epsilon=\frac{\langle\epsilon|Z|+\rangle}{\langle\epsilon|+\rangle}=\frac{\epsilon+\sqrt{1-\epsilon^2}}{\epsilon-\sqrt{1-\epsilon^2}}$, then ϵ should be close to $\frac{1}{\sqrt{2}}$. So $\alpha\approx\frac{\sqrt{2}}{A_w^\epsilon}=\sqrt{2}\frac{\Delta\phi\tau}{2\hbar}$ and $\beta\approx 1$. Therefore, Eq. (A6) becomes

$$\{p^{(0,+)},p^{(1,+)},p^{(-,\chi_\epsilon)},p^{(+,+)}\}=\left\{\frac{1}{2},\frac{1}{2},4\left(\frac{\Delta\phi\tau}{2\hbar}\right)^2,1\right\}. \qquad \text{(A7)}$$

It is easy to verify that the above results can be simulated by a separable state $|+\rangle_A\otimes|+\rangle_B$ (a more accurate model should be in the form of ϱ_{AB}^C) if $4(\frac{\Delta\phi\tau}{2\hbar})^2$ is small, which may be masked by the noise of measurement device.

2. Visibility of measurement for a separable model

Before we calculate the visibility of measurement, we need to find the reduced density matrix $\varrho_B^{\langle 0|_A}$, $\varrho_B^{\langle 1|_A}$, $\varrho_B^{\langle \epsilon|_A}$, and $\varrho_B^{\langle \epsilon^\perp|_A}$ for ϱ_{AB}^C, which is as follows:

$$\varrho_B^{\langle 0|_A} = \frac{\mathrm{Tr}_A(|0\rangle\langle 0|_A \otimes I_B \varrho_{AB}^C)}{\mathrm{Tr}(|0\rangle\langle 0|_A \otimes I_B \varrho_{AB}^C)} = \frac{\frac{1}{2}p^{(0,\phi_+)}|\phi_+\rangle\langle\phi_+| + \frac{1}{2}\epsilon^2 p^{(\epsilon,\chi_\epsilon)}|\chi_\epsilon\rangle\langle\chi_\epsilon| + \frac{1}{2}(1-\epsilon^2)p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|}{\frac{1}{2}p^{(0,\phi_+)} + \frac{1}{2}\epsilon^2 p^{(\epsilon,\chi_\epsilon)} + \frac{1}{2}(1-\epsilon^2)p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}},$$

$$\varrho_B^{\langle 1|_A} = \frac{\mathrm{Tr}_A(|1\rangle\langle 1|_A \otimes I_B \varrho_{AB}^C)}{\mathrm{Tr}(|1\rangle\langle 1|_A \otimes I_B \varrho_{AB}^C)} = \frac{\frac{1}{2}p^{(1,\phi_-)}|\phi_-\rangle\langle\phi_-| + \frac{1}{2}(1-\epsilon^2) p^{(\epsilon,\chi_\epsilon)}|\chi_\epsilon\rangle\langle\chi_\epsilon| + \frac{1}{2}\epsilon^2 p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|}{\frac{1}{2}p^{(1,\phi_-)} + \frac{1}{2}(1-\epsilon^2) p^{(\epsilon,\chi_\epsilon)} + \frac{1}{2}\epsilon^2 p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}},$$

$$\varrho_B^{\langle \epsilon|_A} = \frac{\mathrm{Tr}_A(|\epsilon\rangle\langle \epsilon|_A \otimes I_B \varrho_{AB}^C)}{\mathrm{Tr}(|\epsilon\rangle\langle \epsilon|_A \otimes I_B \varrho_{AB}^C)} = \frac{\frac{1}{2}\epsilon^2 p^{(0,\phi_+)}|\phi_+\rangle\langle\phi_+| + \frac{1}{2}(1-\epsilon^2) p^{(1,\phi_-)}|\phi_-\rangle\langle\phi_-| + \frac{1}{2} p^{(\epsilon,\chi_\epsilon)}|\chi_\epsilon\rangle\langle\chi_\epsilon|}{\frac{1}{2}\epsilon^2 p^{(0,\phi_+)} + \frac{1}{2}(1-\epsilon^2) p^{(1,\phi_-)} + \frac{1}{2} p^{(\epsilon,\chi_\epsilon)}},$$

$$\varrho_B^{\langle \epsilon^\perp|_A} = \frac{\mathrm{Tr}_A(|\epsilon^\perp\rangle\langle \epsilon^\perp|_A \otimes I_B \varrho_{AB}^C)}{\mathrm{Tr}(|\epsilon^\perp\rangle\langle \epsilon^\perp|_A \otimes I_B \varrho_{AB}^C)} = \frac{\frac{1}{2}(1-\epsilon^2) p^{(0,\phi_+)}|\phi_+\rangle\langle\phi_+| + \frac{1}{2}\epsilon^2 p^{(1,\phi_-)}|\phi_-\rangle\langle\phi_-| + \frac{1}{2} p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|}{\frac{1}{2}(1-\epsilon^2) p^{(0,\phi_+)} + \frac{1}{2}\epsilon^2 p^{(\epsilon,\chi_\epsilon)} + \frac{1}{2} p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}}. \tag{A8}$$

Now, substituting them into the Eq. (6), we get the visibility of measurement of ϱ_{AB}^C:

$$V_{\Pi_0}^C(\varrho_B^{\langle 0|_A}) = \mathrm{Tr}(|\phi_+\rangle\langle\phi_+|\varrho_B^{\langle 0|_A}) = \frac{\frac{1}{2}p^{(0,\phi_+)} + \frac{1}{2}\epsilon^2 p^{(\epsilon,\chi_\epsilon)}\langle\phi_+|\chi_\epsilon\rangle\langle\chi_\epsilon|\phi_+\rangle + \frac{1}{2}(1-\epsilon^2)p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}\langle\phi_+|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|\phi_+\rangle}{\frac{1}{2}p^{(0,\phi_+)} + \frac{1}{2}\epsilon^2 p^{(\epsilon,\chi_\epsilon)} + \frac{1}{2}(1-\epsilon^2)p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}},$$

$$V_{\Pi_1}^C(\varrho_B^{\langle 1|_A}) = \mathrm{Tr}(|\phi_-\rangle\langle\phi_-|\varrho_B^{\langle 1_A}) = \frac{\frac{1}{2}p^{(1,\phi_-)} + \frac{1}{2}(1-\epsilon^2) p^{(\epsilon,\chi_\epsilon)}\langle\phi_-|\chi_\epsilon\rangle\langle\chi_\epsilon|\phi_-\rangle + \frac{1}{2}\epsilon^2 p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}\langle\phi_-|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|\phi_-\rangle}{\frac{1}{2}p^{(1,\phi_-)} + \frac{1}{2}(1-\epsilon^2) p^{(\epsilon,\chi_\epsilon)} + \frac{1}{2}\epsilon^2 p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}},$$

$$V_{\Pi_2}^C(\varrho_B^{\langle \epsilon|_A}) = \mathrm{Tr}(|\chi_\epsilon\rangle\langle\chi_\epsilon|\varrho_B^{\langle \epsilon|_A}) = \frac{\frac{1}{2}\epsilon^2 p^{(0,\phi_+)}\langle\chi_\epsilon|\phi_+\rangle\langle\phi_+|\chi_\epsilon\rangle + \frac{1}{2}(1-\epsilon^2)p^{(1,\phi_-)}\langle\chi_\epsilon|\phi_-\rangle\langle\phi_-|\chi_\epsilon\rangle + \frac{1}{2}p^{(\epsilon,\chi_\epsilon)}}{\frac{1}{2}\epsilon^2 p^{(0,\phi_+)} + \frac{1}{2}(1-\epsilon^2)p^{(1,\phi_-)} + \frac{1}{2}p^{(\epsilon,\chi_\epsilon)}},$$

$$V_{\Pi_3}^C(\varrho_B^{\langle \epsilon^\perp|_A}) = \mathrm{Tr}(|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|\varrho_B^{\langle \epsilon^\perp|_A}) = \frac{\frac{1}{2}(1-\epsilon^2) p^{(0,\phi_+)}\langle\chi_{\epsilon^\perp}|\phi_+\rangle\langle\phi_+|\chi_{\epsilon^\perp}\rangle + \frac{1}{2}\epsilon^2 p^{(1,\phi_-)}\langle\chi_{\epsilon^\perp}|\phi_-\rangle\langle\phi_-|\chi_{\epsilon^\perp}\rangle + \frac{1}{2}p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}}{\frac{1}{2}(1-\epsilon^2) p^{(0,\phi_+)} + \frac{1}{2}\epsilon^2 p^{(\epsilon,\chi_\epsilon)} + \frac{1}{2}p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}}, \tag{A9}$$

where $|\phi_\pm\rangle = \cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle \pm i\sin(\frac{\Delta\phi\tau}{2\hbar})|-\rangle$, $|\chi_\epsilon\rangle = \frac{\cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle_B + i\sin(\frac{\Delta\phi\tau}{2\hbar})A_w^\epsilon|-\rangle}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]^{\frac{1}{2}}}$, and $|\chi_{\epsilon^\perp}\rangle = \frac{\cos(\frac{\Delta\phi\tau}{2\hbar})|+\rangle + i\sin(\frac{\Delta\phi\tau}{2\hbar})A_w^{\epsilon^\perp}|-\rangle}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]^{\frac{1}{2}}}$.

Recall that $\{p^{(0,\phi_+)}, p^{(1,\phi_-)}, p^{(\epsilon,\chi_\epsilon)}, p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}\} = \{\frac{1}{2},\frac{1}{2},\alpha^2\mathrm{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|),\beta^2\mathrm{Tr}(|\tilde{\chi}_{\epsilon^\perp}\rangle\langle\tilde{\chi}_{\epsilon^\perp}|)\}$, $\mathrm{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|) = \cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2$, $\mathrm{Tr}(|\tilde{\chi}_{\epsilon^\perp}\rangle\langle\tilde{\chi}_{\epsilon^\perp}|) = \cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2$, $|\langle\phi_+|\chi_\epsilon\rangle|^2 = \frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})A_w^\epsilon|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]}$, $|\langle\phi_+|\chi_{\epsilon^\perp}\rangle|^2 = \frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})A_w^{\epsilon^\perp}|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]}$, $|\langle\phi_-|\chi_\epsilon\rangle|^2 = \frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar})-\sin^2(\frac{\Delta\phi\tau}{2\hbar})A_w^\epsilon|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]}$, $|\langle\phi_-|\chi_{\epsilon^\perp}\rangle|^2 = \frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar})-\sin^2(\frac{\Delta\phi\tau}{2\hbar})A_w^{\epsilon^\perp}|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]}$, one has

$$V_{\Pi_0}^C(\varrho_B^{\langle 0|_A}) = \frac{\frac{1}{4} + \frac{1}{2}\epsilon^2\alpha^2|\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})A_w^\epsilon|^2 + \frac{1}{2}(1-\epsilon^2)\beta^2|\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})A_w^{\epsilon^\perp}|^2}{\frac{1}{4} + \frac{1}{2}\epsilon^2\alpha^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2] + \frac{1}{2}(1-\epsilon^2)\beta^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]},$$

$$V_{\Pi_1}^C(\varrho_B^{\langle 1|_A}) = \frac{\frac{1}{4} + \frac{1}{2}(1-\epsilon^2)\alpha^2|\cos^2(\frac{\Delta\phi\tau}{2\hbar}) - \sin^2(\frac{\Delta\phi\tau}{2\hbar})A_w^\epsilon|^2 + \frac{1}{2}\epsilon^2\beta^2|\cos^2(\frac{\Delta\phi\tau}{2\hbar}) - \sin^2(\frac{\Delta\phi\tau}{2\hbar})A_w^{\epsilon^\perp}|^2}{\frac{1}{4} + \frac{1}{2}(1-\epsilon^2)\alpha^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2] + \frac{1}{2}\epsilon^2\beta^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]},$$

$$V_{\Pi_2}^C(\varrho_B^{\langle \epsilon|_A}) = \frac{\frac{1}{4}\epsilon^2\frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})A_w^\epsilon|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]} + \frac{1}{4}(1-\epsilon^2)\frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar})-\sin^2(\frac{\Delta\phi\tau}{2\hbar})A_w^\epsilon|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]} + \frac{1}{2}\alpha^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]}{\frac{1}{4}\epsilon^2 + \frac{1}{4}(1-\epsilon^2) + \frac{1}{2}\alpha^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]},$$

$$V_{\Pi_3}^C(\varrho_B^{\langle \epsilon^\perp|_A}) = \frac{\frac{1}{4}(1-\epsilon^2)\frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})A_w^{\epsilon^\perp}|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]} + \frac{1}{4}\epsilon^2\frac{|\cos^2(\frac{\Delta\phi\tau}{2\hbar})-\sin^2(\frac{\Delta\phi\tau}{2\hbar})A_w^{\epsilon^\perp}|^2}{[\cos^2(\frac{\Delta\phi\tau}{2\hbar})+\sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]} + \frac{1}{2}\beta^2\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2}{\frac{1}{4}(1-\epsilon^2) + \frac{1}{4}\epsilon^2 + \frac{1}{2}\beta^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^{\epsilon^\perp}|^2]}. \tag{A10}$$

3. Examples

When the entanglement is extremely weak, $\cos(\frac{\Delta\phi\tau}{2\hbar}) \approx 1$, $\sin(\frac{\Delta\phi\tau}{2\hbar}) = \frac{\Delta\phi\tau}{2\hbar} \sim 0$. The measurement basis $\{|\epsilon\rangle, |\epsilon^\perp\rangle\}$ can realize weak-value amplification (i.e., $A_w^\epsilon \approx \frac{1}{\frac{\Delta\phi\tau}{2\hbar}}$ and $A_w^{\epsilon^\perp} = \frac{1}{A_w^\epsilon} \approx \frac{\Delta\phi\tau}{2\hbar} \ll 1$) when $\epsilon \to \frac{1}{\sqrt{2}}$ and we have $\alpha = \langle\epsilon|+\rangle \approx 0, \beta = \langle\epsilon^\perp|+\rangle \approx 1$. Upon substituting these approximations into Eq. (A10) [discard the second-order small quantity $|\alpha|^2$, $(\frac{\Delta\phi\tau}{2\hbar})^2$ and set $1 \pm (\frac{\Delta\phi\tau}{2\hbar})^2 \approx 1$], we obtain

$$
\begin{aligned}
V^C_{\Pi_0}(\varrho_B^{\langle 0|_A}) &\approx 1, \\
V^C_{\Pi_1}(\varrho_B^{\langle 1|_A}) &\approx 1, \\
V^C_{\Pi_2}(\varrho_B^{\langle \epsilon|_A}) &\approx \frac{1}{2}, \\
V^C_{\Pi_3}(\varrho_B^{\langle \epsilon^\perp|_A}) &\approx 1.
\end{aligned} \tag{A11}
$$

One can see that this result is contradictory to the results analyzed in our main text. The measured visibility of weak entanglement is all equal to 1; however, the separable model has a $\frac{1}{2}$. This is a logical contradiction.

The more general case is that we set $A_w^\epsilon = k\frac{1}{\frac{\Delta\phi\tau}{2\hbar}}$, where is the coefficient. We also assure $\cos(\frac{\Delta\phi\tau}{2\hbar}) \approx 1$, $\sin(\frac{\Delta\phi\tau}{2\hbar}) = \frac{\Delta\phi\tau}{2\hbar}$, $A_w^{\epsilon^\perp} = \frac{1}{A_w^\epsilon} \approx \frac{\Delta\phi\tau}{2k\hbar} \ll 1$, and $\alpha k \ll 1$. One has

$$
\begin{aligned}
V^C_{\Pi_0}(\varrho_B^{\langle 0|_A}) &\approx 1, \\
V^C_{\Pi_1}(\varrho_B^{\langle 1|_A}) &\approx 1, \\
V^C_{\Pi_2}(\varrho_B^{\langle \epsilon|_A}) &\approx \frac{1}{1+k^2}, \\
V^C_{\Pi_3}(\varrho_B^{\langle \epsilon^\perp|_A}) &\approx 1.
\end{aligned} \tag{A12}
$$

APPENDIX B: EXPERIMENTAL CONSIDERATION

We have shown that steering scenarios with weak value amplification are not possible to be simulated classically. In the following, we focus on the analysis of the steered state for weak-value amplification, so as to consider the feasibility of the experiment for detecting weak, gravitationally induced entanglement. Let us set $m_1 = m_2 = m$ and suppose that the coupling strength of gravity is weak. After the weak gravity interaction and projective measurement ($\langle\epsilon|$) on $\mathcal{Q}_\mathcal{A}$, the system $\mathcal{Q}_\mathcal{B}$ becomes $\varrho_B^{\langle\epsilon|_A}(Q) = |\chi_\epsilon\rangle\langle\chi_\epsilon|_B$ with probability $|\alpha|^2[\cos^2(\frac{\Delta\phi\tau}{2\hbar}) + \sin^2(\frac{\Delta\phi\tau}{2\hbar})|A_w^\epsilon|^2]$.

One can see that there is a weak value, which is $A_w^\epsilon = \frac{\langle\epsilon|Z|+\rangle}{\langle\epsilon|+\rangle} = \frac{\epsilon+\sqrt{1-\epsilon^2}}{\epsilon-\sqrt{1-\epsilon^2}}$. If the coupling strength of interaction of gravity is 0, then $\Delta\phi = 0$. In this case, the final state of system $\mathcal{Q}_\mathcal{B}$ is $|+\rangle$, which does not carry any weak-value information. That is, there is no entanglement between these two quantum system. On the contrary, when the coupling strength is not 0 but small; that is, there is a gravitationally induced phase, and the final state of system $\mathcal{Q}_\mathcal{B}$ becomes $|\chi_\epsilon\rangle \approx \frac{1}{[1+|\frac{\Delta\phi}{2\hbar}A_w^\epsilon|^2]^{\frac{1}{2}}}(|+\rangle + i\frac{\Delta\phi}{2\hbar}A_w^\epsilon|-\rangle)$. One can see that the quantum state of $\mathcal{Q}_B$ depends on the weak value A_w^ϵ. Suppose that we measure an observable $\hat{\Pi}_2 = |\chi_\epsilon\rangle\langle\chi_\epsilon|$ on the system $\mathcal{Q}_B$. The observable of the displacement of Π_2 is the expectation value of the final state minus the expectation value of the initial state ($|+\rangle$) of $\mathcal{Q}_B$, and we get

$$
\begin{aligned}
\langle\Delta\hat{\Pi}_2\rangle_Q &= \mathrm{Tr}(\Pi_2\varrho_B^{\langle\epsilon|_A}) - \mathrm{Tr}(\Pi_2|+\rangle\langle+|), \\
&= V^Q_{\Pi_2}(\varrho_B^{\langle\epsilon|_A}) - \frac{1}{[1+|\frac{\Delta\phi}{2\hbar}A_w^\epsilon|^2]}, \\
&= 1 - \frac{1}{[1+|\frac{\Delta\phi}{2\hbar}A_w^\epsilon|^2]}.
\end{aligned} \tag{B1}
$$

One can see that if $\frac{\Delta\phi}{2\hbar}A_w^\epsilon = 1$, $\langle\Delta\hat{\Pi}_2\rangle_Q = \frac{1}{2}$. Even though the heralded probability $2|\alpha|^2 = 1–2\epsilon\sqrt{1-\epsilon^2}$ is small, after many runs of experiment, we can still observe a clear shift of the quantum state, which is a signal of entanglement between two system introduced by gravity (average shift will be 0 without entanglement generation).

For example, in the Ref. [11], the shift of quantity about entanglement is $p_1 = \sin^2(\frac{\Delta\phi}{2\hbar}) \approx \frac{\Delta\phi^2}{4\hbar^2}$ in the case of weak coupling, where p_1 is the probabilities for the mass to emerge on path 1 (R). For showing the clear enhancement of our scheme, we give a simple example. If we set $A_w^\epsilon = 10^4$ and $\frac{\Delta\phi^2}{4\hbar^2} = 10^{-4}$, then we have $\langle\Delta\hat{\Pi}_2\rangle_Q = \frac{1}{2}$, while $p_1 \approx 10^{-4}$. That is, we can enhance the sensitivity and resolution for detect the quantum gravity 0.5×10^4 times by using a weak-value amplification scheme. However, in above example, the steered probability becomes $p = 2 \times 10^{-8}$ in weak-value scheme. Fortunately, for existing quantum technologies, the frequency of experiments can reach MHz and beyond (supposing the time of gravitational interaction to be within microseconds). That is we may have 10^6 runs in one second. The total run of experiments is about $p \times 10^6 \times 3600 \times 24 = 864$ each day. This is enough for us to achieve an accurate experimental estimation. Given a resolution of measurement, we achieved $X = 10^4$, saving for the coupling strength of gravity. In other words, we can reduce the mass of two systems by 10 times, and shorten the coupling time by 100 times. This is a very experiment-friendly scheme, which increases the feasibility of testing gravitationally induced entanglement by using existing technology.

APPENDIX C: OBSERVING QUANTUM GRAVITY USING LIMITED RESOLUTION OF THE MEASUREMENT DEVICE

In a von Neumann–type measurement, the pointer is shifted proportional to the eigenvalues of the measured observable

$$|\psi\rangle \otimes |\phi(q)\rangle \to \sum_a \langle a|\psi\rangle \cdot |a\rangle \otimes |\phi(q - g_0 a)\rangle, \quad \text{(C1)}$$

where Ψ and $\phi(q)$ are the initial states of system and probe, respectively, the index a refers to the eigenbasis of the observable, q is the position of the probe, and $g0$ is a coupling constant. The outcome of the measurement is then provided by reading the position of the probe.

In a ideal projective measurement the probe's initial state is narrower than the distance between the eigenvalues, i.e., $\langle\phi(q-a)|\phi(q-a')\rangle = \delta_{aa'}$; hence, reading the probe's position provides full information of the measured physical quantity and collapses the system into the corresponding eigenstate of the observable. However, it has been shown that the ideal projective measurements cannot be implemented in experiments since they need infinite resource costs [61]. Therefore, the resolution of measurement devices are always limited, that is $|\langle\phi(q-a)|\phi(q-a')\rangle|^2 = \gamma \neq 0$. This is the noise come from the measurement process.

In the previous section, we consider two quantum system in initial state $|+\rangle|+\rangle$ interact each other by gravity, and one of them postselected to an almost completely orthogonal state $|\epsilon\rangle = \epsilon|0\rangle - \sqrt{1-\epsilon^2}|1\rangle$, where ϵ is close to $\frac{1}{\sqrt{2}}$. This postselected operation is exactly limited by the resolution of measurement γ. So the minimal overlap $\langle\epsilon|+\rangle = \sqrt{\gamma}$, which determines the upper limit of the weak value, $Max(A_w^\epsilon) \approx \frac{\sqrt{2}}{\sqrt{\gamma}}$.

Now let consider the case that the square of effective coupling strength $\frac{\Delta\phi^2}{4\hbar^2} = \gamma$. That is one cannot measure the entanglement using traditional entanglement witness methods [11,12], since the signal of gravitationally induced entanglement is covered by the noise of measurement device. In this case, a weak-value-based scheme is still work. One can obtain obvious signal of gravitationally induced entanglement $\langle\Delta\hat{\Pi}_2\rangle_Q = \frac{2}{3} \gg \gamma$.

APPENDIX D: OBSERVING GRAVITATIONALLY INDUCED ENTANGLEMENT WITH DECOHERENCE

In fact, the decoherence exists in experiment. The longer time in single run, the more decoherence. Besides, the evolution of desired initial quantum state is not ideal because the system is inevitably coupled with the environment. Let us consider an environment-induced decoherence model (other decoherence models are out of our analysis) for the system $\mathcal{Q}_\mathcal{A}$ and $\mathcal{Q}_\mathcal{B}$. Without a loss of generality, we consider the action of environment as a partially depolarizing channel, which is given as

$$N_E(\varrho) = (1-q)\varrho + q\frac{I}{d}, \quad \text{(D1)}$$

where q corresponds to the degree of a system that has been decohered. Therefore, the final state before measurement has a mathematical form

$$\varrho_{AB} = (1-q)|\Psi\rangle\langle\Psi| + q\frac{I_A \otimes I_B}{4}, \quad \text{(D2)}$$

where $|\Psi\rangle = \alpha|\epsilon\rangle_A \otimes |\tilde{\chi}_\epsilon\rangle_B + \beta|\epsilon^\perp\rangle_A \otimes |\tilde{\chi}_{\epsilon^\perp}\rangle_B$. Therefore, we can get four steered states

$$\begin{aligned}
\tilde{\varrho}_B^{\langle 0|_A}(Q) &= \frac{(1-q)}{2}|\phi_+\rangle\langle\phi_+|_B + \frac{q}{2}I_B, \\
\tilde{\varrho}_B^{\langle 1|_A}(Q) &= \frac{(1-q)}{2}|\phi_-\rangle\langle\phi_-|_B + \frac{q}{2}I_B, \\
\tilde{\varrho}_B^{\langle \epsilon|_A}(Q) &= (1-q)|\alpha|^2\mathrm{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|)|\chi_\epsilon\rangle\langle\chi_\epsilon|_B + \frac{q}{2}I_B, \\
\tilde{\varrho}_B^{\langle \epsilon^\perp|_A}(Q) &= (1-q)|\beta|^2\mathrm{Tr}(|\tilde{\chi}_{\epsilon^\perp}\rangle\langle\tilde{\chi}_{\epsilon^\perp}|)|\chi_{\epsilon^\perp}\rangle\langle\chi_{\epsilon^\perp}|_B + \frac{q}{2}I_B.
\end{aligned} \quad \text{(D3)}$$

Further, we can project quantum states of $\mathcal{Q}_\mathcal{B}$ to $|\phi_+\rangle, |\phi_-\rangle, |\chi_\epsilon\rangle, |\chi_{\epsilon^\perp}\rangle$ with probabilities

$$\begin{aligned}
&\{p^{(0,\phi_+)}, p^{(1,\phi_-)}, p^{(\epsilon,\chi_\epsilon)}, p^{(\epsilon^\perp,\chi_{\epsilon^\perp})}\} \\
&= \left\{\frac{1}{2}-\frac{q}{4}, \frac{1}{2}-\frac{q}{4}, (1-q)|\alpha|^2\mathrm{Tr}(|\tilde{\chi}_\epsilon\rangle\langle\tilde{\chi}_\epsilon|)\right. \\
&\quad \left.+\frac{q}{4}, (1-q)|\beta|^2\mathrm{Tr}(|\tilde{\chi}_{\epsilon^\perp}\rangle\langle\tilde{\chi}_{\epsilon^\perp}|) + \frac{q}{4}\right\},
\end{aligned} \quad \text{(D4)}$$

Further, the visibility of measurement for quantum mediator are

$$\begin{aligned}
V_{\Pi_0}^{\text{noise}}(\varrho_B^{\langle 0|_A}) &= \frac{1}{1+q}, \\
V_{\Pi_1}^{\text{noise}}(\varrho_B^{\langle 1|_A}) &= \frac{1}{1+q}, \\
V_{\Pi_2}^{\text{noise}}(\varrho_B^{\langle \epsilon|_A}) &= 1 - \frac{q}{2\mathrm{Tr}(\tilde{\varrho}_B^{\langle\epsilon|_A})}, \\
V_{\Pi_3}^{\text{noise}}(\varrho_B^{\langle \epsilon^\perp|_A}) &= 1 - \frac{q}{2\mathrm{Tr}(\tilde{\varrho}_B^{\langle,\epsilon^\perp|_A})},
\end{aligned} \quad \text{(D5)}$$

where $i = 0, 1, 2, 3$. Similarly, we can construct a separable state (A5) model for comparison. If q is small, i.e., so that $V_{\Pi_2}^{\text{noise}}(Q) > V_{\Pi_2}^{\text{noise}}(C)$, then we can still obtain the amplified signal of entanglement. Therefore, our weak entanglement criterion is applicable to some kinds of mixed states.

APPENDIX E: ATOMIC INTERFEROMETERS WITH A HARMONIC OSCILLATOR

Here we analyze a possible experimental proposal based on atomic interferometers with a harmonic oscillator [27]. We analyze how to apply our weak entanglement criterion to this scenario. Consider a harmonic oscillator $\mathcal{Q}_\mathcal{A}$ (a mechanical resonator) coupled to a two-state system $\mathcal{Q}_\mathcal{B}$ (an atom trapped in a double-well potential). Since $\mathcal{Q}_\mathcal{B}$ is a qubit state, one can set the position operator of the atom $\mathcal{Q}_\mathcal{B}$ to the Pauli matrix σ_z with the eigenstates $|L\rangle$ and $|R\rangle$, which represent the location of atom occupying, respectively. The gravitationally driven Hamiltonian of these two systems is given as ($\hbar = 1$) [27]

$$H = wa^\dagger a + g(a + a^\dagger)\sigma_z, \tag{E1}$$

where w, $a^\dagger$, and a denote the frequency, creation, and annihilation operators of the harmonic oscillator, respectively. The coupling coefficient g correspond to the gravitational interaction between atom and oscillator, satisfying $g \ll w$. Up to a global phase, the time-evolution operator can be rewritten as

$$U(t) = D^\dagger(\sigma_z\lambda)e^{-iwa^\dagger at}D(\sigma_z\lambda), \tag{E2}$$

where $D(\zeta) \equiv \exp\{\zeta a^\dagger - \zeta^* a\}$ is the usual displacement operator and $\lambda = \frac{g}{w}$. Consider this: the oscillator is initialized in its ground state $|0\rangle_A$ and the atom is in the superposition of $|L\rangle_B$ and $|R\rangle_B$. After the time evolution, the composite quantum state becomes $|\Psi(t)\rangle = U(t)|0\rangle_A \otimes \frac{1}{\sqrt{2}}(|L\rangle_B + |R\rangle_B) = \frac{1}{\sqrt{2}}(|\eta\rangle_A \otimes |L\rangle_B + |-\eta\rangle_A \otimes |R\rangle_B)$, where the evolved states of the oscillator are coherent states $|\pm\eta\rangle_A = D(\pm\lambda(e^{-iwt} - 1))$. If we implement the Hadamard gate to the two-level system $\mathcal{Q}_\mathcal{B}$, then we have

$$|\Psi(t)\rangle = \frac{1}{2c_+}|\mathrm{cat}_{\eta_+}\rangle_A \otimes |L\rangle_B + \frac{1}{2c_-}|\mathrm{cat}_{\eta_-}\rangle_A \otimes |R\rangle_B, \tag{E3}$$

where $|\mathrm{cat}_{\eta_+}\rangle_A = c_+(|\eta\rangle_A + |-\eta\rangle_A)$ and $|\mathrm{cat}_{\eta_-}\rangle_A = c_-(|\eta\rangle_A - |-\eta\rangle_A)$ are the Schrödinger's cat state with $c_+ = \frac{1}{\sqrt{2(1+e^{-2|\eta|^2})}}$ and $c_- = \frac{1}{\sqrt{2(1-e^{-2|\eta|^2})}}$. Since $\langle \mathrm{cat}_{\eta_-}|\mathrm{cat}_{\eta_+}\rangle = 0$, one can address Eq. (E3) as a two-qubit entangled state. However, the entanglement $|\Psi(t)\rangle$ is very weak ($\frac{1}{2c_-}$ is small) due to the fact that $|\eta|$ is very small (determined by gravitational interaction). Similar to Eq. (2), one may choose a suitable basis for $\mathcal{Q}_\mathcal{A}$ ($\mathcal{Q}_\mathcal{B}$) to a new form of $|\Psi(t)\rangle$, which has the amplified signal. For example, let us expand $\mathcal{Q}_\mathcal{A}$ to the basis $\{|v\rangle = \sin(\theta)|\mathrm{cat}_{\eta_+}\rangle + \cos(\theta)|\mathrm{cat}_{\eta_-}\rangle, |v^\perp\rangle = \cos(\theta)|\mathrm{cat}_{\eta_+}\rangle - \sin(\theta)|\mathrm{cat}_{\eta_+}\rangle\}$ with $\theta \ll 1$, and one has

$$|\Psi(t)\rangle = |v\rangle_A \otimes \left(\frac{\langle v|\mathrm{cat}_{\eta_+}\rangle}{2c_+}|L\rangle_B + \frac{\langle v|\mathrm{cat}_{\eta_-}\rangle}{2c_-}|R\rangle_B\right) + |v^\perp\rangle_A \otimes \left(\frac{\langle v^\perp|\mathrm{cat}_{\eta_+}\rangle}{2c_+}|L\rangle_B + \frac{\langle v^\perp|\mathrm{cat}_{\eta_-}\rangle}{2c_-}|R\rangle_B\right). \tag{E4}$$

One can see that compared to the components of $|L\rangle_B$, the component of $|R\rangle_B$ is enlarged when $\mathcal{Q}_\mathcal{A}$ is projected to $|v\rangle_A$. If θ is small enough, then one may has $\frac{\langle v|L\rangle}{2c_+} \sim \frac{\langle v|R\rangle}{2c_-}$ such that the component of $|R\rangle_B$ in the steered quantum state of $\mathcal{Q}_\mathcal{B}$ is boosted. According to weak entanglement criteria we proposed, this basis is the most significant ingredient to amplify the entangled signal. Similarly, another measurement basis needs to be selected, which may be $\{|+\rangle = \frac{1}{\sqrt{2}}|\mathrm{cat}_{\eta_+}\rangle + \frac{1}{\sqrt{2}}|\mathrm{cat}_{\eta_-}\rangle, |-\rangle = \frac{1}{\sqrt{2}}|\mathrm{cat}_{\eta_+}\rangle - \frac{1}{\sqrt{2}}|\mathrm{cat}_{\eta_-}\rangle\}$. A random selection of these two measurement bases yields the probability distribution and measurement visibility of system $\mathcal{Q}_\mathcal{B}$, which allows us to witness the gravitationally induced entanglement. In real scenarios, the resonator $\mathcal{Q}_\mathcal{B}$ may be a thermal state close to the ground state. We analyze this case in the following.

In above analysis, we assume that the oscillator is initialized in its ground state $|0\rangle$. In a realistic implementation, due to the finite temperature (may be nK), the oscillator instead starts in a mixed state, such as a thermal state, donated as $\varrho_{th} = \int d^2\zeta \frac{1}{\pi\bar{n}} e^{-|\zeta|^2/\bar{n}}|\zeta\rangle\langle\zeta|$. In this case, the evolving state of the two systems becomes

$$\varrho_{AB} = \int d^2\zeta \frac{1}{\pi\bar{n}} e^{-|\zeta|^2/\bar{n}}|\Psi_\zeta\rangle\langle\Psi_\zeta|, \tag{E5}$$

where $|\Psi_\zeta\rangle = \frac{1}{2}(|\zeta+\eta\rangle_A + |\zeta-\eta\rangle_A) \otimes |L\rangle_B + \frac{1}{2}(|\zeta+\eta\rangle_A - |\zeta-\eta\rangle_A) \otimes |R\rangle_B$. Obviously, if $\zeta = 0$, then $|\Psi_\zeta\rangle$ reduces to Eq. (E3). Here we will be concerned only with the projective measurements $\langle v|_A$ (corresponds to amplified entanglement), since the measurements of the other bases are trivial. The conditional state of the atom becomes

$$\varrho_B^{\langle v|_A} = \frac{\mathrm{Tr}_A(|v\rangle\langle v|_A \otimes I_B\varrho_{AB})}{\mathrm{Tr}(|v\rangle\langle v|_A \otimes I_B\varrho_{AB})}, \tag{E6}$$

leading the measurement visibility

$$V_{\Pi_\mu}(\varrho_B^{\langle v|_A}) = \mathrm{Tr}(\Pi_\mu\varrho_B^{\langle v|_A}), \tag{E7}$$

where $\Pi_\mu = |\mu\rangle\langle\mu|$ and $|\mu\rangle = (\frac{\langle v|\mathrm{cat}_{\eta_+}\rangle}{2c_+}|L\rangle_B + \frac{\langle v|\mathrm{cat}_{\eta_-}\rangle}{2c_-}|R\rangle_B)/\sqrt{|\frac{\langle v|\mathrm{cat}_{\eta_+}\rangle}{2c_+}|^2 + |\frac{\langle v|\mathrm{cat}_{\eta_-}\rangle}{2c_-}|^2}$. When the thermal state is very close to the ground state, one may has $\frac{V_{\Pi_\mu}(\varrho_B^{\langle v|_A}) - V^C_{\Pi_\mu}(\varrho_B^{\langle v|_A})}{\gamma} = k > 1$, where γ is measurement sensitivity. Therefore we may still achieve k-fold magnification compare to the usual ones.

[1] B. De Witt, *A Global Approach to Quantum Field Theory*, International Series of Monographs on Physics (Clarendon Press, Oxford, 2014).

[2] C. M. DeWitt, *Conference on the Role of Gravitation in Physics at the University of North Carolina, Chapel Hill, 1957* (Wright Air Development Center, Air Research and Development Command, United States Air Force, Wright Patterson Air Force Base, Ohio, 1957) (WADC Technical Report No. 57-216, 1957).

[3] A. Peres and D. Terno, Hybrid classical-quantum dynamics, Phys. Rev. A **63**, 022101 (2001).

[4] K. Eppley and E. Hannah, The necessity of quantizing the gravitational field, Found. Phys. **7**, 51 (1977).

[5] C. Marletto and V. Vedral, Why we need to quantise everything, including gravity, npj Quantum Inf. **3**, 29 (2017).

[6] C. M. DeWitt and D. Rickles, *The Role of Gravitation in Physics: Report from the 1957 Chapel Hill Conference*, Vol. 5 (Edition Open Sources, 2011).

[7] G. Gorelik, Matvei Bronstein and quantum gravity: 70th anniversary of the unsolved problem, Phys. Usp. **48**, 1039 (2005).

[8] C. Anastopoulos and B.-L. Hu, Probing a gravitational cat state, Classical Quantum Gravity **32**, 165022 (2015).

[9] M. Derakhshani, C. Anastopoulos, and B.-L. Hu, Probing a gravitational cat state: Experimental Possibilities, J. Phys. Conf. Ser. **701**, 012015 (2016).

[10] C. Marletto and V. Vedral, Witness gravity's quantum side in the lab, Nature (London) **547**, 156 (2017).

[11] C. Marletto and V. Vedral, Gravitationally Induced Entanglement between Two Massive Particles is Sufficient Evidence of Quantum Effects in Gravity, Phys. Rev. Lett. **119**, 240402 (2017).

[12] S. Bose, A. Mazumdar, G. W. Morley, H. Ulbricht, M. Toroš, M. Paternostro, A. A. Geraci, P. F. Barker, M. S. Kim, and G. Milburn, Spin Entanglement Witness for Quantum Gravity, Phys. Rev. Lett. **119**, 240401 (2017).

[13] C. Marletto and V. Vedral, Witnessing nonclassicality beyond quantum theory, Phys. Rev. D **102**, 086012 (2020).

[14] D. Deutsch and C. Marletto, Constructor theory of information, Proc. R. Soc. A **471**, 2174 (2014).

[15] C. Marletto, V. Vedral, and D. Deutsch, Quantum-gravity effects could in principle be witnessed in neutrino-like oscillations, New J. Phys. **20**, 083011 (2018).

[16] A. Belenchia, R. M. Wald, F. Giacomini, E. Castro-Ruiz, C. Brukner, and M. Aspelmeyer, Quantum superposition of massive objects and the quantization of gravity, Phys. Rev. D **98**, 126009 (2018).

[17] R. J. Marshman, A. Mazumdar, and S. Bose, Locality and entanglement in table-top testing of the quantum nature of linearized gravity, Phys. Rev. A **101**, 052110 (2020).

[18] T. Krisnanda, G. Y. Tham, M. Paternostro, and T. Paterek, Observable quantum entanglement due to gravity, npj Quantum Inf. **6**, 1 (2020).

[19] M. J. Hall and M. Reginatto, On two recent proposals for witnessing nonclassical gravity, J. Phys. A **51**, 085303 (2018).

[20] H. Chevalier, A. J. Paige, and M. S. Kim, Witnessing the nonclassical nature of gravity in the presence of unknown interactions, Phys. Rev. A **102**, 022428 (2020).

[21] T. W. van de Kamp, R. J. Marshman, S. Bose, and A. Mazumdar, Quantum gravity witness via entanglement of masses: Casimir screening, Phys. Rev. A **102**, 062807 (2020).

[22] A. Kent and D. Pitalua-Garcıa, Testing the non-classicality of spacetime: What can we learn from Bell-Bose et al.-Marletto-Vedral experiments?, Phys. Rev. D **104**, 126030 (2021).

[23] H. Chau Nguyen and F. Bernards, Entanglement dynamics of two mesoscopic objects with gravitational interaction, Eur. Phys. J. D **74**, 69 (2020).

[24] D. Miki, A. Matsumura, and K. Yamamoto, Entanglement and decoherence of massive particles due to gravity, Phys. Rev. D **103**, 026017 (2021).

[25] Y. Margalit, O. Dobkowski, Z. Zhou, O. Amit, Y. Japha, S. Moukouri, D. Rohrlich, A. Mazumdar, S. Bose, C. Henkel *et al.*, Realization of a complete stern-gerlach interferometer: Toward a test of quantum gravity, Sci. Adv. **7**, 2879 (2021).

[26] J. Tilly, R. J. Marshman, A. Mazumdar, and S. Bose, Qudits for witnessing quantum-gravity-induced entanglement of masses under decoherence, Phys. Rev. A **104**, 052416 (2021).

[27] D. Carney, H. Müller, and J. M. Taylor, Using an atom interferometer to infer gravitational entanglement generation, PRX Quantum **2**, 030330 (2021).

[28] R. Howl, V. Vedral, D. Naik, M. Christodoulou, C. Rovelli, and Aditya Iyer, Non-gaussianity as a signature of a quantum theory of gravity, PRX Quantum **2**, 010325 (2021).

[29] H. Miao, D. Martynov, H. Yang, and A. Datta, Quantum correlations of light mediated by gravity, Phys. Rev. A **101**, 063804 (2020).

[30] M. Schut, J. Tilly, R. J. Marshman, S. Bose, and A. Mazumdar, Improving resilience of the Quantum Gravity Induced Entanglement of Masses (QGEM) to decoherence using 3 superpositions, Phys. Rev. A **105**, 032411 (2022).

[31] A. Datta and H. Miao, Signatures of the quantum nature of gravity in the differential motion of two masses, Quantum Sci. Technol. **6**, 045014 (2021).

[32] S. Rijavec, M. Carlesso, A. Bassi, V. Vedral, and C. Marletto, Decoherence effects in non-classicality tests of gravity, New J. Phys. **23**, 043040 (2021).

[33] M. Christodoulou, A. D. Biagio, M. Aspelmeyer, C. Brukner, C. Rovelli, and R. Howl, Locally mediated entanglement through gravity from first principles, arXiv: 2202.03368.

[34] C. Marletto and V. Vedral, Sagnac interferometer and the quantum nature of gravity, J. Phys. Commun. **5**, 051001 (2021).

[35] G. A. Neppoleon, A. Iyer, V. Vedral, and Y. Wang, Quantum signatures of gravity from superpositions of primordial massive particles, Phys. Rev. D **105**, 043505 (2022).

[36] S. Bose, A. Mazumdar, M. Schut, and M. Toros, Two mechanisms for quantum natured gravitons to entangle masses, Phys. Rev. D **105**, 106028 (2022).

[37] S. Kanno, J. Soda, and J. Tokuda, Indirect detection of gravitons through quantum entanglement, Phys. Rev. D **104**, 083516 (2021).

[38] A. Datta and H. Miao, Signatures of the quantum nature of gravity in the differential motion of two masses, Quantum Sci. Technol. **6**, 045014 (2021).

[39] D. L. Danielson, G. Satishchandran, and R. M. Wald, Gravitationally mediated entanglement: Newtonian field vs gravitons, Phys. Rev. D **105**, 086001 (2022).
[40] H. Chevalier, A. J. Paige, and M. S. Kim, Witnessing the nonclassical nature of gravity in the presence of unknown interactions, Phys. Rev. A **102**, 022428 (2020).
[41] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).
[42] M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, Philipp Hyllus, D. Bruß, M. Lewenstein, and A. Sanpera, Experimental Detection of Multipartite Entanglement using Witness Operators, Phys. Rev. Lett. **92**, 087902 (2004).
[43] Y. Aharonov, D. Z. Albert, and L. Vaidman, How the Result of a Measurement of a Component of the Spin of a Spin-$1/2$ Particle Can Turn Out to be 100, Phys. Rev. Lett. **60**, 1351 (1988).
[44] A. G. Kofman, S. Ashhab, and F. Nori, Nonperturbative theory of weak pre- and post-selected measurements, Phys. Rep. **520**, 43 (2012).
[45] J. Dressel, M. Malik, F. M. Miatto, A. N. Jordan, and R. W. Boyd, Colloquium: Understanding quantum weak values: Basics and applications, Rev. Mod. Phys. **86**, 307 (2014).
[46] O. Hosten and P. Kwiat, Observation of the spin Hall effect of light via weak measurements, Science **319**, 787 (2008).
[47] P. B. Dixon, D. J. Starling, A. N. Jordan, and J. C. Howell, Ultrasensitive Beam Deflection Measurement via Interferometric Weak Value Amplification, Phys. Rev. Lett. **102**, 173601 (2009).
[48] D. J. Starling, P. B. Dixon, A. N. Jordan, and J. C. Howell, Precision frequency measurements with interferometric weak values, Phys. Rev. A **82**, 063822 (2010).
[49] X.-Y. Xu, Y. Kedem, K. Sun, L. Vaidman, C.-F. Li, and G.-C. Guo, Phase Estimation with Weak Measurement Using a White Light Source, Phys. Rev. Lett. **111**, 033604 (2013).
[50] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, Phys. Rev. **47**, 777 (1935).
[51] E. Schrödinger, Discussion of probability relations between separated systems, Naturwissenschaften **23**, 807 (1935).
[52] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Steering, Entanglement, Nonlocality, and the Einstein-Podolsky-Rosen Paradox, Phys. Rev. Lett. **98**, 140402 (2007).
[53] J. S. Bell, On the Einstein Podolsky Rosen paradox, Physics (Long Island City, N.Y.) **1**, 195 (1964).
[54] J. Clauser, M. Horne, A. Shimony, and R. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, Phys. Rev. Lett. **23**, 880 (1969).
[55] It should be noted that, traditionally, the general weak value derivation is based on the Taylor approximation under weak coupling, because not all the evolution of the Hamiltonian can be accurately expanded.
[56] R. Uola, A. C. S. Costa, H. C. Nguyen, and O. Gühne, Quantum steering, Rev. Mod. Phys. **92**, 015001 (2020).
[57] J. Chen, H. Su, Z. Xu, and A. K. Pati, Sharp contradiction forlocal-hidden-state model in quantum steering, Sci. Rep. **6**, 32075 (2016).
[58] J. L. Chen, X. J. Ye, C. Wu, H. Y. Su, A. Cabello, L. C. Kwek, and C. H. Oh, All-versus-nothing proof of Einstein-Podolsky-Rosen steering, Sci. Rep. **3**, 2143 (2013).
[59] T. Feng, C. Ren, Q. Feng, M. Luo, X. Qiang, J. Chen, and X. Zhou, Steering paradox for Einstein-Podolsky-Rosen argument and its extended inequality, Photonics Res. **9**, 992 (2021).
[60] Note that if the logical $|0\rangle$ and $|1\rangle$ are redefined, any two-qubit pure state has a mathematical form such as $|\Psi\rangle$, so our analysis later will not lose the generality.
[61] Y. Guryanova, N. Friis, and M. Huber, Ideal projective measurements have infinite resource costs, Quantum **4**, 222 (2020).
[62] U. Delic, M. Reisenbauer, K. Dare, D. Grass, V. Vuletić, N. Kiesel, and M. Aspelmeyer, Cooling of a levitated nanoparticle to the motional quantum ground state, Science **367**, 892 (2020).
[63] C. Whittle *et al.*, Approaching the motional ground state of a 10-kg object, Science **372**, 6548 (2021).
[64] F. Tebben- johanns, M. L. Mattana, M. Rossi, M. Frimmer, and L. Novotny, Quantum control of a nanoparticle optically levitated in cryogenic free space, Nature (London) **595**, 378 (2021).
[65] T. Westphal, H. Hepach, J. Pfaff, and M. Aspelmeyer, Measurement of gravitational coupling between millimeter-sized masses, Nature (London) **591**, 225 (2021).
[66] J. S. Pedernales, K. Streltsov, and M. B. Plenio, Enhancing Gravitational Interaction between Quantum Systems by a Massive Mediator, Phys. Rev. Lett. **128**, 110401 (2022).
[67] N. Gisin, Hidden quantum nonlocality revealed by local filters, Phys. Lett. A **210**, 151 (1995).
[68] C. Ferrie and J. Combes, How the Result of a Single Coin Toss Can Turn Out to be 100 Heads, Phys. Rev. Lett. **113**, 120404 (2014).
[69] A. Romito, A. N. Jordan, Y. Aharonov, and Y. Gefen, Weak values are quantum: You can bet on it, Quantum Stud. Math. Found. **3**, 1 (2016).
[70] K. Kirkpatrick, Classical three-box, “paradox”, J. Phys. A **36**, 4891 (2003).
[71] J. Dressel and A. N. Jordan, Sufficient conditions for uniqueness of the weak value, J. Phys. A **45**, 015304 (2012).
[72] M. F. Pusey, Anomalous Weak Values Are Proofs of Contextuality, Phys. Rev. Lett. **113**, 200401 (2014).