Shear viscosity from black holes in generalized scalar-tensor theories in arbitrary dimensions

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In higher dimensions, we study degenerate-higher-order-scalar-tensor theories and we derive solutions that resemble the Schwarzschild anti–de Sitter black holes. We compute their thermodynamic quantities following the Wald formalism, satisfying the first law of thermodynamics and a higher dimensional Smarr relation. Constructing a Noether charge with a suitable choice of a spacelike Killing vector, we obtain the shear viscosity of the nongravitational dual field theory, where for a suitable choice of the couplings functions, the Kovtun-Son-Starinets bound is violated. These results are corroborated by the calculation of the Green's functions following the Kubo formalism.

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I. INTRODUCTION

General relativity (GR) provides the standard description of gravity. Extensions of GR have been considered in the literature as gravitational alternatives, for instance, for unified descriptions of inflation and cosmic acceleration. Various proposals include $F(R)$ gravity, scalar-tensor theories, and string-inspired and Gauss-Bonnet theories [1–7]. Some of these models might be consistent with local tests, and the occurrence of finite-time future singularities in modified gravity may be cured by the addition of higher-derivative terms. For a review, see, e.g., [8]. In particular, in the 1970s, Horndeski constructed a four-dimensional scalar-tensor theory wherein the equations of motions are at most of the second order in the derivatives of the field functions [9]. Motivated with the above, in recent years a new class of scalar-tensor theories of gravity that extend Horndeski, or "generalized Galileon," models have been proposed. Despite possessing equations of motion of higher-order derivatives, the propagating degrees of freedom satisfy second-order equations of motion and are thus free from Ostrogradski instabilities [10].

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Astrophysical implications [11–13] have yielded further motivation to study theories of gravity beyond Horndeski proposal, channeling in a model denominated as degenerate-higher-order-scalar-tensor (DHOST) theory, also avoiding Ostrogradsky instability due to its degeneracy property [14,15]. It is important to note that these degenerate theories have allowed the exploration of four-dimensional regular black holes [16,17], rotating black holes stealth [18], and three-dimensional spinning configurations [19]. As far as we know, the extension of these solutions for higher-dimensional theories is still an open problem, which we address in this work.

On the other hand, the AdS/CFT correspondence [20–22] is a relation between a gravitational theory in D-spacetime dimensions and a field theory in flat $(D-1)$ dimensions (without gravity). In its most general form, it is known as a gauge/gravity duality. One nice property of this duality is that when the field theory is strongly coupled the gravitational dual is weakly coupled and vice versa. This property opens a large window of applications in many different areas. In particular, it gives support to study the dynamics of tightly coupled systems, especially the transport coefficients from condensed matter and hydrodynamics to the quark-gluon plasma formed at relativistic heavy-ion collisions [23–31]. One of these coefficients is the well-known shear viscosity η [27,28], calculated from holographic bottom-up models. Under this scenario, it is possible to compute the ratio between η and the entropy density s, arising a conjecture about a universal bound, known as the Kovtun-Son-Starinets (KSS) bound, which reads [27,28,32,33]

$$\frac{\eta}{s} \geq \frac{1}{4\pi}, \tag{1}$$

being a support in a variety of gravity dual models [34–37].

On the contrary, for some examples—which include unconventional superconducting systems [25,27], the Einstein-Hilbert Gauss-Bonnet action in five dimensions [38,39], the Horndeski theory [40–44], as well as the DHOST theories in four dimensions [45]—the KSS bound is violated, at least for some choices of the relevant parameters of these models.

In this work, we consider DHOST theories in higher dimensions ($D \geq 3$). We construct hairy black hole solutions with planar geometry under this scenario and study their thermodynamic properties. Together with the above, in order to obtain the shear viscosity of the dual gauge theories living in lower-dimensional flat spaces, we employ two different methods for $D > 3$. The first one is performed via the Wald formalism, with the construction of a Noether charge and an election for the spacelike Killing vector [46]. For the second, we use the more traditional methods present in [33,39], with the calculation of Green's functions and the use of the Kubo formula. The two formalisms generate the same expression for the shear viscosity of the dual gauge field theories, allowing violation of the KSS bound.

This paper is organized as follows: In Sec. II, we consider DHOST theories in higher dimensions ($D \geq 3$) and obtain hairy black hole solutions with planar base manifolds in these spacetimes. In Sec. III, through the Wald formalism [47,48] the thermodynamics of that solutions are explored, and in Sec. IV we obtain the viscosity/entropy density ratio of the corresponding dual field theories for $D > 3$, showing that the KSS bound could be violated in these theories. Some details of the equations of motion are presented in Appendix A. Further, in Appendix B, we reobtain the viscosity/entropy density ratio using Green's functions and the Kubo formula, corroborating the results of Sec. IV. Finally, Sec. V is devoted to our conclusions and discussions.

II. DHOST THEORIES AND HAIRY BLACK HOLE SOLUTIONS IN HIGHER DIMENSIONS

To our knowledge, DHOST theories in $D = 3$ and $D = 4$ spacetime dimensions have been studied in Refs. [7,10,14–19]. Here we generalize these previous approaches to the D-dimensional case ($D \geq 3$) defining the action as

$$S[g_{\mu\nu}, \phi] = \int d^D x \sqrt{-g} \mathcal{L}, \tag{2}$$

where the Lagrangian $\mathcal{L}$ reads

$$\mathcal{L} = \lambda_0 Z(X) + [1 + \lambda_1 G(X)]R + \sum_{i=2}^{5} \lambda_i A_i(X) \mathcal{L}_i, \tag{3}$$

with $X \coloneqq \partial_\mu \phi \partial^\mu \phi$ being the kinetic term of the scalar field ϕ, and

$$\mathcal{L}_2 \coloneqq (\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}, \qquad \mathcal{L}_3 \coloneqq \Box\phi \phi^\mu \phi_{\mu\nu} \phi^\nu, \tag{4}$$

$$\mathcal{L}_4 \coloneqq \phi^\mu \phi_{\mu\nu} \phi^{\nu\rho} \phi_\rho, \qquad \mathcal{L}_5 \coloneqq (\phi^\mu \phi_{\mu\nu} \phi^\nu)^2. \tag{5}$$

Here R is the scalar curvature, λ_m, with $m \in \{0, 1, 2, 3, 4, 5\}$, are convenient parameters to control the couplings between the functions $Z(X)$, $G(X)$, and $A_i(X)$, with $i \in \{2, 3, 4, 5\}$, while we have defined $\phi_\mu \coloneqq \nabla_\mu \phi$ and $\phi_{\mu\nu} \coloneqq \nabla_\mu \nabla_\nu \phi$. For later convenience, we also define the derivative with respect to X, $F_X \coloneqq dF/dX$, so that, for instance, $Z_X \coloneqq dZ/dX$, $G_X \coloneqq dG/dX$, and $A_{iX} \coloneqq dA_i/dX$. The equations of motions with respect to the metric $g_{\mu\nu}$ and the scalar field ϕ are given by

$$\mathcal{E}_{\mu\nu} \coloneqq \mathcal{G}^Z_{\mu\nu} + \mathcal{G}^G_{\mu\nu} + \sum_{i=2}^{5} \mathcal{G}^{(i)}_{\mu\nu} = 0, \tag{6}$$

$$\mathcal{E}_\phi = \nabla_\mu \mathcal{J}^\mu = \nabla_\mu \left[\frac{\delta \mathcal{L}}{\delta(\phi_\mu)} - \nabla_\nu \left(\frac{\delta \mathcal{L}}{\delta(\phi_{\mu\nu})} \right) \right] = 0, \tag{7}$$

where the expressions given in $\mathcal{E}_{\mu\nu}$ and $\mathcal{J}^\mu$ are reported in the Appendix A.

For the generalized scalar-tensor configuration Eqs. (2)–(5), we consider the following higher-dimensional metric ansatz:

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \sum_{i=1}^{D-2} dx_i^2, \tag{8}$$

$$\phi(t, r) = \psi(r), \tag{9}$$

where only a radial dependence for the scalar field ϕ is required, given that we are working on a planar base manifold. In order to simplify our computations, we also suppose that the kinetic term X is a constant. This hypothesis implies that

$$X = g^{rr} (\psi')^2, \tag{10}$$

and the square of the derivative of the scalar field ϕ can be cast as

$$(\psi')^2 = \frac{X}{f}, \tag{11}$$

where ($'$) denotes the derivative with respect to the radial coordinate r. Following the steps performed in [19,45], we fix the function A_5 as

$$\lambda_5 A_5 = \frac{(2\lambda_2 A_2 + X\lambda_3 A_3 + 4\lambda_1 G_X)^2}{2X(1 + \lambda_1 G + \lambda_2 X A_2)} - \left(\frac{\lambda_3 A_3 + \lambda_4 A_4}{X} \right), \tag{12a}$$

or

$$\lambda_5 A_5 = \frac{1}{X}\left(\frac{[\mathcal{Z}_2(X)]^2}{2\mathcal{Z}_1(X)} - \mathcal{Z}_3(X)\right), \tag{12b}$$

defining the functions

$$\mathcal{Z}_1(X) = 1 + \lambda_1 G(X) + X\lambda_2 A_2(X), \tag{12c}$$

$$\mathcal{Z}_2(X) = 2\lambda_2 A_2(X) + \lambda_3 X A_3(X) + 4\lambda_1 G_X(X), \tag{12d}$$

$$\mathcal{Z}_3(X) = \lambda_3 A_3(X) + \lambda_4 A_4(X). \tag{12e}$$

Then, a solution in higher dimensions $D \geq 3$ reads

$$f(r) = h(r) = \frac{\lambda_0 Z r^2}{(D-1)(D-2)\mathcal{Z}_1} - \frac{M}{r^{D-3}}, \tag{13}$$

where M is a positive integration constant, as long as the coupling functions are related in the following form

$$2(D-2)(Z\mathcal{Z}_1)_X = (D-1)\mathcal{Z}_2 Z, \tag{14}$$

and the scalar field from (11) can be obtained as

$$\begin{aligned}\phi(t,r) &= \psi(r)\\ &= \pm\left(\frac{2l}{D-1}\right)\sqrt{X}\ln\left[r^{\frac{D-3}{2}}\left(\frac{r}{l} + \sqrt{\frac{r^2}{l^2} - \frac{M}{r^{D-3}}}\right)\right].\end{aligned} \tag{15}$$

Many commentaries can be carried out with respect to the solution (13)–(15). First, the metric function $h = f$ resembles the well-known Schwarzschild-AdS (anti–de Sitter) black hole in D-spacetime dimensions. Second, Eq. (14) represents the extension of the particular cases found previously in four [45] and three dimensions [19], where the scalar field is well defined on the location of the event horizon $r_h = (Ml^2)^{\frac{1}{D-1}}$ and in order to have a real and nontrivial expression for ϕ, we need $X > 0$ for $r \geq r_h$. Finally, in order to have an asymptotically AdS black hole configuration, we will define the AdS radius l as

$$l^2 = \frac{(D-1)(D-2)\mathcal{Z}_1}{\lambda_0 Z}, \tag{16}$$

and impose the constraint

$$\frac{\mathcal{Z}_1}{\lambda_0 Z} > 0,$$

to have a real expression for l. Summarizing—with the DHOST theory, Eqs. (2)–(5), together with a constant kinetic term X and the coupling functions Z, G, and the A_is satisfying the relation (14)—we can obtain a higher-dimensional hairy solution with a planar base manifold given in Eqs. (11)–(13). In the following section, we will derive the thermodynamic quantities corresponding to this solution.

III. THERMODYNAMICS OF THE HAIRY SOLUTION FROM THE WALD FORMALISM

Given the hairy higher-dimensional black hole solution, found in the previous section, to compute extensive thermodynamic quantities (these are the mass $\mathcal{M}$ and the entropy $\mathcal{S}_W$), we will consider the Wald formalism [47,48]. We start through the variation of the action (2)–(5) with respect to all the dynamical fields, which is

$$\delta S = \sqrt{-g}[\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} + \mathcal{E}_\phi \delta\phi + \nabla_\mu J^\mu(\delta g, \delta\phi)],$$

where, as before, $\mathcal{E}_{\mu\nu}$ and $\mathcal{E}_\phi$ are the equations of motions with respect to the metric and the scalar field. The surface term J^μ reads

$$\begin{aligned}J^\mu = \sqrt{-g}\Big[&2(P^{\mu(\alpha\beta)\gamma}\nabla_\gamma \delta g_{\alpha\beta} - \delta g_{\alpha\beta}\nabla_\gamma P^{\mu(\alpha\beta)\gamma}) + \mathcal{J}^\mu \delta\phi + \frac{\delta\mathcal{L}}{\delta(\phi_{\mu\nu})}\delta(\phi_\nu)\\ &-\frac{1}{2}\frac{\delta\mathcal{L}}{\delta(\phi_{\mu\sigma})}\phi^\rho \delta g_{\sigma\rho} - \frac{1}{2}\frac{\delta\mathcal{L}}{\delta(\phi_{\sigma\mu})}\phi^\rho\delta g_{\sigma\rho} + \frac{1}{2}\frac{\delta\mathcal{L}}{\delta(\phi_{\sigma\rho})}\phi^\mu \delta g_{\sigma\rho}\Big],\end{aligned} \tag{17}$$

with $\mathcal{J}^\mu$ reported in Appendix A. Further, in our case:

$$P^{\mu\nu\sigma\rho} = \frac{\delta\mathcal{L}}{\delta R_{\mu\nu\sigma\rho}} = \frac{1}{2}(1 + \lambda_1 G(X))(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}), \tag{18}$$

while

$$\frac{\delta\mathcal{L}}{\delta\phi_\mu} = 2\lambda_0 Z_X\phi^\mu + 2\lambda_1 G_X R\phi^\mu + 2\lambda_2 A_{2X}\phi^\mu[(\Box\phi)^2 - \phi_{\lambda\rho}\phi^{\lambda\rho}] + 2\lambda_3 A_{3X}\phi^\mu\Box\phi\phi^\lambda\phi_{\lambda\rho}\phi^\rho + 2\lambda_3 A_3\Box\phi\phi^\mu{}_\lambda\phi^\lambda$$
$$+ 2\lambda_4 A_{4X}\phi^\mu\phi^\sigma\phi_{\sigma\rho}\phi^{\rho\lambda}\phi_\lambda + \lambda_4 A_4(X)[\phi^\mu{}_\rho\phi^{\rho\lambda}\phi_\lambda + \phi^\sigma\phi_{\sigma\rho}\phi^{\rho\mu}] + 2\lambda_5 A_{5X}\phi^\mu(\phi^\sigma\phi_{\sigma\rho}\phi^\rho)^2$$
$$+ 2\lambda_5 A_5(X)(\phi^\sigma\phi_{\sigma\rho}\phi^\rho)(\phi^{\mu\sigma}\phi_\sigma + \phi^{\sigma\mu}\phi_\sigma) \tag{19}$$

and

$$\frac{\delta\mathcal{L}}{\delta\phi_{\mu\nu}} = 2\lambda_2 A_2(g^{\mu\nu} - \phi^{\mu\nu}) + \lambda_3 A_3(g^{\mu\nu}\phi^\lambda\phi_{\lambda\rho}\phi^\rho + \Box\phi\phi^\mu\phi^\nu) + \lambda_4 A_4(X)(\phi^\mu\phi^{\nu\rho}\phi_\rho + \phi^\sigma\phi^\mu{}_\sigma\phi^\nu) + 2\lambda_5 A_5(X)\phi^\sigma\phi_{\sigma\rho}\phi^\rho\phi^\mu\phi^\nu. \tag{20}$$

Defining a 1-form $J_{(1)} = J_\mu dx^\mu$ and its corresponding Hodge dual $\Theta_{(D-1)} = (-1)^{D+1} * J_{(1)}$, together with considering a variation induced by an infinitesimal diffeomorphism $\delta x^\mu = \xi^\mu$, and making use of the equations of motions (6)–(7), we have that $J_{(D-1)} = \Theta_{(D-1)} - i_\xi(*\mathcal{L}) = d(*J_{(2)})$, where i_ξ is a contraction of the vector ξ^μ with the first index of $*\mathcal{L}$, and in our notations the subindex "(p)" corresponds to the fact that we are working with p-forms. The above allows the definition of a $(D-2)$-form $Q_{(D-2)} = *J_{(2)}$ such that $J_{(D-1)} = dQ_{(D-2)}$, where

$$Q_{(D-2)} = Q_{\alpha_1\alpha_2\cdots\alpha_{D-2}} = \epsilon_{\alpha_1\alpha_2\cdots\alpha_{D-2}\mu\nu}Q^{\mu\nu},$$

with

$$Q^{\mu\nu} = 2P^{\mu\nu\rho\sigma}\nabla_\rho\xi_\sigma - 4\xi_\sigma\nabla_\rho P^{\mu\nu\rho\sigma} + \frac{\delta\mathcal{L}}{\delta\phi_{\mu\sigma}}\phi^\nu\xi_\sigma - \frac{\delta\mathcal{L}}{\delta\phi_{\nu\sigma}}\phi^\mu\xi_\sigma, \tag{21}$$

and $P^{\mu\nu\rho\sigma}$ and $\delta\mathcal{L}/\delta\phi_{\mu\sigma}$ were given previously in Eqs. (18) and (20), respectively. Concretely, for the action (2)–(5) and using the fact from (11) that $\delta(\phi') = -\sqrt{X}\delta f/(2f^{3/2})$ [here we note that $\mathcal{J}^\mu\delta\phi$ from (17) vanishes after making use of the equations of motion], we find that $i_\xi\Theta_{(D-1)}$ as well as $Q_{(D-2)}$ read

$$i_\xi\Theta_{(D-1)} = r^{D-3}[-(D-2)\mathcal{Z}_1\delta f + 2(D-2)\delta f(1+\lambda_1 G) + r(1+\lambda_1 G)\delta(f')]\Omega_{D-2},$$
$$Q_{(D-2)} = r^{D-3}[r(1+\lambda_1 G)f' + 2(D-2) \times f(1+\lambda_1 G - \mathcal{Z}_1)]\Omega_{D-2},$$

and the variation of $Q_{(D-2)}$ takes the form

$$\delta Q_{(D-2)} = r^{D-3}[r(1+\lambda_1 G)\delta(f') + 2(D-2)\delta f(1+\lambda_1 G - \mathcal{Z}_1)]\Omega_{D-2},$$

where Ω_{D-2} is the finite volume of the $(D-2)$-dimensional compact angular base manifold. Finally, taking ξ^μ as a timelike Killing vector that is null on the location of the event horizon, denoted as r_h, the variation of the Hamiltonian reads

$$\delta\mathcal{H} = \delta\int_{\mathcal{C}} J_{(D-1)} - \int_{\mathcal{C}} d(i_\xi\Theta_{(D-1)}) = \delta\int_{\mathcal{C}} d(Q_{(D-2)}) - \int_{\mathcal{C}} d(i_\xi\Theta_{(D-1)}), = \int_{\Sigma^{(D-2)}}(\delta Q_{(D-2)} - i_\xi\Theta_{(D-1)}),$$

where $\mathcal{C}$ and $\Sigma^{(D-2)}$ are a Cauchy surface and its boundary, respectively, which has two components, located at infinity ($\mathcal{H}_\infty$) and at the horizon ($\mathcal{H}_+$). According to the Wald formalism [47,48], the first law of black holes thermodynamics,

$$\delta\mathcal{M} = T\delta\mathcal{S}_W, \tag{22}$$

is a consequence of $\delta\mathcal{H}_\infty = \delta\mathcal{H}_+$, where $\mathcal{M}$ and $\mathcal{S}_W$ denote the mass as well as the entropy, while that the Hawking temperature T reads

$$T = \frac{\kappa}{2\pi}\bigg|_{r=r_h} = \frac{1}{2\pi}\sqrt{-\frac{1}{2}(\nabla_\mu\xi_\nu)(\nabla^\mu\xi^\nu)}\bigg|_{r=r_h}, = \frac{1}{4\pi}\frac{\lambda_0 Z r_h}{(D-2)\mathcal{Z}_1} = \frac{(D-1)r_h}{4\pi l^2}, \tag{23}$$

where κ is the surface gravity, r_h is the location of the event horizon, and the AdS radius l was defined previously in Eq. (16). With all the above, we have that

$$\delta Q_{(D-2)} - i_\xi\Theta_{(D-1)} = -(D-2)r^{D-3}\mathcal{Z}_1\delta f,$$

where

$$\delta f = -\frac{\delta M}{r^{D-3}}.$$

Computing the respective variation of the solution (11)–(15), at the infinity we can write

$$\delta\mathcal{H}_\infty = \delta\mathcal{M} = (D-2)\mathcal{Z}_1\Omega_{D-2}\delta M \tag{24}$$

so that the mass $\mathcal{M}$ takes the form

$$\mathcal{M} = (D-2)\mathcal{Z}_1 M\Omega_{D-2} = \frac{(D-2)\mathcal{Z}_1 r_h^{D-1}\Omega_{D-2}}{l^2}. \tag{25}$$

Note that the positivity of the physical mass $\mathcal{M}$ implies that $\mathcal{Z}_1 > 0$. This thermodynamic condition will be important, as we will see below, at the moment to study the shear viscosity η. At the horizon, where from the metric function (13)

$$\delta M = \frac{(D-1)r_h^{D-2}\delta r_h}{l^2},$$

where l^2 is the AdS radius (16), we have

$$\delta\mathcal{H}_+ = T\delta\mathcal{S}_W = T\delta(4\pi\Omega_{D-2}\mathcal{Z}_1 r_h^{D-2}), \tag{26}$$

from which the entropy $\mathcal{S}_W$ takes the form

$$\mathcal{S}_W = 4\pi\Omega_{D-2}\mathcal{Z}_1 r_h^{D-2}. \tag{27}$$

Note that the condition $\mathcal{Z}_1 > 0$ from the positivity of the mass $\mathcal{M}$ from Eq. (25) guarantees the positivity of the entropy $\mathcal{S}_W$. It is worth pointing out that besides the fulfillment of the first law (22), a higher-dimensional Smarr relation [49]

$$\mathcal{M} = \left(\frac{D-2}{D-1}\right)T\mathcal{S}_W \tag{28}$$

holds.

IV. THE VISCOSITY/ENTROPY DENSITY RATIO THROUGH THE WALD FORMALISM

After obtaining the thermodynamic quantities from the hairy higher-dimensional black hole solution, in particular for the Wald entropy $\mathcal{S}_W$ from (27) we can obtain the entropy density s in our set up, given by

$$s = \frac{\mathcal{S}_W}{\Omega_{D-2}} = 4\pi r_h^{D-2}\mathcal{Z}_1. \tag{29}$$

In order to calculate the shear viscosity η, according to the procedure performed in [46], we first perform a transverse and traceless perturbation on the metric (8) for $D > 3$ with $h = f$, which reads

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + 2r^2\Psi(t,r)dx_1dx_2 + r^2\sum_{i=1}^{D-2}dx_i^2, \tag{30}$$

with the ansatz

$$\Psi(t,r) = \zeta t + h_{x_1x_2}(r),$$

where ζ is a constant identified as the gradient of the fluid velocity along the x_1 direction. This perturbation yields the following (x_1, x_2) component of the linearized Einstein equations:

$$[\mathcal{Z}_1(X)r^{D-2}f(h_{x_1x_2})']' = 0, \tag{31}$$

and by using a spacelike Killing vector $\partial_{x_1} = \xi^\mu\partial_\mu$, the charge $\sqrt{-g}Q^{rx_2}$, constructed through $Q^{\mu\nu}$ from (21), becomes an integration constant [46], which reads

$$\sqrt{-g}Q^{rx_2} = \mathcal{Z}_1(X)r^{D-2}f(h_{x_1x_2})'. \tag{32}$$

Imposing the ingoing horizon boundary condition

$$h_{x_1x_2} = \zeta\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}\frac{\log(r-r_h)}{4\pi T} + \cdots,$$

as well as a Taylor expansion in the near horizon region r_h

$$h = f = 4\pi T(r-r_h) + \cdots,$$

where T is the Hawking temperature given previously in (23), we have

$$\sqrt{-g}Q^{rx_2} = \zeta\mathcal{Z}_1\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}r_h^{D-2} = \zeta\left(\frac{1}{4\pi}\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}s\right),$$

where s was given in (29). Following the steps from [46], the shear viscosity η can be obtained in the following way

$$\eta = \frac{\partial(\sqrt{-g}Q^{rx_2})}{\partial\zeta} = \frac{1}{4\pi}\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}s. \tag{33}$$

Since the shear viscosity η is real and non-negative, and the fact that $\mathcal{Z}_1 > 0$, from the positivity of the mass $\mathcal{M}$ (25) as well as the entropy $\mathcal{S}_W$ (27), we conclude that $1 + \lambda_1 G \geq 0$. Then, the viscosity/entropy density ratio takes the form

$$\frac{\eta}{s} = \frac{1}{4\pi}\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}} = \frac{1}{4\pi}\sqrt{\frac{1+\lambda_1 G(X)}{1+\lambda_1 G(X)+\lambda_2 X A_2(X)}}, \tag{34}$$

or

$$\frac{\eta}{s} = \frac{1}{4\pi}\sqrt{\frac{\mathcal{Z}_1(X)|_{\lambda_2=0}}{\mathcal{Z}_1(X)}}. \tag{35}$$

Here, we note that there is no a presence of the location of the event horizon r_h on the η/s ratio, and although the dimension of the space-time D is not present in (34) [or (35)], this expression appears actively from the relation of the coupling functions (14). Together with the above, since

$$\mathcal{Z}_1(X) \coloneqq 1 + \lambda_1 G(X) + \lambda_2 X A_2(X) > 0, \tag{36}$$

from the positivity of the mass $\mathcal{M}$ (and the entropy $\mathcal{S}_W$), and since

$$1 + \lambda_1 G(X) \geq 0, \tag{37}$$

from the reality and non-negativity of the shear viscosity η, we conclude that $\lambda_2 X A_2(X) > -(1 + \lambda_1 G(X)) \leq 0$. It is worth pointing out that the above conditions are not new, in fact, according to [50], the condition (36) together with the strict inequality from (37) allow us to obtain necessary conditions of stability in four-dimensional spherical symmetric solutions. Curiously enough, the above is consistent with the stability conditions for linear cosmological perturbations, guaranteeing that both the effective gravitational constant and the squared propagation speed of the tensor modes are positive [51], without the requirement on the sign of $\lambda_2 X A_2(X)$. For our particular situation, and given the explicit expression for the scalar field ϕ found in (15), where $X > 0$, only the possibilities of study are performed by analyzing $\lambda_2 A_2(X)$. First, if $\lambda_2 A_2(X) > 0$, then the KSS bound would be violated even in the limit $\lambda_1 \to 0$. On the other hand, if $\lambda_2 A_2(X) < 0$, and still obeying the condition (36), then one recovers the usual KSS bound where $\eta/s > 1/(4\pi)$. Finally, if $\lambda_2 A_2(X) = 0$, which can be achieved in the limit $\lambda_2 \to 0$, then the KSS bound is saturated [$\eta/s = 1/(4\pi)$].

As concrete examples from the cases showed before, we can see that the Einstein-Hilbert case, together with a cosmological constant (this is for $\lambda_i = 0$ for $i = \{1, 2, 3, 4, 5\}$, and $Z = -2\Lambda/\lambda_0$), is naturally recovered, where we can see that $\eta/s = 1/(4\pi)$. Together with the above, for example, for

$$G(X) = X^j, \qquad A_2(X) = G_X = jX^{j-1},$$

where j is a positive constant, while that Z and $\mathcal{Z}_2$ satisfy the condition (14), we obtain

$$0 < \frac{1 + \lambda_1 X^j}{1 + (\lambda_1 + j\lambda_2)X^j} < \frac{1}{4\pi},$$

as long as $\lambda_2 > 0$.

In resume, with all this information we can to conclude that, for a higher-dimensional scalar-tensor theory (2)–(5) with specific coupling functions, represented via Z, G and the A_is, the explicit expression for the kinetic term X can be obtained through (14), and the solution takes the form (13)–(16). Moreover, in particular, there exists an active presence of the functions G and $\mathcal{Z}_1$ in the η/s ratio, providing a new example of the violation of the KSS bound whose Lagrangian is at most linear in curvature tensor.

V. CONCLUSIONS AND DISCUSSIONS

In the present paper we explored the dimensional continuation of planar hairy black hole solutions found in [19,45], where the theory is given by a model denominated as DHOST theory, Eqs. (2)–(5), constructed by a nontrivial scalar field ϕ and its derivatives, the Ricci scalar R, and coupling functions depending on the kinetic term $X \coloneqq \partial_\mu\phi\partial^\mu\phi$ Eq. (10), which we suppose to be constant. In this case, black holes resemble the well-known Schwarzschild-AdS configurations in arbitrary dimensions, where the integration constant M is related to the mass, and the AdS radius depends on the coupling functions present in the theory, being interpreted as an effective cosmological constant. With these results, via the Wald formalism, we compute their thermodynamical parameters, which satisfy the first law, Eq. (22), as well as a higher-dimensional Smarr relation, Eq. (28).

Motivated by recent concrete examples presented in the literature (see [40,41,43–45]), where the bound for the viscosity/entropy density ratio Eq. (1) can be violated, we analyzed the shear viscosity η for DHOST theory, Eqs. (2)–(5), following two procedures. The first one, through the construction of a conserved charge as well as a suitable election of the Killing vector [46] where in this case it is not necessary to impose any hydrodynamic limit, such as the low frequencies, to define the transport coefficient. On the other hand, in the second one, via Green's functions and Kubo formula, given by [33,39], respectively, and explained in the Appendix B. For both techniques, we obtain the same expression for the η/s ratio. Here we note that these results are not a surprise, because the boundary condition to the effective action for the transverse off-diagonal gravitons $h_{x_1 x_2}$ for the two methods considers the gravity fluctuations around the metric. For higher-dimensional planar black holes in DHOST theories, the presence of the coupling functions G and $\mathcal{Z}_1$ in the η/s ratio, provides a new example of the violation of the KSS bound whose Lagrangian is at most linear in curvature tensor. Note that only with specific coupling functions (these are Z, G and the A_is), the explicit expression for the kinetic term X, obtained through (14), and the solution (13)–(16), are required. In fact, according to the expression (34) [or (35)], the η/s ratio is controlled by the parameters λ_1 and λ_2 as well as the kinetic term X and the coupling functions $A_2(X)$ together with $G(X)$, allowing us to get cases where the KSS bound can be violated, fulfilled, or saturated. It is worth pointing out that although in the expression (34) there is no explicit presence of the dimension of the space-time, this quantity appears implicitly in the relation of the coupling functions obtained in (14).

From this work, some natural extensions can be raised. For example, to simplify our computations, from the beginning we suppose that the kinetic term X is a constant. It would be interesting to explore more general solutions in arbitrary dimensions, where now from (10) $X = X(r)$, which allows us to explore the shear viscosity η on configurations with nonstandard asymptotically behaviors, as was studied for Lifshitz black holes in [41]. Finally, these results deserve further investigation both in their own right, in particular in the context of the AdS/CFT correspondence and their implications.

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APPENDIX A: RELEVANT TENSORS AND VECTORS FOR THE EQUATIONS OF MOTION

For the sake of completeness, in this Appendix we report the expressions for $\mathcal{G}^Z_{\mu\nu}$, $\mathcal{G}^G_{\mu\nu}$, the $\mathcal{G}^{(i)}_{\mu\nu}$s and $\mathcal{J}^\mu$ present in Eqs. (6) and (7)

$$
\begin{aligned}
\mathcal{G}^Z_{\mu\nu} &= \lambda_0\left(-\frac{1}{2}Z(X)g_{\mu\nu} + Z_X\phi_\mu\phi_\nu\right),\\
\mathcal{G}^G_{\mu\nu} &= (1+\lambda_1 G)G_{\mu\nu} + \lambda_1 G_X R\phi_\mu\phi_\nu - \lambda_1\nabla_\nu\nabla_\mu G + \lambda_1 g_{\mu\nu}\nabla_\lambda\nabla^\lambda G,\\
\mathcal{G}^{(2)}_{\mu\nu} &= \lambda_2\Big[-\phi_\mu(A_{2X}\nabla_\nu X)\Box\phi - (A_{2X}\nabla_\mu X)\phi_\nu\Box\phi - A_2\phi_{\nu\mu}\Box\phi - \phi_{\nu\mu}\phi_\lambda(A_{2X}\nabla^\lambda X) + \phi_\nu\phi_{\lambda\mu}(A_{2X}\nabla^\lambda X) + \phi_\mu\phi_{\lambda\nu}(A_{2X}\nabla^\lambda X)\\
&\quad + A_2R_{\nu\lambda}\phi_\mu\phi^\lambda + A_2R_{\mu\lambda}\phi_\nu\phi^\lambda - A_2\phi_{\lambda\nu\mu}\phi^\lambda + \frac{1}{2}A_2g_{\mu\nu}(\Box\phi)^2 + g_{\mu\nu}\phi_\lambda(A_{2X}\nabla^\lambda X)\Box\phi + A_2g_{\mu\nu}\phi^\lambda\phi_\rho{}^\rho{}_\lambda\\
&\quad - A_2g_{\mu\nu}R_{\lambda\rho}\phi^\lambda\phi^\rho + \frac{1}{2}A_2g_{\mu\nu}\phi_{\rho\lambda}\phi^{\rho\lambda} + A_{2X}\phi_\mu\phi_\nu((\Box\phi)^2 - \phi_{\lambda\rho}\phi^{\lambda\rho})\Big],\\
\mathcal{G}^{(3)}_{\mu\nu} &= \lambda_3\Big[-\frac{1}{2}A_3\phi_\mu\phi_\nu(\Box\phi)^2 - \frac{1}{2}\phi_\mu\phi_\nu\phi_\lambda(A_{3X}\nabla^\lambda X)\Box\phi + \frac{1}{2}A_3\phi_\mu\phi_{\lambda\nu}\phi^\lambda\Box\phi + \frac{1}{2}A_3\phi_\nu\phi_{\lambda\mu}\phi^\lambda\Box\phi - \frac{1}{2}A_3\phi_\mu\phi_\nu\phi^\lambda\phi_\rho{}^\rho{}_\lambda\\
&\quad + \frac{1}{2}A_3R_{\lambda\rho}\phi_\mu\phi_\nu\phi^\lambda\phi^\rho - \frac{1}{2}\phi_\mu(A_{3X}\nabla_\nu X)\phi^\lambda\phi_{\rho\lambda}\phi^\rho - \frac{1}{2}(A_{3X}\nabla_\mu X)\phi_\nu\phi^\lambda\phi_{\rho\lambda}\phi^\rho - \frac{1}{2}A_3\phi_\nu\phi^\lambda\phi_{\rho\lambda\mu}\phi^\rho - \frac{1}{2}A_3\phi_\mu\phi^\lambda\phi_{\rho\lambda\nu}\phi^\rho\\
&\quad - A_3\phi_\nu\phi^\lambda\phi_{\rho\lambda}\phi^\rho{}_\mu - A_3\phi_\mu\phi^\lambda\phi_{\rho\lambda}\phi^\rho{}_\nu + \frac{1}{2}g_{\mu\nu}\phi_\lambda(A_{3X}\nabla^\lambda X)\phi^\rho\phi_{\sigma\rho}\phi^\sigma + \frac{1}{2}g_{\mu\nu}A_3\phi^\lambda\phi^\rho\phi_{\sigma\rho\lambda}\phi^\sigma + g_{\mu\nu}A_3\phi^\lambda\phi^\rho\phi_{\sigma\rho}\phi^\sigma{}_\lambda\\
&\quad + A_{3X}\phi_\mu\phi_\nu(\Box\phi)\phi^\rho\phi_{\sigma\rho}\phi^\sigma\Big],\\
\mathcal{G}^{(4)}_{\mu\nu} &= \lambda_4\Big[-A_4\phi_\mu\phi_\nu\phi^\lambda\phi_\rho{}^\rho{}_\lambda + A_4\phi_{\lambda\mu}\phi^\lambda\phi_{\rho\nu}\phi^\rho - \phi_\mu\phi_\nu(A_{4X}\nabla^\lambda X)\phi_{\rho\lambda}\phi^\rho - A_4\phi_\mu\phi_\nu\phi_{\rho\lambda}\phi^{\rho\lambda}\\
&\quad - \frac{1}{2}A_4g_{\mu\nu}\phi^\lambda\phi^\rho\phi_{\sigma\rho}\phi^\sigma{}_\lambda + A_{4X}\phi_\mu\phi_\nu\phi_{\lambda\rho}\phi^\lambda\phi^{\rho\sigma}\phi_\sigma\Big],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}^{(5)}_{\mu\nu} &= \lambda_5\Big[-A_5\phi_\mu\phi_\nu\phi^\lambda\phi_{\rho\lambda}\phi^\rho(\Box\phi) - \phi_\mu\phi_\nu\phi_\lambda(A_{5X}\nabla^\lambda X)\phi^\rho\phi_{\sigma\rho}\phi^\sigma + A_5\phi_\nu\phi_{\lambda\mu}\phi^\lambda\phi^\rho\phi_{\sigma\rho}\phi^\sigma + A_5\phi_\mu\phi_{\lambda\nu}\phi^\lambda\phi^\rho\phi_{\sigma\rho}\phi^\sigma\\
&\quad - A_5\phi_\mu\phi_\nu\phi^\lambda\phi^\rho\phi_{\sigma\rho\lambda}\phi^\sigma - 2A_5\phi_\mu\phi_\nu\phi^\lambda\phi^\rho\phi_{\sigma\rho}\phi^\sigma{}_\lambda - \frac{1}{2}A_5g_{\mu\nu}\phi^\lambda\phi_{\rho\lambda}\phi^\rho\phi^\sigma\phi_{\tau\sigma}\phi^\tau + A_{5X}\phi_\mu\phi_\nu\phi^\lambda\phi^\rho\phi_{\rho\lambda}\phi^\sigma\phi^\tau\phi_{\tau\sigma}\Big],
\end{aligned}
$$

while that

$$\mathcal{J}^{\mu} = \mathcal{J}^{\mu}_{Z} + \mathcal{J}^{\mu}_{G} + \sum_{i=2}^{5} \mathcal{J}^{\mu}_{(i)},$$

with

$$\begin{aligned}
\mathcal{J}^{\mu}_{Z} &= 2\lambda_0 Z_X \phi^{\mu},\\
\mathcal{J}^{\mu}_{G} &= 2\lambda_1 G_X R \phi^{\mu},\\
\mathcal{J}^{\mu}_{(2)} &= \lambda_2\{2A_{2X}\phi^{\mu}[(\Box\phi)^2 - \phi_{\lambda\rho}\phi^{\lambda\rho}] - 2\nabla_{\nu}[A_2(g^{\mu\nu} - \phi^{\mu\nu})]\},\\
\mathcal{J}^{\mu}_{(3)} &= \lambda_3\{2A_{3X}\phi^{\mu}\Box\phi\phi^{\lambda}\phi_{\lambda\rho}\phi^{\rho} + 2A_3\Box\phi\phi^{\mu}{}_{\lambda}\phi^{\lambda} - \nabla_{\nu}[A_3(g^{\mu\nu}\phi^{\lambda}\phi_{\lambda\rho}\phi^{\rho} + \Box\phi\phi^{\mu}\phi^{\nu})]\},\\
\mathcal{J}^{\mu}_{(4)} &= \lambda_4\{2A_{4X}\phi^{\mu}\phi^{\sigma}\phi_{\sigma\rho}\phi^{\rho\lambda}\phi_{\lambda} + A_4(X)[\phi^{\mu}{}_{\rho}\phi^{\rho\lambda}\phi_{\lambda} + \phi^{\sigma}\phi_{\sigma\rho}\phi^{\rho\mu}] - \nabla_{\nu}[A_4(X)(\phi^{\mu}\phi^{\nu\rho}\phi_{\rho} + \phi^{\sigma}\phi^{\mu}{}_{\sigma}\phi^{\nu})]\},\\
\mathcal{J}^{\mu}_{(5)} &= \lambda_5\{2A_{5X}\phi^{\mu}(\phi^{\sigma}\phi_{\sigma\rho}\phi^{\rho})^2 + 2A_5(X)(\phi^{\sigma}\phi_{\sigma\rho}\phi^{\rho})(\phi^{\mu\sigma}\phi_{\sigma} + \phi^{\sigma\mu}\phi_{\sigma}) - 2\nabla_{\nu}[A_5(X)\phi^{\sigma}\phi_{\sigma\rho}\phi^{\rho}\phi^{\mu}\phi^{\nu}]\}.
\end{aligned}$$

APPENDIX B: SHEAR VISCOSITY FROM GREEN'S FUNCTIONS AND KUBO FORMULA

In order to corroborate the above computation, we perform in this section transverse and traceless perturbations following the steps of [1,41,52–54]. In the gravity side, we have that the black hole in generalized scalar-tensor theories in arbitrary dimensions plays the role of the gravitational dual of a certain fluid. Besides, to compute the shear viscosity through the holographic correspondence it is necessary to linearize the field equations, as in [27,28,30], so that the effective hydrodynamics in the boundary field theory can be constructed using conserved currents and the energy-momentum tensor. In this sense, we do not consider the scalar field perturbations, i.e., $\delta\phi = 0$. Thus, the original Ricci tensor of the background metric acquires a single nonvanishing correction at linear order in Ψ, as in Eq. (30)

$$R^{(1)}_{x_1x_2} = \frac{1+\lambda_1 G}{\mathcal{Z}_1}\left(-\frac{r^2}{2}\Box\Psi - rf'\Psi - (D-2)f\Psi\right), \quad \text{(B1)}$$

where we can identify that

$$R^{(0)}_{xx} = -rf' - (D-2)f. \quad \text{(B2)}$$

Here, the Eq. (B2) denotes any of the (diagonal) components of the zeroth-order Ricci tensor of the background metric. Combining Eqs. (B1) and (B2), we have

$$R^{(1)}_{x_1x_2} = \frac{1+\lambda_1 G}{\mathcal{Z}_1}\left(-\frac{r^2}{2}\Box\Psi + R^{(0)}_{xx}\Psi\right), \quad \text{(B3)}$$

and we can write the perturbed Einstein tensor to first order as

$$\begin{aligned}
G^{(1)}_{x_1x_2} &= \frac{1+\lambda_1 G}{\mathcal{Z}_1}\left(-\frac{r^2}{2}\Box\Psi + R^{(0)}_{xx}\Psi - \frac{1}{2}R^{(0)}\Psi\right), &\text{(B4)}\\
&= \frac{1+\lambda_1 G}{\mathcal{Z}_1}\left(-\frac{r^2}{2}\Box\Psi + G^{(0)}_{xx}\Psi\right). &\text{(B5)}
\end{aligned}$$

Together with the above and considering the Einstein tensor as given by $G^{(0)}_{xx} = T^{(0)}_{xx}/2$, we can write Eq. (B5) as

$$\frac{1+\lambda_1 G}{\mathcal{Z}_1}\left(\Box\Psi - \frac{1}{r^2}\left(T^{(0)}_{xx} + T^{(1)}_{x_1x_2}\right)\right) = \sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}\left(\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}\Box\Psi - \frac{1}{r^2}\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}\left(T^{(0)}_{xx} + T^{(1)}_{x_1x_2}\right)\right) = 0, \quad \text{(B6)}$$

where $T^{(1)}_{x_1x_2} = (\delta T_{x_1x_2}/\delta g_{x_1x_2})\delta g_{x_1x_2}$, with

$$T^{(0)}_{\mu\nu} = -\frac{2}{\mathcal{Z}_1}\frac{\delta}{\delta g^{\mu\nu}}\left(\sum_{i=2}^{5}\lambda_i A_i \mathcal{L}_i\right) + \frac{g_{\mu\nu}}{\mathcal{Z}_1}\sum_{i=2}^{5}\lambda_i A_i \mathcal{L}_i. \quad \text{(B7)}$$

Now, for the component $T^{(0)}_{xx}$ we have

$$T^{(0)}_{xx} = -\frac{2}{\mathcal{Z}_1}\frac{\delta}{\delta g^{xx}}\left(\sum_{i=2}^{5}\lambda_i A_i \mathcal{L}_i\right) + \frac{g_{xx}}{\mathcal{Z}_1}\sum_{i=2}^{5}\lambda_i A_i \mathcal{L}_i, \quad \text{(B8)}$$

and using the Lagrangians (4) and (5) we can see that all the contributions from $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$, and $\mathcal{L}_5$ becomes null, because the kinetic coupling is a function of the radial component, namely, $\phi = \phi(r)$. In this case, $T^{(0)}_{xx} = 0 = T^{(1)}_{x_1x_2}$ and we can write

$$\mathcal{Z}_1\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}\Box\Psi = 0, \quad \text{(B9)}$$

where in the metric background (8) with $h=f$, and considering Eq. (B9), we have

$$\mathcal{Z}_1\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}f\Psi'' + \mathcal{Z}_1\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}\left(f' + \frac{(D-2)f}{r}\right)\Psi' + \mathcal{Z}_1\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}\frac{\ddot{\Psi}}{f} = 0. \tag{B10}$$

Now, we consider the following ansatz

$$\Psi = \int \frac{d^{(D-1)}k}{(2\pi)^{(D-1)}} e^{ikx}\chi(r,k). \tag{B11}$$

Here $x=(t,\vec{x})$ and $k=(\omega,\vec{q})$, where as we know in general for the mass term Eq. (B11) contains contributions $k^2=q^2-\omega^2$. Nevertheless, considering the case $\omega\to 0$ and spatial momentum $q=0$, we can find that $\chi(r,k)=\chi(r)$. In addition, we can write Eq. (B11) in terms of a Klein-Gordon-like equation [41,54] as follows

$$\frac{1}{\sqrt{-g}}\partial_\alpha\left(\mathcal{Z}_1\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}\sqrt{-g}g^{\alpha\beta}\partial_\beta\Psi\right) = 0, \tag{B12}$$

and effective action for Eq. (B12) can be written as

$$S = -\int d^{(D-1)}k\left(\frac{N(r)}{2}\frac{d\chi(r,k)}{dr}\frac{d\chi(r,-k)}{dr}\right), \tag{B13}$$

where $N(r)=\mathcal{Z}_1\sqrt{(1+\lambda_1 G)/\mathcal{Z}_1}\sqrt{-g}g^{rr}$. This on-shell action reduces to the surface term

$$S = -\int d^{(D-1)}k\left(\frac{1}{2}N(r)\chi(r,k)\partial_r\chi(r,-k)\right)\Big|_{r_h}^{\infty}. \tag{B14}$$

Following the procedure of [41,54], we can extract the retarded Green's function, which reads

$$G^R_{x_1x_2,x_1x_2}(\omega,0) = -2\left(\mathcal{Z}_1\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}\sqrt{-g}g^{rr}\Big|_{r_h}\right) \times \chi(r,-\omega)\partial_r\chi(r,\omega)|_{r_h}. \tag{B15}$$

This expression for the Green's function diverges at the horizon. In order to remove this divergence we implement a regularity condition such that the derivative of χ is given in terms of $\chi(r_h)$ at leading order in $\omega\to 0$. This low frequency limit corresponds to the hydrodynamical limit and has physical importance to define transport coefficients, such as the shear viscosity, in our case [55]. From Eq. (B15), we have

$$G^R_{x_1x_2,x_1x_2}(\omega,0) = -2i\omega r_h^{D-2}\mathcal{Z}_1\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}} = -2i\omega\left(\frac{1}{4\pi}\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}\right)s, \tag{B16}$$

where the entropy density s was given in (29). Finally, the shear viscosity η [32,33] is given by

$$\eta = -\lim_{\omega\to 0}\frac{1}{2\omega}\mathrm{Im}G^R_{x_1x_2,x_1x_2} = \left(\frac{1}{4\pi}\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}\right)s \Rightarrow \frac{\eta}{s} = \left(\frac{1}{4\pi}\sqrt{\frac{1+\lambda_1 G}{\mathcal{Z}_1}}\right), \tag{B17}$$

recovering the viscosity/entropy density ratio obtained in Eq. (34). According to [28], the right-hand side of Eq. (B17) is related to the absorption cross-section of low-energy gravitons. From usual theories that consider the low-frequency regime, the relation η/s is not violated but has a universal bound. The main idea behind the computations of the viscosity/entropy density ratio is to characterize how close a given fluid is to be perfect. However, in our study we show that for the generalized scalar-tensor theories in arbitrary dimensions, Eqs. (2)–(5), this limit is violated, due to the coupling constants that control the influence of the kinetic term in this relation.

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