Regular black holes: Entropy products and central charges

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In this paper, for a variety of types of regular black hole solutions, we investigate the entropy products of inner and outer horizons. Similar to singular black holes, for the regular ones we find that universality (mass independence) of the entropy product is true for some solutions, and it fails for some others. In the case of regular black holes that respect the universality, we read central charges of the dual CFTs from the entropy product, according to the thermodynamics method introduced in chen *et al.* [J. High Energy Phys. 03 (2013) 102]. For these solutions, we also calculate central charges, using the asymptotic symmetry group formalism. The results of these two approaches are the same, which means that universality of the entropy product provides a simple method to find central charges of the dual CFTs.

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I. INTRODUCTION

According to the singularity theorem [1], the formation of singularities is unavoidable (under some circumstances) in general relativity. The existence of singularities represents the failure of general relativity, and to protect against it, Penrose proposed that singularities are covered by horizons (the weak cosmic censorship conjecture). It is believed that singularities appear in the classical theory of gravity, and by taking into account the quantum effects, they can be avoided [2,3]. Inspired by this idea, Bardeen proposed [4] a regular black hole (RBH) solution in which the singularity is replaced by a de Sitter core. After the seminal work of Bardeen, a large number of RBHs have been constructed [5–19]. Recently, some models for RBHs with a Minkowski core have also been proposed [13,14,20,21]. However, a main concern with RBHs is the violation of energy conditions. In fact, it has been shown [17,22,23] that the energy conditions do not hold for many RBH solutions.

Black holes (BHs) are good locations to explore the relation of gravity and quantum mechanics. It has been observed [24–36] that the horizons' entropy product for many BH solutions is universal (mass independent), as $S_+S_- = 4\pi^2 N$, where N is related to the quantized charges of the solution, like angular momentum and electric/magnetic charge. It has been shown [30] that the condition $T_+S_+ = T_-S_-$, where T_+ (T_-) is the Hawking temperature of the outer (inner) horizon, is equivalent to the mass independence of S_+S_-. This condition also implies that the central charges of the left-moving and right-moving sectors in the dual conformal field theory (CFT) are the same ($c_L = c_R$). It is discussed in Ref. [37] that for the solutions with a universal entropy product, one can read the central charge(s) of the dual CFT(s) as

$$c_i = \frac{6}{4\pi^2}\frac{\partial(S_+S_-)}{\partial N_i}, \tag{1}$$

which is the thermodynamics method for finding the central charge.

There are also BH solutions in which the entropy product is not universal. For instance, it has been observed [38] that the universality of the entropy product fails for some BH solutions of higher-curvature gravity. The entropy product is also mass dependent for the BHs with Newman-Unti-Tamburino (NUT) charge [39,40]. Although in the case of 5D Myers-Perry and the BTZ black hole the universality is true, it fails for the Myers-Perry BHs in $D \geq 6$ and for Kerr-AdS BHs in $D \geq 4$ [30].

In the case of RBHs, it has been shown [41] that universality fails for the ABG (Ayón-Beato and García) BH. In Ref. [17], it is shown that for RBHs of Einstein gravity coupled to extended nonlinear electrodynamics, by choosing adequate parameters, S_+S_- can be mass independent. However, for a large number of RBHs (maybe because of the complex form of the metric), the entropy product has not yet been studied.

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This paper is organized as follows: In Sec. II, we briefly review the thermodynamics method for finding central charge(s) of the dual CFT(s). Then, we investigate the entropy product for RBHs where the universality fails for some of them (solutions in Sec. III), and it works for some others (Sec. IV). In Sec. V, using the thermodynamics method, we find the central charges of the dual CFTs for RBHs with a universal entropy product, studied in Sec. IV. We then recalculate the central charges using the asymptotic symmetry group formalism to check the result of the thermodynamics method. We conclude the paper with a discussion on the universality of the entropy product and its importance in the BH solutions.

Throughout the paper, we set Newton's gravitational constant to unity—i.e., $G = 1$.

II. UNIVERSALITY OF ENTROPY PRODUCT AND DUAL CFTs

In this section, we review some BH solutions with a universal entropy product. For these solutions, it is possible to find central charges of the dual CFTs using the thermodynamics method.

A. The Kerr-Newman BH

The Kerr-Newman BH [42] is determined by three conserved charges: M (mass), J (angular momentum), and Q (electric charge). In Ref. [26], it has been shown that the entropy product of the inner and outer horizons for this solution is universal, as

$$S_+S_- = \pi^2(4J^2 + Q^4). \tag{2}$$

The dual CFT for this solution is studied in Refs. [43,44], where it is shown that the J-picture central charge c^J and the Q-picture central charge c^Q are given by

$$c^J = 12\,J, \qquad c^Q = 6Q^3. \tag{3}$$

Due to the universality of the entropy product, one can also find these central charges by using the thermodynamics method [Eq. (1)]:

$$\begin{aligned} c^J &= \frac{3}{2}\frac{\partial}{\partial J}(4J^2 + Q^4) = 12\,J, \\ c^Q &= \frac{3}{2}\frac{\partial}{\partial Q}(4J^2 + Q^4) = 6Q^3, \end{aligned} \tag{4}$$

which are in agreement with Eq. (3).

B. The Myers-Perry BH

The Myers-Perry BH [45] in five dimensions is characterized by its two angular momenta J_ϕ, J_ψ and mass M. Applying the Kerr/CFT analysis, it is shown [46–48] that for the Myers-Perry BH, there are two dual CFT descriptions associated with the rotations along the ϕ and ψ directions. The central charges of these dual CFTs are

$$c_\phi = 6J_\psi, \qquad c_\psi = 6J_\phi. \tag{5}$$

It is also shown [24,30] that the horizons' entropy product for this solution is universal—i.e.,

$$S_+S_- = 4\pi^2 J_\phi J_\psi. \tag{6}$$

Now, using the thermodynamics method [Eq. (1)], one can easily find the central charges [Eq. (5)] from the above entropy product.

There are also more examples for the relation between the entropy product and central charges; we refer the reader to Refs. [30,36,37,49]. These examples tell us that the universality of the entropy product implies the existence of dual CFTs. In the following, we check the validity of the thermodynamics method in the case of RBH solutions. To this end, we should find solutions with a universal entropy product, so in the following sections we investigate the universality for RBHs.

III. RBHs WITH A NONUNIVERSAL ENTROPY PRODUCT

As we mentioned, the product of horizon entropies is mass independent if the condition $T_+S_+ = T_-S_-$ is satisfied [30]. In this section and the next one, by explicit calculations we investigate this universality condition for RBHs. We consider a variety of types of regular solutions in four and higher dimensions, both with and without angular momentum and charge. We also consider RBHs with both de Sitter and Minkowski cores. We find BH horizons by solving $g^{rr} = 0$, which may have negative or complex roots. By the outer (inner) horizon, we mean the largest (smallest) positive real root.

A. The Bardeen BH

The Bardeen BH is the first known RBH solution [4]. The matter source which supplies this solution was unknown for many years till Ayón-Beato and Garsía showed that the Bardeen BH is a magnetic solution of Einstein equations coupled to a nonlinear electrodynamics [5,6]. The metric of a Bardeen BH is in the form

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \\ f &= 1 - \frac{2Mr^2}{(r^2+q^2)^{3/2}}, \end{aligned} \tag{7}$$

where M is the mass and q is the magnetic monopole charge. In a Bardeen BH, the singularity at $r = 0$ is replaced by a de Sitter core. For this solution, there are three horizons (roots of $f = 0$) which are too long to be written here. We denote the outer and inner horizons by r_+ and r_-, respectively. The entropy and temperature of Bardeen BH on the outer and inner horizons are

$$S_\pm = \pi r_\pm^2, \qquad T_\pm = \pm \frac{M r_\pm ({r_\pm}^2 - 2q^2)}{2\pi (q^2 + {r_\pm}^2)^{5/2}}. \tag{8}$$

It is straightforward to check that $T_+ S_+ \neq T_- S_-$, which means that entropy product is not universal for the Bardeen BH.

B. The Hayward BH

Another RBH solution with a de Sitter core is the Hayward BH [7]. For this solution, the metric is in the static spherically symmetric form of Eq. (7) with the function

$$f = 1 - \frac{2Mr^2}{r^3 + q^3}. \tag{9}$$

Similarly to the Bardeen BH, we find the entropy and temperature on the outer and inner horizons as

$$S_\pm = \pi r_\pm^2, \qquad T_\pm = \pm \frac{M r_\pm (r_\pm^3 - 2q^3)}{2\pi (r_\pm^3 + q^3)}. \tag{10}$$

One can check that the condition $T_+ S_+ = T_- S_-$ is not satisfied here, so the universality of the entropy product is not valid for the Hayward BH.

C. The rotating Bardeen and Hayward BHs

The extension of the Bardeen and Hayward RBHs to the solutions containing angular momentum is done in Ref. [8]. The line element for these solutions is

$$ds^2 = -F(r,\theta)dt^2 + \frac{dr^2}{G(r,\theta)} + \Sigma(r,\theta)d\theta^2 + H(r,\theta)d\phi^2 - 2K(r,\theta)dtd\phi, \tag{11}$$

with the functions

$$F(r,\theta) = \frac{\Pi(r) - a^2 \sin^2\theta}{\Sigma(r,\theta)}, \qquad G(r,\theta) = \frac{\Pi(r)}{\Sigma(r,\theta)},$$
$$\Sigma(r,\theta) = r^2 + a^2 \cos^2\theta, \qquad K(r,\theta) = \frac{2rm(r)a\,\sin^2\theta}{\Sigma(r,\theta)},$$
$$H(r,\theta) = \frac{(r^2 + a^2)^2 - \Pi(r) a^2 \sin^2\theta}{\Sigma(r,\theta)} \sin^2\theta. \tag{12}$$

The function $\Pi(r)$ in the above is introduced as $\Pi(r) = r^2 + a^2 - 2rm(r)$, where a is the rotation parameter. In order to remove the singularity, $m(r)$ is given in the form

$$m(r) = M \left(\frac{r^p}{r^p + r_0^p} \right)^{3/p}. \tag{13}$$

The rotating Bardeen and Hayward BHs are constructed by setting $p = 2$ and $p = 3$ in Eq. (13), respectively. Note that for $r \gg r_0$, the above mass reaches the Kerr BH mass M. Moreover, in the case of $r_0 = 0$, one finds $m(r) = M$ and Eq. (11) reduces to the Kerr solution; in other words, one can interpret r_0 as the deviation parameter from the Kerr BH. Horizons of the solution are roots of $\Pi(r) = 0$. In order to find entropy and temperature on the inner and outer horizons, it is more convenient to solve $\Pi(r_+) = 0$, $\Pi(r_-) = 0$ and find a, M in terms of r_- and r_+. Noticing this point, we calculate entropy and temperature for the rotating Hayward BH ($p = 3$). The result is

$$S_+ = \frac{\pi {r_+}^4 (r_- + r_+)({r_0}^3 + {r_-}^3)}{(r_- + r_+)({r_-}^2 + {r_+}^2){r_0}^3 + {r_-}^3 {r_+}^3}, \qquad S_- = \frac{\pi {r_-}^4 (r_- + r_+)({r_0}^3 + {r_+}^3)}{(r_- + r_+)({r_-}^2 + {r_+}^2){r_0}^3 + {r_-}^3 {r_+}^3},$$
$$T_+ = \frac{(r_- - r_+)[2r_0^6 (r_+ + r_-)^2 - r_0^3 r_+^2 (4r_-^3 + 4r_-^2 r_+ + 2r_- r_+^2 + r_+^3) - r_-^3 r_+^5]}{4\pi r_+^3 (r_+ + r_-)(r_0^3 + r_-^3)(r_0^3 + r_+^3)},$$
$$T_- = \frac{(r_- - r_+)[2r_0^6 (r_+ + r_-)^2 - r_0^3 r_-^2 (4r_+^3 + 4r_+^2 r_- + 2r_+ r_-^2 + r_-^3) - r_+^3 r_-^5]}{4\pi r_-^3 (r_+ + r_-)(r_0^3 + r_-^3)(r_0^3 + r_+^3)}. \tag{14}$$

In the case of a rotating Bardeen BH ($p = 2$), $S_\pm$ are messy and we do not show them here. We check that for both rotating Bardeen and Hayward solutions $T_+ S_+ \neq T_- S_-$, which means that the entropy product is not universal for them. In the case of small values for the deviation parameter r_0, by keeping the first order in Taylor expansion around $r_0 = 0$, it is possible to find the perturbed (around the Kerr) solution and then calculate the entropy and temperature. For instance, we find the entropy and temperature for the Bardeen BH with small r_0 to be

$$S_+ = \frac{\pi r_-(r_- + r_+)(3r_0^2 - 2r_+^2)}{3r_0^2 + 2r_- r_+},$$
$$S_- = \frac{\pi r_+(r_- + r_+)(3r_0^2 - 2r_-^2)}{3r_0^2 + 2r_- r_+},$$
$$T_+ = \frac{(r_+ - r_-)[r_0^2(r_- + 2r_+) + \frac{2}{3}r_- r_+^2]}{4\pi r_+ r_-(r_+ + r_-)(r_0^2 - \frac{2}{3}r_+^2)},$$
$$T_- = \frac{(r_+ - r_-)[r_0^2(r_+ + 2r_-) + \frac{2}{3}r_+ r_-^2]}{4\pi r_+ r_-(r_+ + r_-)(r_0^2 - \frac{2}{3}r_-^2)}. \tag{15}$$

It is straightforward to check that, due to violation of $T_+S_+ = T_-S_-$, the entropy product of the rotating Bardeen solution with small r_0 is not universal. In other words, adding rotation to the Bardeen and Hayward BHs does not lead to the universality of the entropy product.

D. The Frolov-Zelnikov BH

In Ref. [9], an evaporating (time-evolving) RBH with a de Sitter core is introduced. The metric is in the form

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$f = 1 - \frac{2M(t)r^2}{r^3 + 2M(t)\ell^2 + \ell^3}. \tag{16}$$

In the static case, this solution differs from the Hayward BH by an extra term ℓ^3 in the denominator of function f. This term guarantees the metric smoothness at $r = 0$ in the limit $M(t) \to 0$. For this solution, one can find the horizons by solving $f = 0$ [in the static case $M(t) = M$] as

$$r_+ = \frac{\ell}{2} + M + \frac{1}{2}\sqrt{4M(M - \ell) - 3\ell^2},$$
$$r_- = \frac{\ell}{2} + M - \frac{1}{2}\sqrt{4M(M - \ell) - 3\ell^2}, \quad r_0 = -\ell. \tag{17}$$

For Hayward and Frolov solutions, ℓ is positive, so the third root ($r_0 = -\ell$) is a negative-valued radius, which is not relevant to our study. Now it is possible to find the entropy and temperature as

$$S_+ = \frac{\pi}{2}(\ell + 2M)(2M - \ell + \sqrt{4M(M - \ell) - 3\ell^2}),$$
$$S_- = \frac{\pi}{2}(\ell + 2M)(2M - \ell - \sqrt{4M(M - \ell) - 3\ell^2}),$$
$$T_+ = \frac{Mr_+(\frac{r_+^3}{2} - \ell^3 - 2M\ell^2)}{\pi(r_+^3 + 2\ell^2 M + \ell^3)^2},$$
$$T_- = \frac{Mr_-(\frac{r_-^3}{2} - \ell^3 - 2M\ell^2)}{\pi(r_-^3 + 2\ell^2 M + \ell^3)^2}. \tag{18}$$

It is easy to check that $T_+S_+ \neq T_-S_-$, which means that the entropy product is mass dependent in the case of a Frolov-Zelnikov BH.

E. Five-dimensional RBHs

In this part, we investigate the entropy product for some five-dimensional BHs. We consider regular solutions with both Minkowski and de Sitter cores.

1. Static RBH

A five-dimensional static RBH is introduced in Ref. [12]. The line element is in the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2),$$
$$f(r) = 1 - \frac{M}{r^2}e^{-k/r^2}. \tag{19}$$

This solution, which satisfies the equations of motion of 5D Einstein gravity coupled to nonlinear electrodynamics, is a RBH with a Minkowski core at $r = 0$. Here, M is the black hole mass, and k is related to the magnetic monopole charge (q) via $q^2 = Mk$. The solution (19) reduces to a 5D Schwarzschild-Tangherlini BH when $k = 0$, and one can also find the Minkowski spacetime by setting $M = 0$. For this RBH, the temperatures on the outer and inner horizons are [12]

$$T_+ = \frac{1}{4\pi r_+}\left(1 - \frac{k}{r_+^2}\right), \qquad T_- = \frac{-1}{4\pi r_-}\left(1 - \frac{k}{r_-^2}\right), \tag{20}$$

and the entropies are

$$S_+ = \frac{\pi^2}{2}r_+^3, \qquad S_- = \frac{\pi^2}{2}r_-^3. \tag{21}$$

This solution is not universal, since $T_+S_+ \neq T_-S_-$.

2. Magnetically charged Myers-Perry solution

Applying the Newman-Janis algorithm on the static solution of the previous subsection, a 5D rotating RBH is constructed in Ref. [12]. This solution is a magnetically charged Myers-Perry BH with a Minkowski core. The metric is

$$ds^2 = -\left(1 - \frac{Me^{-k/r^2}}{\rho^2}\right)dt^2 + \frac{r^2\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 - \frac{2Ma\sin^2\theta e^{-k/r^2}}{\rho^2}dtd\phi - \frac{2Mb\cos^2\theta e^{-k/r^2}}{\rho^2}dtd\psi$$
$$+\sin^2\theta\left(r^2 + a^2 + \frac{Ma^2\sin^2\theta e^{-k/r^2}}{\rho^2}\right)d\phi^2 + \cos^2\theta\left(r^2 + b^2 + \frac{Mb^2\cos^2\theta e^{-k/r^2}}{\rho^2}\right)d\psi^2 + \frac{2Mab\sin^2\theta\cos^2\theta e^{-k/r^2}}{\rho^2}d\phi d\psi, \quad (22)$$

where ρ and Δ are defined as

$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta,$$
$$\Delta = (r^2 + a^2)(r^2 + b^2) - Mr^2 e^{-k/r^2}. \quad (23)$$

The above solution, which is a RBH with a Minkowski core at $r = 0$, is characterized by four parameters: mass M and rotation parameters a, b, and k, the latter of which is related to the magnetic charge. By setting $k = 0$ in Eq. (22), one can recover the 5D Myers-Perry BH, so k can be interpreted as a deviation from the Myers-Perry geometry. Moreover, by inserting $a = b = 0$, the solution reduces to a Schwarzschild-Tangherlini BH.

Although one may find the horizons by solving $\Delta = 0$, it is more convenient to find M and a in terms of $r_\pm$. For the regular solution with a small charge parameter k, we calculate the entropy and temperature on the horizons. The result is

$$S_+ = \frac{\pi^2(k - r_+^2)(b^2 + r_+^2)(b^2 + r_-^2)}{2r_+(k + b^2)},$$
$$S_- = \frac{\pi^2(k - r_-^2)(b^2 + r_+^2)(b^2 + r_-^2)}{2r_-(k + b^2)},$$
$$T_+ = \frac{r_+^4(r_-^2 - b^2 - 2k) - k^2(b^2 + r_-^2)}{2\pi r_+^3(b^2 + r_-^2)(k - r_+^2)},$$
$$T_- = \frac{r_-^4(r_+^2 - b^2 - 2k) - k^2(b^2 + r_+^2)}{2\pi r_-^3(b^2 + r_+^2)(r_-^2 - k)}. \quad (24)$$

Now, it is straightforward to check that the condition $T_+S_+ = T_-S_-$ does not hold, and so the entropy product for the above solution is not universal.

3. Electrically charged rotating Bardeen BH

The 5D Bardeen BH with electric charge and its rotating version are introduced in Refs. [10,11]. The line element for an electrically charged rotating Bardeen BH is [11]

$$ds^2 = -dt^2 + \frac{m(r)}{\rho^2}(dt - a\sin^2\theta d\phi + b\cos^2\theta d\psi)^2$$
$$+ \frac{r^2\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2$$
$$+ (r^2 + b^2)\cos^2\theta d\psi^2, \quad (25)$$

where ρ, $m(r)$, and Δ are as follows:

$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \qquad m(r) = \mu\left(\frac{r^3}{r^3 + q_e^3}\right)^{4/3},$$
$$\Delta = (r^2 + a^2)(r^2 + b^2) - m(r)r^2. \quad (26)$$

The above solution is characterized by the four parameters a, b, μ, and q_e, where a and b are rotation parameters around the ϕ, ψ axes, and μ and q_e are related to the mass and electric charge of the BH, respectively. By setting $q_e = 0$ in Eq. (25), the Myers-Perry BH is recovered, so q_e determines the deviation of a charged rotating Bardeen BH from the Myers-Perry solution.

Horizons of this solution are the roots of $\Delta = 0$, and due to the complicated form of the metric, one finds messy terms for the entropies and temperatures on the horizons. In the following, for the sake of simplicity, we consider a single rotating Bardeen BH with a small value of charge, which is obtained by setting $b = 0$ and performing Taylor expansion for $m(r)$. Denoting the ranges of angles in this solution as $\theta \in [0, \pi/2]$ and $\phi, \psi \in [0, 2\pi]$, we calculate the entropy and temperature on the inner and outer horizons. The result is

$$S_+ = \frac{\pi^2 r_+ r_-^3(r_- + r_+)(4q_e^3 - 3r_+^3)}{8q_e^3(r_-^2 + r_-r_+ + r_+^2)},$$
$$S_- = \frac{\pi^2 r_- r_+^3(r_- + r_+)(4q_e^3 - 3r_-^3)}{8q_e^3(r_-^2 + r_-r_+ + r_+^2)},$$
$$T_+ = \frac{-3r_-^4 - 3r_-^3 r_+ + 2r_-^2 r_+^2 + 2r_- r_+^3 + 2r_+^4}{3\pi r_-^3 r_+^4(r_- + r_+)}q_e^3,$$
$$T_- = \frac{-3r_+^4 - 3r_+^3 r_- + 2r_-^2 r_+^2 + 2r_+ r_-^3 + 2r_-^4}{3\pi r_+^3 r_-^4(r_- + r_+)}q_e^3, \quad (27)$$

and one can easily check that $T_+S_+ \neq T_-S_-$, which means the universality of the entropy product fails in this solution.

IV. RBHs WITH A UNIVERSAL ENTROPY PRODUCT

Until now, we have observed that the universality of the entropy product fails for many RBHs. However, there are some regular solutions which respect the universality.

A. The Kerr-like RBH

References [13,14] have introduced a regularized Kerr BH with modified mass $m \to m(r) = me^{-\ell/r}$. The metric is given by

$$ds^2 = -\frac{\Delta'}{\Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\Sigma}{\Delta'}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}[(r^2+a^2)d\phi - adt]^2,$$
$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta' = r^2 + a^2 - 2mre^{-\ell/r}. \quad (28)$$

This RBH is characterized by its mass m, the rotation parameter a, and the regularization parameter ℓ. Setting $\ell = 0$, Eq. (28) reduces to the Kerr metric, so ℓ can be also viewed as the deviation form Kerr. This solution at $r \to \infty$ reaches the Kerr BH, while the ring singularity of Kerr is replaced by an asymptotically Minkowski spacetime [14]. For small values of ℓ, we keep the first order in Taylor expansion of the metric. We also rewrite a and m in terms of $r_\pm$. Now, the entropy and temperature for this perturbed solution take the following forms:

$$S_+ = \pi(r_+ + r_-)(\ell - r_+), \qquad S_- = \pi(r_+ + r_-)(\ell - r_-),$$
$$T_+ = \frac{r_- - r_+}{4\pi(r_- + r_+)(\ell - r_+)}, \qquad T_- = \frac{r_- - r_+}{4\pi(r_- + r_+)(\ell - r_-)}. \quad (29)$$

It is easy to check that $T_+S_+ = T_-S_-$ is satisfied, which means that the entropy product is universal. We also calculate the angular momentum for the solution with small ℓ as

$$J = \frac{1}{2}(r_+ + r_-)\sqrt{r_-r_+ - \ell(r_+ + r_-)} = ma. \quad (30)$$

Noticing Eqs. (29) and (30) and ignoring the ℓ^2, ℓ^3 ..., terms, one finds the universality of the entropy product:

$$S_+S_- = 4\pi^2J^2. \quad (31)$$

In the next section, we will use this result to find the central charge of the dual CFT for the regular Kerr-like BH.

B. Reissner-Nordström outside a de Sitter core

In Ref. [15], a regular solution is constructed by connecting a de Sitter space to the core of a Reissner-Nordström BH. The metric is written in the static, spherically symmetric form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (32)$$

In order to remove the singularity at $r = 0$, we suppose a sphere with radius r_0, centered at $r = 0$, which is filled by a static charged perfect fluid distribution with spherical symmetry. Solving the equations of motion, the functions $A(r)$ and $B(r)$ are obtained as [15]

$$B(r) = A^{-1}(r) = \begin{cases} 1 - \frac{r^2}{R^2}, & r \le r_0, \\ 1 - \frac{2m}{r} + \frac{q^2}{r^2}, & r \ge r_0. \end{cases} \quad (33)$$

The above solution is determined by its mass (m), charge (q), the radius of de Sitter space at the core (R), and the radius of matter distribution (r_0). In the region $r > r_0$, the solution is given by the Reissner-Nordström metric, which has two horizons at $r_\pm = m \pm \sqrt{m^2 - q^2}$. It has been shown in Ref. [15] that $r_0 \le r_-$; in other words, the region of matter distribution lies inside the inner horizon, and the spacetime for $r \ge r_-$ is given by the Reissner-Nordström metric. The entropy and temperature on the inner and outer horizons for this RBH are just like those of the Reissner-Nordström BH:

$$S_+ = \pi r_+^2, \qquad S_- = \pi r_-^2,$$
$$T_+ = \frac{r_+ - r_-}{4\pi r_+^2}, \qquad T_- = \frac{r_+ - r_-}{4\pi r_-^2}. \quad (34)$$

The universality is true, since the condition $T_+S_+ = T_-S_-$ is satisfied. Moreover, the entropy product for this solution is mass independent:

$$S_+S_- = \pi^2q^4. \quad (35)$$

C. The topological star

The topological star is a solution which is obtained from dimensional reduction of a 5D solution in Einstein-Maxwell theory [16]. The starting point is the static, spherically symmetric metric in five dimensions with a magnetic flux F:

$$ds^2 = -f_S(r)dt^2 + f_B(r)dy^2 + \frac{dr^2}{f_S(r)f_B(r)} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \qquad F = P\sin\theta d\theta \wedge d\phi. \quad (36)$$

Coordinate y in the above equation parametrizes a circle of perimeter $2\pi R_y$, and the functions $f_S(r)$, $f_B(r)$, and P are given by

$$f_B(r) = 1 - \frac{r_B}{r}, \qquad f_S(r) = 1 - \frac{r_S}{r},$$
$$P = \pm\frac{1}{\kappa_5^2}\sqrt{\frac{3r_Sr_B}{2}}, \quad (37)$$

where κ_5 is the gravitational coupling. By Kaluza-Klein reduction along y, one finds the following solution:

$$ds_5^2 = e^{2\Phi}ds_4^2 + e^{-4\Phi}dy^2, \qquad e^{2\Phi} = f_B^{-\frac{1}{2}},$$
$$ds_4^2 = f_B^{\frac{1}{2}}\left[-f_Sdt^2 + \frac{dr^2}{f_Bf_S} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right]. \quad (38)$$

In the case of $r_B = 0$, the above metric reduces to the Schwarzschild BH. Note also that the coefficient $f_B^{1/2}$ is imaginary for $r < r_B$, which means that the spacetime ends at $r = r_B$. In other words, the singularity at $r = 0$ is excluded from the four-dimensional BH. There are two horizons (roots of the term $f_B f_S$) for this solution at r_B and r_S with the entropy and temperature

$$S_S = \frac{\pi r_S^2}{G}, \qquad S_B = \frac{\pi r_B^2}{G},$$
$$T_S = \frac{1}{4\pi r_S}, \qquad T_B = \frac{r_S}{4\pi r_B^2}. \tag{39}$$

The universality is held, since the condition $T_+S_+ = T_-S_-$ is respected. For this solution, the mass and magnetic charge take to the form

$$M = \frac{2\pi}{\kappa_4^2}(2r_S + r_B), \qquad Q_m = \frac{1}{\kappa_4}\sqrt{\frac{3}{2} r_B r_S}, \tag{40}$$

where κ_4 is the 4D gravitational constant, which is related to κ_5 as $\kappa_4 = \frac{\kappa_5}{\sqrt{2\pi R_y}}$. One can check that the entropy product for this regular solution is mass independent as [50]

$$S_S S_B = \pi^2 Q_m^4. \tag{41}$$

D. BHs in Einstein gravity coupled to extended nonlinear electrodynamics

A physical source for the RBHs is nonlinear electrodynamics [5,6]. For the Einstein gravity coupled to the nonlinear electrodynamics

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}[R - \mathcal{L}(\mathcal{F})], \tag{42}$$

where $F_{\mu\nu} = dA_\nu$ is the field strength tensor, $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$, and Lagrangian density $\mathcal{L}$ is a function of $\mathcal{F}$, it has been shown [17] that one can construct a variety of classes of RBHs by choosing different forms of $\mathcal{L}(\mathcal{F})$. In the following, we review three classes of these RBHs.

1. Bardeen class

Considering the Lagrangian density

$$\mathcal{L} = \frac{4\mu(\alpha\mathcal{F})^{5/4}}{\alpha(1 + \sqrt{\alpha\mathcal{F}})^{1+\mu/2}}, \tag{43}$$

where $\mu > 0$ and $\alpha > 0$ are some constants, the general form for the two-parameter (mass and magnetic charge) BH solutions is

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$f = 1 - \frac{2M}{r} - \frac{2\alpha^{-1}q^3 r^{\mu-1}}{(r^2 + q^2)^{\mu/2}}, \tag{44}$$

in the above, q is related to the black hole magnetic charge, and the ADM mass can be read off from the asymptotic behavior of the metric

$$f = 1 - \frac{2(M + \alpha^{-1}q^3)}{r} + \cdots, \tag{45}$$

which implies that $M_{\rm ADM} = M + M_{em}$, where $M_{em} = \alpha^{-1}q^3$. In other words, the BH mass receives contributions from the Schwarzschild mass M and the nonlinear effects that are denoted by M_{em}. Since by setting $M = 0$ and $\mu = 3$ the Bardeen BH [Eq. (7)] is recovered, Eq. (44) is called the "Bardeen class." It has been observed [17] that for a solution with $M = 0$ and $\mu = 2$, the entropy product is universal, as

$$S_+S_- = \pi^2 q^4. \tag{46}$$

In the next section, we will use the above result to find the central charge of dual CFT.

2. Hayward class

The nonsingular BHs of the Hayward class are solutions to Einstein gravity coupled to a nonlinear electrodynamics with Lagrangian density [17]

$$\mathcal{L} = \frac{4\mu(\alpha\mathcal{F})^{(\mu+3)/4}}{\alpha[1 + (\alpha\mathcal{F})^{\mu/4}]^2}. \tag{47}$$

Similarly to the Bardeen class, for these BHs, the metric is in the static, spherically symmetric form [Eq. (44)] with the function

$$f = 1 - \frac{2M}{r} - \frac{2\alpha^{-1}q^3 r^{\mu-1}}{r^\mu + q^\mu}. \tag{48}$$

By setting $M = 0$ and $\mu = 3$, one finds the Hayward solution; for this reason it is called the "Hayward class." It has been found [17] that in the case of vanishing Schwarzschild mass ($M = 0$) and $\mu = 2$, the entropy product is universal, as

$$S_+S_- = \pi^2 q^4. \tag{49}$$

3. New class

By choosing the Lagrangian density [17]

$$\mathcal{L} = \frac{4\mu\mathcal{F}}{[1 + (\alpha\mathcal{F})^{1/4}]^{\mu+1}}, \tag{50}$$

a “new class” of RBHs can be obtained. The metric is given by Eq. (44) with the new function

$$f = 1 - \frac{2M}{r} - \frac{2\alpha^{-1} q^3 r^{\mu-1}}{(r+q)^{\mu}}. \tag{51}$$

For a solution with $M = 0$ and $\mu = 2$, the entropy product is universal, as with Eq. (49).

V. CENTRAL CHARGES AND THE ENTROPY PRODUCT

In Secs. III and IV, we studied the universality of the entropy product in the case of RBHs. We found that universality is true for some of them and it fails for some others. According to the thermodynamics method (reviewed in Sec. II), when the entropy product is universal, one can easily read the central charge of the dual CFT from Eq. (1). In the following, for RBHs in Sec. IV which respect universality, we find central charges using Eq. (1). Then we compare this result with the central charge calculated from the asymptotic symmetry group (ASG) method [51,52] to check the validity of the thermodynamics method.

A. Central charge of the Kerr-like RBH

In Sec. IV A, we observed that for the small deviation parameter ℓ, the entropy product of the Kerr-like RBHs is universal, as

$$S_+ S_- = 4\pi^2 J^2. \tag{52}$$

Now, using Eq. (1), it is easy to read off the central charge from this entropy product:

$$\begin{aligned} c &= \frac{6}{4\pi^2}\frac{\partial(4\pi^2 J^2)}{\partial J} = 12J \\ &= 6(r_+ + r_-)\sqrt{r_- r_+ - \ell(r_+ + r_-)}, \end{aligned} \tag{53}$$

where in the second line we use Eq. (30). We are going to verify the above result by performing the Kerr/CFT analysis [52]. The first step is to find the near-horizon metric of the extremal solution. Remember that we have written parameters a and m in the Kerr-like BH [Eq. (28)] in terms of $r_\pm$, so one obtains the extremal solution by setting $r_- = r_+$. Following the procedure as in Refs. [31,49], we find the near-horizon extremal metric. The result is

$$\begin{aligned} ds^2 &= \alpha(\theta)\left[-r^2 dt^2 + \frac{dr^2}{r^2}\right] + \beta(\theta) d\theta^2 \\ &\quad + \gamma(\theta)(d\phi + f^{\phi} r dt)^2, \end{aligned} \tag{54}$$

where f^{ϕ} and the functions $\alpha(\theta)$, $\beta(\theta)$, and $\gamma(\theta)$ are

$$\begin{aligned} f^{\phi} &= \frac{r_+(r_+ - 2\ell)}{\ell - r_+}, \qquad \gamma(\theta) = \frac{4r_+(\ell - r_+)^2 \sin^2\theta}{r_+ - (2\ell - r_+)\cos^2\theta}, \\ \alpha(\theta) &= \beta(\theta) = -2r_+\left[\left(\ell - \frac{r_+}{2}\right)\cos^2\theta - \frac{r_+}{2}\right]. \end{aligned} \tag{55}$$

By choosing the adequate boundary conditions given in Ref. [52] and performing the calculations, we find the central charge to be

$$c = \frac{3}{2\pi} f^{\phi} \int d\theta d\phi \sqrt{\beta(\theta)\gamma(\theta)} = 12\sqrt{r_+^3 (r_+ - 2\ell)} = 12 J_{\mathrm{ext}}, \tag{56}$$

where J_{ext} is the angular momentum of the extremal solution. To compare the central charges in Eqs. (53) and (56), it is important to note that in the Kerr/CFT analysis we have inserted extremality into the solution. Now, applying the extremality condition $r_- = r_+$ to Eq. (53), it is obvious that the result is in complete agreement with Eq. (56), which means that the thermodynamics method is valid. It is worth mentioning that using the thermodynamics method, one finds the central charge for the generic (nonextremal) solution, while in the Kerr/CFT analysis, the central charge is obtained for the extremal solution.

B. Central charge of the charged RBHs

For a regular Reissner-Nordström and topological star in Secs. IV B and IV C, we find that the entropy product is universal:

$$S_+ S_- = \pi^2 q^4, \tag{57}$$

where q is the charge (electric or magnetic) of the solution. The same relation is obtained in the case of RBHs of Einstein gravity coupled to nonlinear electrodynamics (Sec. IV D). Now, it is easy to find the central charge of the dual CFT using the thermodynamics method [Eq. (1)]:

$$c = \frac{6}{4\pi^2}\frac{\partial(\pi^2 q^4)}{\partial q} = 6q^3. \tag{58}$$

One can also find central charges for charged solutions using the ASG formalism. In fact, this is done in Ref. [53], and the result is

$$c = 6q^3, \tag{59}$$

which is in agreement with the result of the thermodynamics method.

VI. CONCLUSIONS

In order to extend our intuition about the entropy product law and its relation to dual CFTs of the BHs, we investigated the entropy product for RBHs. We considered a variety of types of regular solutions, including the rotating Bardeen and Hayward BHs, the Frolov-Zelnikov solution, the Kerr-like RBH, the regular Reissner-Nordström solution, topological stars, 5D charged rotating RBHs, and regular solutions in extended nonlinear electrodynamics.

It seems that there is no rule for the RBHs that makes their entropy product universal. In other words, regardless of the charges or asymptotic behavior at the $r \to 0$ limit or even the theory that RBH satisfies its equations of motion, the universality of entropy product is true for some RBHs but fails for some others. A similar pattern is observed for the singular BHs. For instance, the universality is true for the Kerr BH and 5D Myers-Perry BH, but it fails for the Myers-Perry in $D \geq 6$. The same situation applies to the BTZ (which is universal) and Kerr-AdS in $D \geq 4$ (nonuniversal) BHs. One can deduce from these observations that in general, the universality is a characteristic for the "solutions" and not for the "theories" [54].

So, what is the advantage of entropy product universality? One may find the answer in the relation between the entropy products and dual CFTs of the BHs. It has been discussed [37] that when the entropy product is universal, the central charges of the dual CFTs can be easily read off from the entropy product according to the thermodynamics method [Eq. (1)].

In the case of RBHs with a universal entropy product, we found the central charges using the thermodynamics method. We then found the central charges from the ASG method. These two results are the same, which shows the validity of the thermodynamics method for RBHs. In other words, universality of the entropy product provides an easy way to find central charges of the dual CFTs. It is worth mentioning that in the ASG method, one takes the extremal limit of the solutions, and so the central charges are obtained for the extremal BHs, while the thermodynamics method yields the central charges of the generic (nonextremal) BH solutions.

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