Minimal model for the Bekenstein-Hawking entropy

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In this work we derive the Bekenstein-Hawking entropy formula, $S = \frac{A}{4l_p^2}$, from the following minimal assumptions: (i) there is a minimum area, $A_{\rm min}$, proportional to l_p^2; (ii) the event horizon area, A, is tessellated by $N = A/A_{\rm min}$ distinguishable units; and (iii) the internal structure of these units is that of an infinite tower of internal levels. Although our results are model independent, this internal structure can be realized as the excitations of more fundamental entities such as, for instance, strings or loop quantum gravity spin networks. Even more, once the microstates of the black hole are taken to be singlets formed within the infinite tower of states describing the whole event horizon, the correction term $-\frac{3}{2}\log A$ emerges from our model. Finally, some comments regarding the applicability of the present model to extremal black holes, as well as possible relationships with spectral geometry and other approaches are pointed out. Our results are independent of the dimension of the black hole and whether it is rotating or not.

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I. INTRODUCTION

Identifying the physical degrees of freedom underlying the Schwarzschild black hole prototype and counting its quantum microstates is a major problem of theoretical physics since the early 1970s [1,2]. Although various attempts to address this issue beyond general relativity have come from string theory [3], loop quantum gravity (LQG) [4], and the AdS-CFT correspondence [5], a definite answer is still lacking.

The discovery of Hawking radiation, together with the whole black hole thermodynamics, paved the way toward a new understanding of space and time. In particular, and following Boltzmann's dictum, "If you can heat it, it has a microstructure," the mere existence of Hawking radiation implies that black holes must have an internal structure. The quest for this structure is supposed to reach to and end when a full quantum gravity theory will be uncovered.

Meanwhile, the emergent approach, which can be stated going from thermodynamics (continuum) to statistical mechanics (emergence of molecules as a first discretization) to finally uncover some internal structure (molecular energy levels as a second discretization), when applied to gravity, shows us some light in order to deepen our knowledge of space and time. Specifically, Padmanabhan's holographic equipartition [6,7], shows the emergence of the so-called *holographic degrees of freedom*, which are important because they have a clear thermodynamic meaning. The Komar energy of a spherically symmetric black hole spacetime, E, is given by [6,7] $E = \frac{1}{2}NT$, where T stands for the local Hawking temperature and $N = \frac{A}{l_p^2}$, being A the area of the event horizon. Even more, both the variables E and T have valid interpretations in the continuum, thermodynamic limit, but the N has no meaning in the same limit. In fact, the N *spacetime atoms* [6,7] count the microscopic degrees of freedom and, therefore, holographic equipartition provides a direct link between the macroscopic and microscopic descriptions. Even more, it is possible to fully reconstruct AdS black hole thermodynamics starting from statistical mechanics principles applied to the aforementioned N *spacetime atoms* introducing some *ad hoc* postulates [8,9]. In this sense, N can be considered as the *Avogadro number of the spacetime* (for a recent account of the emergent paradigm program and related issues see, for example, [10]).

In essence, these holographic degrees of freedom correspond to some bits of information encoded in each Planck area on the horizon, which is a realization of Wheeler's "it from bit" proposal [11]. This, together with the Bekenstein-Mukhanov area spectrum [12], has inspired a large number of quantum black hole models (see, for example, [13]).

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Having gone one step further in the intimate description of spacetime, now the question is to complete the emergent approach toward a microscopic understanding of black hole entropy. Specifically, we would like to know how to descend from black hole thermodynamics (continuum) to the picture of spacetime atoms (holographic equipartition as a first discretization) to (hopefully) finally reach some internal structure describing spacetime, following the thermodynamic analogy.

Then, what is the internal structure (if any) of these holographic degrees of freedom? This question will keep us busy throughout the whole manuscript. In particular, we look for a minimal model which could summarize the main features of such an internal structure without having to choose a particular candidate for quantum gravity.

As commented in the introduction, the ideal candidate in order to prove our ideas is the elusive Bekenstein-Hawking entropy, namely

$$S = \frac{A}{4l_p^2}, \tag{1}$$

which is expected to hold within a semiclassical quantum gravity regime when macroscopic black holes ($A \gg l_p^2$) are considered.

II. A MINIMAL MODEL

The following *minimal model* suffices to obtain the Bekenstein-Hawking entropy formula. We consider N cells which tessellate any event horizon, such that

$$N = \frac{A}{A_{\min}} \tag{2}$$

Therefore, a minimum area is assumed to exist.

Then, we associate to each one of these cells a (quantum or classical) system such that its corresponding *phase space volume is finite*. This occurs for *compact phase spaces* (see the next section for more comments about it).

Let ν be the volume of this phase space. If the N cells are taken as *distinguishable*, then the total phase space volume will be

$$\nu^N. \tag{3}$$

Finally, the entropy ascribed to the horizon will be

$$S = \log \nu^N = \frac{A}{A_{\min}} \log \nu. \tag{4}$$

If

$$A_{\min} = \alpha l_p^2, \tag{5}$$

being α an arbitrary constant which, in principle, should be determined by any specific quantum gravity model we are working with, we obtain

$$S = \frac{\log \nu}{\alpha} \frac{A}{l_p^2}. \tag{6}$$

Therefore, after an appropriate choice of both the phase space volume and the proportionality factor, α, is done, the Bekenstein-Hawking entropy formula is recovered.

At this point, let us remind the reader the main ingredients of this minimal model: (i) there is a minimum area proportional to l_p^2; (ii) the horizon is tessellated by $N = A/A_{\min}$ units; (iii) these units are distinguishable; and (iv) to each one of these units we associate a compact phase space. Interestingly, a direct consequence of (i) and (ii) is that the area is quantized to be an integer multiple of the minimal area. This remarkable feature of area quantization has observational consequences which include differences between the usual Hawking behavior from that of LQG [14], a distorted ringdown signal, or the presence of late–time echoes. Indeed, the ringdown signal from the merger of a binary black hole, as computed from general relativity, is described very well by the quasinormal modes of the final Kerr black hole (see, for example, [15–17]).

III. A NATURALLY COMPACT PHASE SPACE

As described in [18], both IR and UV cutoffs can be realized from globally deformed Hamiltonian systems that are defined on compact symplectic manifolds, concluding that quantum gravity cutoffs are global (topological) properties of the corresponding symplectic manifolds. In fact, the expectation is that compact phase space extension of general relativity may resolve the problem of singularities [19].

Although a phase space with finite volume can be obtained by appropriate compactification of either position or momentum (or both) spaces, here we will consider that only *internal degrees of freedom* are relevant in the description of the N cells. We would like to emphasize that we have not specified (and, indeed, we do not need it) the nature of such an internal degrees of freedom but the only requirement is that their phase space must be finite. Indeed, our proposal is model–independent but it resembles well–established models for quantum gravity. For example, our construction is somewhat similar to the extended LQG approach by Rovelli and Vidotto [20]. The idea put forward by them is to generalize the theory with $su(2) \times SU(2)$ phase space per link of the spin network, where only the part of the phase space associated with the $SU(2)$ holonomies, is already compact, to the compact phase space $SU(2) \times SU(2)$.

Specifically, in this section we will be interested in the $\alpha = 4\pi$ case, which corresponds to $\nu = e^{\pi}$. As it will be clear, in this case, microstate counting can be interpreted in terms of the dimensions of complex projective space, which acts as the *quantum phase space* of k-level systems, which we assume to be a plausible description of the internal structure of the N holographic degrees of freedom.

Let us begin with some mathematical preliminaries. Complex projective space, $\mathbb{C}P^n$, is defined as the set of lines in $\mathbb{C}^{n+1}$, i.e.,

$$\mathbb{C}P^n = \frac{S^{2n+1}}{S^1} = \frac{S^{2n+1}}{U(1)} = \frac{U(n+1)}{U(1)\times U(n)} = \frac{SU(n+1)}{U(1)}. \tag{7}$$

We define the volume of $\mathbb{C}P^n$ as the quotient

$$\mathrm{Vol}\,(\mathbb{C}P^n) = \frac{\mathrm{Vol}(S^{2n+1})}{\mathrm{Vol}(S^1)} \tag{8}$$

and, therefore,

$$\mathrm{Vol}\,(\mathbb{C}P^n) = \frac{\frac{2\pi^{n+1}}{n!}}{2\pi} = \frac{\pi^n}{n!}. \tag{9}$$

We note that

$$\nu \equiv \sum_{n=0}^{\infty} \mathrm{Vol}\,(\mathbb{C}P^n) = e^{\pi}. \tag{10}$$

Even more, the quantity defined as $S = \log \nu^N$ ($N = 1$) is given by

$$S = \log \nu = \pi = \mathrm{Vol}\,(\mathbb{C}P^1) = \frac{\mathrm{Vol}\,(S^2)}{4}. \tag{11}$$

At this point, a couple of comments are in order: (i) we note that pure states are given by vectors in a Hilbert space, $\mathcal{H}$. If a finite dimensional Hilbert space is considered, then $\mathcal{H} = \mathbb{C}^n$ equipped with a scalar product (a Hermitian form). However, given that two states are related by multiplication by any complex number, it is not the space of physical spaces. Therefore, the true quantum phase space (the space of pure states) is the space of rays in the Hilbet space, namely, for the special case in which $\mathcal{H}$ is $\mathbb{C}^{n+1}$, the space of rays is $\mathbb{C}P^n$ (the so-called projective Hilbert space, $\mathcal{H}^{\mathcal{P}}$.) [21]; (ii) $n = k - 1$, where k is the number of relevant (energy, spin,…) levels which are needed to describe the physics under study; (iii) if we define a quantum state, $|\sigma\rangle$, such that its quantum phase space is ν, then its statistical entropy satisfies an area law. This is somehow reminiscent of the so-called black hole-qubit correspondence (see Ref. [22] for a review), which links the structure of the Bekenstein-Hawking entropy of certain black hole solutions in string theory with certain multipartite entanglement measures in quantum information.

Interestingly, this $|\sigma\rangle$ state can be written as

$$|\sigma\rangle = \bigotimes_{k\in\mathbb{N}^+} |k\rangle = |\underline{\quad}\rangle \otimes |\underline{\underline{\quad}}\rangle \otimes |\underline{\underline{\underline{\quad}}}\rangle \otimes \cdots \tag{12}$$

where $|k\rangle$ stands for a k-level system. Note that $|\sigma\rangle$ is a *separable* state.

Although we have emphasized that our approach is model independent, we can try to interpret our results in the framework of other models. Among all possible interpretations of $|\sigma\rangle$, particularly appealing is either a *tower of string or LQG spin network excitations*.

Regarding the string case, let us note that the string functional is a composite of all possible string configurations, which we consider here for simplicity a tower on increasing k-level systems. A possible drawback regarding this realization is that string kinetic levels are not considered. Although this could be surpassed by considering a Bose-Einstein condensation of strings, we think that this argument, which is essentially *ad hoc*, opens the window for an alternative interpretation *via* LQG.

In the LQG case, "the quanta (the modes of the spin network) carry no quantum number such as momentum or position. Rather, they carry quantum numbers that define a quantized geometry" [23]. Therefore, a minimal model compatible with the main ideas of LQG (but independent of the full LQG machinery) is to consider the full projective Hilbert space of this toy model as the sum

$$\mathcal{H}^{\mathcal{P}} = \mathcal{H}^{\mathcal{P}}{}_1 \oplus \mathcal{H}^{\mathcal{P}}{}_2 \oplus \mathcal{H}^{\mathcal{P}}{}_3 \cdots = \mathbb{C}P^0 \oplus \mathbb{C}P^1 \oplus \mathbb{C}P^2 \oplus \cdots \tag{13}$$

A consequence of the appearance of the aforementioned tower of states, $\nu = \mathrm{Vol}\,(\mathcal{H}^{\mathcal{P}}) = e^{\pi}$ and, therefore, the minimum area is fixed as $A_{\min} = 4\pi l_p^2$ in order to recover the Bekenstein-Hawking formula using Eq. (6). Although this feature contrasts with the usual LQG findings, where $A_{\min} = 4\pi\sqrt{3}\gamma l_p^2$, where γ is the Barbero-Immirzi prameter, it is interesting to note that LQG predicts the Bekenstein-Hawking entropy when $\gamma = \gamma_0 = 0.274067\ldots$ for different kind of (macroscopic) black holes and, in the same spirit, the approach here introduced predicts the Bekenstein-Hawking formula when $\alpha = 4\pi$ for all (macroscopic) black holes. Specifically, the entropy is given in the LQG framework by

$$S = \frac{\gamma_0}{\gamma}\frac{A}{4l_p^2}, \tag{14}$$

where γ is the Barbero-Immirzi parameter and γ_0 takes an order one numerical value depending on the specific model considered. For our minimal model, we have

$$S = \frac{4\pi}{\alpha}\frac{A}{4l_p^2} \tag{15}$$

and, therefore, the simplest way to make our result compatible with the Bekenstein-Hawking entropy is by choosing $\alpha = 4\pi$. However, we could relax the above constraint by relating α with the Barbero-Immirzi parameter through Eqs. (14) and (15) from where

$$\alpha = \frac{4\pi\gamma}{\gamma_0}. \tag{16}$$

However, we would like to emphasize that as our result is model–independent, the relation between α and γ is not compulsory (the interested reader find a detailed discussion on the role of the Barbero-Immirzi parameter in Sec. 8 of [24]).

In any case, we would like to mention that the minimal model here proposed cannot be directly applied to LQG (at least in the standard formulation which does not support a compact phase space). However, Ref. [19] considers a two-dimensional minisuperspace model of compact phase space gravity which is a two-sphere. Since $S^2 \sim \mathbb{C}P^1$, we could interpret the internal structure of the N cells as described by two-level systems, which corresponds to Makela's approach [25], where the only allowed values of the quantum numbers at the punctures of the spin network on the spacelike two-surfaces of spacetime are 0 and $1/2$.

The compactness of $\mathbb{C}P^n$, in addition to providing an efficient mechanism for getting a finite phase space volume, implies discreteness for the spectrum of the Laplacian which is, in some way, a desirable feature in the quantum realm. Even more, $\mathrm{Vol}\,(\mathbb{C}P^n)$ can be spectrally determined *via* the Laplacian on scalars by the well-known Minakshisundaram-Pleijel formula [26]. Therefore, as the spectra only depend on the Riemannian structure, black hole microstates would be easily identified as some set of eigenvalues and, therefore, their dynamics and quantization would be, in principle, free of difficulties [27]. Similar spectral-geometric techniques have been very recently employed [28] in order to show that the introduction of an appropriate cutoff in the spectra of the Laplacian on a spherically symmetric and static black hole reveals an equivalence between shape degrees of freedom [29] and the number of holographic degrees of freedom).

In addition, we would like to mention that, although other choices for specific microscopic degrees of freedom describing the N cells partitioning the horizon are certainly possible (see, for example, a graph-theoretic model recently proposed by Davidson [30], a mapping to the equivalent statistical mechanical problem of counting of conformations of a closed polymer chain [31] or the previously mentioned Makela's two-level system of LQG punctures [25]), the model here proposed only assumes the mere existence of an infinite tower of discrete levels describing, with more depth, holographic degrees of freedom. Although we have shown that a toy model based on LQG can accommodate the aforementioned tower, it would be interesting to look for other realizations.

IV. EXTREMAL BLACK HOLES AND LOGARITHMIC CORRECTIONS

Regarding the applicability and scope of our approach, a couple of words are in order: (i) the present model does not predict any corrections to the Bekenstein-Hawking entropy and (ii) what about extremal black holes?

Let us remind the reader that extremal black holes are topologically disconnected from the nonextremal ones [32]. In particular, there is an interesting connection between topology and entropy for gravitational instantons [33], which reads

$$S = \frac{\chi A}{8}, \tag{17}$$

where χ is the Euler characteristic of the (Riemannian, compact) gravitational instanton (including boundaries if necessary). By using Eqs. (6) and (17) implies an interesting connection between topology and microscopic physics through

$$\chi = \frac{8\log\nu}{\alpha}. \tag{18}$$

This expression gives $S = 0$ for extremal BHs, as expected. Even more, it recovers the Bekenstein-Hawking entropy because the Euler number is $\chi = 2$ in most of standard black hole solutions such as, for example, in the Schwarzschild and Kerr cases.

Now, let us assume a finite α (which makes sense because $A_{\min} = \alpha l_p^2$). Then, for an extremal black hole we have that $S = \chi = 0$, implying that $\nu = 1$. Therefore, remembering that $\nu = \sum_{n=0}^{\infty} \mathrm{Vol}\,(\mathbb{C}P^n)$ any holographic degree of freedom is described by $\mathbb{C}P^0$, which implies $k = 1$, i.e., the first excited state of the infinite tower. On the contrary, a nonextremal black hole is described by N holographic degrees of freedom, each one described by an infinite tower of states, ranging from $k = 2$ to ∞. Intriguingly, the role played by the vacuum state remains to be explored.

Interestingly, our minimal model can be extended to include corrections to the Bekenstein-Hawking entropy as we shall explain in what follows. It is now accepted that the semiclassical Bekenstein-Hawking area acquires a fully quantum correction which, interestingly, seems to be model-independent, and reads $-\frac{3}{2}\log A$. Specifically, both the CFT-based Cardy formula [34], stringy calculations [35] based on it and a LQG-based counting [36] give place to the aforementioned correction. Given the radical differences between these approaches, one could ask whether some underlying model is trying to tell us something at a, let us say, deep (model-independent) level. As the key idea is combinatorics (counting microstates), and having into account that most of calculations require a large number of these microstates, one could look for a key asymptotic formula related, for instance, with an specific integer partition (or other equivalent realizations). We propose that this formula should read: "when the number of microstates, $n(N)$, is very large (remember that N is the number of patches tessellating the horizon), and given some constraints between these microstates, their asymptotic distribution is given by the Catalan numbers". For us, the appearance of Catalan numbers is of fundamental importance [37]. Let us elaborate.

We demand that the entropy of a black hole is ($l_p = 1$):

$$S = \frac{A}{4} - \frac{3}{2}\log A + \cdots \tag{19}$$

If a simple horizon partition such as, for example, the one proposed in our work, is performed, then the number of cells tilling the horizon is $N = A/A_{\min}$. Now, instead of assigning a quantum phase space to any of these N cells, we define the number of black holes microstates, n such that

$$\log\, n \sim \frac{N}{4} - \frac{3}{2}\log N + \cdots \tag{20}$$

This implies that

$$n(N) \sim \frac{e^{N/4}}{N^{3/2}}. \tag{21}$$

At this point, we note that the n-Catalan number is defined as ($n > 0$)

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}. \tag{22}$$

Even more, the Catalan numbers grow asymptotically as

$$C_N \sim \frac{4^N}{\sqrt{\pi}N^{3/2}}. \tag{23}$$

Therefore,

$$\log C_N \sim N - \frac{3}{2}\log N \sim S. \tag{24}$$

Then, we conclude the following: if black hole microstates are distributed as Catalan numbers, then their asymptotic growing is exactly what is needed in order to obtain black hole entropy including the logarithmic correction. Note that, at this point, our conclusion does not say anything about the physical mechanism which makes Catalan numbers to appear. However, our conclusion is somehow natural within LQG, as shown explicitly, for example, by Kaul in [38].

In addition, Carlip shown [34] that the density of states for a CFT with central charge c and eigenvalues Δ grows as

$$\rho(\Delta) \sim \left(\frac{c}{96\Delta^3}\right)^{1/4} e^{2\pi\sqrt{\frac{c\Delta}{6}}}. \tag{25}$$

Intriguingly, both the application to the Banados, Teilteilboim and Zanelli case (where a Virasoro algebra is well known to exist) and also to a generic black holelike metric in arbitrary dimensions by treating the horizon as a boundary and considering the behavior of the algebra of diffeomorphisms of the $r-t$ plane near the horizon [39] gives place to a generic density of states given by:

$$\rho(\Delta) \sim \frac{c}{12}\left(\frac{A}{8\pi}\right)^{-3/2} e^{A/4}, \tag{26}$$

which is exactly the asymptotic expansion of the aforementioned Catalan numbers.

Interestingly, we can consider a minimal extension of our previous model which encodes the Catalan number as follows. First, let us assume that, besides the tensorial product of k-level systems associated to each fundamental cell, we can append a label $i = 1, 2$ to each $|\sigma\rangle$, namely

$$|\sigma\rangle \to |\sigma, i\rangle. \tag{27}$$

Now, let us assign a fundamental $SU(2)$ representation for each value of i so the whole horizon is tilled by $2n$ doublets. Note that the above construction is reminiscent of what we obtain through the relativistic correction of the Hamiltonian of an atom where for each electron we assign an internal two–level system, namely spins up and down. Note that the number of singlet states that can be formed using $2n$ doublets is exactly given by:

$$\frac{(2n)!}{(n+1)!n!}, \tag{28}$$

which coincides with the definition of the nth Catalan number. At this point, *we define the microstates of the black hole*, $g(N)$, *as the* $SU(2)$*–invariant states, namely the singlet states formed within the event horizon.* Then, it follows that $g(N) = C_N$ and, therefore, when N is large,

$$S = \log\, g(N) = \log\, C_N = N\, \log\, 4 - \frac{3}{2}\log\, N + \cdots \tag{29}$$

Finally, if a minimum area

$$A_{\min} = 4\, \log\, 4\, l_p^2 \tag{30}$$

is assumed, then our Eq. (29) gives exactly

$$S = \log\, g(N) = \log\, C_N = \frac{A}{4} - \frac{3}{2}\log\, A + \cdots \quad (31)$$

At this point some comments are in order. First, note that the isolated horizon approach of LQG realizes that the set of states to be counted must obey the constraint that they are $SU(2)$ singlets [40]. Besides, in Ref. [41] the entropy was explicitly calculated in terms of the number of spin–valued punctures on a 2-sphere tessellated by conformal blocks in the framework of the Wess-Zumino model. More precisely, under the assumption that the spin of the punctures are all set to $1/2$, then the authors inferred that the contribution from other spins is negligible and, therefore, under these approximations, they obtain $S = \log\, C_N$ (although not explicitly alluding to the Catalan numbers). In this regard, we think that the approach followed in [41] should be considered as an "approximation." Then, although the counting we are here performing formally coincides with that reported in [41], we consider that our result is exact in the precise sense that the label i is not associated to any "geometrical" realization. In summary, we arrive to the very same conclusion from a different point of view. Second, note that the minimal area given by (30) is smaller than the previously obtained for the black hole entropy without the logarithmic correction. Finally, it is clear that the introduction of the label i should lead to a modification of the volume of the total phase space of the form

$$\nu^N \to \frac{\nu^{\beta N}}{N^{3/2}}, \quad (32)$$

which accounts for the logarithmic correction in the entropy with $\beta = \log 4/\pi$.

V. FINAL COMMENTS

In this work we have developed a minimal model which accounts for the Bekenstein-Hawking entropy starting from the basic assumption that the area of the black hole horizon is tessellated by minimal cells (areas of the order of the Planck length squared). We found that, provided each one of these cells encodes some internal degrees of freedom spanning a compact phase space, microstate counting can be straightforwardly performed to obtain $S = A/4l_p^2$. Although our findings are model-independent, we have been able to interpret our results in terms of other well-established approaches such as loop quantum gravity and string theory. We have also extended our results to include a logarithmic correction to black hole entropy (including the $-\frac{3}{2}$ factor) by assuming that black hole microstates are nothing but singlet states formed within the event horizon. Remarkably, our results are independent of the dimension of the black hole and whether it is rotating or not. We hope to report on the computation of Hawking radiation spectra within our model and on some similarities between our approach and different Quantum Gravity models in a near future.

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[37] During the writing of this manuscript we learned that A. Davidson published a nice idea connecting Catalan numbers and the holographic shell model in A. Davidson, Int. J. Mod. Phys. D **23**, 1450041 (2014). Even more, Davidson recognizes in some of his recent work [30] that if the graphs he was considering to enumerate black hole microstates were rooted trees, the $-\frac{3}{2}\log A$ would have appeared. Here we note that, within Davidson's context, rooted trees implies one of the contexts where the Catalan numbers appear.
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