Quantum complexity as hydrodynamics

Pablo Basteiro,[1] Johanna Erdmenger,[1] Pascal Fries,[1] Florian Goth,[1] Ioannis Matthaiakakis,[1,2] and René Meyer[1,*]

[1] *Institut für Theoretische Physik und Astrophysik and Würzburg-Dresden Cluster of Excellence ct.qmat, Julius-Maximilians-Universität Würzburg, Am Hubland, 97074 Würzburg, Germany*

[2] *Dipartimento di Fisica, Università di Genova and I.N.F.N.—Sezione di Genova, via Dodecaneso 33, I-16146 Genova, Italy*

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As a new step toward defining complexity for quantum field theories, we map Nielsen operator complexity for $SU(N)$ gates to two-dimensional hydrodynamics. We develop a tractable large N limit that leads to regular geometries on the manifold of unitaries as N is taken to infinity. To achieve this, we introduce a basis of noncommutative plane waves for the $\mathfrak{su}(N)$ algebra and define a metric with polynomial penalty factors. Through the Euler-Arnold approach we identify incompressible inviscid hydrodynamics on the two-torus as a novel effective theory of large-qudit operator complexity. For large N, our cost function captures two essential properties of holographic complexity measures: ergodicity and conjugate points.

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I. INTRODUCTION

Quantum computational complexity [1], referred to as complexity hereafter, quantifies the number of simple gates required to synthesize a given unitary operation in quantum computing. In recent years, the geometric approach to complexity of Nielsen *et al.* [2–4] has proven immensely useful for investigating the complexity of n-qubit systems, independently of the particular state of the system. In its original manifestation, this geometric framework relied on the $SU(2^n)$ manifold of unitaries acting on n-qubits. In this paper, we consider a generalization of Nielsen's approach that incorporates quantum circuits acting on N-level systems, i.e., qudits of dimension N. For qudit systems, the unitaries of interest belong to $SU(N)$. The key ingredient in Nielsen's approach is the choice of metric on this manifold, assigning penalty factors to unitaries departing from the identity operator I. The complexity of $U \in SU(N)$ is then identified with the length of the minimal geodesic connecting I and U. These unitaries are generated by a control Hamiltonian $\mathcal{H}$ tangent to $SU(N)$. The corresponding group algebra characterizes fully these Hamiltonians and their geodesics, through the Euler-Arnold method [5–8].

Recent progress on complexity measures via the AdS/CFT correspondence (or holography) [9–11] further motivates the present work. In general, holographic complexity measures should be ergodic and exhibit conjugate points: Ergodicity ensures all points on the group manifold can be reached in finite time, thus implying a linear [12] growth of complexity with time [14,15]. In contrast, conjugate points, i.e., the meeting points of equal-length geodesics, provide bounds on this growth [4]. This conjecture is supported by work on the curvatures of complexity measures in holographic CFT's [7,8].

Based on Nielsen's approach, different definitions for complexity both for discrete systems [14–19] and quantum field theories [6–8,20–25] have been investigated. However, how these notions of complexity are related in the limit of infinite Hilbert space dimensions is an open question. This limit is in general not well-defined, because desirable features of quantum circuits, such as k-locality [15,17–19,26], require penalty factors typically scaling exponentially with the system size for every direction on the manifold [2]. In the $N \to \infty$ limit, this leads to singular geometries on the manifold of unitaries, impeding the definition of complexity as geodesic length.

In this paper, we show how a judicious choice of basis and metric for the $\mathfrak{su}(N)$ algebra with *polynomial* penalty factors leads to well-defined nonsingular geometry for $SU(N \to \infty)$ [27]. We show that, on the $SU(N)$ manifold at infinite N and at low energies, Nielsen complexity can be equivalently evaluated on the manifold of volume-preserving diffeomorphisms $\mathrm{SDiff}(\mathbb{T}^2)$ of the torus. We show that the Euler-Arnold equation on the resulting manifold coincides with the Euler equation of a two-dimensional ideal fluid [29].

*Corresponding author.
rene.meyer@physik.uni-wuerzburg.de

This permits identifying control Hamiltonians with diffeomorphism generators in two-dimensional hydrodynamics. Moreover, it suggests a natural cost function, based on the two-dimensional Laplacian, with a smooth dependence on N. This smooth dependence suggests the prevalence of particular characteristics of the hydrodynamic theory even at finite N. In particular, two-dimensional ideal hydrodynamics is classically chaotic due to the hyperbolic geometry of its phase space [30]. We quantify this instability at finite, large N by numerically computing the sectional and Ricci curvatures of $SU(N)$.

Finally, we find that our formulation of complexity exhibits both ergodicity and conjugate points, whose presence is necessary for a proper holographic complexity measure (although see [27]). In this way, our results constitute a new step toward understanding quantum complexity in QFT's with a holographic dual.

II. THE ALGEBRA OF N-LEVEL QUDITS

At the heart of our construction lies a new and nontrivial choice of anti-Hermitian generators for the $\mathfrak{su}(N)$ Lie algebra. We employ known results for $\mathfrak{su}(N)$ to investigate the large N limit of our basis and explain the subtleties it carries. The corresponding structure constants determine the Riemannian curvature of $SU(N)$, which we compute in Sec. V. We outline our construction below [31].

We first introduce the $N\times N$ "shift" matrix $h_{kl}=\delta_{k+1,l}$ and "clock" matrix $g_{kl}=\omega^l\delta_{k,l}$, with $\omega=\exp\frac{2\pi i}{N}$ a primitive Nth root of unity, and $k,l=0,\ldots,N-1$ mod N. These matrices commute up to a phase, i.e., $hg=\omega gh$. Then, following [32–35], we define a basis of unitary, but not necessarily anti-Hermitian, matrices $J_{\vec{m}}=\omega^{\frac{m_1m_2}{2}}g^{m_1}h^{m_2}$, indexed by a two-vector $\vec{m}=(m_1,m_2)$ on the $\mathbb{Z}^2$-lattice. These can be thought of as a noncommutative version of plane waves, with the vector index $\vec{m}$ playing the role of the wave vector, and h and g the momentum and position modes, respectively [34]. Their commutator is given by

$$[J_{\vec{m}},J_{\vec{n}}]=-2i\sin\left(\frac{\pi}{N}(\vec{m}\times\vec{n})\right)J_{\vec{m}+\vec{n}},\qquad(1)$$

where $\vec{m}\times\vec{n}\equiv m_1n_2-m_2n_1$. In [34] it was shown that the algebra (1) is, in the $N\to\infty$ limit, isomorphic to the algebra $\mathrm{SVect}(\mathbb{T}^2)$ of the group $\mathrm{SDiff}(\mathbb{T}^2)$ of volume-preserving diffeomorphisms on the standard two-torus $\mathbb{T}^2$. To see this, note that $\mathrm{SVect}(\mathbb{T}^2)$ admits a symplectic structure in terms of divergence-free vector fields which, in two-dimensions, are Hamiltonian vector fields X_f. The X_f are uniquely determined by their associated *stream function* f [36], which can be expanded in plane waves on $\mathbb{T}^2$ as $f_{\vec{m}}\propto\exp(i(m_1x+m_2p))$ [30]. The isomorphism then between $\mathrm{SVect}(\mathbb{T}^2)$ and $\mathfrak{su}(N)$ in the large N limit is obtained by expanding the sine in (1) to first order in $1/N$ and identifying

$$J_{\vec{m}}\overset{N\to\infty}{\longrightarrow}\frac{2\pi}{iN}X_{\vec{m}}.\qquad(2)$$

The isomorphism (2) thus relates Hamiltonian vector fields [i.e., elements of $\mathrm{SVect}(\mathbb{T}^2)$] with the basis elements of $\mathfrak{su}(N)$ given by (1). A detail not addressed in [32–35], but already mentioned in [37], is that the Taylor expansion truncation is invalid for several classes of vectors $\vec{m},\vec{n}$; There are vector pairs defined for all N, e.g., $\vec{m}=(\frac{N-1}{2},0)$ and $\vec{n}=(0,\frac{N-1}{2})$, for which the cross product is of order $\mathcal{O}(N^2)$ or $\mathcal{O}(N)$. We must restrict the isomorphism to only the pairs of $\mathcal{O}(1)$. These turn out to be precisely the low-momentum modes relevant for hydrodynamics, see Sec. IV and [31] for more details.

We now proceed with the definition of anti-Hermitian basis elements, capable of constructing $SU(N)$ operators through exponentiation, as required by Nielsen's approach. To this end, we introduce for each $\vec{m}$,

$$C_{\vec{m}}\equiv i(J_{\vec{m}}+J^\dagger_{\vec{m}}),\qquad S_{\vec{m}}\equiv(J_{\vec{m}}-J^\dagger_{\vec{m}}).\qquad(3)$$

The generators (3) obey commutation relations inherited from (1) [31]. The structure constants thus obtained are more involved than those in (1) due to the overcompleteness of (3). However, we can make this basis complete via modularity symmetries and linear dependences enjoyed by the generators [31]. Most importantly, due to linearity, the isomorphism (2) carries over to this basis, which hence exhibits a well-defined large N limit. We exploit this in our curvature computations in Sec. V.

III. EULER-ARNOLD FRAMEWORK

Nielsen's approach identifies complexity with the length of the minimal geodesic $U(s)$, with s parametrizing the position along the trajectory [38], connecting the identity element with the desired U on the manifold of unitaries. Each geodesic $U(s)$ is generated by an (anti-Hermitian) *control Hamiltonian* $\mathcal{H}(s)$ via the Schrödinger equation $\frac{dU}{ds}=\mathcal{H}(s)U(s)$. The Euler-Arnold formalism [30,39] exploits the group structure by identifying $\mathcal{H}$ with a Lie algebra element $\mathcal{H}(s)=U^{-1}(s)\dot{U}(s)\in\mathfrak{su}(N)$, i.e., with the pullback of the vector $\dot{U}(s)$ onto the tangent space at the identity. The time-evolution of $\mathcal{H}(s)$ within this approach is then given by the *Euler-Arnold* (EA) equation

$$\dot{\mathcal{H}}=\kappa(\mathcal{H},\mathcal{H}),\qquad(4)$$

where κ is a quadratic bilinear two-form defined via $\langle[X,Y],Z\rangle=\langle\kappa(Z,X),Y\rangle$, for $X,Y,Z\in\mathfrak{su}(N)$ and $\langle\cdot,\cdot\rangle$ the Lie algebra inner product [39]. Distinct κ-forms are induced by different inner products on the algebra. This is equivalent to choosing a metric on $SU(N)$, and hence a cost function in Nielsen's setup. Using the EA equation is advantageous, since it can be easier to solve (4) than to compute the nested commutators appearing in a solution of the Schrödinger equation [20]. In fact, we explain in the next section how the EA equation drastically simplifies in the large N limit, which allows for the direct calculation of the control Hamiltonian.

IV. INNER PRODUCT AND PENALTY FACTORS

We now show how at large N, $\mathfrak{su}(N)$ leads within the EA framework to the ideal-fluid Euler equation. Solutions to this equation are control Hamiltonians in the sense of Nielsen, which allows for a hydrodynamic interpretation of the standard computation of Nielsen complexity [31]. We also discuss in detail how this suggests a natural extrapolation of the hydrodynamic cost function to finite N.

In the large N limit and for low-energy $\mathcal{O}(1)$ generators, $SU(N)$ is identified via (2) with the manifold of volume-preserving diffeomorphisms $\mathrm{SDiff}(\mathbb{T}^2)$ [40]. We consider the standard inner product on its algebra $\mathrm{SVect}(\mathbb{T}^2)$, given by the L^2-inner product between Hamiltonian vector fields. This can be rewritten in terms of the Laplacian acting on stream functions as

$$\langle X_f, X_g\rangle = \int_\omega X_f \cdot X_g = -\int_\omega f\Delta g, \tag{5}$$

with ω the symplectic form on $\mathbb{T}^2$ [31]. For $f = g$, this is (twice) the kinetic energy of a flow. The inner product (5) induces a metric on $\mathrm{SDiff}(\mathbb{T}^2)$ which defines the length, and thus the cost, of geodesics. Consequently, it defines a κ form given by $\kappa(f,f) = -\Delta^{-1}\{f, \Delta f\}$, with $\{\cdot,\cdot\}$ the usual Poisson bracket [31]. This κ form leads to the following EA equation for the control Hamiltonian $\mathcal{H}$

$$\Delta\dot{\mathfrak{h}} = -\{\mathfrak{h}, \Delta\mathfrak{h}\}, \tag{6}$$

with $\mathfrak{h}$ the "control stream function" associated to $\mathcal{H}$ via the symplectic form [31]. Equation (6) constitutes a main result of this work. Considering the large N limit of the $\mathfrak{su}(N)$ algebra, we obtain the stream function form of the Euler equation for a $(2+1)$-dimensional ideal fluid [30].

We find that the Nielsen complexity of a large-qudit unitary can be evaluated via this effective hydrodynamic theory, for which the control Hamiltonian can be straightforwardly computed (6). Moreover, the computation of Nielsen complexity can be recast in the hydrodynamic setting: The Schrödinger equation defining the target unitary U in terms of the control Hamiltonian, is now the equation relating the Eulerian and Lagrangian frames of reference of the fluid $\frac{df}{ds} = \mathcal{H}(s, f(s))$ [41]. $\mathcal{H}(s)$ is the control Hamiltonian obtained from the solution $\mathfrak{h}$ of (6) and f the corresponding diffeomorphism. Imposing the boundary conditions $f(0) = \mathbf{id}$, the identity map, and $f(1) = f_{\mathrm{target}}$, the target diffeomorphism [42], yields initial velocities $v^{\vec{m}} \equiv v^{\vec{m}}(s, f_{\mathrm{target}})$ [43] for the geodesic as functions of f_{target}. These are inserted into the length functional $\ell = \int_0^1 ds\sqrt{\tilde{G}_{\vec{m}\,\vec{n}}v^{\vec{m}}v^{\vec{n}}}$. Here, $\tilde{G}_{\vec{m}\,\vec{n}}$ is the metric induced by the inner product (5). Minimizing this functional over all solutions (6) yields the complexity of f_{target}, $\mathcal{C}(f_{\mathrm{target}})$. In summary, the Nielsen complexity of large-qudit unitaries is given by the length of the minimal geodesic, generated by a solution to the EA equation (6), that connects the Lie algebra $\mathrm{SVect}(\mathbb{T}^2)$ with the desired target element of $\mathrm{SDiff}(\mathbb{T}^2)$. Our construction, hence, provides a smooth geometry at large N, the manifold $\mathrm{SDiff}(\mathbb{T}^2)$, on which complexity can be calculated. Thus, we avoid the singular geometries encountered in previous manifestations of large N complexity models. Additionally, $\mathrm{SDiff}(\mathbb{T}^2)$ has a clear physical interpretation as the phase space of a well-known theory, namely two-dimensional hydrodynamics.

Motivated by our construction for complexity at $N \to \infty$, and exploiting the generator isomorphism (2), we now formulate new results for the complexity geometry at finite N by ensuring a smooth transition between the finite- and infinite-dimensional setups. We adapt the inner product (5) at $N \to \infty$ to finite N by defining the action of the Laplacian on noncommutative waves as $\Delta J_{\vec{m}} = -m^2 J_{\vec{m}}$. This choice transitions smoothly to infinite N, where the action of the standard Laplacian on plane waves $f_{\vec{k}}$ is $\Delta f_{\vec{k}} = (\partial_x^2 + \partial_p^2) f_{\vec{k}} = -k^2 f_{\vec{k}}$ [44]. We define an inner product on $\mathfrak{su}(N)$ for finite N as

$$\langle \mathcal{T}_{\vec{m}}, \mathcal{T}_{\vec{n}}\rangle \coloneqq -\frac{1}{2N}\mathrm{Tr}(\mathcal{T}_{\vec{m}}\Delta\mathcal{T}^\dagger_{\vec{n}}), \tag{7}$$

with $\mathcal{T} \in \{C, S\}$. By means of group translation, this inner product induces a right-invariant metric $G_{\vec{m}\,\vec{n}}$ on $SU(N)$. Its components are the penalty factors for different directions on the tangent space, given by the eigenvalues of the Laplacian acting on the generators, i.e., $G_{\vec{m}\,\vec{n}} = m^2\delta_{\vec{m}\,\vec{n}}$, with $m = |\vec{m}|$. These penalties render the metric homogeneous but *not* isotropic, since not all directions get penalized equally. Equal penalty factors are assigned only to those vectors related by parity or conjugation e.g., $\vec{m} = (1,2)$, $\vec{n} = (2,1)$ and $\vec{l} = (-2,-1)$. This reflects the Hamiltonian structure of the problem, with the Laplacian being invariant under symplectic transformations, and is visually manifest in our curvature results shown in Fig. 1. Due to this symmetry, every direction on the Lie algebra gets assigned a different penalty with at most eight-fold degeneracy, yielding a maximally anisotropic metric for the manifold of unitaries. This is an essential property, since anisotropy leads to negative curvature on the manifold and negative curvature is a strong indicator of ergodic geodesic flow [16]. In terms of N-level qudits, our choice of metric ensures high-energy excitations with large wave vector receive larger penalty

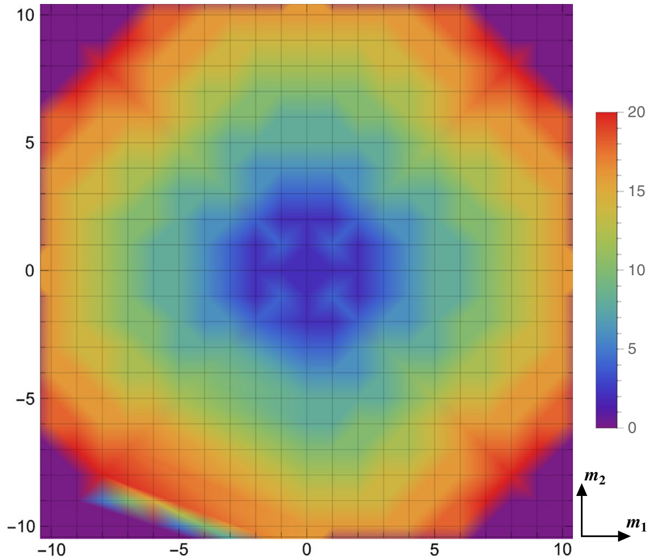

FIG. 1. Color density plot of the critical value N_c at which the normalized Ricci curvature of a given direction $\vec{m}$ in $\mathfrak{su}(N)$ becomes negative over the $\mathbb{Z}^2$ lattice, spanned by the vector components m_1, m_2. The interpolation between the integer points of the lattice is there to guide the eye. The color flare at the lower left corner is an artifact of this interpolation.

factors. These high-energy sectors effectively decouple in the strict large N limit.

Our choice of penalty factors is fundamentally different from the majority of previous work on the subject, e.g., [4,15,17,18]. In particular, the penalty factors in these setups grow exponentially $p \sim \alpha^k \sim e^{k \ln \alpha}$ instead of polynomially [45]. Here, k is the Pauli weight of the many-qubit gate in the Pauli basis [46]. Although exponential penalty factors are well motivated from the point of view of local quantum operations, they typically lead to singular geometries in the $N = 2^n \to \infty$ limit [17–19]. Instead, our penalty factors remain finite in the large N limit, by transitioning to the position and momentum modes of Hamiltonian vector fields on $\mathbb{T}^2$ via (2). This identification with the hydrodynamical phase-space is naturally restricted to the low-energy sector of $\mathcal{O}(1)$ vectors, the so-called *admissible* directions. Thus, there are no infinitely penalized directions on $\mathrm{SDiff}(\mathbb{T}^2)$, resulting in a regular geometry. The remaining directions of the large $SU(N \to \infty)$ manifold containing the high-energy sectors with $\mathcal{O}(N)$ vectors are *inadmissible* and effectively decouple from the geometry since they are assigned penalty factors that are *at least* infinite. This situation is captured in the framework of sub-Riemannian geometry [47], also recently mentioned in the context of complexity in [48]. A fundamental theorem due to Chow and Rashevskii [49,50] asserts that geodesics can still reach every point by only accessing admissible directions [51,52]. That is, trajectories from the hydrodynamic phase space can still reach every large qudit unitary. Since it is infinitely expensive to move in inadmissible directions, the hydrodynamic trajectories have an overall smaller cost, i.e., smaller complexity, cf. [31].

V. AVERAGE RICCI CURVATURE

Curvature computations in recent literature regarding Nielsen complexity geometries [15–19,26] typically focus on the sectional curvatures of the manifold of unitaries. The sign of the sectional curvatures is an indicator for convergence (positive sign) or divergence (negative sign) of nearby geodesics [53]. However, the stability of a geodesic and, hence, the emergence of ergodic behavior does not only depend on the sign of the sectional curvature in the direction parallel to its velocity, but rather on the sign of the sectional curvatures of all two-planes containing its velocity vector [30]. For this reason, we believe that a more telling quantity to describe the stability of a geodesic with velocity vector v is given by the *normalized Ricci curvature* [54],

$$\mathrm{Ric}(v) = \lim_{N\to\infty} \frac{1}{N^2 - 2} \sum_{\vec{m}} K(v, \mathcal{T}_{\vec{m}}), \tag{8}$$

with K the sectional curvature tensor and $\vec{m}$ running over the algebra directions. Equation (8) can be thought of as an average sectional curvature across an orthonormal basis for the tangent space and is well defined as $N \to \infty$. Importantly, $\mathrm{Ric}(v) \leq 0$ for $\mathrm{SDiff}(\mathbb{T}^2)$ [54], which is the quantitative reason for the chaotic behavior of two-dimensional hydrodynamics. The smooth limit of our $\mathfrak{su}(N)$ basis for large N, given by $\mathrm{SVect}(\mathbb{T}^2)$, indicates we can compute Ricci curvatures of $SU(N)$ at large N and compare to the hydrodynamic result. We evaluate the Ricci curvature for every direction $\vec{m}$ in $\mathfrak{su}(N)$ for odd values of $N \in [3, 39]$ by first calculating the corresponding sectional curvatures [31]. Note that, since the dimensionality of the tangent space grows with N, a given velocity $\vec{m}$ can be defined only after it appears within the distribution of directions at $N = N_0(\vec{m})$. Its corresponding Ricci curvature $\mathrm{Ric}(\vec{m})$ is thus defined only after $N = N_0(\vec{m})$ and will continue to change with N as more and more directions contribute to the average in (8). Our numerical data shows that Ricci curvatures of newly introduced directions at a given $N = N_0$ are always positive, but all eventually turn negative at some *critical value* $N = N_c(\vec{m})$. The resulting data for N_c as a function of the direction $\vec{m} \in \mathfrak{su}(N)$ is shown in Fig. 1 and constitutes a second main result of our work. We interpret this figure and its extrapolation at large N as a visual definition of the $\mathcal{O}(1)$ subsector (the blue region) from which the hydrodynamic theory emerges in the strict large N limit [31].

Our results have the following implications for the complexity geometry of $SU(N)$ at large N. The large N geometry of the low-energy sector has negative Ricci curvature, thus numerically confirming previous mathematical results [54], as well as indicating emergent chaotic behavior [55]. This implies geodesics can reach every point of the manifold in finite time, resulting in an ergodic geodesic flow. Therefore, our canonical cost function has a property characteristic of any proper holographic complexity measure as conjectured by [14,15].

Furthermore, the numerically evaluated distribution of sectional curvatures of our model always contains positive sectional curvatures at large N [31]. The presence of positively sectional curvatures is also a necessary property of our cost function from two points of view: First, while a strictly negative geometry is indeed ergodic [18,56], complexity metrics on Lie groups without positive curvatures are necessarily flat [19,57]. Our choice of metric thus combines the negative average Ricci curvature beneficial for ergodicity with the necessity of having positively curved directions. Second, strictly negative geometries lack an important feature of geometric complexity, namely conjugate points [58]. These are points on a manifold where a geodesic ceases to be globally minimizing, e.g., antipodal points on the sphere. Conjugate points seem to be a necessary feature of complexity geometries, since they forbid complexity from exhibiting an unbounded linear growth with time [4,17–19]. It is well-known that $\mathrm{SDiff}(\mathbb{T}^2)$, our effective geometry at $N \to \infty$, indeed exhibits conjugate points [30,59]. All in all, our results indicate that even though the large N limit considered here is more similar to the vector large N limit than to the matrix large N limit relevant in holography [28], our setup still exhibits desirable properties of a holographic complexity measure for large enough values of N.

Finally, our results for the sectional curvatures [31] indicate the existence of a universality class of $SU(N)$ metrics, as defined in [48], indexed by N^2. These $SU(N)$ metrics are conjectured to be equivalent, i.e., leading to the same complexity, at late geodesic times. We infer from this conjecture that the complexity of $SU(N)$ scales at large N and at large geodesic distances as $\mathcal{C}(SU(N)) \simeq \mathcal{C}(\mathrm{SDiff}(\mathbb{T}^2)) + \mathcal{O}(1/N)$. Moreover, this implies a finite critical $N = N_C$ such that $\mathcal{C}(SU(N_C))$ at short distances equals $\mathcal{C}(\mathrm{SDiff}(\mathbb{T}^2))$ at long distances, even if the manifolds are not isomorphic.

VI. DISCUSSION AND OUTLOOK

For the first time, we provide a definition for Nielsen operator complexity of $SU(N)$ with a well-defined large N limit. This is realized by using the ideal hydrodynamics equation as the geodesic equation on the low-momentum sector of $SU(N \to \infty) \cong \mathrm{SDiff}(\mathbb{T}^2)$. The natural choice of cost function is the kinetic energy of the fluid, which we derive within the Euler-Arnold approach and adapt for every value of N. Our construction provides a simple way of computing the control Hamiltonian, thus simplifying one of the main obstacles in the computation of Nielsen complexity. In particular, our setup allows to reach every point of the large $SU(N \to \infty)$ manifold via admissible geodesics within the hydrodynamical phase space, thus drastically simplifying the complexity geometry.

From the perspective of quantum information, we find a basis for qudit unitaries that scales nicely with the qudit size. This scalability allows for the synthesis of large qudit gates as long as they only implement $\mathcal{O}(1)$ transitions, with respect to N. This corresponds to a locality property of our basis, implying its usefulness for constructing qudit lattices. This is particularly interesting in view of computing complexity of fault-tolerant quantum error-correction qudit architectures [60–62].

It is possible to include $1/N$ corrections for large, but finite, qudit unitaries in order to confirm the conjectured scaling of $\mathcal{C}(SU(N))$ with N in terms of $\mathcal{C}(\mathrm{SDiff}(\mathbb{T}^2))$. This will also allow to derive the critical N_C at which the complexities for the two manifolds coincide. This approach is closely related to integrable systems in noncommutative geometry [37,63–65], see [31] for a first step in this direction.

Finally, our cost function captures two essential properties in view of holographic complexity, ergodicity and conjugate points. Both are consequences of the phase space geometry of hydrodynamics. This suggests that cost functions based on the Laplacian acting on infinitesimal gates are a promising new avenue for describing operator complexity also in holographic CFTs.

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