Symmetry restoration, tunneling, and the null energy condition

Jean Alexandre
Theoretical Particle Physics and Cosmology, King's College London WC2R 2LS, United Kingdom

Janos Polonyi
Theory Group, Institut Pluridisciplinaire Hubert Curien, University of Strasbourg, BP 28, 67037 Strasbourg cedex 2, France

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A finite volume allows tunneling between degenerate vacua in quantum field theory, and leads to remarkable energetic features, arising from the competition of different saddle points in the partition function. We describe this competition for finite temperature at equilibrium, taking into account both static and (Euclidean) time-dependent saddle points. The effective theory for the homogeneous order parameter yields a nonextensive vacuum energy at low temperatures, implying a dynamical violation of the null energy condition.

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I. INTRODUCTION

Spontaneous symmetry breaking (SSB) is an important and peculiar feature of many body systems. Its importance stretches from solid state physics to the Standard Model and its peculiarity is that it is not truly realized. In fact, SSB is valid in strictly infinite systems, a condition which one never meets in physics. However the order of magnitude of Avogadro's number and the ratio of the macroscopic and microscopic characteristic scales make SSB an excellent approximation in most cases. This work shows how fundamental energetic features are modified in finite systems, where SSB is not a good approximation. More specifically, the null energy condition (NEC [1]) can be violated, as shown in [2–4], and the present article describes the finite-temperature equilibrium state obtained from tunneling between two vacua.

In the thermodynamical limit, involving an infinite volume, a first order phase transition is characterized by the Maxwell cut, signalling the degeneracy of the ground state of a system. The corresponding flat effective potential is consistent with convexity, obtained as a consequence of the competition of different key configurations in the partition function [5], the saddle points of the model. For finite systems though, symmetry is restored by tunneling, and the effective potential obtained from the interplay of different saddle points has a nontrivial dependence on the volume. This featured was noticed in [6], where the one-particle-irreducible (1PI) effective potential is calculated for an $O(N)$-symmetric scalar field, taking into account homogeneous saddle points. In [6] fluctuation factors are ignored in the semiclassical approximation for the partition function, but taking these factors into account leads to identical conclusions [4]. In these works, the Maxwell cut is recovered in the limit where the spacetime volume goes to infinity.

We briefly comment here on the Wilsonian running potential, obtained from exact functional renormalization group studies [7]. While this potential is not necessarily convex along the renormalization trajectory, it approaches a convex effective potential at the infrared end point [8]. But the Wilsonian effective potential is identical to the 1PI effective potential at the IR end point only for infinite spacetime volumes, whereas the present study focuses on finite volumes.

NEC violation is known in quantum field theory, with the typical example of the Casimir effect (see [9] for a review). This effect has been used in the context of Early Cosmology to induce a spacetime expansion [10], where NEC violation is obtained from a massless scalar field in a 3-torus. The Casimir effect is not only suppressed by some inverse power of the volume though, but it is also exponentially suppressed by the scalar field mass, and we neglect it in the present work. In the case we study here—a massive scalar field in the presence of different vacua—NEC violation is expected by recalling the variational method of quantum mechanics, where the ground state energy is lowered by taking into account the mixing of the degenerate potential wells [11]. In the present work, the NEC violation is a consequence of a nontrivial volume dependence of the effective action $S_{\rm eff}$, hence of nonextensive thermodynamical potentials. The energy density ρ

and the pressure p in the vacuum are obtained from the free energy $F = VU_{\rm eff}(0) = TS_{\rm eff}(0)$, and

$$\rho + p = \frac{1}{V}\left(F - T\frac{\partial F}{\partial T}\right) - \frac{\partial F}{\partial V} = -T\frac{\partial U_{\rm eff}(0)}{\partial T} - V\frac{\partial U_{\rm eff}(0)}{\partial V}. \quad (1)$$

The first term in the second line is the usual one, arising from temperature-driven quantum fluctuations, and is positive. The nontrivial second term arises from tunneling and is a consequence of the nonextensive feature of the effective action $S_{\rm eff}(0)$. Note that, if the partition function is based on one saddle point only, then the effective action is extensive and this second term vanishes. In the situation of several saddle points though, we show in this article that this term is negative, and quantum fluctuations dominates over thermal fluctuations at low enough temperature. In our work, NEC violation is therefore a finite volume effect. As shown in [2], the corresponding mechanism could be relevant for the generation a cosmological bounce [12], without the need for modified gravity or exotic matter.

We stress here that this work is not related to the Kibble-Zurek mechanism [13]. The latter necessitated a high temperature in order to restore symmetry, whereas the mechanism we present here is valid for any temperature, including zero-temperature, where SSB would indeed occur in an infinite volume. This is detailed in Sec. II, where we also explain that we do not take into account bubbles of different vacua, because we work at finite volume. In Sec. III we describe the different saddle points relevant here, which are the static ones and the time-dependent instanton/anti-instantons pairs, hence going further than the work [14], which take into account static saddle points only. The contribution of these saddle points to the semiclassical expansion is discussed in Sec. IV. We study then the intermediate-temperature regime in Sec. V, where the Euclidean time is not large enough to allow the formation of instanton/anti-instanton pairs, and only the static saddle points contribute to the partition function. Although the effective potential has a nonextensive field dependence, the vacuum energy is intensive, such that the NEC is satisfied in the vacuum. Section VI describes the low temperature regime, dominated by a dilute gas of instanton/anti-instantons, which allow for a nontrivial volume-dependence of the vacuum energy. As a result, for a fixed volume, we show that we can always find a temperature small enough for the NEC to be violated.

II. TUNNELING VERSUS THERMAL SYMMETRY RESTORATION

We consider the bare Euclidean action for a scalar field $\phi(t,\vec{x})$ at temperature $T = 1/\beta$ and three dimensional spatial volume $V = L^3$

$$S_{\rm bare}[\phi] = \int_0^\beta dt \int_V d^3x \left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{m_b^2}{2}\phi^2 + \frac{\lambda_b}{4!}\phi^4\right]. \quad (2)$$

The parameters m_b^2, λ_b of the bare theory are chosen in such a manner that the vacuum displays the spontaneously broken symmetry $\phi \to -\phi$. Therefore the order parameter, the space-time average of the field, develops nonvanishing expectation value in the thermodynamical limit $V \to \infty$, and for temperatures below the critical temperature which appears in the Kibble-Zurek mechanism

$$T < T_c = 2v_0, \quad (3)$$

where v_0 is the dressed vacuum expectation value (vev) for vanishing temperature.

Exact SSB may occur only in a strictly infinite system which is a formal construct since one always encounters finite objects in physics. A finite physical system is said to display broken symmetry if SSB becomes a better and better approximation as the volume increases without limit. Hence SSB is a particular asymptotic finite size scaling law.

Let us start with an infinite system in a symmetry broken vacuum where the SSB is exact and distinguish two symmetry restoration mechanisms. The symmetry is restored thermally if thermal fluctuations have sufficient energy to spread the density matrix over different degenerate vacua. Such a symmetry restoration mechanism is active even in the strictly infinite volume case, and happens for $T > T_c$. The alternative symmetry restoration process, tunneling under the potential barrier separating the degenerate vacua, operates only in finite volume. As was observed in [2–4], the nonextensive feature of the thermodynamical potential due to such a symmetry restoration in finite systems leads to NEC violation. Extensivity of a quantity is defined for lengths large compared to the characteristic length of a system though, such that our main interest in this work is to find the volume dependence of the thermodynamical potential for large volume.

The improvement of the SSB approximation with increasing volume is related to a tunneling time $\tau_t(V)$ which increases exponentially with the volume V. In addition, the tunneling dynamics depends on the temperature $T = 1/\beta$, in such a way that the thermodynamical properties of the system are controlled by the dimensionless ratio $\beta/\tau_t(V)$, cf. Eq. (46) below. Tunneling ceases to be active for $V \gg V_\beta$ where V_β is defined by $\beta = \tau_t(V_\beta)$. As shown in Sec. VI, the latter identity leads to a temperature $T(V)$ which decreases exponentially with the volume, and below which extensive properties are recovered.

We are interested in the effective dynamics of the order parameter because it may have an important role in violating NEC. This effective dynamics is defined by the Wilsonian action $S[\phi]$ for the spatially homogeneous field component, $\phi(t)$. It is important to distinguish the full dynamics of the field $\phi(t,\vec{x})$ and the effective dynamics of the order parameter $\phi(t)$: The spontaneous breakdown or the restoration of the symmetry is driven by the former, whereas the latter only provides a diagnostic of the status of symmetry.

To probe the possible nonextensive feature of the thermodynamical potentials in the symmetry broken phase we need the partition function of the full theory as the function of the volume and the temperature. The status of symmetry impacts the partition function, with an effect encoded by the effective potential for $\phi(t)$. To obtain the effective potential in the one-loop saddle point expansion, we assume the following action to describe the effective dynamics (see Appendix for details)

$$S[\phi] = V\int_0^\beta dt\left(\frac{1}{2}(\partial_0\phi)^2 + \frac{\lambda}{24}(\phi^2 - v^2)^2 + j\phi\right), \qquad (4)$$

which features a double well structure. It is argued below that the qualitative aspects of our results remain valid beyond this particular form of the action.

We end this section with a comment on the choice of a homogeneous order parameter $\phi(t)$. For a nonvanishing source j, the vacuum degeneracy is shifted, and one should in principle consider coexisting domains of true and false vacuum. Following [15], it is reasonable to assume spherical domains of radius R, whose actions are of the form

$$S_{\rm bubble} = -ajR^3 + bR^2, \qquad (5)$$

where $a > 0$ and $b > 0$. In the previous expression, the volume contribution is proportional to the energy difference between the vacua, and competes with the surface term arising from surface tension. The resulting critical bubble radius is

$$R_{cr} = \frac{2b}{3aj}, \qquad (6)$$

which diverges for the ground state of the system, obtained for $j \to 0$ because of symmetry restoration. As a consequence, finite volumes on which the present study is based do not allow the formation of bubbles, as long as one focuses on the vicinity of the true ground state. This justifies the study of homogeneous and time-dependent saddle points in this work.

III. SADDLE POINTS FOR HOMOGENEOUS DYNAMICS

We introduce the dimensionless variables

$$\tau = \omega t, \qquad \varphi \equiv \sqrt{\frac{\lambda}{6}}\frac{\phi}{\omega}, \qquad k \equiv \sqrt{\frac{\lambda}{6}}\frac{j}{\omega^3}, \qquad (7)$$

with $\omega \equiv v\sqrt{\lambda/6}$, we are lead to the bare action

$$S = B\int_{-\omega\beta/2}^{\omega\beta/2} d\tau\left((\varphi')^2 + \frac{1}{2}(\varphi^2 - 1)^2 + 2k\varphi\right), \qquad (8)$$

where a prime denotes a derivative with respect to τ and

$$B \equiv \frac{3}{\lambda}(L\omega)^3. \qquad (9)$$

The equation of motion obtained from the action (8) is

$$\varphi'' + \varphi = \varphi^3 + k. \qquad (10)$$

Below we study the different solutions of this equation, which are relevant to finite-temperature field theory, where periodic boundary conditions are imposed on field configurations. We note that there are other solutions of the equation of motion, which are relevant in the context of false vacuum decay, as the "shot" [16], which does not satisfy periodic boundary conditions though, and is not considered in the present study.

The partition function is an analytic function of k for finite volume $V < \infty$ and finite temperature $\beta < \infty$. As a consequence, the formal symmetry $\phi \to -\phi$ and $k \to -k$ assures that the partition function depends on k^2, although individual saddle points are not necessarily even in the source.

In what follows we consider an infinitesimal source k, since we focus on the ground state of the system: Because of tunneling, this ground state corresponds to a vanishing classical field, which in finite volume maps to a vanishing source through the Legendre transform.

A. Static saddle points

Static vacua satisfy

$$\varphi^3 - \varphi + k = 0, \qquad (11)$$

and the number of solutions depends on the dimensionless source k: if we define $k_c \equiv 2/3\sqrt{3} \simeq 0.385$, we have the following two regimes:

(i) For $|k| > k_c$, the model has only one (real) static vacuum, which is

$$\varphi_{s0} = -\mathrm{sign}(k)\frac{2}{\sqrt{3}}\cosh\left(\frac{1}{3}\cosh^{-1}(|k|/k_c)\right). \qquad (12)$$

(ii) For $|k| < k_c$ the model has two static vacua, which are

$$\varphi_{s1}(k) = \varphi_s(k) \quad \text{and} \quad \varphi_{s2}(k) = -\varphi_s(-k) \qquad (13)$$

where

$$\varphi_s(k) = \frac{2}{\sqrt{3}}\cos\left(\frac{\pi}{3} - \frac{1}{3}\cos^{-1}(k/k_c)\right), \qquad (14)$$

and which corresponds to the regime we focus on in this article. The actions for these saddle points are

$$S_1 \equiv S[\varphi_{s1}] = \Sigma(k), \qquad S_2 \equiv S[\varphi_{s2}] = \Sigma(-k), \quad (15)$$

where

$$\Sigma(k) = B\omega\beta\left(2k - \frac{k^2}{2} - \frac{k^3}{4} - \frac{k^4}{4} - \frac{21}{64}k^5 - \frac{k^6}{2} + \mathcal{O}(k^7)\right), \quad (16)$$

where we note that they differ only by the sign of the odd powers of the source k. This will be important to recover the appropriate parity of the effective potential.

B. Time-dependent saddle points

The Euclidean time-dependence of a saddle point can be found from the Minkowski time-dependence after the sign flip $U(\phi) \to -U(\phi)$. There are many periodic solutions of Eq. (10), whose qualitative features can better be understood by following their period length as the function of the maximal value, $\varphi_m = \max_\tau(\varphi(\tau))$. The solutions with the shortest period $2\pi/\omega$ are harmonic oscillations around $\varphi = 0$ with infinitesimal φ_m. Hence there are no time-dependent saddle points at high temperature, $\beta < 2\pi/\omega$. The increase of φ_m leads longer periods, since the quartic part of the potential weakens the restoring force to the equilibrium position $\varphi = 0$. As φ_m approaches $\max\{\varphi_{s1}, \varphi_{s2}\}$, the trajectory spends most of the time around one of these static saddle points and the period length diverges. The action is a decreasing function of φ_m hence our interest lies mainly in low temperature time-dependent saddle points. The periodic saddle points are instanton/anti-instanton pairs, and each instanton or anti-instanton has the approximate width $1/\omega$. The number of pairs allowed in the Euclidean total time β is not a continuous function of β, and their maximum number is

$$N_\beta \sim \frac{\omega\beta}{2\pi}. \quad (17)$$

We are thus led to two saddle point regimes, that we will study separately:

(i) The intermediate-temperature regime $\beta_c < \beta < 2\pi/\omega$, where only static saddle point are present. This is a wide temperature interval if $\omega\beta_c \ll 2\pi$;

(ii) The low-temperature regime $2\pi/\omega \ll \beta$, where instanton/anti-instanton pairs can develop.

1. Instanton/anti-instanton pair

For large but finite $\omega\beta$, a instanton/anti-instanton pair spends most of the Euclidean time close to the static saddle points $\pm v$ when $k = 0$. Figure 1 shows such a pair, which is exactly symmetric and spends the same time close to the two vacua. This pair can be approximated by

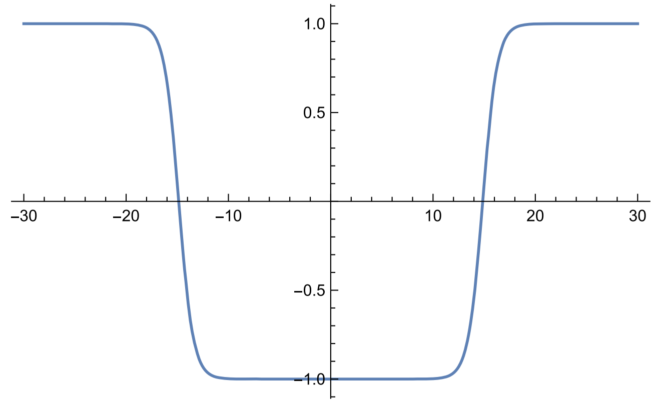

FIG. 1. An instanton/anti-instanton pair within the period $\omega\beta = 60$, obtained with $\varphi_0(0) \simeq -0.999999997$. In the absence of source, the configuration spends the same time close to both static saddle points.

$$\varphi_0 \simeq \tanh\left(\frac{\tau - \frac{\omega\beta}{4}}{\sqrt{2}}\right)\tanh\left(\frac{\tau + \frac{\omega\beta}{4}}{\sqrt{2}}\right), \quad (18)$$

since, for both signs of τ, one of the hyperbolic tangent factors is very close to ± 1, and the other hyperbolic tangent factor satisfies the equation of motion. Apart from a time approximately equal to $2/\omega$, the saddle point is exponentially close to the vacuum hence the action for the configuration arises mainly from the jumps only and can be approximated by

$$S_0 = \frac{8\sqrt{2}}{3}B. \quad (19)$$

If one introduces an infinitesimal source $k \neq 0$, the action for one instanton/anti-instanton pair is still dominated half of the Euclidean time β by the action of the saddle point ϕ_{s1}, and the other half by the action of the saddle point ϕ_{s2}. The subdominant part of the action arises from the jumps of the instanton and anti-instanton, and the pair action is

$$S_p(k) \simeq S_0 + \frac{1}{2}\Sigma(k) + \frac{1}{2}\Sigma(-k), \quad (20)$$

where S_0 and Σ are given by Eqs. (19) and (16) respectively.

2. Anti-ferromagnetic saddle point configurations

The regime $\omega\beta \gg 1$ allows multiple instanton/anti-instanton pairs exact saddle points, which are regularly distributed in the Euclidean period β. The corresponding action is even in k and has the generic form

$$S_p^{\rm multi}(k) = \sum_{l=0}^{\infty} s_{2l}k^{2l}. \quad (21)$$

If the instantons and anti-instantons are far enough from each other though, the configuration still spends most of the time close to one saddle point, and the total contribution from the n pairs is approximately nS_0. The action is thus, for an infinitesimal source,

$$S_p^{(n)}(k) \simeq nS_0 + \frac{1}{2}\Sigma(k) + \frac{1}{2}\Sigma(-k), \tag{22}$$

and a discussion on dilute gas of instanton/ant-instanton is given in the next section.

IV. SEMICLASSICAL APPROXIMATION

In the presence of multiple saddle points ϕ_n and if one does not allow SSB, the partition function can be approximated by the sum of path integrals, each integrating fluctuations perturbatively around ϕ_n

$$Z[j] \simeq \sum_i \int \mathcal{D}[\xi] \exp(-S[\phi_i + \xi]), \tag{23}$$

where the action $S[\phi_i + \xi]$ involves the source term. This semiclassical approximation becomes better as the space-time volume $V\beta$ is large, since fluctuations around each saddle point are suppressed exponentially and do not communicate. We have thus

$$\int \mathcal{D}[\xi] \exp(-S[\phi_i + \xi]) \simeq F_i e^{-S[\phi_i]}, \tag{24}$$

where the factors F_i take into account fluctuations determinants for time-dependent modes, but also the eventual zero mode corresponding to the translational invariance of instantons/anti-instantons, as explained below.

A. Static saddle points

The fluctuation factor arising from time-dependent fluctuations over a static saddle point is calculated in Appendix, and we have

$$F_1 = F_\beta(k) \quad \text{and} \quad F_2 = F_\beta(-k), \tag{25}$$

where

$$F_\beta(k) = \exp\Big(-B \ln \sinh\Big((\omega\beta/2)\sqrt{3\varphi_s^2(k) - 1}\Big)\Big), \tag{26}$$

and $\varphi_s(k)$ is given in Eq. (14). Note that, for $\omega\beta \gtrsim 2$ we have

$$F_\beta(k) \simeq \exp\left(-\frac{B\omega\beta}{2}\sqrt{3\varphi_s^2(k) - 1}\right), \tag{27}$$

which will be used further on.

B. Instanton/anti-instanton pair

Following [11], the fluctuation factor for an instanton or an anti-instanton spending time β_1 close to one static saddle point and the time β_2 close to the other static saddle point, with $\beta_1 + \beta_2 \simeq \beta$ is

$$F_{\text{inst}} \simeq \sqrt{\frac{S_0}{2}} F_{\beta_1}(k) F_{\beta_2}(-k)\omega\beta, \tag{28}$$

where the factor $\sqrt{S_0/2}\omega\beta$ arises from the zero mode corresponding to the translational invariance of the configuration over the total length $\omega\beta$.

For an instanton/anti-instanton pair, with negligible correlation for $\beta\omega \gg 1$, the overall saddle point spends some time β_1 close to one static saddle point, then some time β_2 close to the other static saddle point, and back close to the first one for some time β_3, with $\beta_1 + \beta_2 + \beta_3 \simeq \beta$. The center of the instanton can be placed freely over a time interval of length $\omega\beta$ and the center of the anti-instanton can be placed in the remaining time, leading to the zero-mode factor

$$\sqrt{\frac{S_0}{2}} \int_{-\omega\beta/2}^{\omega\beta/2} d\tau_1 \sqrt{\frac{S_0}{2}} \int_{\tau_1}^{\omega\beta/2} d\tau_2 = \frac{1}{4} S_0 (\omega\beta)^2. \tag{29}$$

On average $\beta_1 + \beta_3 = \beta_2 = \beta/2$ such that the total fluctuation factor for an instanton/anti-instanton pair is

$$F_p^{(1)} \simeq \frac{S_0}{4}(\omega\beta)^2 F_{\beta_1}(k) F_{\beta_2}(-k) F_{\beta_3}(k), \tag{30}$$

and, given the asymptotic form (27),

$$F_p^{(1)} \simeq \frac{S_0}{4}(\omega\beta)^2 F_{\beta/2}(k) F_{\beta/2}(-k). \tag{31}$$

C. Instanton/anti-instanton gas

The exact solutions of the equation of motion (10) involving several pairs of instanton/anti-instanton form an "antiferromagnetic crystal," with a rigid structure and therefore with a small configuration-space measure in the partition function. The approximate saddle points made of weakly coupled instanton/anti-instanton, where the later are free to be moved around in time, have a huge degeneracy arising form translational zero modes, without increasing much the action of the total configuration. Because of this degeneracy, at low enough temperature they dominate over the exact crystal-structured saddle points. The zero mode for n instanton/anti-instanton pairs is then

$$\sqrt{\frac{S_0}{2}} \int_{-\omega\beta/2}^{\omega\beta/2} d\tau_{2n} \sqrt{\frac{S_0}{2}} \int_{\tau_{2n}}^{\omega\beta/2} d\tau_{2n-1} \cdots \sqrt{\frac{S_0}{2}} \int_{\tau_2}^{-\omega\beta/2} d\tau_1 = \frac{S_0^n}{2^n} \frac{(\omega\beta)^{2n}}{(2n)!}. \tag{32}$$

On average, the antiferromagnetic saddle point spends the overall time $\beta/2$ close to both static saddle point and, using the asymptotic form (27), we obtain for the total fluctuation factor for n pairs

$$F_p^{(n)} \simeq \frac{S_0^n}{2^n}\frac{(\omega\beta)^{2n}}{(2n)!}F_{\beta/2}(k)F_{\beta/2}(-k). \tag{33}$$

D. Comments on the bounce saddle point

In the present study we focus on an infinintesimal source k, which is why the relevant time-dependent saddle points consist in pairs of instanton and anti-instantons. The latter configurations do not generate negative fluctuation modes, which is why the above fluctuation factors are all real.

If one considers a larger source though, the time-dependent saddle point cannot be decomposed as dilute pairs of instantons and anti-instantons, and the basic building block of the dominant configurations is a bounce instead, as shown in Fig. 2. Such a bounce does induce an imaginary fluctuation determinant, arising from one negative mode which should be treated appropriately with an analytical continuation, for the integration over quadratic fluctuations [11]. This feature is used to determine the decay rate of a false vacuum in $O(4)$-symmetric Euclidean coordinates [15], and does not appear in our study, which focused on the vicinity of $k = 0$.

Should we include higher orders in the classical field for the effective potential, we would need to take a larger source, and thus consider the bounce imaginary fluctuation factor. The partition function is real though, and so must be the effective potential which describes an equilibrium state, such that no imaginary part should appear in the effective dynamics. As explained in [16], when integrating quadratic fluctuation, the analytic continuation to avoid the bounce negative mode requires a contour of integration which goes through the other saddle points. The resulting closed contour necessarily involves additional imaginary parts, which exactly cancel the imaginary part arising from the bounce, indeed leading to real effective dynamics.

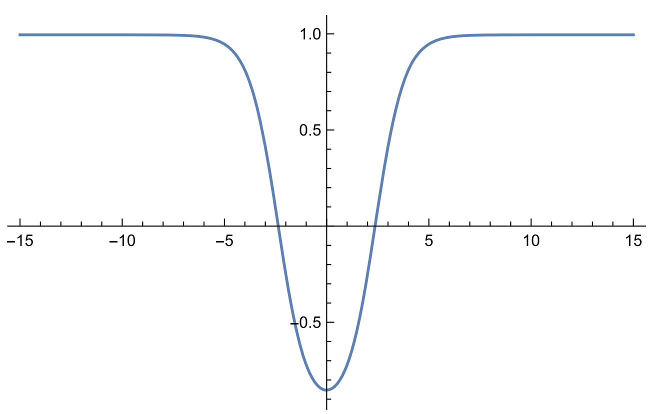

FIG. 2. A single-bounce configuration within the period $\omega\beta = 30$, obtained with the source $k = 0.01$ and $\varphi_B(0) \simeq -0.853183042684$. A finite source breaks the symmetry between the vacua, and the bounce cannot be decomposed as a dilute instanton/anti-instanton pair.

V. INTERMEDIATE-TEMPERATURE REGIME

We consider here the intermediate temperature regime $\beta_c < \beta \lesssim 2\pi/\omega$, dominated by static saddle points.

A. Effective potential

Although in this work we ignore spatial fluctuations, we rescale the partition function $Z_\beta[k]$ by a source-independent term R_β, in order to reproduce the usual ground state $(3+1)$-thermal fluctuations contribution, at each static saddle point. This contribution is

$$U_0 = \frac{T^4}{2\pi^2}I(\sqrt{2}\omega\beta), \tag{34}$$

where

$$I(a) \equiv \int_0^\infty dx x^2 \ln\left(1 - e^{-\sqrt{x^2+a^2}}\right), \tag{35}$$

and for high temperatures leads to the Stefan-Boltzmann law

$$U_0 \sim -\frac{\pi^2 T^4}{90} \quad \text{for } \omega\beta \ll 1. \tag{36}$$

In order to achieve this, we need to take

$$R_\beta = \frac{1}{F_\beta(0)}\exp\left(-\frac{VI(\sqrt{2}\omega\beta)}{2\pi^2\beta^3}\right), \tag{37}$$

such that the partition function is

$$\begin{aligned} Z_\beta(k) &\simeq R_\beta[F_1(k)\exp(-S_1(k)) + F_2(k)\exp(-S_2(k))] \\ &= R_\beta[F_\beta(k)\exp(-\Sigma(k)) + F_\beta(-k)\exp(-\Sigma(-k))] \\ &= \exp\left(-\frac{VI(\sqrt{2}\omega\beta)}{2\pi^2\beta^3}\right)\left(2 + \frac{B\omega\beta A_\beta}{32}k^2\right) + \mathcal{O}(k^4), \end{aligned} \tag{38}$$

where

$$\begin{aligned} A_\beta = \frac{1}{\sinh^2(\omega\beta/\sqrt{2})}(&-32 + 18\omega\beta - 119B\omega\beta \\ &+ (32 + 137B\omega\beta)\cosh(\sqrt{2}\omega\beta) \\ &+ 3\sqrt{2}(7 - 16B\omega\beta)\sinh(\sqrt{2}\omega\beta)). \end{aligned} \tag{39}$$

As expected the partition function is even in k, and one can check that $A_\beta > 0$ for all β. The classical field, the homogeneous source generated saddle point, is

$$\phi_c \equiv -\frac{\delta \ln(Z_\beta)}{\delta j} \to -\frac{1}{V\beta}\frac{\partial \ln(Z_\beta)}{\partial j} = \frac{-v}{2B\omega\beta}\frac{\partial \ln(Z_\beta)}{\partial k}, \quad (40)$$

which, in terms of the original source j is

$$\phi_c = -\frac{A_\beta j}{64\omega^2} + \mathcal{O}(j^3). \quad (41)$$

Because we have a finite volume, there is a one-to-one mapping between the source j and the classical field ϕ_c, as in the situation where quantization is based on one saddle point. This feature is essential to determine the Legendre transform leading to the 1PI effective potential. Also, one can see from Eq. (41) that the classical field vanishes when $j \to 0$, which shows that the true vacuum of the model is at $\phi_c = 0$.

The effective potential is finally obtained by integrating $U'_{\rm eff}(\phi_c) = -j$, hence

$$U_{\rm eff}(\phi_c) = U_0 + \frac{32\omega^2}{A_\beta}\phi_c^2 + \mathcal{O}(\phi_c^4), \quad (42)$$

where U_0 is given in Eq. (34). Hence the effective potential is indeed convex, with its minimum at $\phi_0 = 0$, and with volume-dependent coupling constants.

B. Null energy condition

For a perfect fluid with energy density ρ and pressure p, the NEC $\rho + p \geq 0$ is considered a universal feature for known matter [1], and is one of the key ingredients for singularity theorems in cosmology [17]. Few quantum effects are known for violating the NEC though [18], and among these the Casimir effect has been shown to provide a cosmological expansion [10], arising from a dynamical NEC violation.

The competition between two vacua and the resulting tunneling effect have proven to violate the NEC dynamically at zero temperature and for an $O(4)$-symmetric Euclidean spacetime [2–4]. For the present intermediate-temperature regime though, we have $U_{\rm eff}(0) = U_0$, which is independent of the volume, such that

$$\begin{aligned} \rho + p &= -T\frac{\partial U_0}{\partial T} \\ &= \frac{T^4}{2\pi^2}\left(a\frac{dI(a)}{da} - 4I(a)\right) > 0, \end{aligned} \quad (43)$$

where $a = \sqrt{2}\omega\beta$: the vacuum satisfies the NEC.

C. Exact SSB limit

The results derived in this section are based on expansions which assume $B\omega\beta k \ll 1$, such that in principle one cannot consider a large volume ($B \gg 1$). But it is interesting to note that Eq. (42) remains useful for $B \gg 1$ to indicate that the flatness of the effective potential, the Maxwell cut, is realized for $B = \infty$ when the SSB approximation is exact.

One can predict a flat effective potential in the thermodynamical limit though, without any expansion of the partition function. Indeed, for $(B\omega\beta)^{-1} \ll k \ll 1$ we have

$$\begin{aligned} Z_\beta &\to R_\beta F_\beta(-\epsilon k)\exp(-\Sigma(-\epsilon k)) \\ &\quad \text{where } \epsilon = \mathrm{sign}(k) = \mathrm{sign}(j). \end{aligned} \quad (44)$$

such that, when $j \to 0$,

$$\phi_c \to \epsilon v\left(1 - \frac{3}{8\sqrt{2}}\right), \quad (45)$$

leading to the discontinuity $\Delta\phi_c \simeq 2v$. Hence the one-to-one mapping between j and ϕ_c is lost for an infinite volume: $j = 0$ for all $|\phi_c| \lesssim v$, leading to a constant effective potential and thus to the Maxwell cut.

VI. LOW TEMPERATURE REGIME

For large $B\omega\beta$ but a source small enough to satisfy $B\omega\beta|k| \ll 1$, the two static saddle points play a similar role, and the instanton/anti-instanton pairs are most of the time asymptotically close to one static saddle point or the other—see Fig. 1. In the corresponding approximate saddle point, these instantons and anti-instantons can be translated individually: they form a dilute gas when $\omega\beta$ is large enough for instanton and anti-instantons to be far enough from each other, and the action for an n-pairs configuration can be approximated by the expression (22). The probability per unit time for an instanton or anti-instanton to form is $\omega\sqrt{S_0/2}e^{-S_0/2}$, so that the average number $\bar{N}$ of these configurations during the Euclidean time β is

$$\bar{N} \simeq \omega\beta\sqrt{\frac{S_0}{2}}e^{-S_0/2}. \quad (46)$$

We note that $\bar{N}$ is just β/τ_t introduced in Sec. II. The average distance $\Delta\beta$ between instantons and anti-instantons can be expressed in terms of $\bar{N}$

$$\Delta\beta \equiv \frac{\beta}{\bar{N}} \simeq \sqrt{\frac{2}{S_0}}\frac{e^{S_0/2}}{\omega}, \quad (47)$$

and is large compared to the width $1/\omega$ of an instanton or anti-instanton in the dilute gas assumption.

A. Effective potential

Taking into account the fluctuation factor (33), the rescaled partition function including the static saddle points and the gas of instanton/anti-instanton pairs is

$$
\begin{aligned}
Z_\beta(k) &= R_\beta\bigg[F_\beta(k)e^{-\Sigma(k)} + F_\beta(-k)e^{-\Sigma(-k)} \\
&\quad + F_{\beta/2}(k)F_{\beta/2}(-k)\left(\sum_{n=1}^{\infty}\frac{(\omega\beta)^{2n}}{(2n)!}\left(\frac{S_0}{2}\right)^n e^{-nS_0}\right) \\
&\quad \times e^{-\Sigma(k)/2-\Sigma(-k)/2}\bigg] \\
&= R_\beta[F_\beta(k)e^{-\Sigma(k)} + F_\beta(-k)e^{-\Sigma(-k)} \\
&\quad + F_{\beta/2}(k)F_{\beta/2}(-k)(\cosh(\bar{N})-1)e^{-\Sigma(k)/2-\Sigma(-k)/2}].
\end{aligned} \tag{48}
$$

This leads to the classical field

$$\phi_c = -\frac{\hat{A}_\beta j}{64\omega^2} + \mathcal{O}(j^3), \tag{49}$$

where, if we assume $\exp(-\omega\beta/\sqrt{2}) \ll 1$,

$$\hat{A}_\beta = 32 + 21\sqrt{2} + 2B\omega\beta\frac{137-48\sqrt{2}}{1+\cosh(\bar{N})}, \tag{50}$$

and $\bar{N}$ is given by Eq. (46). Finally, the convex effective potential is

$$U_{\rm eff}(\phi_c) = U_1 + \frac{32\omega^2}{\hat{A}_\beta}\phi_c^2 + \mathcal{O}(\phi_c^4), \tag{51}$$

where

$$U_1 = U_0 - \frac{1}{V\beta}\ln(\cosh(\bar{N})+1), \tag{52}$$

and U_0 is given by Eq. (34).

B. Violation of the null energy condition

From the expression (52) for the vacuum energy, one can consider two asymptotic cases:

$\bar{N} \ll 1$ This situation corresponds to the suppression of instanton/anti-instanton pairs, such that we expect to recover the same results as in the intermediate temperature regime. Indeed, we have

$$U_1 \simeq U_0 - \frac{\ln 2}{V\beta}, \tag{53}$$

and $(T\partial/\partial T + V\partial/\partial V)(V\beta) = 0$. The sum $\rho + p$ is thus identical to the expression (43), and the NEC is satisfied. Note that

$$\lim_{T\to 0}(\rho + p) = 0, \tag{54}$$

which is the expected result for a zero-temperature theory in infinite volume, as long as the limits $T \to 0$ and $V \to \infty$ are taken in such a way that $\bar{N} \ll 1$.

$1 \ll \bar{N}$ In this case we have

$$U_1 \simeq U_0 - \frac{\bar{N}}{V\beta} + \frac{\ln 2}{V\beta}, \tag{55}$$

such that

$$\rho + p = \frac{T^4}{2\pi^2}\left(a\frac{dI(a)}{da} - 4I(a)\right) - \frac{1+S_0}{2}\frac{\omega}{V}\sqrt{\frac{S_0}{2}}e^{-S_0/2}, \tag{56}$$

where $a = \sqrt{2}\omega\beta$. One can clearly see here the competition between:

(i) temperature-driven quantum corrections (first term), which depend on T only and vanish for $T \to 0$;
(ii) tunneling (second term) which depends on V only and vanishes for $V \to \infty$. The regime $1 \ll \bar{N}$ allows to fix V and take $T \to 0$, such that it is always possible to find a temperature small enough for which $\rho + p < 0$, and the NEC is violated in a given volume.

VII. CONCLUSION

This article describes finite-temperature and finite-volume tunneling between degenerate vacua of a scalar theory. Taking care of the appropriate order of the different limits to consider (large volume or/and large inverse temperature), we showed that the NEC can be violated dynamically by a nonextensive vacuum energy, generated by competing vacua at any finite temperature and finite volume.

One motivation for this work is the generation of a cosmological bounce in early Universe cosmology [2]. In the vicinity of a bounce, spacetime is effectively flat, allowing the tunneling process described here. Which finite volume should then be considered is still an open question: one might naively think of the Hubble volume, but the latter becomes infinite at the bounce, such that one needs to define another causal volume to describe tunneling. Furthermore, in order to apply the mechanism described here to curved spacetimes, one needs to find the energy-momentum tensor which takes into account the nonextensive nature of the matter effective action. The first step in this direction is detailed in [3], but it needs to be further developed in future work.

The extension of this mechanism to curved spacetime, specifically to cosmology, should involve a comparison between timescales for spacetime expansion and for tunneling. Hence a study of real-time tunneling would be complementary, in order to allow a real-time-dependent process (see [19] for a review), unlike the present study which is done at equilibrium.

Finally, the energetic effects derived here could have analogue condensed matter systems, as those used to study false vacuum decay [20], which is an avenue to explore in the longer term.

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APPENDIX: ONE-LOOP DYNAMICS

1. Effective theory for the order parameter

The Wilsonian action for the order parameter is a highly involved nonlocal functional, to be approximated here in four steps. First the functional form (4) of the local potential approximation is assumed, yielding

$$U^{(1)}(\phi) = \frac{m_B^2}{2}\phi^2 + \frac{\lambda_B}{4!}\phi^4 + \frac{1}{2V\beta}\mathrm{Tr}\ln((2\pi n\beta^{-1})^2 + k^2 + U''_{\mathrm{bare}}(\phi)) \tag{A1}$$

in the one-loop approximation where the trace represents the sum over Mastsubara frequency $2\pi n\beta^{-1}$ and nonvanishing three-momentum. Next the zero temperature local potential is truncated into a quartic form (without the source term)

$$U_0^{(1)}(\phi) = \frac{\lambda_0}{24}(\phi^2 - v_0^2)^2, \tag{A2}$$

where the index 0 denotes $T = 0$, involving the cutoff independent zero temperature renormalized parameters. According to the strategy of the saddle point expansion the saddle point gives the tree-level contribution and $v = \mathcal{O}(\lambda^0)$. The third approximation is to restrict the finite temperature corrections to $\mathcal{O}(\lambda)$, $m^2 = m_0^2 - \lambda T^2/24$ and $\lambda = \lambda_0$. This leads to action (4) with the known temperature-dependent vev

$$v^2 = v_0^2 - \frac{1}{4\beta^2}, \tag{A3}$$

from which one can see that symmetry is restored above the temperature $2v_0$, where $v = 0$. In this article we assume that the temperature is lower than $2v_0$ (or equivalently $\beta > \beta_c = 1/2v_0$), and thermal fluctuations do not restore the symmetry.

The low energy modes are not always perturbative in the symmetry boken phase. In fact, the modes with momentum $|\vec{k}| < \omega = v\sqrt{\lambda/6}$ develop nonvanishing saddle point, e.g., domains of the false vacuum, when the order parameter is brought in between the degenerate minimas of one-loop potential by the help of an external source.

It is instructive to consider these approximations from the point of view of the Landua-Ginzburg double expansion in the amplitude and the derivative of the order parameter. In fact, our scheme corresponds to the $\mathcal{O}(\phi^4)$ and ∂_t^2 order where the wave function renormalization constant is frozen to one. Higher order terms in the effective action should be taken into account when the order parameter assumes larger and faster changing value, at smaller volume, higher temperature and strong external source.

2. Fluctuation determinant for a static saddle point

Starting from the action

$$S = B\int_{-\omega\beta/2}^{\omega\beta/2} d\tau\left((\varphi')^2 + \frac{1}{2}(\varphi^2 - 1)^2 + 2k\varphi\right), \tag{A4}$$

expressed in terms of dimensionless quantities only, one can see that B plays the role of a dimensionless volume. Also, $\omega\beta$ plays the role of a dimensionless finite total Euclidean time, imposing a discrete set of dimensionless frequencies $\nu_n = 2\pi n/(\omega\beta)$. The integration over frequencies should be replaced by summation

$$\int d\nu\, f(\nu) \to \frac{1}{\omega\beta}\sum_n f(\nu_n), \tag{A5}$$

and the Dirac distribution for frequencies becomes $\delta(\nu - \nu') \to \omega\beta\delta_{n,n'}$. The trace of a time-dependent operator $\mathcal{O}$ is

$$\mathrm{Tr}\{\mathcal{O}\} = \int d^4x \int d^4x'\delta^{(4)}(x - x')\mathcal{O}_{\tau,\tau'} = B\int d\tau \int d\tau'\delta(\tau - \tau')\mathcal{O}_{\tau,\tau'}, \tag{A6}$$

which, in terms of the Fourier components $\tilde{\mathcal{O}}_{n,n'}$, gives

$$\mathrm{Tr}\{\mathcal{O}\} = \frac{B}{\omega\beta}\sum_n\sum_{n'}\delta_{n,n'}\tilde{\mathcal{O}}_{n,n'}. \tag{A7}$$

The second derivative of the action (8) evaluated at a static saddle point φ_s is, in terms of the discrete Fourier components,

$$\frac{\partial^2 S}{\partial\tilde{\varphi}(\nu_n)\partial\tilde{\varphi}(\nu_{n'})} = 2B(\nu_n^2 + 3\varphi_s^2 - 1)\omega\beta\delta_{n,n'}, \tag{A8}$$

and is diagonal in n, n', such that its logarithm is also diagonal

$$\ln\left(\frac{\partial^2 S}{\partial\tilde{\varphi}(\nu_n)\partial\tilde{\varphi}(\nu_{n'})}\right) = \ln[2B(\nu_n^2 + 3\varphi_s^2 - 1)]\omega\beta\delta_{n,n'}. \tag{A9}$$

The fluctuation determinant is

$$F(k) = \frac{1}{\sqrt{\det(\delta^2 S)}} = C\exp\left(-\frac{B}{2}\sum_{n=-\infty}^{\infty}\ln(n^2 + \Phi^2(k))\right), \tag{A10}$$

where C is a (source-independent) constant which can be set to 1, and

$$\Phi(k) = \frac{\omega\beta}{2\pi}\sqrt{3\varphi_s^2(k) - 1}. \tag{A11}$$

For the sum over the modes n, we note that

$$\frac{d}{d\Phi}\sum_{n=-\infty}^{\infty}\ln(n^2+\Phi^2) = 2\Phi\sum_{n=-\infty}^{\infty}\frac{1}{n^2+\Phi^2} = 2\pi\coth(\pi\Phi), \tag{A12}$$

such that

$$\sum_{n=-\infty}^{\infty}\ln(n^2+\Phi^2) = 2\,\ln|\sinh(\pi\Phi)| + \text{constant}, \tag{A13}$$

where the constant does not depend on any parameter appearing in Φ, and is therefore disregarded. The fluctuation determinant is finally

$$F(k) = \exp(-B\ln|\sinh(\pi\Phi(k))|). \tag{A14}$$

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