


Classical and quantum reconciliation of electromagnetic radiation: Vector Unruh modes and zero-Rindler-energy photons

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A great deal of evidence has been mounting over the years, showing a deep connection between acceleration, radiation, and the Unruh effect. Indeed, the fact that the Unruh effect can be codified in the Larmor radiation emitted by the charge was used to propose an experiment to experimentally confirm the existence of the Unruh thermal bath. However, such connection has two puzzling issues: (1) how the quantum Unruh effect can be codified in the classical Larmor radiation and (2) the fundamental role played by zero-Rindler-energy modes of the Unruh thermal bath in such a context. Here we generalize a recent analysis made for the scalar case to the more realistic case of Maxwell electrodynamics and settle these two puzzling issues.

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I. INTRODUCTION

Acceleration and radiation are deeply interconnected phenomena and their relationship has intrigued physicists for decades, dating back to the publication of Larmor's seminal work [1]. Although Larmor's formula played an instrumental role in establishing the relationship between acceleration and radiation, there were disagreements about its interpretation and validity (in particular, in light of Einstein's equivalence principle, see, e.g., Refs. [2,3]). The works of Rohrlich [4,5] and Boulware [6] shed some light on the classical aspects of this question. They found that radiation is not a covariant concept, as the perception of radiative phenomena is intrinsically associated with the observer's state of motion: if an inertial observer detects radiation coming from a uniformly accelerated charge, another observer in the coaccelerated frame of the charge will not. However interesting and surprising that such classical results may be, it is in the context of quantum field theory (QFT) that the interplay between acceleration and radiation presents its most interesting hues.

In 1976, Unruh [7] discovered that uniformly accelerated observers in the inertial vacuum perceive themselves immersed in a thermal bath of (Rindler) particles at a temperature

$$T_U = \frac{\hbar a}{2\pi c k_B}, \quad (1.1)$$

known as the Unruh temperature. The Unruh effect vindicated Fulling's previous discovery that the particle concept in

quantum field theory is observer dependent [8] and provided the correct explanation to the observations by Davies [9] that an accelerated observer registers radiation with a temperature proportional to his acceleration (see Ref. [10] for a review on the Unruh effect).

The connection between the Unruh effect and bremsstrahlung has been analyzed in Ref. [11] (see also Refs. [12,13]), finding that the emission and absorption rate of zero-energy Rindler photons in the accelerated frame agrees with the emission rate of the accelerated charge seen by the inertial perspective. Such an agreement can only be achieved by taking into account the existence of the thermal bath at temperature T_U in the accelerated frame. The deep link between the Unruh effect and bremsstrahlung has been strengthened by Ref. [14], where the authors found, rather surprisingly, that classical Larmor radiation codifies the purely quantum Unruh effect and provided an experimental procedure whose result can be directly interpreted in terms of the Unruh effect. More recently, in Ref. [15], the authors analyzed the scalar version of the radiation emission problem to clarify (1) the central role played by the zero-energy Rindler photons in the QFT calculations and (2) how the classical Larmor radiation can codify the quantum Unruh effect.

In this work, we expand Ref. [15] to the more realistic case of Maxwell electrodynamics. After extending the definition of Unruh modes to vector-valued solutions of the electromagnetic field equation (while retaining the important characteristics that made them useful for the scalar analysis), we use them to expand the retarded potential associated with a uniformly accelerated charge and find the corresponding expansion amplitudes. With such an expansion, we define and compute the classical

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number of photons emitted by the charge and show the fundamental role played by the zero-energy Rindler photons of the Unruh thermal bath in such a process. Next, we proceed to the quantum analysis of the radiation emission by relating the field state in the asymptotic past and future, to find that not only the number of emitted photons will coincide with the classical case, but also the expectation values of the field and energy-momentum tensors agree with their classical counterparts. As in the classical case, it is made explicit how the radiation seen by the inertial observers can be traced back to zero-energy Rindler photons absorbed/emitted from/to the Unruh thermal bath perceived in accelerated frame.

The paper is structured as follows. In Sec. II, we describe the details corresponding to the radiation emitted by an accelerated charge. In Sec. III, we define vector Rindler and Unruh modes. Section IV is dedicated to finding the amplitudes associated with the Unruh mode decomposition of the classical field, while Sec. V is reserved for the quantum calculations with a classical source. Some final remarks and discussion are presented in Sec. VI. We work with a four-dimensional spacetime with metric signature $(+, -, -, -)$, along with Heaviside-Lorentz units for the electromagnetic quantities, and set $\hbar = c = 1$ throughout the paper.

II. THE FOUR CURRENT AND FOUR POTENTIAL FROM A UNIFORMLY ACCELERATED CHARGE

The Lagrangian density for the electromagnetic field that allows for the field quantization in a globally hyperbolic spacetime (\mathcal{M}, g_{ab}) is given by

$$\mathcal{L} = -\sqrt{-g} \left(\frac{1}{4} F^{ab} F_{ab} + \frac{1}{2\alpha} (\nabla_a A^a)^2 + j_a A^a \right), \quad (2.1)$$

where $F_{ab} \equiv 2\nabla_{[a} A_{b]}$, A_a is the four potential, j_a is the four-current source of the electromagnetic field, ∇_a corresponds to the covariant derivative compatible with the Lorentzian metric g_{ab} , and g indicates the determinant of g_{ab} in some arbitrary coordinate system [16]. The field equation for a Ricci-flat ($R_{ab} = 0$) spacetime in the Feynman gauge $\alpha = 1$ is given by

$$\nabla_b \nabla^b A_a = j_a, \quad (2.2)$$

which can also be achieved by imposing

$$\nabla_a A^a = 0, \quad (2.3)$$

known as the Lorenz condition, *a priori*.

Let us consider an initially inertial charge q in Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$, which is accelerated with constant proper acceleration a for a finite proper time $2T$ and then becomes inertial again. Here, η_{ab} indicates the Minkowski metric whose line element in usual inertial Minkowski coordinates (t, x, y, z) takes the form

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \quad (2.4)$$

In such coordinates, the worldline of the charge is given by (see Fig. 1)

$$z^a(t) = \begin{cases} (t, 0, 0, a^{-1} \cosh(aT) - \tanh(aT)[t + a^{-1} \sinh(aT)]) & \text{if } t \leq -a^{-1} \sinh(aT), \\ (t, 0, 0, \sqrt{a^{-2} + t^2}) & \text{if } |t| < a^{-1} \sinh(aT), \\ (t, 0, 0, a^{-1} \cosh(aT) + \tanh(aT)[t - a^{-1} \sinh(aT)]) & \text{if } t \geq a^{-1} \sinh(aT), \end{cases} \quad (2.5)$$

and the four current associated with this worldline is given by

$$j^a(x) = \begin{cases} j_I^a(x) & \text{if } |t| \geq a^{-1} \sinh(aT), \\ j_A^a(x) & \text{if } |t| < a^{-1} \sinh(aT). \end{cases} \quad (2.6)$$

Here, j_I^a is the current associated with the inertial part of the motion and has components

$$j_I^a(x) = \begin{cases} q \cosh(aT) \delta^2(\mathbf{x}_\perp) \Delta_+(z, t) & \text{if } t \leq -a^{-1} \sinh(aT), \\ q \cosh(aT) \delta^2(\mathbf{x}_\perp) \Delta_-(z, t) & \text{if } t \geq a^{-1} \sinh(aT), \end{cases} \quad (2.7)$$

$$j_I^z(x) = \begin{cases} -q \sinh(aT) \delta^2(\mathbf{x}_\perp) \Delta_+(z, t) & \text{if } t \leq -a^{-1} \sinh(aT), \\ q \sinh(aT) \delta^2(\mathbf{x}_\perp) \Delta_-(z, t) & \text{if } t \geq a^{-1} \sinh(aT), \end{cases} \quad (2.8)$$

along with $j_I^x(x) = j_I^y(x) = 0$. Here we have defined the auxiliary functions

$$\Delta_\pm(z, t) \equiv \delta(z - a^{-1} \text{sech}(aT) \pm t \tanh(aT)). \quad (2.9)$$

The current j_A^a describes the uniformly accelerated part of the motion. In the right Rindler wedge (RRW), i.e., the

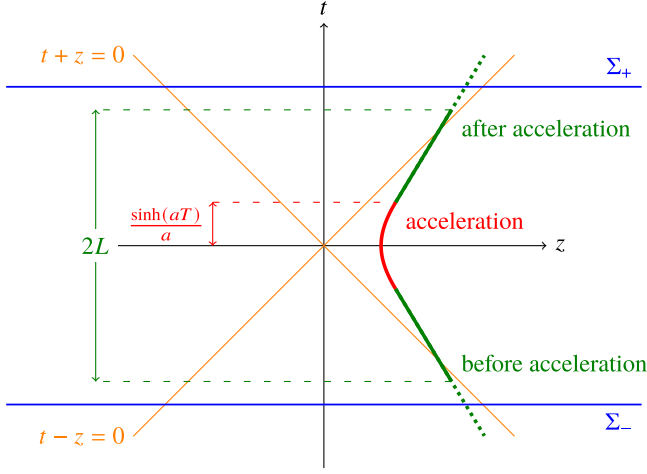


FIG. 1. Spacetime diagram describing the worldline of an initially inertial charge q which is uniformly accelerated for a finite proper time $2T$ and then becomes inertial again. The slices Σ_- and Σ_+ are Cauchy surfaces to the past and future, respectively, of the support of the compactified four current j_L^a .

spacetime region defined by $z > |t|$, $j_L^a(x)$ can be cast in a simple form by using Rindler coordinates (λ, ξ, x, y) , where x and y are left unaltered, while the other two inertial coordinates are written as

$$t = a^{-1} e^{a\xi} \sinh(a\lambda), \quad z = a^{-1} e^{a\xi} \cosh(a\lambda), \quad (2.10)$$

yielding

$$j_A^a = q\delta(\xi)\delta^2(\mathbf{x}_\perp)\theta(T - |\lambda|)(\partial_\lambda)^a. \quad (2.11)$$

Here, δ is the Dirac delta distribution, θ is the Heaviside step function, and $\mathbf{x}_\perp \equiv (x, y) \in \mathbb{R}^2$. We have included the step function to reinforce the fact that this part of the current is limited to the region defined by $|t| < a^{-1} \sinh(aT)$. Notice the trajectory (2.5) is not constrained to the RRW (like an infinitely accelerated particle would): before the acceleration, the charge moves in the contracting degenerate Kasner universe (CDKU, $t < -|z|$), it then crosses the Killing horizon $t + z = 0$ to the RRW, and after the acceleration it crosses the surface $t - z = 0$ into the expanding degenerate Kasner universe (EDKU, $t > |z|$).

As we are interested in describing radiation, let us recall from the classical theory that the energy radiated by a point charge over a unit solid angle per emission time is given by [17]

$$\begin{aligned} \frac{d^2 H}{d\Omega dt_{\text{ret}}} &= \lim_{R \rightarrow \infty} (1 - \mathbf{v}(t_{\text{ret}}) \cdot \mathbf{n}(t_{\text{ret}})) R^2 \mathbf{S}(t) \cdot \mathbf{n}(t_{\text{ret}}) \\ &= \frac{q^2}{16\pi^2} \left[\frac{|\mathbf{n} \times [(\mathbf{n} - \mathbf{v}) \times \mathbf{a}]|^2}{(1 - \mathbf{v} \cdot \mathbf{n})^5} \right]_{\text{ret}}, \end{aligned} \quad (2.12)$$

where \mathbf{n} is a unitary vector that points from the point of emission to the point of evaluation, \mathbf{v} is the velocity of the charge, and \mathbf{a} is its acceleration. We can see from Eq. (2.12) that the only contribution to the radiated energy comes from the parts of the trajectory where acceleration is nonzero. Indeed, by using the worldline (2.5) of our charge together with $\mathbf{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, where the angles θ and φ are defined from the point of emission, Eq. (2.12) reduces to

$$\frac{d^2 H}{d\Omega dt_{\text{ret}}} = \frac{(16\pi^2)^{-1} a^2 q^2 \sin^2 \theta}{\sqrt{1 + a^2 t^2} (\sqrt{1 + a^2 t^2} - at \cos \theta)^5} \Big|_{\text{ret}} \quad (2.13)$$

if $|t_{\text{ret}}| < a^{-1} \sinh(aT)$, and to zero otherwise.

The Liénard-Wiechert potential produced by the current (2.6) is given by [17,18]

$$Rj^a(x) = \frac{q}{2\pi} \int_{-\infty}^{\infty} d\tau \frac{dz^a}{d\tau} \theta(x^0 - z^0(\tau)) \delta([x - z(\tau)]^2), \quad (2.14)$$

where τ is the charge's proper time. Born [19] used this expression for the case where the charge is uniformly accelerated for an infinite amount of time (result that was further discussed and refined in works like [6,20]), yielding

$$Rj^t = \frac{q}{4\pi(t^2 - z^2)} \left[\frac{az(a^{-2} - t^2 + r^2)}{2\rho_0(x)} - t \right] \theta(t + z), \quad (2.15)$$

$$Rj^z = \frac{q}{4\pi(t^2 - z^2)} \left[\frac{at(a^{-2} - t^2 + r^2)}{2\rho_0(x)} - z \right] \theta(t + z), \quad (2.16)$$

$$Rj^x = Rj^y = 0, \quad (2.17)$$

where $r^2 = x^2 + y^2 + z^2$ and

$$\rho_0(x) \equiv \frac{a}{2} \sqrt{\frac{4}{a^2} (t^2 - z^2) + \left[\frac{1}{a^2} - t^2 + r^2 \right]^2}. \quad (2.18)$$

This solution codifies all the information about the charge q following the worldline (2.5) in the limit $T \rightarrow \infty$. Later in the paper, we will show how this solution is built entirely from zero-energy Rindler modes.

For the sake of some calculations, it will be useful to consider the following compactified version of the current (2.6):

$$j_L^a \equiv \theta(L - |t|) j^a, \quad (2.19)$$

which will enable us to use some useful mathematical identities. Here, we have introduced a ‘‘compactification parameter’’ $L > a^{-1} \sinh(aT)$. We note that j_L^a is not locally conserved as

$$\nabla_a j_L^a = j^t \delta(L - |t|) \neq 0, \quad (2.20)$$

where we used the current j^a defined in Eq. (2.6), which does satisfy $\nabla_a j^a = 0$. However, as L is a free parameter, we can compute the quantities of interest using this current of compact support and then recover the physically relevant results, where charge is locally conserved, by taking the limit $L \rightarrow \infty$. As our aim is to study the radiation emitted by a charge accelerated during a finite amount of its proper time, and the inertial portions of the motion of this charge will not contribute to the radiation emitted (nor detected), the analysis of the radiation using j_L^a (eventually taking $L \rightarrow \infty$) will be enough for our purposes. It is important to note that we will also present the corresponding correction terms coming from the inertial parts of the motion and the compactification, to guarantee that it does not contribute when the limit $L \rightarrow \infty$ is taken.

III. RINDLER AND UNRUH MODES

The main idea behind this study is to decompose both the classical and quantum electromagnetic four potential in terms of vector Unruh modes, the definition of which is based on Rindler modes that appear on the description of a scalar field on both Rindler wedges. In the following, we recall the discussion for the scalar case as it is fundamental to our procedure.

Let us consider a free scalar field ϕ in $(\mathbb{R}^4, \eta_{ab})$, satisfying the Klein-Gordon (KG) equation

$$\nabla_a \nabla^a \phi = 0. \quad (3.1)$$

As the RRW is a globally hyperbolic static spacetime on its own right, with timelike Killing field $(\partial_\lambda)^a$, one can define the set of positive-frequency right Rindler modes $\{v_{\omega \mathbf{k}_\perp}^R\}$, which are solutions of Eq. (3.1) vanishing in the left Rindler Wedge (LRW), i.e., the spacetime region defined by $t < -|z|$, and taking the form

$$v_{\omega \mathbf{k}_\perp}^R = \sqrt{\frac{\sinh(\pi\omega/a)}{4\pi^4 a}} K_{i\omega/a} \left(\frac{k_\perp e^{a\xi}}{a} \right) e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} e^{-i\omega\lambda}, \quad (3.2)$$

in the RRW using Rindler coordinates. Here, $K_\nu(z)$ is the modified Bessel function of the second kind, $\mathbf{k}_\perp \equiv (k_x, k_y) \in \mathbb{R}^2 \setminus \{0\}$, and $\omega \in \mathbb{R}^+$. We can also define left Rindler modes $v_{\omega \mathbf{k}_\perp}^L$ by means of the relation

$$v_{\omega \mathbf{k}_\perp}^L(t, \mathbf{x}_\perp, z) \equiv v_{\omega \mathbf{k}_\perp}^{R*}(-t, \mathbf{x}_\perp, -z). \quad (3.3)$$

Hence, they vanish in the RRW and take the form (3.2) in Rindler coordinates covering the LRW. The set of modes $\{v_{\omega \mathbf{k}_\perp}^R, v_{\omega \mathbf{k}_\perp}^L\}$, together with their Hermitian conjugates, forms a complete set of solutions of the Klein-Gordon equation in Minkowski spacetime [10].

By using the Rindler modes (3.2) and (3.3), we can define another suitable set of solutions of the KG equation, the so-called Unruh modes $\{w_{\omega \mathbf{k}_\perp}^1, w_{\omega \mathbf{k}_\perp}^2\}$. They are defined as

$$w_{\omega \mathbf{k}_\perp}^1 \equiv \frac{v_{\omega \mathbf{k}_\perp}^R + e^{-\pi\omega/a} v_{\omega - \mathbf{k}_\perp}^{L*}}{\sqrt{1 - e^{-2\pi\omega/a}}}, \quad (3.4)$$

$$w_{\omega \mathbf{k}_\perp}^2 \equiv \frac{v_{\omega \mathbf{k}_\perp}^L + e^{-\pi\omega/a} v_{\omega - \mathbf{k}_\perp}^{R*}}{\sqrt{1 - e^{-2\pi\omega/a}}}, \quad (3.5)$$

and, although they are labeled by the Rindler energy ω and transverse momentum \mathbf{k}_\perp , they are positive frequency with respect to the *inertial* time t and form (together with their Hermitian conjugate) a complete set of orthonormal solutions of the KG equation. This makes them suitable to investigate the relation between radiation seen by inertial observers and the physics of uniformly accelerated observers.

Having defined the scalar Unruh modes, we can now turn our attention to the electromagnetic case. The solutions of the homogeneous electromagnetic field equation in $(\mathbb{R}^4, \eta_{ab})$,

$$\nabla_b \nabla^b A_a = 0, \quad (3.6)$$

can be decomposed in terms of four independent polarization modes on the RRW given by [11]

$$V_{\omega \mathbf{k}_\perp a}^{R(1)} = \frac{1}{k_\perp} (0, 0, k_y v_{\omega \mathbf{k}_\perp}^R, -k_x v_{\omega \mathbf{k}_\perp}^R), \quad (3.7)$$

$$V_{\omega \mathbf{k}_\perp a}^{R(2)} = \frac{1}{k_\perp} (\partial_\xi v_{\omega \mathbf{k}_\perp}^R, \partial_\lambda v_{\omega \mathbf{k}_\perp}^R, 0, 0), \quad (3.8)$$

$$V_{\omega \mathbf{k}_\perp a}^{R(G)} = \frac{1}{k_\perp} \nabla_a v_{\omega \mathbf{k}_\perp}^R, \quad (3.9)$$

$$V_{\omega \mathbf{k}_\perp a}^{R(L)} = \frac{1}{k_\perp} (0, 0, k_x v_{\omega \mathbf{k}_\perp}^R, k_y v_{\omega \mathbf{k}_\perp}^R), \quad (3.10)$$

where $v_{\omega \mathbf{k}_\perp}^R$ are the scalar right Rindler modes given in Eq. (3.2). These have been selected in such a way that they are orthonormalized with respect to the generalized Klein-Gordon inner product

$$\langle A^{(1)}, A^{(2)} \rangle_{\text{gKG}} = \int_\Sigma d\Sigma_a \Xi^a [A^{(1)}, A^{(2)}] \quad (3.11)$$

between two modes, $A^{(1)}$ and $A^{(2)}$, of the electromagnetic field. Here, the integration is done over any Cauchy surface Σ , with proper-vector-valued-volume element $d\Sigma_a$ and conserved current $\Xi^a [A^{(1)}, A^{(2)}]$ given by

$$\Xi^a [A^{(1)}, A^{(2)}] \equiv \frac{i}{\sqrt{-g}} (A_b^{(1)*} \pi^{(2)ab} - A_b^{(2)} \pi^{(1)ab*}), \quad (3.12)$$

where we have used the generalized momenta of the electromagnetic potential defined as $\pi^{ab} \equiv \partial \mathcal{L} / \partial (\partial_a A_b)$.

On the Feynman gauge, these are computed explicitly yielding

$$\pi^{ab} = \sqrt{-g}(\nabla^b A^a - \nabla^a A^b - g^{ab}\nabla_c A^c). \quad (3.13)$$

The left electromagnetic Rindler modes are defined, as in the scalar case, as

$$V_{\omega\mathbf{k}_\perp a}^{L(\kappa)}(t, \mathbf{x}_\perp, z) \equiv V_{\omega\mathbf{k}_\perp a}^{R(\kappa)*}(-t, \mathbf{x}_\perp, -z), \quad (3.14)$$

with $\kappa = 1, 2, G, L$. Here, the modes (both left and right) labeled with $\kappa = 1, 2$ are physical modes, while the ones labeled with $\kappa = G, L$ are nonphysical, as they are pure gauge and do not satisfy the Lorenz condition, respectively.

By making use of $V_{\omega\mathbf{k}_\perp a}^{R(\kappa)}$ and $V_{\omega\mathbf{k}_\perp a}^{L(\kappa)}$, we can extend the definition of the Unruh modes to the electromagnetic case, in analogous fashion to the definitions of Eqs. (3.4) and (3.5),

$$W_{\omega\mathbf{k}_\perp b}^{1(\kappa)} \equiv \frac{V_{\omega\mathbf{k}_\perp b}^{R(\kappa)} + e^{-\pi\omega/a} V_{\omega-\mathbf{k}_\perp b}^{L(\kappa)*}}{\sqrt{1 - e^{-2\pi\omega/a}}}, \quad (3.15)$$

$$W_{\omega\mathbf{k}_\perp b}^{2(\kappa)} \equiv \frac{V_{\omega\mathbf{k}_\perp b}^{L(\kappa)} + e^{-\pi\omega/a} V_{\omega-\mathbf{k}_\perp b}^{R(\kappa)*}}{\sqrt{1 - e^{-2\pi\omega/a}}}. \quad (3.16)$$

These, by using Eqs. (3.4) and (3.5); (3.7)–(3.10), can be expressed directly in terms of the scalar Unruh modes as

$$W_{\omega\mathbf{k}_\perp a}^{\sigma(1)} = \frac{1}{k_\perp} (0, k_y w_{\omega\mathbf{k}_\perp}^\sigma, -k_x w_{\omega\mathbf{k}_\perp}^\sigma, 0), \quad (3.17)$$

$$W_{\omega\mathbf{k}_\perp a}^{\sigma(2)} = \frac{1}{k_\perp} (\partial_z w_{\omega\mathbf{k}_\perp}^\sigma, 0, 0, \partial_t w_{\omega\mathbf{k}_\perp}^\sigma), \quad (3.18)$$

$$W_{\omega\mathbf{k}_\perp a}^{\sigma(G)} = \frac{1}{k_\perp} \nabla_a w_{\omega\mathbf{k}_\perp}^\sigma, \quad (3.19)$$

$$W_{\omega\mathbf{k}_\perp a}^{\sigma(L)} = \frac{1}{k_\perp} (0, k_x w_{\omega\mathbf{k}_\perp}^\sigma, k_y w_{\omega\mathbf{k}_\perp}^\sigma, 0). \quad (3.20)$$

Hence, they satisfy

$$\langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, W_{\omega'\mathbf{k}'_\perp}^{\sigma'(\kappa')} \rangle_{\text{gKG}} = \delta_{\sigma\sigma'} \delta_{\kappa\kappa'} \delta(\omega - \omega') \delta^2(\mathbf{k}_\perp - \mathbf{k}'_\perp) \quad (3.21)$$

and form (together with their Hermitian conjugate) a complete set of solutions for the homogeneous electromagnetic field equation (3.6), which are positive frequency with respect to the inertial time t .

IV. UNRUH MODE DECOMPOSITION OF THE CLASSICAL RETARDED POTENTIAL

Let us consider two Cauchy surfaces Σ_+ and Σ_- in Minkowski spacetime with $\Sigma_+ \subset I^+(\Sigma_-)$ and

$\Sigma_\pm \subset \mathbb{R}^4 - \text{supp}(j_L^a)$ (see Fig. 2). Here, $I^+(A)$ denotes the chronological future of a subset $A \subset \mathbb{R}^4$.

The advanced and retarded solution of the field equation (2.2) with source j_L^a are given by

$$A j_L^a(x) = \int_{\mathbb{R}^4} d^4 x' \sqrt{-g'} G_{\text{adv}}(x, x') j_L^a(x'), \quad (4.1)$$

$$R j_L^a(x) = \int_{\mathbb{R}^4} d^4 x' \sqrt{-g'} G_{\text{ret}}(x, x') j_L^a(x'), \quad (4.2)$$

respectively, where G_{adv} and G_{ret} are the advanced and retarded Green's functions for the electromagnetic field [18]. Given our choice for Cauchy surfaces, we find that

$$R j_L^a(x) = -E j_L^a(x) \quad \forall x \in \Sigma_+, \quad (4.3)$$

where we have defined $E j_L^a \equiv A j_L^a - R j_L^a$ and we have used that $A j_L^a$ vanishes in $\mathbb{R}^4 - J^-(\text{supp} j_L^a)$.

In order to analyze the role played by the Unruh thermal bath (with particular interest in the zero-energy Rindler modes) in building the radiation emitted by the charge as seen by inertial observers, it will be important to decompose $R j_L^a$ in terms of Unruh modes (3.17)–(3.20). To this end, we will use our compactified current j_L^a to compute $R j_L^a$ and take the limit $L \rightarrow \infty$ to recover the physical current case. Such a task will be greatly simplified by using the identity (proven in Appendix A)

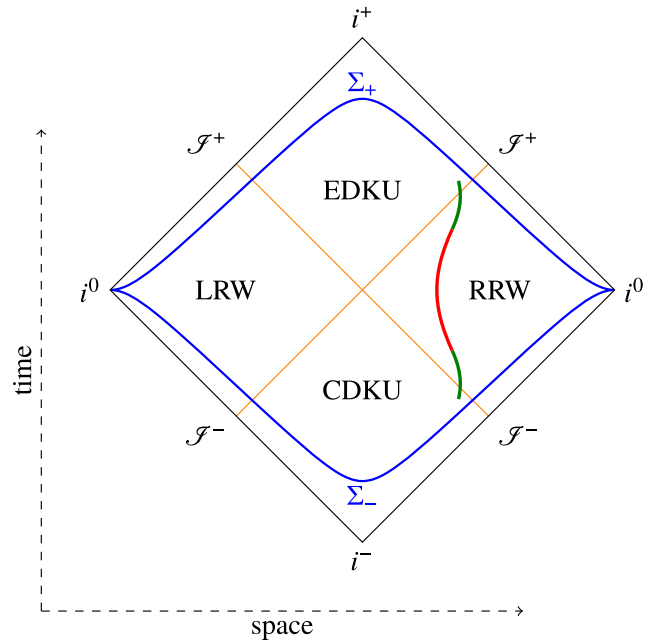


FIG. 2. Conformal diagram of our setup. The red portion of the worldline displays the support of the accelerated part of the compactified current j_L^a , while the supports of the inertial parts are shown in green. The orange lines depict the Killing horizons and the blue ones the Cauchy surfaces.

$$\langle \tilde{A}, E\tilde{j} \rangle_{\text{gKG}} = -i \int_{\mathcal{M}} d^4x \sqrt{-g} \tilde{A}_a^* \tilde{j}^a, \quad (4.4)$$

which is valid for any Ricci-flat ($R_{ab} = 0$) globally hyperbolic spacetime (\mathcal{M}, g_{ab}), with \tilde{A}^a being any solution of the *homogeneous* electromagnetic field equation (3.6) and \tilde{j}^a any compact-support current.

Let us now expand Eq. (4.3) in terms of Unruh modes on Σ_+ as

$$Rj_{La} = -\sum_{\sigma, \kappa} \int_0^\infty d\omega \int_{\mathbb{R}^2} d^2\mathbf{k}_\perp \times (\langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, E j_L \rangle_{\text{gKG}} W_{\omega\mathbf{k}_\perp a}^{\sigma(\kappa)} + \text{c.c.}), \quad (4.5)$$

where c.c. represents the complex conjugate of the previous expression. The coefficients $\langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, E j_L \rangle_{\text{gKG}}$ can be computed using Eq. (4.4). We immediately find that $\langle W_{\omega\mathbf{k}_\perp}^{\sigma(1)}, E j_L \rangle_{\text{gKG}} = \langle W_{\omega\mathbf{k}_\perp}^{\sigma(L)}, E j_L \rangle_{\text{gKG}} = 0$, as the current does not couple with the corresponding modes. As the support of j_A^a is contained in the RRW (even on the limit $T \rightarrow \infty$), by using Eqs. (2.6) and (2.19) we can cast the other coefficients as

$$\langle W_{\omega\mathbf{k}_\perp}^{1(2)}, E j_L \rangle_{\text{gKG}} = -\frac{i q e^{\pi\omega/2a} \sin(\omega T)}{\sqrt{2\pi^4 a \omega}} K'_{i\omega/a}(k_\perp/a) + \mathcal{I}_{\omega\mathbf{k}_\perp}^{1(2)}(a, T, L), \quad (4.6)$$

$$\langle W_{\omega\mathbf{k}_\perp}^{1(G)}, E j_L \rangle_{\text{gKG}} = \frac{q \sin(\omega T) e^{\pi\omega/2a}}{\sqrt{2\pi^4 a k_\perp}} K_{i\omega/a}(k_\perp/a) + \mathcal{I}_{\omega\mathbf{k}_\perp}^{1(G)}(a, T, L), \quad (4.7)$$

$$\langle W_{\omega\mathbf{k}_\perp}^{2(2)}, E j_L \rangle_{\text{gKG}} = -\frac{i q \sin(\omega T)}{\sqrt{2\pi^4 a e^{\pi\omega/a} \omega}} K'_{i\omega/a}(k_\perp/a) + \mathcal{I}_{\omega\mathbf{k}_\perp}^{2(2)}(a, T, L), \quad (4.8)$$

and

$$\langle W_{\omega\mathbf{k}_\perp}^{2(G)}, E j_L \rangle_{\text{gKG}} = -\frac{q \sin(\omega T)}{\sqrt{2\pi^4 a e^{\pi\omega/a} k_\perp}} K_{i\omega/a}(k_\perp/a) + \mathcal{I}_{\omega\mathbf{k}_\perp}^{2(G)}(a, T, L), \quad (4.9)$$

where we have used the prime to denote differentiation with respect to the argument. Here, we have written the amplitudes in Eqs. (4.6)–(4.9) separating two different contributions: the first one coming from the uniformly accelerated part of the current j_L^a and the second one

$$\mathcal{I}_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)} \equiv -i q [\cosh(aT) \mathcal{A}_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)} + \sinh(aT) \mathcal{B}_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}], \quad (4.10)$$

with

$$\mathcal{A}_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)} \equiv \int_{a^{-1} \sinh(aT)}^L dt \int_{\mathbb{R}} dz [W_{\omega\mathbf{k}_\perp t}^{\sigma(\kappa)*}(t, 0, 0, z) \Delta_-(z, t) + W_{\omega\mathbf{k}_\perp t}^{\sigma(\kappa)*}(-t, 0, 0, z) \Delta_+(z, -t)], \quad (4.11)$$

$$\mathcal{B}_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)} \equiv \int_{a^{-1} \sinh(aT)}^L dt \int_{\mathbb{R}} dz [W_{\omega\mathbf{k}_\perp z}^{\sigma(\kappa)*}(t, 0, 0, z) \Delta_-(z, t) - W_{\omega\mathbf{k}_\perp z}^{\sigma(\kappa)*}(-t, 0, 0, z) \Delta_+(z, -t)], \quad (4.12)$$

coming from the inertial part of j_L^a . Now, after computing the above integrals, we take the limit $L \rightarrow \infty$ to obtain the complete result for our physical current, i.e., the current given in Eq. (2.6) which satisfies $\nabla_a j^a = 0$. In Appendix B we present the details of such calculation.

Having determined the amplitudes for our physical current, let us now find their form in the limit where the charge accelerates for an infinite proper time, i.e., $T \rightarrow \infty$. By using

$$\lim_{T \rightarrow \infty} \frac{\sin(\omega T)}{\omega} = \pi \delta(\omega), \quad (4.13)$$

the identity $K'_\nu(z) = \nu K_\nu(z)/z - K_{\nu+1}(z)$ for the modified Bessel functions of the second kind, and the fact (proven in Appendix B) that $\mathcal{I}_{\omega\mathbf{k}_\perp}^{\sigma(2)}$ and $\mathcal{I}_{\omega\mathbf{k}_\perp}^{\sigma(G)}$ vanish for $T \rightarrow \infty$, we find that the amplitudes of Eqs. (4.6)–(4.9) can be cast as

$$\langle W_{\omega\mathbf{k}_\perp}^{1(2)}, E j \rangle_{\text{gKG}} = \langle W_{\omega\mathbf{k}_\perp}^{2(2)}, E j \rangle_{\text{gKG}} = \frac{i q}{\sqrt{2a\pi}} K_1\left(\frac{k_\perp}{a}\right) \delta(\omega), \quad (4.14)$$

$$\langle W_{\omega\mathbf{k}_\perp}^{1(G)}, E j \rangle_{\text{gKG}} = \langle W_{\omega\mathbf{k}_\perp}^{2(G)}, E j \rangle_{\text{gKG}} = 0. \quad (4.15)$$

We can now define the total number of classical photons emitted by the charge (as seen by inertial observers) as

$$N_{\text{cU}} \equiv \langle KRj, KRj \rangle_{\text{gKG}} \equiv \|KRj\|_{\text{gKG}}^2, \quad (4.16)$$

where KRj_a is the (inertial) positive-energy part of the retarded solution given in Eq. (2.2),

$$KRj_a = -\sum_{\sigma, \kappa} \int_0^\infty d\omega \int_{\mathbb{R}^2} d^2\mathbf{k}_\perp \langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, E j \rangle_{\text{gKG}} W_{\omega\mathbf{k}_\perp a}^{\sigma(\kappa)}. \quad (4.17)$$

By using the orthonormality of the Unruh modes, we can write N_{cU} explicitly in terms of the coefficients of the expansion (3.21) as

$$N_{\text{cU}} = \sum_{\sigma, \kappa} \int_0^\infty d\omega \int_{\mathbb{R}^2} d^2\mathbf{k}_\perp |\langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, Ej \rangle_{\text{gKG}}|^2, \quad (4.18)$$

which enables us to interpret $|\langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, Ej \rangle_{\text{gKG}}|^2$ as the number of photons associated with the Unruh mode σ , per polarization mode κ , per transverse momentum \mathbf{k}_\perp and Rindler energy ω . To avoid divergences, we will deal with the number of Unruh photons per transverse momentum

$$\frac{dN_{\text{cU}}}{d^2\mathbf{k}_\perp} \equiv \sum_{\sigma, \kappa} \int_0^\infty d\omega |\langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, Ej \rangle_{\text{gKG}}|^2. \quad (4.19)$$

In the limit $T \rightarrow \infty$, we can use Eqs. (4.14) and (4.15) in Eq. (4.19), together with the property $W_{\omega\mathbf{k}_\perp a}^{2(2)} = W_{-\omega\mathbf{k}_\perp a}^{1(2)}$ to write

$$\frac{dN_{\text{cU}}}{d^2\mathbf{k}_\perp} = \frac{q^2}{4\pi^3 a} |K_1(k_\perp/a)|^2 T_{\text{tot}}, \quad (4.20)$$

where we have used that $T_{\text{tot}} = 2\pi\delta(\omega)|_{\omega=0}$. This gives exactly the total rate of emission and absorption of zero-energy Rindler photons in the Unruh thermal bath seen by an accelerated observer and the rate of emission detected by an inertial one, as computed in Ref. [11] using tree-level QFT. It is interesting to note how this (classical) radiation is built from zero-Rindler-energy Unruh modes, as can be seen explicitly from the form of the amplitudes in Eq. (4.14).

Now, let us explicitly compute the expansion (4.5) and show it reduces to the usual and well-known solution for the electromagnetic four potential (15) in the asymptotic future. For this purpose, let us focus on region EDKU in Fig. 2, the so-called expanding degenerate Kasner universe (the region where $t > |z|$). In the limit $T \rightarrow \infty$, we can see from Eqs. (4.14) and (4.15) that the only modes that couple with our current are $W_{\omega\mathbf{k}_\perp a}^{\sigma(2)}$. As a result, by using again that $W_{\omega\mathbf{k}_\perp a}^{2(2)} = W_{-\omega\mathbf{k}_\perp a}^{1(2)}$ to perform the integration in ω , Eq. (4.5) can be expressed as

$$Rj_a = -\frac{q}{\sqrt{2a\pi^2}} \int_{\mathbb{R}^2} d^2\mathbf{k}_\perp [iK_1(k_\perp/a)W_{0\mathbf{k}_\perp a}^{2(2)} + \text{c.c.}]. \quad (4.21)$$

On the EDKU let us define the coordinates $(\eta, \zeta, \mathbf{x}_\perp)$ by

$$t = a^{-1}e^{a\eta} \cosh(a\zeta), \quad z = a^{-1}e^{a\eta} \sinh(a\zeta), \quad (4.22)$$

while keeping the transverse position coordinates \mathbf{x}_\perp unaltered. In such coordinates, the vector Unruh mode (3.18) reads

$$W_{\omega\mathbf{k}_\perp a}^{2(2)} = \frac{1}{k_\perp} (\partial_\zeta w_{\omega\mathbf{k}_\perp}^2, \partial_\eta w_{\omega\mathbf{k}_\perp}^2, 0, 0),$$

with the scalar Unruh mode $w_{\omega\mathbf{k}_\perp}^2$ having the rather simple form in the EDKU

$$w_{\omega\mathbf{k}_\perp}^2 = -\frac{ie^{\pi\omega/(2a)}}{\sqrt{32\pi^2 a}} e^{i(\mathbf{k}_\perp \cdot \mathbf{x}_\perp + \omega\zeta)} H_{i\omega/a}^{(2)}(k_\perp e^{a\eta}/a). \quad (4.23)$$

From Eq. (4.23) we can see that $\partial_\zeta w_{\omega\mathbf{k}_\perp}^2 = 0$ and, therefore, the only nonzero component of the four potential will be

$$Rj_\zeta(x) = \frac{qe^{a\eta}}{8\pi^2 a} \int_{\mathbb{R}^2} d^2\mathbf{k}_\perp [K_1(k_\perp/a)H_1^{(2)}(k_\perp e^{a\eta}/a)e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} + \text{c.c.}], \quad (4.24)$$

where we have used the identity $H_0^{(2)'}(z) = -H_1^{(2)}(z)$ for the Hankel functions. To carry out the rest of the integration, let us define polar coordinates for the transverse momentum vector by $\mathbf{x}_\perp \equiv (x_\perp \cos \vartheta, x_\perp \sin \vartheta)$ and $\mathbf{k}_\perp \equiv (k_\perp \cos(\vartheta + \vartheta), k_\perp \sin(\vartheta + \vartheta))$, with $k_\perp > 0$ and $0 \leq \vartheta < 2\pi$. As a result, $\mathbf{k}_\perp \cdot \mathbf{x}_\perp = k_\perp x_\perp \cos \vartheta$ and $d^2\mathbf{k}_\perp = d\vartheta dk_\perp k_\perp$, so we can rewrite the nonzero component of the retarded solution in Eq. (4.24) as

$$Rj_\zeta(x) = \frac{qe^{a\eta}}{8\pi^2 a} \int_0^\infty k_\perp dk_\perp \left[K_1(k_\perp/a)H_1^{(2)}(k_\perp e^{a\eta}/a) \times \int_0^{2\pi} d\vartheta \exp(ik_\perp x_\perp \cos \vartheta) + \text{c.c.} \right].$$

Here, we recognize one of the integral forms of the Bessel function of order zero [21]

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} d\vartheta \exp(iz \cos \vartheta)$$

and use the definition of the second Hankel functions $H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z)$ to show that

$$Rj_\zeta = \frac{qe^{a\eta}}{2\pi a} \int_0^\infty k_\perp K_1(k_\perp/a) J_1(k_\perp e^{a\eta}/a) J_0(k_\perp x_\perp) dk_\perp. \quad (4.25)$$

We can now apply the identity [22,23]

$$\int_0^\infty K_1(\alpha\vartheta) J_1(i\beta\vartheta) J_0(\gamma\vartheta) \vartheta d\vartheta = \frac{i}{2\alpha\beta} \left[\frac{\alpha^2 + \beta^2 + \gamma^2}{\sqrt{(\alpha^2 + \beta^2 + \gamma^2)^2 - 4\alpha^2\beta^2}} - 1 \right], \quad (4.26)$$

which holds for the $\text{Re}\alpha > 0$, $\text{Re}\beta = 0$, and $\gamma > 0$. If we identify $\alpha = a^{-1}$, $\beta = -ie^{a\eta}/a$, and $\gamma = x_\perp$, it is immediate to find

$$Rj_\zeta(x) = -\frac{qa}{4\pi} \left[\frac{a(a^{-2} - a^{-2}e^{2a\eta} + x_\perp^2)}{2\rho_0(x)} - 1 \right]. \quad (4.27)$$

This four potential is gauge equivalent to the Born solution in Eq. (15). To prove this, we can define a scalar function

$\Lambda(x) \equiv -qa(\zeta + \eta)/4\pi$ and apply the gauge transformation $Rj_a \rightarrow Rj_a + \nabla_a \Lambda$ to obtain exactly the solution originally found by Born, but written in Rindler coordinates. Again, it is interesting to note how the usual retarded solution Eqs. (2.15)–(2.17) is built entirely from zero-Rindler-energy Unruh modes, as can be seen from Eq. (4.21).

V. UNRUH MODE DECOMPOSITION OF THE QUANTUM POTENTIAL

Let us now analyze the quantum aspects of the radiation emitted by the charge and its relation with the Unruh thermal bath. To this end, we will focus on the quantum four-potential operator \hat{A}^a , defined as a solution to the field equation

$$\nabla_b \nabla^b \hat{A}^a = j_L^a \hat{\mathbb{I}}. \quad (5.1)$$

We can write a this operator in different ways, depending on the boundary/initial conditions chosen. One suitable choice is

$$\hat{A}_a = \hat{A}_a^{\text{in}} + Rj_{La} \hat{\mathbb{I}}, \quad (5.2)$$

where \hat{A}_a^{in} is the solution to the homogeneous field equation

$$\nabla_b \nabla^b \hat{A}_a^{\text{in}} = 0, \quad (5.3)$$

and we recall that Rj_L^a is the retarded solution associated with the current j_L^a . As a result, we can expand \hat{A}_a^{in} as

$$\hat{A}_a^{\text{in}}(t, \mathbf{x}) = \sum_j (u_{(j)a}(t, \mathbf{x}) \hat{a}_{\text{in}}(u_{(j)}^*) + \text{H.c.}), \quad (5.4)$$

where $\{u_{(j)}\}_{j \in \mathfrak{F}}$, with \mathfrak{F} being a suitable set of quantum numbers, is a complete set of (Minkowski) positive-frequency modes. We can define $|0_{\text{in}}^M\rangle$ as the state such that $\hat{a}_{\text{in}}(u_{(j)}^*)|0_{\text{in}}^M\rangle = 0$, for all $j \in \mathfrak{F}$. As Rj_L^a vanishes in the asymptotic past (for instance, on the Cauchy surface Σ_- in Fig. 2), one can interpret $|0_{\text{in}}^M\rangle$ as the vacuum state as seen by inertial observers in the asymptotic past. The Fock space describing particle states as seen by such observers is generated by the states

$$|n_{j_1}, n_{j_2}, \dots\rangle_{\text{in}} = \bigotimes_{j=j_1}^{\infty} \frac{[\hat{a}_{\text{in}}^{\dagger}(u_{(j)})]^{n_j}}{\sqrt{n_j!}} |0_{\text{in}}^M\rangle, \quad (5.5)$$

where $n_j \in \mathbb{N}$ for each $j \in \mathfrak{F}$.

Alternatively, we can also write a solution of Eq. (5.1) as

$$A_a = \hat{A}_a^{\text{out}} + Aj_{La} \hat{\mathbb{I}}, \quad (5.6)$$

where \hat{A}_a^{out} is a solution of the homogeneous field equation,

$$\nabla_b \nabla^b \hat{A}_a^{\text{out}} = 0, \quad (5.7)$$

and we recall that Aj_L^a is the advanced solution associated with the current j_L^a . Hence, we can expand \hat{A}_a^{out} as

$$\hat{A}_a^{\text{out}}(t, \mathbf{x}) = \sum_j (v_{(j)a}(t, \mathbf{x}) \hat{a}^{\text{out}}(v_{(j)}^*) + \text{H.c.}), \quad (5.8)$$

where $\{v_{(j)}\}_{j \in \mathfrak{R}}$, with \mathfrak{R} being a suitable set of quantum numbers, is any set of (Minkowski) positive-energy modes. We can then define $|0_{\text{out}}^M\rangle$ as the state such that $\hat{a}^{\text{out}}(v_{(j)}^*)|0_{\text{out}}^M\rangle = 0$, for all $j \in \mathfrak{R}$. As Aj_L^a vanishes in the asymptotic future (for example, in the Cauchy surface Σ_+ shown in Fig. 2), we can interpret it as the vacuum state seen by an inertial observer in the asymptotic future. We can also construct the Fock space describing the particle states of the field by successive applications of the “creation operators” $\hat{a}^{\text{out}\dagger}(v_{(j)})$ on $|0_{\text{out}}^M\rangle$, hence

$$|n_{j_1}, n_{j_2}, \dots\rangle_{\text{out}} = \bigotimes_{j=j_1}^{\infty} \frac{[\hat{a}^{\text{out}\dagger}(v_{(j)})]^{n_j}}{\sqrt{n_j!}} |0_{\text{out}}^M\rangle. \quad (5.9)$$

We can connect the in and out Fock spaces via the S matrix [24]

$$\hat{S} \equiv \exp\left(-i \int_{\mathbb{R}^4} d^4x \sqrt{-g} \hat{A}_a^{\text{out}}(t, \mathbf{x}) j_L^a(t, \mathbf{x})\right), \quad (5.10)$$

which, in particular, relates the two vacua by

$$|0_{\text{in}}^M\rangle = \hat{S} |0_{\text{out}}^M\rangle. \quad (5.11)$$

In order to compute this operator explicitly, let us expand the out field in terms of Unruh modes (3.20),

$$\begin{aligned} \hat{A}_a^{\text{out}} = & \sum_{\sigma, \kappa} \int_0^{\infty} d\omega \int_{\mathbb{R}^2} d^2\mathbf{k}_{\perp} [W_{\omega\mathbf{k}_{\perp}a}^{\sigma(\kappa)} \hat{a}_{\text{out}}(W_{\omega\mathbf{k}_{\perp}}^{\sigma(\kappa)*}) \\ & + W_{\omega\mathbf{k}_{\perp}a}^{\sigma(\kappa)*} \hat{a}_{\text{out}}^{\dagger}(W_{\omega\mathbf{k}_{\perp}}^{\sigma(\kappa)})]. \end{aligned} \quad (5.12)$$

In order to cast Eq. (5.12) in a more convenient form, let us first define the smearing of the quantum four potential with the classical current as

$$\hat{A}_a^{\text{out}}(j_L) \equiv \int_{\mathbb{R}^4} d^4x \sqrt{-g} \hat{A}_a^{\text{out}}(x) j_L^a(x). \quad (5.13)$$

Next, by using Eqs. (5.12) and (4.4) we can write $\hat{A}_a^{\text{out}}(j_L)$ as

$$\begin{aligned}
 i\hat{A}_a^{\text{out}}(j_L) &= \sum_{\sigma,\kappa} \int_0^\infty d\omega \int_{\mathbb{R}^2} d^2\mathbf{k}_\perp \\
 &\times [\langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, E_{j_L} \rangle_{\text{gKG}}^* \hat{a}_{\text{out}}(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)*}) \\
 &- \langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, E_{j_L} \rangle_{\text{gKG}} \hat{a}_{\text{out}}^\dagger(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)})]. \quad (5.14)
 \end{aligned}$$

This motivates the definitions of the creation and annihilation operators associated with the (inertial) positive energy part of the expansion as

$$\begin{aligned}
 \hat{a}_{\text{out}}^\dagger(KEj_L) \\
 &\equiv \sum_{\sigma,\kappa} \int_0^\infty d\omega \int_{\mathbb{R}^2} d^2\mathbf{k}_\perp \langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, E_{j_L} \rangle_{\text{gKG}} \hat{a}_{\text{out}}^\dagger(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}) \quad (5.15)
 \end{aligned}$$

and

$$\begin{aligned}
 \hat{a}_{\text{out}}(KEj_L^*) \\
 &\equiv \sum_{\sigma,\kappa} \int_0^\infty d\omega \int_{\mathbb{R}^2} d^2\mathbf{k}_\perp \langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, E_{j_L} \rangle_{\text{gKG}}^* \hat{a}_{\text{out}}(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)*}), \quad (5.16)
 \end{aligned}$$

respectively. By means of Eqs. (5.15) and (5.16), the S matrix (5.10) can be cast as

$$\hat{S} = \exp(\hat{a}_{\text{out}}^\dagger(KEj_L) - \hat{a}_{\text{out}}(KEj_L^*)). \quad (5.17)$$

Now, by using the canonical commutation relation

$$[\hat{a}_{\text{out}}(KEj_L^*), \hat{a}_{\text{out}}^\dagger(KEj_L)] = \|KEj_L\|_{\text{gKG}}^2 \hat{\mathbb{I}}, \quad (5.18)$$

together with Zassenhaus formula

$$e^{\hat{X}+\hat{Y}} = e^{\hat{X}} e^{\hat{Y}} e^{-[\hat{X},\hat{Y}]/2}, \quad (5.19)$$

where $[\hat{X}, [\hat{X}, \hat{Y}]] = [\hat{Y}, [\hat{X}, \hat{Y}]] = 0$, we can rewrite Eq. (5.17) as

$$\hat{S} = e^{-\|KEj_L\|_{\text{gKG}}^2/2} e^{\hat{a}_{\text{out}}^\dagger(KEj_L)} e^{-\hat{a}_{\text{out}}(KEj_L^*)}. \quad (5.20)$$

Note that the above S matrix is completely determined, as we have already found the expansion amplitudes $\langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, E_{j_L} \rangle_{\text{gKG}}$, as well as the norm $\|KEj_L\|_{\text{gKG}}$ (which gives the classical number of photons N_{cl}) in Sec. IV. After we recover our physical current (i.e., take $L \rightarrow \infty$) and take the limit where our charge accelerates forever $T \rightarrow \infty$, the nonvanishing amplitudes are given by Eq. (4.14) and the norm by

$$\|KEj\|_{\text{gKG}}^2 = \frac{q^2}{2a\pi^2} T_{\text{tot}} \int_0^\infty dk_\perp k_\perp |K_1(k_\perp/a)|^2, \quad (5.21)$$

where we recall that $T_{\text{tot}} \equiv 2\pi\delta(\omega)|_{\omega=0}$ and we are using j to denote the physical current describing the charge accelerating forever.

Now, by using Eqs. (4.14) and (4.15) in Eqs. (5.15) and (5.16), together with the fact that $W_{\omega\mathbf{k}_\perp}^{2(\kappa)} = W_{-\omega\mathbf{k}_\perp}^{1(\kappa)}$, we can cast $\hat{a}_{\text{out}}(KEj^*)$ and $\hat{a}_{\text{out}}^\dagger(KEj)$ as

$$\hat{a}_{\text{out}}(KEj^*) = \frac{iq}{\sqrt{2a\pi}} \int_{\mathbb{R}^2} d^2\mathbf{k}_\perp K_1(k_\perp/a) \hat{a}_{\text{out}}(W_{0\mathbf{k}_\perp}^{2(2)*}) \quad (5.22)$$

and

$$\hat{a}_{\text{out}}^\dagger(KEj) = \frac{-iq}{\sqrt{2a\pi}} \int_{\mathbb{R}^2} d^2\mathbf{k}_\perp K_1(k_\perp/a) \hat{a}_{\text{out}}^\dagger(W_{0\mathbf{k}_\perp}^{2(2)}), \quad (5.23)$$

respectively. If we now use Eqs. (5.20)–(5.23) in Eq. (5.11), we can write the in vacuum $|0_{\text{in}}^M\rangle$ as

$$\begin{aligned}
 |0_{\text{in}}^M\rangle &= \bigotimes_{\mathbf{k}_\perp \in \mathbb{R}^2} \exp\left(-\frac{q^2 |K_1(k_\perp/a)|^2 T_{\text{tot}}}{8a\pi^3}\right) \\
 &\times \exp\left(\frac{iq}{2\pi} \sqrt{\frac{2}{a}} K_1(k_\perp/a) \hat{a}_{\text{out}}^\dagger(W_{0\mathbf{k}_\perp}^{2(2)})\right) |0_{\text{out}}^M\rangle, \quad (5.24)
 \end{aligned}$$

which explicitly shows that the radiation emitted by the charge in the asymptotic future is built entirely by zero-Rindler-energy Unruh photons when the field is initially in the Minkowski vacuum. Moreover, Eq. (5.24) is a (multimode) coherent state with respect to out-Unruh modes. To see this, let us show that $|0_{\text{in}}^M\rangle$ is an eigenstate of $\hat{a}_{\text{out}}(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)*})$ eigenvalue $\langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, E_j \rangle$. To this end, let us first note that Eq. (5.20) is equivalent to

$$|0_{\text{in}}^M\rangle = e^{-\|KEj\|_{\text{gKG}}^2/2} e^{\hat{a}_{\text{out}}^\dagger(KEj)} |0_{\text{out}}^M\rangle, \quad (5.25)$$

where we recall that $\|KEj\|_{\text{gKG}}$ and $\hat{a}_{\text{out}}^\dagger(KEj)$ are given in Eqs. (5.21) and (5.23), respectively. Now, let us apply $\hat{a}_{\text{out}}(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)*})$ to Eq. (5.25) to find

$$\begin{aligned}
 \hat{a}_{\text{out}}(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)*}) |0_{\text{in}}^M\rangle &= e^{-\|KEj\|_{\text{gKG}}^2/2} e^{\hat{a}_{\text{out}}^\dagger(KEj)} \\
 &\times (e^{-\hat{a}_{\text{out}}^\dagger(KEj)} \hat{a}_{\text{out}}(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)*}) e^{\hat{a}_{\text{out}}^\dagger(KEj)}) |0_{\text{out}}^M\rangle. \quad (5.26)
 \end{aligned}$$

By using the identity

$$e^{-\hat{X}} \hat{Y} e^{\hat{X}} = \hat{Y} - [\hat{X}, \hat{Y}] + \frac{1}{2!} [\hat{X}, [\hat{X}, \hat{Y}]] + \dots \quad (5.27)$$

and the canonical commutation relation

$$[\hat{a}_{\text{out}}(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)*}), \hat{a}_{\text{out}}^\dagger(KEj)] = \langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, E_j \rangle_{\text{gKG}} \hat{\mathbb{I}} \quad (5.28)$$

one can see that Eq. (5.26) can be rewritten as

$$\hat{a}_{\text{out}}(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)*})|0_{\text{in}}^M\rangle = \langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, Ej \rangle_{\text{gKG}}|0_{\text{in}}^M\rangle, \quad (5.29)$$

which proves the in vacuum is an eigenstate of the out-annihilation operator associated with *any* Unruh mode, i.e., a multimode coherent state.

The previous result gives us a straightforward way to show that the expectation value of the out potential (5.12) in the in vacuum in the asymptotic future is given by

$$\langle 0_{\text{in}}^M | \hat{A}_a^{\text{out}}(x) | 0_{\text{in}}^M \rangle = -Rj(x), \quad (5.30)$$

where we have used that $Ej = -Rj$ in the asymptotic future. This implies that the out field has an expectation value given by

$$\langle 0_{\text{in}}^M | \hat{F}_{ab}^{\text{out}} | 0_{\text{in}}^M \rangle = -RF_{ab}, \quad (5.31)$$

where $RF_{ab} = \nabla_a Rj_b - \nabla_b Rj_a$ is the classical retarded Faraday tensor.

The coherent state structure of (5.29) is also useful to find the number of photons radiated by the charge as seen by inertial observers in the asymptotic future. For this purpose, let us note that Eq. (5.29) implies that

$$\langle 0_{\text{in}}^M | \hat{N}_{\text{out}}(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}) | 0_{\text{in}}^M \rangle = |\langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, Ej \rangle_{\text{gKG}}|^2, \quad (5.32)$$

where we used the number operator

$$\hat{N}_{\text{out}}(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}) \equiv \hat{a}_{\text{in}}^\dagger(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}) \hat{a}_{\text{out}}(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)*}). \quad (5.33)$$

The result of Eq. (5.32) coincides with our interpretation of the classical number of Unruh photons (4.18). We can now add for all polarizations κ , Unruh modes σ , and integrate them with respect to Rindler energy ω to obtain the quantum number of photons per transverse momentum \mathbf{k}_\perp ,

$$\begin{aligned} \frac{dN_{\text{qU}}^{\text{out}}}{d^2\mathbf{k}_\perp} &\equiv \sum_{\sigma,\kappa} \int_0^\infty d\omega \langle 0_{\text{in}}^M | \hat{N}_{\text{out}}(W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}) | 0_{\text{in}}^M \rangle \\ &= \sum_{\sigma,\kappa} \int_0^\infty d\omega |\langle W_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}, Ej \rangle_{\text{gKG}}|^2, \end{aligned} \quad (5.34)$$

which we can compute explicitly in the $T \rightarrow \infty$ case by using Eq. (4.20), yielding

$$\frac{dN_{\text{qU}}^{\text{out}}}{d^2\mathbf{k}_\perp} = \frac{q^2}{4a\pi^2} |K_1(k_\perp/a)|^2 T_{\text{tot}}. \quad (5.35)$$

This is consistent with our previous result in the classical case and thus also with the result obtained by Higuchi *et al.* [11] in the tree-level QFT calculations using the Unruh thermal bath.

To conclude, we can now study the *normal ordered* stress-energy tensor for the out field,

$$:T_{\text{out}}^{ab}: \equiv g_{cd} :F_{\text{out}}^{ac} F_{\text{out}}^{bd}: - \frac{1}{4} g^{ab} :F_{\text{out}}^{cd} F_{\text{out}}^{cd}:. \quad (5.36)$$

From Eq. (5.29) it is simple to prove that $:F_{\text{out}}^{ab} F_{\text{out}}^{cd}:$ will coincide with the product between two retarded classical Faraday tensors,

$$\langle 0_{\text{in}}^M | :F_{\text{out}}^{ab} F_{\text{out}}^{cd}: | 0_{\text{in}}^M \rangle = RF^{ab} RF^{cd}, \quad (5.37)$$

and, therefore, the expectation value of the stress-energy-momentum tensor (5.36) is given by

$$\langle 0_{\text{in}}^M | :T_{\text{out}}^{ab}: | 0_{\text{in}}^M \rangle = g_{cd} RF^{ac} RF^{bd} - \frac{1}{4} g^{ab} RF^{cd} RF_{cd}, \quad (5.38)$$

which is exactly the classical value of the stress-energy tensor for the out field. This implies, in particular, that the energy flux integrated over a large sphere S^2 in the asymptotic future gives

$$\int_{S^2} dS^a \langle 0_{\text{in}}^M | :T_{ab}^{\text{out}}: (\partial_t)^b | 0_{\text{in}}^M \rangle = \frac{2}{3} \left(\frac{q^2 a}{4\pi} \right), \quad (5.39)$$

which is the usual Larmor formula. Here, dS^a is the vector-valued-volume element on the sphere and $(\partial_t)^a$ is the Killing field associated with the global inertial congruence.

VI. FINAL DISCUSSION

In this work we have studied the classical and quantum radiation emitted by a classical charge. In the classical realm, we have shown, by extending the definition of the well-known Unruh modes used in scalar electrodynamics to the vector case, that only zero-Rindler-energy Unruh modes contribute to the decomposition for the retarded potential and the corresponding amplitudes for our classical current in the limit where it accelerates for an infinite amount of proper time. We have also shown that the number of (classical) photons recover the usual results for the photon emission computed using tree-level quantum field theory [11].

For the quantum analysis, we have computed the S matrix connecting the quantum field theory construction made by inertial observers in the asymptotic past and future. We have found that if the state is prepared in the (Minkowski) vacuum associated with the past inertial observers, the future ones will detect this as a multimode coherent state built entirely from zero-Rindler-energy Unruh modes in the limit of infinite proper acceleration. In addition, the expectation values of the four potential, Faraday field tensor, and stress-energy tensor are in agreement with the corresponding classical counterparts that arise from the retarded solution described in the classical analysis.

As a result, we were able to extend the analysis made previously by one of the authors of the (classical and quantum aspects of the) radiation emitted by a scalar source coupled to a real scalar field to the more realistic case of a classical charge coupled to the electromagnetic field. This enabled us to settle the two puzzling aspects of the interplay between the Unruh effect and Larmor radiation: (1) how the (quantum) Unruh effect can be codified in the classical Larmor radiation and (2) the key role played by the zero-energy Rindler modes in such context.

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APPENDIX A: DERIVATION OF EQ. (4.4)

Let us prove the validity of the identity written in Eq. (4.4). For this purpose, let (\mathcal{M}, g_{ab}) be a four-dimensional globally hyperbolic spacetime such that $R_{ab} = 0$. Let us consider solutions \tilde{A}^a of the homogeneous electromagnetic equation (3.6) and let j^a be a compactly supported four current in (\mathcal{M}, g_{ab}) . Hence, we can choose a Cauchy surface Σ_- in the causal past of the support of j^a , which will enable us to write the functional

$$I[\tilde{A}, j] \equiv \int_{\mathcal{M}} d^4x \sqrt{-g} \tilde{A}_a^* j^a \quad (\text{A1})$$

as

$$I[\tilde{A}, j] = \int_{J^+(\Sigma_-)} d^4x \sqrt{-g} \tilde{A}_a^* j^a, \quad (\text{A2})$$

where $J^+(\Sigma_-)$ denotes the causal future of Σ_- . As the advanced solution Aj^a satisfies Eq. (4.1), we can cast Eq. (A1) as

$$I[\tilde{A}, j] = \int_{J^+(\Sigma_-)} d^4x \sqrt{-g} \tilde{A}_a^* (\nabla_b \nabla^b A j^a). \quad (\text{A3})$$

Now, the integrand of Eq. (A3) can be cast as

$$\tilde{A}_a^* (\nabla_b \nabla^b A j^a) = \nabla_b (\tilde{A}_a^* \overset{\leftrightarrow}{\nabla}^b A j^a), \quad (\text{A4})$$

which, from the properties of the covariant derivative and the definition of the current $\Xi^b[\tilde{A}, Aj]$ given in Eq. (3.12), can be written as

$$\tilde{A}_a^* (\nabla_b \nabla^b A j^a) = i \nabla_a \Xi^a[\tilde{A}, Aj] + 2R_{ab} \tilde{A}^{a*} A j^b. \quad (\text{A5})$$

Here, we have used that one can write Eq. (3.12) as

$$\Xi^b[\tilde{A}, Aj] = 2i \nabla_a (\tilde{A}^{[a*} A j^{b]}) - i \tilde{A}_a^* \overset{\leftrightarrow}{\nabla}^b A j^a, \quad (\text{A6})$$

as well as $2\nabla_{[a} \nabla_{b]} A^c \equiv -R_{abd}{}^c A^d$ and $R_{ab} \equiv R_{acb}{}^c$ [16].

If we now use Eqs. (A4) and (A5) in Eq. (A3) together with the fact that we are dealing with Ricci-flat spacetimes ($R_{ab} = 0$), we can write $I[\tilde{A}, j]$ using Gauss's theorem as

$$I[\tilde{A}, j] = i \int_{\Sigma_-} d\Sigma_a \Xi^a[\tilde{A}, Aj]. \quad (\text{A7})$$

Now, by using Eqs. (3.11) and (A1) and noting that $Aj = Ej$ on Σ_- , since $Rj^a = 0$ in this region, Eq. (A7) can be cast as

$$\langle \tilde{A}, Ej \rangle_{\text{gKG}} = -i \int_{\mathcal{M}} d^4x \sqrt{-g} \tilde{A}_a^* j^a, \quad (\text{A8})$$

which is exactly the form of Eq. (4.4), as we wanted to prove.

APPENDIX B: COMPACTIFIED CURRENT CALCULATIONS

We need to find the amplitudes $\mathcal{A}_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}$ and $\mathcal{B}_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}$ in order to determine the form of the corrections $\mathcal{T}_{\omega\mathbf{k}_\perp}^{\sigma(\kappa)}$ appearing in the expansion coefficients defined in Eqs. (4.6)–(4.9). For simplicity we shall only compute these for $\kappa = 2$, as the amplitudes corresponding to the pure gauge mode labeled with $\kappa = G$ can be omitted. To this end, let us first note that we can use the fact that $\Delta_+(z, -t) = \Delta_-(z, t)$ to rewrite Eqs. (4.11) and (4.12) as

$$\begin{aligned} \mathcal{A}_{\omega\mathbf{k}_\perp}^{\sigma(2)} &= \int_{a^{-1} \sinh(aT)}^L dt \int_{\mathbb{R}} dz [W_{\omega\mathbf{k}_\perp t}^{\sigma(2)*}(t, 0, 0, z) \\ &+ W_{\omega\mathbf{k}_\perp t}^{\sigma(2)*}(-t, 0, 0, z)] \Delta_-(z, t), \end{aligned} \quad (\text{B1})$$

$$\begin{aligned} \mathcal{B}_{\omega\mathbf{k}_\perp}^{\sigma(2)} &= \int_{a^{-1} \sinh(aT)}^L dt \int_{\mathbb{R}} dz [W_{\omega\mathbf{k}_\perp z}^{\sigma(2)*}(t, 0, 0, z) \\ &- W_{\omega\mathbf{k}_\perp z}^{\sigma(2)*}(-t, 0, 0, z)] \Delta_-(z, t). \end{aligned} \quad (\text{B2})$$

Next, we can use the integral form of the scalar Unruh modes [22]

$$\begin{aligned} w_{\omega\mathbf{k}_\perp}^\sigma &= \frac{e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp}}{(2\pi)^2 \sqrt{2a}} \int_{-\infty}^{\infty} d\vartheta e^{i(-1)^\sigma \vartheta \omega/a} \\ &\times \exp[ik_\perp(z \sinh \vartheta - t \cosh \vartheta)], \end{aligned}$$

to find integral expressions for the components of the vector Unruh modes we need,

$$W_{\omega \mathbf{k}_\perp t}^{\sigma(2)*} = -\frac{ie^{-i\mathbf{k}_\perp \cdot \mathbf{x}_\perp}}{(2\pi)^2 \sqrt{2a}} \int_{-\infty}^{\infty} d\vartheta e^{-i(-1)^\sigma \vartheta \omega/a} \sinh \vartheta \exp[-ik_\perp(z \sinh \vartheta - t \cosh \vartheta)], \quad (\text{B3})$$

$$W_{\omega \mathbf{k}_\perp z}^{\sigma(2)*} = +\frac{ie^{-i\mathbf{k}_\perp \cdot \mathbf{x}_\perp}}{(2\pi)^2 \sqrt{2a}} \int_{-\infty}^{\infty} d\vartheta e^{-i(-1)^\sigma \vartheta \omega/a} \cosh \vartheta \exp[-ik_\perp(z \sinh \vartheta - t \cosh \vartheta)]. \quad (\text{B4})$$

Now, by using Eqs. (B3) and (B4) into Eqs. (B1) and (B2), we find that

$$\mathcal{A}_{\omega \mathbf{k}_\perp}^{\sigma(2)} = -\frac{i}{2\pi^2 \sqrt{2a}} \int_{a^{-1} \sinh(aT)}^L dt \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} d\vartheta e^{-i(-1)^\sigma \vartheta \omega/a} \sinh \vartheta e^{-ik_\perp z \sinh \vartheta} \cos(k_\perp t \cosh \vartheta) \Delta_-(z, t), \quad (\text{B5})$$

$$\mathcal{B}_{\omega \mathbf{k}_\perp}^{\sigma(2)} = -\frac{1}{2\pi^2 \sqrt{2a}} \int_{a^{-1} \sinh(aT)}^L dt \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} d\vartheta e^{-i(-1)^\sigma \vartheta \omega/a} \cosh \vartheta e^{-ik_\perp z \sinh \vartheta} \sin(k_\perp t \cosh \vartheta) \Delta_-(z, t). \quad (\text{B6})$$

The integrals in the z variable in the above expressions can be computed immediately, yielding

$$\mathcal{A}_{\omega \mathbf{k}_\perp}^{\sigma(2)} = -\frac{i}{2\pi^2 \sqrt{2a}} \int_{a^{-1} \sinh(aT)}^L dt \int_{-\infty}^{\infty} d\vartheta e^{-i[(-1)^\sigma \vartheta \omega + k_\perp \text{sech}(aT) \sinh \vartheta]/a} \sinh \vartheta e^{-ik_\perp t \tanh(aT) \sinh \vartheta} \cos(k_\perp t \cosh \vartheta), \quad (\text{B7})$$

$$\mathcal{B}_{\omega \mathbf{k}_\perp}^{\sigma(2)} = -\frac{1}{2\pi^2 \sqrt{2a}} \int_{a^{-1} \sinh(aT)}^L dt \int_{-\infty}^{\infty} d\vartheta e^{-i[(-1)^\sigma \vartheta \omega + k_\perp \text{sech}(aT) \sinh \vartheta]/a} \cosh \vartheta e^{-ik_\perp t \tanh(aT) \sinh \vartheta} \sin(k_\perp t \cosh \vartheta). \quad (\text{B8})$$

For the sake of notation, let us define

$$f_a^A(\vartheta, L, T) \equiv \int_{a^{-1} \sinh(aT)}^L dt e^{-ik_\perp t \tanh(aT) \sinh \vartheta} \cos(k_\perp t \cosh \vartheta), \quad (\text{B9a})$$

$$f_a^B(\vartheta, L, T) \equiv \int_{a^{-1} \sinh(aT)}^L dt e^{-ik_\perp t \tanh(aT) \sinh \vartheta} \sin(k_\perp t \cosh \vartheta), \quad (\text{B9b})$$

which we can use to cast $\mathcal{A}_{\omega \mathbf{k}_\perp}^{\sigma(2)}$ and $\mathcal{B}_{\omega \mathbf{k}_\perp}^{\sigma(2)}$ as

$$\mathcal{A}_{\omega \mathbf{k}_\perp}^{\sigma(2)} = -\frac{i}{2\pi^2 \sqrt{2a}} \int_{-\infty}^{\infty} d\vartheta e^{-i[(-1)^\sigma \vartheta \omega + k_\perp \text{sech}(aT) \sinh \vartheta]/a} f_a^A(\vartheta, L, T) \sinh \vartheta, \quad (\text{B10a})$$

$$\mathcal{B}_{\omega \mathbf{k}_\perp}^{\sigma(2)} = +\frac{i}{2\pi^2 \sqrt{2a}} \int_{-\infty}^{\infty} d\vartheta e^{-i[(-1)^\sigma \vartheta \omega + k_\perp \text{sech}(aT) \sinh \vartheta]/a} f_a^B(\vartheta, L, T) \cosh \vartheta. \quad (\text{B10b})$$

We can see now from Eqs. (B10a) and (B10b) that, if we want to study the behavior of the physical current (recovered when $L \rightarrow \infty$) and of the infinite acceleration proper time (i.e., $T \rightarrow \infty$), it is enough to analyze f_a^A and f_a^B in such limits.

Direct integration of equations of Eqs. (B9) yields

$$f_a^A(\vartheta, L, T) = \frac{i \sinh \vartheta \tanh(aT)}{k_\perp [\sinh^2 \vartheta \tanh^2(aT) - \cosh^2 \vartheta]} \left\{ e^{ik_\perp L \tanh(aT) \sinh \vartheta} \left[\cos(k_\perp L \cosh \vartheta) + \frac{i \coth \vartheta}{\tanh(aT)} \sin(k_\perp L \cosh \vartheta) \right] \right. \\ \left. + e^{i(k_\perp/a) \sinh(aT) \tanh(aT) \sinh \vartheta} \left[\cos\left(\frac{k_\perp}{a} \sinh(aT) \cosh \vartheta\right) + \frac{i \coth \vartheta}{\tanh(aT)} \sin\left(\frac{k_\perp}{a} \sinh(aT) \cosh \vartheta\right) \right] \right\}, \quad (\text{B11a})$$

$$f_a^B(\vartheta, L, T) = \frac{i \sinh \vartheta \tanh(aT)}{k_\perp [\sinh^2 \vartheta \tanh^2(aT) - \cosh^2 \vartheta]} \left\{ e^{ik_\perp L \tanh(aT) \sinh \vartheta} \left[\sin(k_\perp L \cosh \vartheta) - \frac{i \coth \vartheta}{\tanh(aT)} \cos(k_\perp L \cosh \vartheta) \right] \right. \\ \left. + e^{i(k_\perp/a) \sinh(aT) \tanh(aT) \sinh \vartheta} \left[\sin\left(\frac{k_\perp}{a} \sinh(aT) \cosh \vartheta\right) - \frac{i \coth \vartheta}{\tanh(aT)} \cos\left(\frac{k_\perp}{a} \sinh(aT) \cosh \vartheta\right) \right] \right\}. \quad (\text{B11b})$$

Let us now take the limit $L \rightarrow \infty$. To this end, we can use (4.13) and the identity $\cos(\Omega L) = \sin[\Omega(L + \pi/2\Omega)]$ to see that, if $\Omega \neq 0$, we can set

$$\lim_{L \rightarrow \infty} \cos(\Omega L) = \lim_{L \rightarrow \infty} \sin(\Omega L) = \pi \Omega \delta(\Omega) \stackrel{!}{=} 0.$$

As $\cosh \vartheta \geq 1$ for all values of ϑ , we have

$$\begin{aligned} \lim_{L \rightarrow \infty} f_a^A(\vartheta, L, T) &= \frac{i \sinh \vartheta \tanh(aT)}{k_{\perp} [\sinh^2 \vartheta \tanh^2(aT) - \cosh^2 \vartheta]} e^{i(k_{\perp}/a) \sinh(aT) \tanh(aT) \sinh \vartheta} \\ &\times \left[\cos\left(\frac{k_{\perp}}{a} \sinh(aT) \cosh \vartheta\right) + \frac{i \coth \vartheta}{\tanh(aT)} \sin\left(\frac{k_{\perp}}{a} \sinh(aT) \cosh \vartheta\right) \right], \end{aligned} \quad (\text{B12a})$$

$$\begin{aligned} \lim_{L \rightarrow \infty} f_a^B(\vartheta, L, T) &= \frac{i \sinh \vartheta \tanh(aT)}{k_{\perp} [\sinh^2 \vartheta \tanh^2(aT) - \cosh^2 \vartheta]} e^{i(k_{\perp}/a) \sinh(aT) \tanh(aT) \sinh \vartheta} \\ &\times \left[\sin\left(\frac{k_{\perp}}{a} \sinh(aT) \cosh \vartheta\right) - \frac{i \coth \vartheta}{\tanh(aT)} \cos\left(\frac{k_{\perp}}{a} \sinh(aT) \cosh \vartheta\right) \right]. \end{aligned} \quad (\text{B12b})$$

The same arguments can be used in the limit $T \rightarrow \infty$ to see that

$$\lim_{T \rightarrow \infty} \lim_{L \rightarrow \infty} f_a^A(\vartheta, L, T) = \lim_{T \rightarrow \infty} \lim_{L \rightarrow \infty} f_a^B(\vartheta, L, T) = 0,$$

which, directly implies that

$$\lim_{T \rightarrow \infty} \lim_{L \rightarrow \infty} \mathcal{I}_{\omega \mathbf{k}_{\perp}}^{\sigma(2)} = 0.$$

The analysis for $\mathcal{I}_{\omega \mathbf{k}_{\perp}}^{\sigma(G)}$ follows the same reasoning, as direct calculations will show that these depend on the same integrals $f_a^A(\vartheta, L, T)$ and $f_a^B(\vartheta, L, T)$.

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- [1] J. Larmor, On the theory of the magnetic influence on spectra; and on the radiation from moving ions, *Philos. Mag.* **44**, 503 (1897).
 - [2] W. Pauli, Relativitätstheorie, in *Encyklopädie der Mathematischen Wissenschaften*, edited by A. Sommerfeld (B. G. Teubner, Leipzig, 1918), Vol. 5, p. 539; translated as *Theory of Relativity* (Pergamon Press, New York, 1958).
 - [3] R. P. Feynman, F. B. Morinigo, and W. G. Wagner, *Feynman Lectures on Gravitation*, edited by B. Hatfield (Addison-Wesley, Reading, MA, 1995), pp. 123–124.
 - [4] F. Rohrlich, The definition of electromagnetic radiation, *Nuovo Cimento* **21**, 811 (1961).
 - [5] F. Rohrlich, The principle of equivalence, *Ann. Phys. (N.Y.)* **22**, 169 (1963).
 - [6] D. G. Boulware, Radiation from a uniformly accelerated charge, *Ann. Phys. (N.Y.)* **124**, 169 (1980).
 - [7] W. G. Unruh, Notes on black-hole evaporation, *Phys. Rev. D* **14**, 870 (1976).
 - [8] S. A. Fulling, Nonuniqueness of canonical field quantization in Riemannian space-time, *Phys. Rev. D* **7**, 2850 (1973).
 - [9] P. C. W. Davies, Scalar production in Schwarzschild and Rindler metrics, *J. Phys. A* **8**, 609 (1975).
 - [10] L. C. B. Crispino, A. Higuchi, and G. E. Matsas, The Unruh effect and its applications, *Rev. Mod. Phys.* **80**, 787 (2008).
 - [11] A. Higuchi, G. E. A. Matsas, and D. Sudarsky, Bremsstrahlung and Fulling-Davies-Unruh thermal bath, *Phys. Rev. D* **46**, 3450 (1992).
 - [12] W. G. Unruh and R. M. Wald, What happens when an accelerating observer detects a Rindler particle, *Phys. Rev. D* **29**, 1047 (1984).
 - [13] K. Paithankar and S. Kolekar, Role of the Unruh effect in bremsstrahlung, *Phys. Rev. D* **101**, 065012 (2020).
 - [14] G. Cozzella, A. G. S. Landulfo, G. E. A. Matsas, and D. A. T. Vanzella, Proposal for Observing the Unruh Effect Using Classical Electrodynamics, *Phys. Rev. Lett.* **118**, 161102 (2017).
 - [15] A. G. S. Landulfo, S. A. Fulling, and G. E. A. Matsas, Classical and quantum aspects of the radiation emitted by a uniformly accelerated charge: Larmor-Unruh reconciliation and zero-frequency Rindler modes, *Phys. Rev. D* **100**, 045020 (2019).
 - [16] R. M. Wald, *General Relativity* (Chicago University Press, Chicago, 1984).

- [17] A. Zangwill, *Modern Electrodynamics* (Cambridge University Press, Cambridge, England, 2013).
- [18] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999).
- [19] M. Born, Die Theorie des starren Elektrons in der Kinematik des Relativitätsprinzips, *Ann. Phys. (Leipzig)* **335**, 1 (1909).
- [20] T. Fulton and F. Rohrlich, Classical radiation from a uniformly accelerated charge, *Ann. Phys. (N.Y.)* **9**, 499 (1960).
- [21] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 2014).
- [22] A. Higuchi, S. Iso, K. Ueda, and K. Yamamoto, Entanglement of the vacuum between left, right, future, and past: The origin of entanglement-induced quantum radiation, *Phys. Rev. D* **96**, 083531 (2017).
- [23] W.N. Bailey, Some infinite integrals involving bessel functions, *Proc. London Math. Soc.* **s2-40**, 37 (1936).
- [24] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).