Timing-residual power spectrum of a polarized stochastic gravitational-wave background in pulsar-timing-array observation

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(Received 25 January 2022; accepted 19 August 2022; published 2 September 2022)

We study the observation of a stochastic gravitational-wave background (SGWB) made by a pulsar-timing array in the spherical harmonic space of the observable. The observable is a timing residual which is the time-averaged redshift fluctuation of a pulsar over the duration of observation. Using the Sachs-Wolfe line-of-sight integral for the redshift fluctuation, we derive the power spectrum of the timing residual, from which we develop a fast algorithm to compute the overlap reduction functions for the SGWB intensity and polarization anisotropies. We find that the algorithm is less complicated and more efficient than our previous work which is based on the expansion of the polarization basis tensors in terms of spherical harmonics. Also, we use the power spectrum to construct the bipolar spherical harmonic coefficients that characterize the statistical isotropy of the SGWB. In particular, the coefficients for the linear-polarization anisotropy are worked out for the first time. Our harmonic-space method is useful for the data analysis in future pulsar-timing-array observation on a large number of pulsars as well as for the measurement of the statistical isotropy of the SGWB.

DOI: 10.1103/PhysRevD.106.064004

I. INTRODUCTION

The detection of gravitational waves (GWs) emitted from the coalescence of binary black holes by the LIGO-Virgo experiment opens up a new era of GW astronomy and cosmology [1,2]. The current Advanced LIGO [3], Advanced Virgo [4], KAGRA [5], and GEO600 [6] are ground-based laser interferometers that each has two perpendicular detector-arms to measure hundred-hertz GW strain amplitudes. Proposed GW interferometry experiments include Einstein Telescope, Cosmic Explorer, as well as space missions such as LISA, DECIGO, Taiji, and TianQin [7], aiming to measure GWs at frequencies ranging from kilohertz to millihertz. Pulsar timing is another method to detect GWs by monitoring the arrival times of radio pulses from pulsars with ground-based radio telescopes [8]. The line of sight from a telescope to the observed pulsar is like a detector-arm which is sensitive to propagating nanohertz GWs through the space. Current pulsar timing array (PTA) experiments, monitoring roughly 100 Galactic millisecond pulsars, include European Pulsar Timing Array (EPTA) [9], NANOGrav [10], and Parkes Pulsar Timing Array (PPTA) [11]. They are joined by new efforts such as MeerKAT [12], FAST [13], Indian Pulsar Timing Array (InPTA) [14], and the combined International Pulsar Timing Array (IPTA) [15]. The future Square Kilometre Array (SKA) project will observe about 6000 Galactic millisecond pulsars at a sensitivity three to four orders of magnitude better than the current PTAs [16,17].

Detecting a stochastic gravitational-wave background (SGWB) is a key science goal in GW experiments. GWs are very weakly interacting. The observation of the SGWB enables us to probe directly the physical processes that produce GWs in the early Universe, such as distant compact binary coalescences, early-time phase transitions, cosmic string or defect networks, second-order primordial scalar perturbations, and inflationary GWs [18]. The SGWB is predicted to be highly isotropic; however, it has been proposed that it could be anisotropic and even circularly or linearly polarized [19–27].

In the search of a SGWB in an interferometry network, the responses of a pair of detectors to the GW strain amplitude are correlated so as to filter out detector noises and increase the signal-to-noise ratio [18]. In PTA observation, quadrupolar spatial correlations between pulsar pairs are used to identify the presence of a SGWB [8]. Recently, the NANOGrav Collaboration [28] has found strong evidence of a stochastic common-spectrum process across 45 millisecond pulsars, indicating a SGWB with the spectral energy density of $\Omega_{\rm GW}(f) \simeq 5.0\times 10^{-9}$ at a reference frequency of $f = 32$ nHz for a f^{α} power-law spectrum with a spectral index of $\alpha = 2/3$. The common-spectrum process has also been found in the second data release of the Parkes Pulsar Timing Array (PPTA) [29], as well as in the data release 2 of the European Pulsar Timing

Array (EPTA) covering a timespan up to 24 years [30] and the second data release of the International Pulsar Timing Array (IPTA) synthesizing decadal-length pulsar-timing campaigns [31]. However, all of these observations have not found statistically significant quadrupolar interpulsar correlations in this common-spectrum process. Lately, using the data from Advanced LIGO's and Advanced Virgo's third observing run (O3) combined with the earlier O1 and O2 runs, upper limits have been derived on an isotropic SGWB, $\Omega_{\rm GW}(f) \leq 3.4\times10^{-9}$ at $f=25$ Hz for $\alpha=2/3$ [32], and on an anisotropic SGWB, ranging from $\Omega_{\rm GW}(f) < (0.57\text{–}9.3)\times10^{-9}$ sr^{-1} at $f=25$ Hz, depending on the spectral index [33].

In this paper, we will study the observation of SGWB intensity and polarization anisotropies in pulsar-timing-array experiments. There has been a lot of theoretical studies on the pulsar-timing observation of SGWB [8]. The observable is a timing residual which is the time-averaged redshift fluctuation of a pulsar over the duration of observation. The redshift fluctuation due to the presence of a SGWB is given by the Shapiro time delay between the Earth and the observed pulsar. Previous works have dealt with the spatial correlation functions of the timing residuals for the intensity anisotropy [34–36] and for the circular-polarization anisotropy [37]. In Ref. [38], instead of relying on the Shapiro time delay, the authors have derived the angular power spectrum of a line-of-sight integral for the timing residual for the SGWB intensity and circular-polarization anisotropies in the total-angular-momentum formulation of chiral spherical gravitational waves. In all of these works only the contribution from the Earth term has been considered, whereas modifications of the intensity correlation functions by the pulsar term have been discussed in Ref. [39]. Adopting the technique to expand the polarization basis tensors in terms of spin-weighted spherical harmonics [40], a numerical scheme has been developed to compute the correlation functions, which are extended to including the linear-polarization anisotropy [41]. Here we will tackle the problem in the spherical harmonic space of the observable. Following Refs. [41,42], we will use the Sachs-Wolfe line-of-sight integral for the timing residual of an observed pulsar to derive the timing-residual power spectrum.

In fact, there have been many schemes for measuring the SGWB intensity and polarization anisotropies in the pulsar-timing data analyses, such as the Bayesian parameter-estimation pipeline [43,44], the spherical harmonic power spectrum estimators [36,45,46], and the Fisher matrix of observed pulsar pairs [47,48]. Some of the limitations of using the spherical harmonic approach such as being computationally demanding in the analysis pipeline and an inhomogeneous sky coverage of pulsars in current PTA observation have been discussed [47,48]. Our formalism will be useful for future pulsar-timing arrays that observe a large number of pulsars on the full sky. The algorithm developed to compute the time-residual power spectrum will help reduce the computational load in data fitting processes. Lastly, using the Sachs-Wolfe line-of-sight integral can conveniently incorporate the effects of pulsar terms with various pulsar distances [41,42].

The paper is organized as follows. We will introduce a polarized SGWB and its Stokes parameters in the next section, followed by a brief account of the pulsar timing in Sec. III. In Secs. IV and V, the power spectrum of the timing residual will be derived. We will obtain the overlap reduction functions in the celestial coordinates in Sec. VI and in the computational frame in Sec. VII. We will briefly mention the bipolar spherical harmonic coefficients in Sec. VIII. Section IX is our conclusion.

II. POLARIZED SGWB

In the Minkowskian spacetime $(t,\vec{x})$, the metric perturbation h_{ij} in the transverse traceless gauge depicts traveling GWs at the speed of light $c=\omega/k$. It can be expanded by Fourier modes as

$$h_{ij}(t,\vec{x}) = \sum_A \int_{-\infty}^{\infty} \mathbf{d}f \int_{S^2} \mathbf{d}\hat{k}\, h_A(f,\hat{k})\mathbf{e}^A_{ij}(\hat{k}) e^{-2\pi i f(t-\hat{k}\cdot\vec{x}/c)}, \tag{1}$$

where A stands for the polarization of GWs with basis tensors $\mathbf{e}^A_{ij}(\hat{k})$, which are transverse to the propagation direction, $\hat{k}$. Here h_{ij} is treated as real, so the Fourier components with negative frequencies are given by $h_A(-f,\hat{k}) = h^*_A(f,\hat{k})$ for all $f\geq0$. We define a SGWB as a collection of GWs satisfying the condition that h_{ij} are random Gaussian fields with a statistical behavior completely characterized by the two-point correlation function $\langle h_{ij}(t,\vec{x}_1)h_{ij}(t,\vec{x}_2)\rangle$, where the angle brackets denote their ensemble averages. The ensemble averages of the Fourier modes have the following form

$$\langle h_A(f,\hat{k})h^*_{A'}(f',\hat{k}')\rangle = \delta(f-f')\delta(\hat{k}-\hat{k}')P_{AA'}(f,\hat{k}), \tag{2}$$

where the spatial translational invariance dictates the delta function of their three-momenta, $\delta(\vec{k}-\vec{k}')$. Note that the power spectra $P_{AA'}(f,\hat{k})$ remain to be direction dependent.

For GWs coming from the sky direction $-\hat{k}$ with wave vector $\vec{k}$, it is customary to write the polarization basis tensors in terms of the basis vectors in the spherical coordinates,

$$\mathbf{e}^+(\hat{k}) = \hat{\mathbf{e}}_\theta\otimes\hat{\mathbf{e}}_\theta - \hat{\mathbf{e}}_\phi\otimes\hat{\mathbf{e}}_\phi,$$
$$\mathbf{e}^\times(\hat{k}) = \hat{\mathbf{e}}_\theta\otimes\hat{\mathbf{e}}_\phi + \hat{\mathbf{e}}_\phi\otimes\hat{\mathbf{e}}_\theta, \tag{3}$$

in which $\hat{\mathbf{e}}_\theta$, $\hat{\mathbf{e}}_\phi$, and $\hat{k}$ form a right-handed orthonormal basis. Also, we can define the complex circular polarization basis tensors as

$$\mathbf{e}_R = \frac{(\mathbf{e}_+ + i\mathbf{e}_\times)}{\sqrt{2}}, \qquad \mathbf{e}_L = \frac{(\mathbf{e}_+ - i\mathbf{e}_\times)}{\sqrt{2}}, \tag{4}$$

where $\mathbf{e}_R$ stands for the right-handed GW with a positive helicity while $\mathbf{e}_L$ stands for the left-handed GW with a negative helicity. The corresponding amplitudes in Eq. (1) in the two different bases are related to each other via

$$h_R = \frac{(h_+ - ih_\times)}{\sqrt{2}}, \qquad h_L = \frac{(h_+ + ih_\times)}{\sqrt{2}}. \tag{5}$$

Analogous to the case in electromagnetic waves [49], the coherency matrix $P_{AA'}$ in Eq. (2) is related to the Stokes parameters, I, Q, U, and V as

$$\begin{aligned} I &= [\langle h_R h_R^* \rangle + \langle h_L h_L^* \rangle]/2, \\ Q + iU &= \langle h_L h_R^* \rangle, \\ Q - iU &= \langle h_R h_L^* \rangle, \\ V &= [\langle h_R h_R^* \rangle - \langle h_L h_L^* \rangle]/2, \end{aligned} \tag{6}$$

which are functions of the frequency f and the propagation direction $\hat{k}$. I is the intensity, Q and U represent the linear polarization, and V is the circular polarization.

III. PULSAR TIMING

In the pulsar-timing observation, radio pulses from an array of roughly 100 Galactic millisecond pulsars are being monitored with ground-based radio telescopes. The redshift fluctuation of a pulsar in the pointing direction $\hat{e}$ on the sky is given by the Sachs-Wolfe effect [50],

$$z(\hat{e}) = -\frac{1}{2}\int_{\eta_e}^{\eta_r} \mathbf{d}\eta\, \hat{e}^i \hat{e}^j \frac{\partial}{\partial \eta} h_{ij}(\eta, \vec{x}), \tag{7}$$

where the lower (upper) limit of integration in the line-of-sight integral represents the point of emission (reception) of the radio pulse. The physical distance of the pulsar from the Earth is

$$D = c(\eta_r - \eta_e), \tag{8}$$

which is of order 1 kpc.

The quantity that is actually observed in the pulsar-timing observation is the timing residual counted as

$$r(t) = \int_0^t \mathbf{d}t' z(t'), \tag{9}$$

where t' denotes the laboratory time and t is the duration of the observation. Using the laboratory time t', we rewrite Eq. (7) as

$$z(t', \hat{e}) = -\frac{1}{2}\int_{t'+\eta_e}^{t'+\eta_r} \mathbf{d}\eta\, d^{ij} \frac{\partial}{\partial \eta} h_{ij}(\eta, \vec{x}), \tag{10}$$

where the detector tensor is

$$d^{ij} = \hat{e}^i \hat{e}^j. \tag{11}$$

IV. TIMING-RESIDUAL POWER SPECTRUM

Then, replacing $\vec{x}$ by $c(\eta_r - \eta)\hat{e}$ in Eq. (10) and using the spherical wave expansion (A7) for the plane wave (1), Eq. (9) becomes

$$r(t, \hat{e}) = 2\pi \sum_A \int_{-\infty}^{\infty} \mathbf{d}f \int_{S^2} \mathbf{d}\hat{k} \int_{\eta_e}^{\eta_r} \mathbf{d}\eta (1 - e^{-2\pi i f t}) h_A(f, \hat{k}) d^{ij} \mathbf{e}_{ij}^A(\hat{k}) e^{-2\pi i f \eta} \sum_{LM} i^L j_L[2\pi f(\eta_r - \eta)] Y_{LM}^*(\hat{k}) Y_{LM}(\hat{e}). \tag{12}$$

We expand

$$r(t, \hat{e}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{e}). \tag{13}$$

Defining $x = 2\pi f(\eta_r - \eta)$, we have

$$\begin{aligned} a_{\ell m} &= \int_{-\infty}^{\infty} \frac{\mathbf{d}f}{2\pi f} (1 - e^{-2\pi i f t}) e^{-2\pi i f \eta_r} \sum_A \int_{S^2} \mathbf{d}\hat{k}\, h_A(f, \hat{k}) J_{\ell m}^A(fD, \hat{k}), \\ J_{\ell m}^A(fD, \hat{k}) &\equiv \sum_{LM} 2\pi i^L Y_{LM}^*(\hat{k}) \int_0^{2\pi f D/c} \mathbf{d}x\, e^{ix} j_L(x) \int_{S^2} \mathbf{d}\hat{e}\, d^{ij} \mathbf{e}_{ij}^A(\hat{k}) Y_{LM}(\hat{e}) Y_{\ell m}^*(\hat{e}). \end{aligned} \tag{14}$$

The timing-residual correlation between a pair of Galactic pulsars a and b is constructed as

$$\langle r(t_a,\hat{e}_a)r(t_b,\hat{e}_b)\rangle = \int_0^{t_a}\mathbf{d}t'\int_0^{t_b}\mathbf{d}t''\langle z(t',\hat{e}_a)z(t'',\hat{e}_b)\rangle = \sum_{\ell_1 m_1 \ell_2 m_2}\langle a_{\ell_1 m_1}a^*_{\ell_2 m_2}\rangle Y_{\ell_1 m_1}(\hat{e}_a)Y^*_{\ell_2 m_2}(\hat{e}_b), \tag{15}$$

where the ensemble average is given by Eq. (2) as

$$\langle a_{\ell_1 m_1}a^*_{\ell_2 m_2}\rangle = \int_{-\infty}^{\infty}\frac{\mathbf{d}f}{(2\pi f)^2}(1-e^{-2\pi i f t_a})(1-e^{2\pi i f t_b})\sum_{A_1 A_2}\int_{S^2}\mathbf{d}\hat{k}\, P_{A_1A_2}(f,\hat{k})J^{A_1}_{\ell_1 m_1}(fD_a,\hat{k})J^{A_2 *}_{\ell_2 m_2}(fD_b,\hat{k}). \tag{16}$$

In terms of the Stokes parameters in Eq. (6) and the definitions,

$$\begin{aligned}
\mathbb{J}^{I}_{\ell_1 m_1 \ell_2 m_2}(fD_a,fD_b,\hat{k}) &\equiv J^{R}_{\ell_1 m_1}(fD_a,\hat{k})J^{R*}_{\ell_2 m_2}(fD_b,\hat{k}) + J^{L}_{\ell_1 m_1}(fD_a,\hat{k})J^{L*}_{\ell_2 m_2}(fD_b,\hat{k}),\\
\mathbb{J}^{V}_{\ell_1 m_1 \ell_2 m_2}(fD_a,fD_b,\hat{k}) &\equiv J^{R}_{\ell_1 m_1}(fD_a,\hat{k})J^{R*}_{\ell_2 m_2}(fD_b,\hat{k}) - J^{L}_{\ell_1 m_1}(fD_a,\hat{k})J^{L*}_{\ell_2 m_2}(fD_b,\hat{k}),\\
\mathbb{J}^{Q+iU}_{\ell_1 m_1 \ell_2 m_2}(fD_a,fD_b,\hat{k}) &\equiv J^{L}_{\ell_1 m_1}(fD_a,\hat{k})J^{R*}_{\ell_2 m_2}(fD_b,\hat{k}),\\
\mathbb{J}^{Q-iU}_{\ell_1 m_1 \ell_2 m_2}(fD_a,fD_b,\hat{k}) &\equiv J^{R}_{\ell_1 m_1}(fD_a,\hat{k})J^{L*}_{\ell_2 m_2}(fD_b,\hat{k}),
\end{aligned} \tag{17}$$

we have

$$\langle a_{\ell_1 m_1}a^*_{\ell_2 m_2}\rangle = \int_{-\infty}^{\infty}\frac{\mathbf{d}f}{(2\pi f)^2}(1-e^{-2\pi i f t_a})(1-e^{2\pi i f t_b})\sum_{X=\{I,V,Q\pm iU\}}\int_{S^2}\mathbf{d}\hat{k}\, X(f,\hat{k})\mathbb{J}^{X}_{\ell_1 m_1 \ell_2 m_2}(fD_a,fD_b,\hat{k}). \tag{18}$$

We further expand the Stokes parameters in terms of ordinary and spin-weighted spherical harmonics as

$$\begin{aligned}
I(f,\hat{k}) &= \sum_{\ell m} I_{\ell m}(f)Y_{\ell m}(\hat{k}),\\
V(f,\hat{k}) &= \sum_{\ell m} V_{\ell m}(f)Y_{\ell m}(\hat{k}),\\
(Q+iU)(f,\hat{k}) &= \sum_{\ell m}(Q+iU)_{\ell m}(f)\,{}_{+4}Y_{\ell m}(\hat{k}),\\
(Q-iU)(f,\hat{k}) &= \sum_{\ell m}(Q-iU)_{\ell m}(f)\,{}_{-4}Y_{\ell m}(\hat{k}),
\end{aligned} \tag{19}$$

where the specific combinations, $Q\pm iU$, make them become spin ±4 objects so that we can expand them nicely by the corresponding spin-weighted spherical harmonics. A brief introduction to the spin-weighted spherical harmonics is found in the Appendix.

Hence, we can express the timing-residual correlation in the following form

$$\langle r(t_a,\hat{e}_a)r(t_b,\hat{e}_b)\rangle = \int_{-\infty}^{\infty}\frac{\mathbf{d}f}{(2\pi f)^2}(1-e^{-2\pi i f t_a})(1-e^{2\pi i f t_b})\sum_{X=\{I,V,Q\pm iU\}}\sum_{\ell m}X_{\ell m}(f)\gamma^{X}_{\ell m}(fD_a,fD_b;\hat{e}_a,\hat{e}_b), \tag{20}$$

where the overlap reduction functions (ORFs) are given by

$$\gamma^{I,V}_{\ell m}(fD_a,fD_b;\hat{e}_a,\hat{e}_b) = \sum_{\ell_1 m_1 \ell_2 m_2}Y_{\ell_1 m_1}(\hat{e}_a)Y^*_{\ell_2 m_2}(\hat{e}_b)\int_{S^2}\mathbf{d}\hat{k}Y_{\ell m}(\hat{k})\mathbb{J}^{I,V}_{\ell_1 m_1 \ell_2 m_2}(fD_a,fD_b,\hat{k}), \tag{21}$$

$$\gamma^{Q\pm iU}_{\ell m}(fD_a,fD_b;\hat{e}_a,\hat{e}_b) = \sum_{\ell_1 m_1 \ell_2 m_2}Y_{\ell_1 m_1}(\hat{e}_a)Y^*_{\ell_2 m_2}(\hat{e}_b)\int_{S^2}\mathbf{d}\hat{k}\,{}_{\pm4}Y_{\ell m}(\hat{k})\mathbb{J}^{Q\pm iU}_{\ell_1 m_1 \ell_2 m_2}(fD_a,fD_b,\hat{k}). \tag{22}$$

Equations (21) and (22) are the most general ORFs for a pair of Galactic pulsars a and b, respectively, at distances D_a and D_b from the Earth.

V. CALCULATION OF $J^A_{\ell m}(fD,\hat{k})$

Now we calculate the contribution of a k-mode to the redshift fluctuation of a pulsar, namely $J^A_{\ell m}(fD,\hat{k})$ in Eq. (14). For convenience, we first assume that $\hat{k}$ points to the direction of the polar axis or $\hat{\mathbf{z}}$-axis. In this case, the basis vectors (3) become

$$\mathbf{e}^{+}(\hat{k}) = \hat{\mathbf{x}}\otimes\hat{\mathbf{x}} - \hat{\mathbf{y}}\otimes\hat{\mathbf{y}},$$
$$\mathbf{e}^{\times}(\hat{k}) = \hat{\mathbf{x}}\otimes\hat{\mathbf{y}} + \hat{\mathbf{y}}\otimes\hat{\mathbf{x}}, \tag{23}$$

and we have

$$\hat{e} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}, \tag{24}$$

$$Y_{LM}(\hat{k}) = \sqrt{\frac{2L+1}{4\pi}}\delta_{M0}. \tag{25}$$

This gives

$$d^{ij}\mathbf{e}^{R,L}_{ij}(\hat{k}) = 4\sqrt{\frac{\pi}{15}}Y_{2\pm2}(\hat{e}), \tag{26}$$

where the helicity R takes the value of 2 and L the value of -2. Hence, we obtain

$$J^{R,L}_{\ell m}(fD,\hat{\mathbf{z}}) = \sum_L 4\pi i^L\sqrt{\frac{2L+1}{15}}\int_0^{2\pi fD/c}\mathbf{d}x\, e^{ix}j_L(x) \times\int_{S^2}\mathbf{d}\hat{e}\, Y_{2\pm2}(\hat{e})Y_{L0}(\hat{e})Y^*_{\ell m}(\hat{e}). \tag{27}$$

From Eq. (A8), we have

$$\int_{S^2}\mathbf{d}\hat{e}\, Y_{2\pm2}(\hat{e})Y_{L0}(\hat{e})Y^*_{\ell m}(\hat{e}) = (-1)^m\sqrt{\frac{5(2L+1)(2\ell+1)}{4\pi}}\begin{pmatrix}2 & L & \ell\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}2 & L & \ell\\ \pm2 & 0 & -m\end{pmatrix}, \tag{28}$$

which vanishes unless $m=\pm2$ and $L=\ell-2,\ell,\ell+2$. These nonzero integral values are given by

$$\int_{S^2}\mathbf{d}\hat{e}\, Y_{2\pm2}(\hat{e})Y_{\ell-2,0}(\hat{e})Y^*_{\ell\pm2}(\hat{e}) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\left[\frac{(\ell-1)\ell(\ell+1)(\ell+2)}{(2\ell-3)(2\ell-1)^2(2\ell+1)}\right]^{\frac{1}{2}},$$
$$\int_{S^2}\mathbf{d}\hat{e}\, Y_{2\pm2}(\hat{e})Y_{\ell0}(\hat{e})Y^*_{\ell\pm2}(\hat{e}) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\left[\frac{(\ell-1)\ell(\ell+1)(\ell+2)}{(2\ell-1)^2(2\ell+3)^2}\right]^{\frac{1}{2}},$$
$$\int_{S^2}\mathbf{d}\hat{e}\, Y_{2\pm2}(\hat{e})Y_{\ell+2,0}(\hat{e})Y^*_{\ell\pm2}(\hat{e}) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\left[\frac{(\ell-1)\ell(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)^2(2\ell+5)}\right]^{\frac{1}{2}}. \tag{29}$$

Substituting them in Eq. (27), we obtain

$$J^{R,L}_{\ell m}(fD,\hat{\mathbf{z}}) = -\delta_{m\pm2}2\pi i^\ell\sqrt{\frac{(2\ell+1)}{8\pi}\frac{(\ell+2)!}{(\ell-2)!}}\int_0^{2\pi fD/c}\mathbf{d}x\, e^{ix}\left[\frac{j_{\ell-2}(x)}{(2\ell-1)(2\ell+1)}+\frac{2j_\ell(x)}{(2\ell-1)(2\ell+3)}+\frac{j_{\ell+2}(x)}{(2\ell+1)(2\ell+3)}\right]. \tag{30}$$

This can be cast into a compact form by using the recursion relation, $j_\ell(x)/x = [j_{\ell-1}(x)+j_{\ell+1}(x)]/(2\ell+1)$, which gives

$$J^{R,L}_{\ell m}(fD,\hat{\mathbf{z}}) = -\delta_{m\pm2}2\pi i^\ell\sqrt{\frac{(2\ell+1)}{8\pi}\frac{(\ell+2)!}{(\ell-2)!}} \times\int_0^{2\pi fD/c}\mathbf{d}x\, e^{ix}\frac{j_\ell(x)}{x^2}. \tag{31}$$

Through a three-dimensional rotation that takes the $\hat{\mathbf{z}}$-axis into the direction $\hat{k}$, we can relate [40]

$$J^{R,L}_{\ell m}(fD,\hat{k}) = \sum_{m'}D^\ell_{m'm}(-\alpha,-\theta,-\phi)J^{R,L}_{\ell m'}(fD,\hat{\mathbf{z}}), \tag{32}$$

where $\hat{k}=(\theta,\phi)$. Here, the Wigner-D matrix is given by

$$D^\ell_{m'm}(-\alpha,-\theta,-\phi) = \sqrt{\frac{4\pi}{2\ell+1}}{}_{-m'}Y_{\ell m}(\theta,\phi)e^{im'\alpha}, \tag{33}$$

where $e^{im'\alpha}$ is a redundant phase that reflects a remaining degree of freedom in the rotation about the $\hat{k}$-direction. Hence, we have

$$J^R_{\ell m}(fD,\hat{k}) = D^{\ell *}_{2m}(-\alpha,-\theta,-\phi)J^R_{\ell 2}(fD,\hat{\mathbf{z}}),$$
$$J^L_{\ell m}(fD,\hat{k}) = D^{\ell *}_{-2m}(-\alpha,-\theta,-\phi)J^L_{\ell -2}(fD,\hat{\mathbf{z}}), \quad (34)$$

noting that $J^R_{\ell 2}(fD,\hat{\mathbf{z}}) = J^L_{\ell -2}(fD,\hat{\mathbf{z}})$ and $e^{im'\alpha}$ will not appear in physical observables.

VI. OVERLAP REDUCTION FUNCTIONS IN THE CELESTIAL COORDINATES

Inserting the results (34) into Eq. (17), we have

$$\mathbb{J}^{I,V}_{\ell_1 m_1 \ell_2 m_2}(fD_a,fD_b,\hat{k}) = (-1)^{m_1}[{}_2Y_{\ell_1 -m_1}(\hat{k})_{-2}Y_{\ell_2 m_2}(\hat{k}) \pm {}_{-2}Y_{\ell_1 -m_1}(\hat{k})_2Y_{\ell_2 m_2}(\hat{k})]J_{\ell_1}(fD_a)J^*_{\ell_2}(fD_b),$$
$$\mathbb{J}^{Q\pm iU}_{\ell_1 m_1 \ell_2 m_2}(fD_a,fD_b,\hat{k}) = (-1)^{m_1}{}_{\mp 2}Y_{\ell_1 -m_1}(\hat{k})_{\mp 2}Y_{\ell_2 m_2}(\hat{k})e^{\pm i4\alpha}J_{\ell_1}(fD_a)J^*_{\ell_2}(fD_b), \quad (35)$$

where we have defined the function

$$J_\ell(fD) = \sqrt{2}\pi i^\ell \sqrt{\frac{(\ell+2)!}{(\ell-2)!}} \int_0^{2\pi fD/c} \mathbf{d}x\, e^{ix}\frac{j_\ell(x)}{x^2}, \quad (36)$$

and the phase factors $e^{\pm i4\alpha}$ are resulted from a rotation of angle α about the $\hat{k}$-direction on the spin-4 objects, $\mathbb{J}^{Q\pm iU}_{\ell_1 m_1 \ell_2 m_2}$, respectively. Simultaneously, the spin-4 spherical harmonics ${}_{\pm 4}Y_{\ell m}(\hat{k})$ in Eq. (22) are augmented by the same phase factors of opposite signs $e^{\mp i4\alpha}$ under the rotation, which exactly cancel $e^{\pm i4\alpha}$ from $\mathbb{J}^{Q\pm iU}_{\ell_1 m_1 \ell_2 m_2}$. Using Eq. (A8) and the property (A9), the ORFs in Eqs. (21) and (22) become

$$\gamma^{I,V}_{\ell m}(fD_a,fD_b;\hat{e}_a,\hat{e}_b) = \sum_{\ell_1 m_1 \ell_2 m_2}(-1)^{m_1}[1\pm(-1)^{\ell+\ell_1+\ell_2}]J_{\ell_1}(fD_a)J^*_{\ell_2}(fD_b)Y_{\ell_1 m_1}(\hat{e}_a)Y^*_{\ell_2 m_2}(\hat{e}_b)$$
$$\times\sqrt{\frac{(2\ell+1)(2\ell_1+1)(2\ell_2+1)}{4\pi}}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ 0 & -2 & 2\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ m & -m_1 & m_2\end{pmatrix}, \quad (37)$$

$$\gamma^{Q\pm iU}_{\ell m}(fD_a,fD_b;\hat{e}_a,\hat{e}_b) = \sum_{\ell_1 m_1 \ell_2 m_2}(-1)^{m_1}J_{\ell_1}(fD_a)J^*_{\ell_2}(fD_b)Y_{\ell_1 m_1}(\hat{e}_a)Y^*_{\ell_2 m_2}(\hat{e}_b)$$
$$\times\sqrt{\frac{(2\ell+1)(2\ell_1+1)(2\ell_2+1)}{4\pi}}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ \mp 4 & \pm 2 & \pm 2\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ m & -m_1 & m_2\end{pmatrix}. \quad (38)$$

The two lowest moments, γ^I_{00} and γ^V_{00}, select the unpolarized and the circularly polarized components of an isotropic SGWB, respectively. When $\ell = m = 0$, the Wigner-3j symbol is proportional to $\delta_{\ell_1\ell_2}\delta_{m_1 m_2}$. It immediately gives us

$$\gamma^V_{00} = 0, \quad (39)$$

whereas

$$\gamma^I_{00} = \sum_\ell \frac{2l+1}{4\pi}C_\ell P_\ell(\hat{e}_a\cdot\hat{e}_b) \quad \text{with} \quad (40)$$

$$C_\ell \equiv \frac{1}{\sqrt{\pi}}J_\ell(fD_a)J^*_\ell(fD_b), \quad (41)$$

which depends solely on the separation angle as expected for an isotropic SGWB. The power spectrum C_ℓ has an analytic form under the limit that $fD_a \gg c$ and $fD_b \gg c$ [38,42]. Using the integral result

$$\int_0^\infty \mathbf{d}x\, e^{ix}\frac{j_\ell(x)}{x^2} = 2i^{\ell-1}\frac{(\ell-2)!}{(\ell+2)!}, \quad (42)$$

Eq. (36) can be approximated as

$$J_\ell(fD)|_{fD/c\to\infty} = 2^{3/2}\pi i^{2\ell-1}\sqrt{\frac{(\ell-2)!}{(\ell+2)!}}, \quad (43)$$

which gives

$$C_\ell = \frac{8\pi^{3/2}}{(\ell+2)(\ell+1)\ell(\ell-1)}. \quad (44)$$

It was shown [36] that the γ^I_{00} (40) with this C_ℓ reproduces the Hellings and Downs curve for the quadrupolar interpulsar correlations [51]. For finite fD_a and fD_b, one needs to calculate numerically the power spectrum (41) to get corrections to the Hellings and Downs curve [42], where it was found that the pulsar term adds power to the power spectrum at higher-ℓ and hence modifies the Hellings and Downs curve at small angular separations. This small-scale modification from the pulsar term has also been discussed in Refs. [39,41].

For higher multipole moments, we can easily set up a numerical scheme, similar to that in Ref. [41], to compute the ORFs in Eqs. (37) and (38) for any pair of Galactic pulsars on the sky with known distances and coordinates, (D_a, θ_a, ϕ_a) and (D_b, θ_b, ϕ_b). The factor $J_{\ell_1}(fD_a)J^*_{\ell_2}(fD_b)$ is generally a complex number. When $fD_a \gg c$ and $fD_b \gg c$, we can approximate it as

$$J_{\ell_1}(fD_a)J^*_{\ell_2}(fD_b) \simeq 8\pi^2(-1)^{\ell_1+\ell_2}\sqrt{\frac{(\ell_1-2)!}{(\ell_1+2)!}\frac{(\ell_2-2)!}{(\ell_2+2)!}}. \tag{45}$$

VII. OVERLAP REDUCTION FUNCTIONS IN THE COMPUTATIONAL FRAME

To compare the present method with previous works [34–37,39,41], we also compute the ORFs in the so-called computational frame: pulsar a is placed along the $\hat{\mathbf{z}}$-axis while pulsar b is in the $\hat{\mathbf{x}} - \hat{\mathbf{z}}$ plane. Then, their polar coordinates are given by

$$\hat{e}_a = (0, 0), \qquad \hat{e}_b = (\zeta, 0), \tag{46}$$

where ζ is their separation angle, and we have

$$\begin{aligned} Y_{\ell_1 m_1}(\hat{e}_a) &= \sqrt{\frac{2\ell_1+1}{4\pi}}\delta_{m_1 0}, \\ Y^*_{\ell_2 m_2}(\hat{e}_b) &= Y^*_{\ell_2 m_2}(\zeta, 0) = Y_{\ell_2 m_2}(\zeta, 0). \end{aligned} \tag{47}$$

Hence, Eqs. (37) and (38) simplify to

$$\begin{aligned} \gamma^{I,V}_{\ell m}(fD_a, fD_b, \zeta) = \sum_{\ell_1\ell_2}(-1)^m\frac{2\ell_1+1}{4\pi}[1 \pm (-1)^{\ell+\ell_1+\ell_2}]J_{\ell_1}(fD_a)J^*_{\ell_2}(fD_b)Y_{\ell_2 m}(\zeta, 0) \\ \times\sqrt{(2\ell+1)(2\ell_2+1)}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ 0 & -2 & 2\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ m & 0 & -m\end{pmatrix}, \end{aligned} \tag{48}$$

$$\begin{aligned} \gamma^{Q\pm iU}_{\ell m}(fD_a, fD_b, \zeta) = \sum_{\ell_1\ell_2}(-1)^m\frac{2\ell_1+1}{4\pi}J_{\ell_1}(fD_a)J^*_{\ell_2}(fD_b)Y_{\ell_2 m}(\zeta, 0) \\ \times\sqrt{(2\ell+1)(2\ell_2+1)}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ \mp 4 & \pm 2 & \pm 2\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_2\\ m & 0 & -m\end{pmatrix}. \end{aligned} \tag{49}$$

Using the property (A9), it is straightforward to show that the ORFs in the computational frame have the conjugate relations:

$$\gamma^I_{\ell -m} = (-1)^m\gamma^I_{\ell m}, \tag{50}$$

$$\gamma^V_{\ell -m} = (-1)^{m+1}\gamma^V_{\ell m}, \tag{51}$$

$$\gamma^{Q\pm iU}_{\ell -m} = (-1)^m\gamma^{Q\mp iU}_{\ell m}. \tag{52}$$

We have computed numerically the ORF multipole moments in Eqs. (48) and (49). For $fD_a \gg c$ and $fD_b \gg c$, we reproduce the results for the intensity and circular-polarization ORFs in Refs. [34–37]. For $fD_a = fD_b = 10c$, we confirm the contribution of the pulsar term to the ORFs on small angular scales [39,41] and reproduce the results for the linear-polarization ORFs [41].

Although the main results in Eqs. (48) and (49) look similar to those in the previous work [41] (Eqs. (33) and (34) there), they are basically derived from different approaches. Here, we expand the timing residual by a single $a_{\ell m}$, whereas the latter uses spherical harmonic expansion of the polarization basis tensors, which is essentially harmonic expansion in the timing-residual correlation function or something that behaves as a combination of two $a_{\ell m}$'s. The approach in the previous work is based on a method adopted in GW interferometry that has fixed baselines connecting different detectors [40]. The setting in PTA observation is different; however, the method is still workable as shown in the previous work [41], allowing one to develop an algorithm to compute the ORF multipole moments. Nevertheless, for the method in this previous work, it is impossible to extract individual $a_{\ell m}$ from the correlation function, and it is quite involved to derive the C_ℓ's from that harmonic expansion. We note that the present approach should be more natural for PTA observation in the sense that it is an harmonic expansion of the observable. In fact, the algorithm in the present work is less complicated and more efficient than that in the previous work. Furthermore, with $a_{\ell m}$'s, we can construct the bipolar spherical harmonic coefficients as follows.

VIII. BIPOLAR SPHERICAL HARMONIC COEFFICIENTS

Bipolar spherical harmonic (BiPoSH) coefficients have been introduced and widely discussed in the context of cosmic microwave background observation, providing us with an efficient way to measure the statistical isotropy of the cosmic microwave background [52]. The methodology is equally well applicable to PTA observation of the SGWB.

We have derived the ORFs using the harmonic-space method. Let us go back to the power spectrum in the timing-residual correlation function (15), which can be also expanded in terms of bipolar spherical harmonics: $\{Y_{\ell_1}(\hat{e}_a) \otimes Y_{\ell_2}(\hat{e}_b)\}_{\ell m}$ as [53]

$$\langle r(t_a, \hat{e}_a) r(t_b, \hat{e}_b) \rangle = \sum_{\ell_1 \ell_2 \ell m} A^{\ell m}_{\ell_1 \ell_2} \{Y_{\ell_1}(\hat{e}_a) \otimes Y_{\ell_2}(\hat{e}_b)\}_{\ell m}, \tag{53}$$

where $A^{\ell m}_{\ell_1 \ell_2}$ are the expansion coefficients and

$$\{Y_{\ell_1}(\hat{e}_a) \otimes Y_{\ell_2}(\hat{e}_b)\}_{\ell m} = \sum_{m_1 m_2} (-1)^{m_1} \sqrt{2\ell+1} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ m & -m_1 & m_2 \end{pmatrix} Y_{\ell_1 m_1}(\hat{e}_a) Y^*_{\ell_2 m_2}(\hat{e}_b), \tag{54}$$

which satisfy the orthogonal condition,

$$\int_{S^2} \mathbf{d}\hat{e}_a \int_{S^2} \mathbf{d}\hat{e}_b \{Y_{\ell_1}(\hat{e}_a) \otimes Y_{\ell_2}(\hat{e}_b)\}_{\ell m} \{Y_{\ell_1'}(\hat{e}_a) \otimes Y_{\ell_2'}(\hat{e}_b)\}^*_{\ell' m'} = \delta_{\ell_1 \ell_1'} \delta_{\ell_2 \ell_2'} \delta_{\ell \ell'} \delta_{m m'}, \tag{55}$$

where we have used the relation (A10). The BiPoSH coefficients is thus related to the power spectrum by

$$A^{\ell m}_{\ell_1 \ell_2} = \sum_{m_1 m_2} (-1)^{m_1} \sqrt{2\ell+1} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ m & -m_1 & m_2 \end{pmatrix} \langle a_{\ell_1 m_1} a^*_{\ell_2 m_2} \rangle. \tag{56}$$

From Eqs. (15) and (20), we have

$$\langle a_{\ell_1 m_1} a^*_{\ell_2 m_2} \rangle = \int_{-\infty}^{\infty} \frac{\mathbf{d}f}{(2\pi f)^2} (1 - e^{-2\pi i f t_a})(1 - e^{2\pi i f t_b}) \sum_{X=\{I,V,Q\pm iU\}} \sum_{\ell m} X_{\ell m}(f) \tilde{\gamma}^X_{\ell m, \ell_1 m_1 \ell_2 m_2}, \tag{57}$$

where

$$\gamma^X_{\ell m}(fD_a, fD_b; \hat{e}_a, \hat{e}_b) = \sum_{\ell_1 m_1 \ell_2 m_2} \tilde{\gamma}^X_{\ell m, \ell_1 m_1 \ell_2 m_2} Y_{\ell_1 m_1}(\hat{e}_a) Y^*_{\ell_2 m_2}(\hat{e}_b). \tag{58}$$

Using the ORFs in Eqs. (37) and (38), we can then explicitly write the BiPoSH coefficients as

$$\begin{aligned} A^{\ell m}_{\ell_1 \ell_2} = & \int_{-\infty}^{\infty} \frac{\mathbf{d}f}{(2\pi f)^2} (1 - e^{-2\pi i f t_a})(1 - e^{2\pi i f t_b}) \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi}} J_{\ell_1}(fD_a) J^*_{\ell_2}(fD_b) \\ & \times \left\{ I_{\ell m}[1 + (-1)^{\ell+\ell_1+\ell_2}] \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & -2 & 2 \end{pmatrix} + V_{\ell m}[1 - (-1)^{\ell+\ell_1+\ell_2}] \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & -2 & 2 \end{pmatrix} \right. \\ & \left. + (Q + iU)_{\ell m} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ -4 & 2 & 2 \end{pmatrix} + (Q - iU)_{\ell m} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 4 & -2 & -2 \end{pmatrix} \right\}, \end{aligned} \tag{59}$$

which shows that the intensity and circular-polarization anisotropies induce even-parity and odd-parity BiPoSHs respectively, whose parity is defined by the sum, $\ell + \ell_1 + \ell_2$. This selection rule has been pointed out in Refs. [45,46], though the authors considered a limiting case with $fD_a \gg c$ and $fD_b \gg c$. Using the property (A9), under the approximation (45) we have $A^{\ell m}_{\ell_1 \ell_2} = A^{\ell m}_{\ell_2 \ell_1}$. The BiPoSH coefficients induced by the linear-polarization anisotropy are derived for the first time in this work. Similar to the works in cosmic microwave background, these BiPoSH coefficients can be used to construct optimal estimators for testing the statistical isotropy of the SGWB intensity and polarization [45,46].

IX. CONCLUSION

We have studied the pulsar-timing-array observation of the Stokes parameters of an anisotropic stochastic gravitational-wave background, based on the spherical harmonic expansion of the pulsar timing residual. A numerical scheme to compute the overlap reduction functions (ORFs) and the bipolar spherical harmonic (BiPoSH) coefficients has been developed. We have used the Sachs-Wolfe line-of-sight integral for the timing residual of a Galactic millisecond pulsar, which properly takes into account the contribution of the pulsar term to the ORF multipoles at small angular separation of the pulsar pair. Using the spherical-harmonic method, we can compute an ORF multipole with a required angular resolution. The method also allows us to compute ORFs for Galactic pulsar pairs at different distances from the Earth. Indeed, the Sachs-Wolfe line-of-sight integral can be generalized to deal with extragalactic pulsars at high redshifts. In future pulsar-timing-array observation on a large number of pulsars, our method can provide a fast algorithm for computing accurate ORF multipoles and BiPoSH coefficients.

ACKNOWLEDGMENTS

This work was supported in part by the Ministry of Science and Technology (MOST) of Taiwan, Republic of China, under Grants No. MOST 110-2112-M-032-007 (G. C. L.), No. MOST 110-2123-M-007-002 (G. C. L.), and No. MOST 110-2112-M-001-036 (K. W. N.).

APPENDIX: SPIN-WEIGHTED SPHERICAL HARMONICS

The explicit form of the spin-weighted spherical harmonics that we use is

$$ {}_sY_{\ell m}(\theta,\phi) = (-1)^m e^{im\phi}\sqrt{\frac{(2\ell+1)}{(4\pi)}\frac{(\ell+m)!(\ell-m)!}{(\ell+s)!(\ell-s)!}}\sin^{2\ell}\left(\frac{\theta}{2}\right)\sum_r \binom{\ell-s}{r}\binom{\ell+s}{r+s-m}(-1)^{\ell-r-s}\cot^{2r+s-m}\left(\frac{\theta}{2}\right). \tag{A1}$$

When $s=0$, it reduces to the ordinary spherical harmonics,

$$ Y_{\ell m}(\hat{n}) = \sqrt{\frac{(2\ell+1)}{(4\pi)}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi}. \tag{A2}$$

Spin-weighted spherical harmonics satisfy the orthogonal relation,

$$ \int_{S^2}\mathbf{d}\hat{n}\,{}_sY^*_{\ell m}(\hat{n})\,{}_sY_{\ell' m'}(\hat{n}) = \delta_{\ell\ell'}\delta_{mm'}, \tag{A3}$$

and the completeness relation,

$$ \sum_{\ell m}{}_sY^*_{\ell m}(\hat{n})\,{}_sY_{\ell m}(\hat{n}') = \delta(\hat{n}-\hat{n}') = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta'). \tag{A4}$$

Its complex conjugate is

$$ {}_sY^*_{\ell m}(\hat{n}) = (-1)^{s+m}\,{}_{-s}Y_{\ell -m}(\hat{n}), \tag{A5}$$

and its parity is given by

$$ {}_sY_{\ell m}(-\hat{n}) \equiv {}_sY_{\ell m}(\pi-\theta,\phi+\pi) = (-1)^\ell\,{}_{-s}Y_{\ell m}(\hat{n}). \tag{A6}$$

Also, we have the spherical wave expansion:

$$ e^{i\vec{k}\cdot\vec{r}} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^\ell j_\ell(kr)Y^*_{\ell m}(\hat{k})Y_{\ell m}(\hat{r}), \tag{A7}$$

where $j_\ell(x)$ is the spherical Bessel function.

We can calculate the integral of a product of three spin-weighted spherical harmonics using the formula:

$$ \int_{S^2}\mathbf{d}\hat{e}\,{}_{s_1}Y_{\ell_1 m_1}(\hat{e})\,{}_{s_2}Y_{\ell_2 m_2}(\hat{e})\,{}_{s_3}Y_{\ell_3 m_3}(\hat{e}) = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ -s_1 & -s_2 & -s_3\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}, \tag{A8}$$

which involves two Wigner-3j symbols representing the coupling coefficients between different spherical harmonics [53]. The Wigner-3j symbol is zero unless it satisfies: ℓ_1, ℓ_2, and ℓ_3 have to meet the triangular condition, i.e.,

$\ell_1 + \ell_2 \geq \ell_3 \geq |\ell_1 - \ell_2|$, while $m_1 + m_2 + m_3 = 0$; when $m_1 = m_2 = m_3 = 0$, $\ell_1 + \ell_2 + \ell_3$ is even. The Wigner-3j symbols have the reflection property and the summation relation:

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ s_1 & s_2 & s_3 \end{pmatrix} = (-1)^{\ell_1+\ell_2+\ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -s_1 & -s_2 & -s_3 \end{pmatrix} = (-1)^{\ell_1+\ell_2+\ell_3} \begin{pmatrix} \ell_1 & \ell_3 & \ell_2 \\ s_1 & s_3 & s_2 \end{pmatrix}, \tag{A9}$$

$$(2\ell + 1) \sum_{m_1 m_2} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ m & -m_1 & m_2 \end{pmatrix} \begin{pmatrix} \ell' & \ell_1 & \ell_2 \\ m' & -m_1 & m_2 \end{pmatrix} = \delta_{\ell\ell'}\delta_{mm'}. \tag{A10}$$

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