Generalized covariant entropy bound in Lanczos-Lovelock gravity

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In this paper, we investigate the generalized covariant entropy bound in the theory where the Einstein gravity is perturbed by the higher-order Lovelock terms. After choosing the light sheet that is smooth under the perturbation limit, replacing the Bekenstein-Hawking entropy with the Jacobson-Myers entropy, and introducing two reasonable physical assumptions, we show that the corresponding generalized covariant entropy bound is satisfied under a higher-order approximation of the perturbation from the higher-order Lovelock terms. Our result implies that the Jacobson-Myers entropy strictly obeys the entropy bound under the perturbation level, and the generalized second law of Lanczos-Lovelock gravity is also satisfied when the Einstein gravity is perturbed by the higher-order Lovelock terms.

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I. INTRODUCTION

The investigation of black hole thermodynamics has led to some interesting entropy bounds that should be observed to guarantee theoretical consistency. Bekenstein [1] has conjectured that the entropy S and energy E of any stable gravitational thermodynamic system satisfies a universal bound

$$S \leq 2\pi E R, \tag{1}$$

in which R is defined as the circumferential radius of the sphere surrounding the thermodynamical system. This bound is called the Bekenstein bound and it can be indicated by the generalized second law (GSL) of black holes. The Bekenstein bound has been confirmed in many weakly gravitational systems with finite size. In a strongly gravitational system, it is hard to define the energy E and radius R locally. Counterexamples can be found in the process of gravitational collapse [2]. For a spherical system in Einstein gravity, the Bekenstein bound can be simplified as

$$S \leq \frac{A}{4}, \tag{2}$$

in which S and A are the entropy and area of the system. It is worth noting that this bound is not well defined in a strongly gravitational system since the area A is dependent on the choice of the spacelike region in the system and it can always be selected to be arbitrarily small by an almost null hypersurface. It was shown that this bound can be violated in the system for large volume [3].

To find a covariant version of the entropy bound, Bousso considered the entropy across a light sheet and proposed a covariant entropy bound, called the Bousso bound [2], which can be well formulated in arbitrarily curved spacetime. Consider a $(D-2)$-dimensional compact spacelike surface B with area $A(B)$. Let L be a null hypersurface generated by the null geodesics which starts at B and is orthogonal to B. Assume that the expansion of the null congruence is nonpositive (i.e., L is a light sheet) and L is not terminated until a caustic point is reached. Then, the entropy S_L passing through the light sheet L is bounded by the quarter of $A(B)$, i.e.,

$$S_L \leq \frac{A(B)}{4}. \tag{3}$$

This is the covariant bound proposed by Bousso and it is conjectured to be valid in any strongly gravitational system with arbitrary large regions. This bound is shown to hold in various cases [4–10] and it can be regarded as a formulation of holographic principles in spacetime.

Note that the above conjecture requires that the light sheet L ends at a caustic point. Flanagan *et al.* [9] extended this bound in which the light sheet can be terminated at another $(D-2)$-dimensional spatial surface B' before reaching a caustic. Then the entropy bound is modified as

$$S_L \leq \frac{1}{4}|A(B') - A(B)|, \tag{4}$$

in which $A(B')$ is the area of the spatial surface B'. This is called the generalized covariant entropy bound or

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generalized Bousso bound and it has been proved in Einstein gravity under some physical assumptions [9–11].

General relativity (GR) is not a complete theory of gravity due to the lack of a definitive quantum gravity theory and it can only be regarded as an effective theory in a certain region of scale. After considering the quantum effect or string modification, the higher-curvature terms are often added to Einstein-Hilbert action to modify the effective action of the gravitational theory [12–15]. In these cases, the Einstein gravity is perturbed by the higher-curvature terms. In this paper, we focus on the Lanczos-Lovelock gravity, which is the only natural generalization of Einstein gravity to higher-dimensional spacetime if we demand that the equations of motion are second-order differential equations of the metric [16,17]. Moreover, unlike most higher-curvature gravitational theories, Lanczos-Lovelock gravity is a ghost-free theory and admits a consistent initial value formulation [18,19]. As mentioned above, the generalized covariant entropy bound (4) is only valid in Einstein gravity. It is natural to ask whether the higher-curvature corrections can affect the entropy bound of the gravitational theory. Recently, Matsuda *et al.* [20] extended the generalized covariant entropy bound into the modified gravitational theory by replacing the quarter of area $A(B)/4$ with some appropriate black hole entropy $S_{\rm bh}(B)$, such as the Wald entropy [21,22] or Jacobson-Myers (JM) entropy [23]. Under two reasonable assumptions, they proved the entropy bound for Wald entropy in $f(R)$ gravity and canonical scalar-tensor theory. Moreover, they also showed that the bound using JM entropy holds for the GR branch of spherically symmetric configurations in Einstein-Gauss-Bonnet gravity. In the following, we would like to extend their discussion into the case where the Einstein gravity is perturbed by the higher-order Lovelock terms and show that the JM entropy can give a reasonable entropy bound in this theory.

The outline of this paper is as follows. In Sec. II, we briefly review the Lanczos-Lovelock gravity and discuss the features of Wald entropy and JM entropy. In Sec. III, we introduce the generalized entropy bound in Lanczos-Lovelock gravity and show the physical assumptions as well as the key point for proving this bound. In Sec. IV, we prove the generalized entropy bound in the theory where the Einstein gravity is perturbed by the higher-order Lovelock terms and show that the JM entropy strictly obeys the entropy bound under the perturbation level. Finally, the conclusion and discussion are presented in Sec. V.

II. LANCZOS-LOVELOCK GRAVITY

In this paper, we consider the Lanczos-Lovelock gravitational theory with some minimally coupled matter fields. The action of this theory in D-dimensional spacetime is given by

$$I=\frac{1}{16\pi}\int d^Dx\sqrt{g}\left(\sum_{k=0}^{k_{\rm max}}\frac{a_k}{2^k}\mathcal{L}^{(k)}+\mathcal{L}_{\rm mat}\right),\tag{5}$$

in which $\mathcal{L}_{\rm mat}$ is the Lagrangian density of the matter fields, g_{ab} is the Minkowski metric of the spacetime, and

$$\mathcal{L}^{(k)}=\delta^{a_1b_1\cdots a_kb_k}_{c_1d_1\cdots c_kd_k}R^{c_1d_1}_{a_1b_1}\cdots R^{c_kd_k}_{a_kb_k}\tag{6}$$

is the k-order Lovelock term. Here $k_{\rm max}=[(D-1)/2]$[1] and

$$\delta^{a_1b_1\cdots a_kb_k}_{c_1d_1\cdots c_kd_k}=(2k)!\delta^{[a_1}_{c_1}\delta^{b_1}_{d_1}\cdots\delta^{a_k}_{c_k}\delta^{b_k]}_{d_k}\tag{7}$$

is the generalized Kronecker tensor. The equation of motion is given by

$$E_{ab}=8\pi T_{ab},\tag{8}$$

in which T_{ab} is the stress-energy tensor of the matter fields, and

$$E_{ab}=-\sum_{k=0}^{k_{\rm max}}\frac{a_k}{2^{k+1}}\delta^{ba_1b_1\cdots a_kb_k}_{ac_1d_1\cdots c_kd_k}R^{c_1d_1}_{a_1b_1}\cdots R^{c_kd_k}_{a_kb_k}\tag{9}$$

is the generalized Einstein tensor of Lanczos-Lovelock gravity. Employing the Noether charge method of Iyer and Wald [21,22], the Wald entropy of Lanczos-Lovelock gravity can be obtained and it is given by

$$S_W=-2\pi\int_s d^{D-2}x\sqrt{\gamma}P^{abcd}\hat{\epsilon}_{ab}\hat{\epsilon}_{cd},\tag{10}$$

where we have denoted

$$P^{cd}_{ab}=\frac{1}{16\pi}\sum_{k=0}^{k_{\rm max}}\frac{ka_k}{2^k}\delta^{aba_2b_2\cdots a_kb_k}_{cdc_2d_2\cdots c_kd_k}R^{c_2d_2}_{a_2b_2}\cdots R^{c_kd_k}_{a_kb_k}.\tag{11}$$

Here s is a cross section of the event horizon, γ_{ab} is the induced metric on s, and $\hat{\epsilon}_{ab}$ is the binormal to s. The Wald entropy gives the correct first law in the stationary black holes. However, as discussed in Refs. [24–27], the Wald entropy of the Lanczos-Lovelock gravity does not obey the linearized second law and we need to focus on the JM entropy, i.e.,

$$S_{\rm JM}=\frac{1}{4}\int_s d^{D-2}x\sqrt{\gamma}\rho_{\rm JM}\tag{12}$$

with

$$\rho_{\rm JM}=\sum_{k=1}^{k_{\rm max}}\frac{ka_k}{2^{k-1}}\delta^{a_2b_2\cdots a_kb_k}_{c_2d_2\cdots c_kd_k}\hat{R}^{c_2d_2}_{a_2b_2}\cdots\hat{R}^{c_kd_k}_{a_kb_k},\tag{13}$$

in which $\hat{R}^{cd}_{ab}$ is the Riemann tensor of the induced metric γ_{ab} on the cross section s. In the stationary black hole, JM

[1]The square brackets $[x]$ denote the integer part of x.

entropy and Wald entropy give the same result, and therefore the JM entropy also obeys the first law. Considering the relationship between the generalized covariant entropy bound and the generalized second law of the black holes, it is natural to apply the JM entropy to discuss the entropy bound in the Lanczos-Lovelock gravity.

III. GENERALIZED COVARIANT ENTROPY BOUND

In this section, we first introduce the basic setups of the generalized covariant entropy bound in Lanczos-Lovelock gravity. Let L be a null hypersurface generalized by null geodesics, which starts at a compact $(D-2)$-dimensional spatial surface B_0 and ends at another compact $(D-2)$-dimensional spatial surface B_1. Let $k^a = (\partial/\partial u)^a$ be the tangent vector field of the null geodesics, in which u is an affine parameter of the null geodesics such that the spatial surfaces B_0 and B_1 are given by $u=0$ and $u=1$, separately. Any spatial surface B determined by the same u is called the cross section of the null hypersurface. Suppose that the expansion θ associated with k^a is nonpositive everywhere on L, i.e., $\theta \le$ on L. Then, we can choose (u, x) to a coordinate system on the null hypersurface L, in which $x = \{x^1, \cdots x^{D-2}\}$ denotes the coordinate of the cross section and every geodesic is determined by a constant x. Then, the covariant entropy bound in Lanczos-Lovelock gravity demands that the entropy S_L passing through the null hypersurface L should satisfy

$$S_L \le |S_{\mathrm{JM}}(B_0) - S_{\mathrm{JM}}(B_1)|, \tag{14}$$

in which $S_{\mathrm{JM}}(B)$ is evaluated by the JM entropy formula (12) on the cross-section B.

To prove the entropy bound, we first define the generalized expansion Θ of the JM entropy as the change of entropy per unit area, i.e.,

$$\frac{dS_{\mathrm{JM}}}{du} = \frac{1}{4}\int_B d^{D-2}x\sqrt{\gamma}\Theta. \tag{15}$$

Noting that the JM entropy is a purely spatial quantity in the $(D-2)$-dimensional slice B, i.e., it is determined by the induced metric γ_{ab}, we can regard S_{JM} as an action on the cross-section B. Then, the Lie-derivative $\mathcal{L}_k = \partial_u$ can be seen as a variation on S_{JM}. After assuming that B is compact and dropping the surface terms in $\partial_u S_{\mathrm{JM}}$, we can get

$$\begin{aligned}\partial_u S_{\mathrm{JM}} &= -\frac{1}{4}\sum_{k=1}^{k_{\max}} k a_k \int_B d^{D-2}x\sqrt{\gamma}[\hat{E}^{(k-1)}]^{ab}\partial_u\gamma_{ab} \\ &= -\frac{1}{2}\sum_{k=1}^{k_{\max}} k a_k \int_B d^{D-2}x\sqrt{\gamma}[\hat{E}^{(k-1)}]^{ab}K_{ab}\end{aligned} \tag{16}$$

in which

$$[\hat{E}^{(k)}]_a^b = -\frac{1}{2^{k+1}}\delta^{ba_1b_1\cdots a_kb_k}_{ac_1d_1\cdots c_kd_k}\hat{R}^{c_1d_1}_{a_1b_1}\cdots\hat{R}^{c_kd_k}_{a_kb_k}, \tag{17}$$

and

$$K_{ab} = \frac{1}{2}\partial_u\gamma_{ab} \tag{18}$$

is the extrinsic curvature associated with k^a. These results imply that

$$\Theta = K_b^a\sum_{k=1}^{k_{\max}}\frac{k a_k}{2^{k-1}}\delta^{ba_2b_2\cdots a_kb_k}_{ac_2d_2\cdots c_kd_k}\hat{R}^{c_2d_2}_{a_2b_2}\cdots\hat{R}^{c_kd_k}_{a_kb_k} \tag{19}$$

after neglecting the total-derivative terms.

Choose u to be an affine parameter of the null geodesics. Using the equation of motion, we can write the change of Θ as

$$\frac{d\Theta}{du} = -8\pi\mathcal{T} + \mathcal{F}, \tag{20}$$

in which

$$\begin{aligned}\mathcal{T} &= T_{ab}k^ak^b, \\ \mathcal{F} &= E_{ab}k^ak^b + k^a\nabla_a\Theta.\end{aligned} \tag{21}$$

This can be regarded as the Raychaudhuri equation in Lanczos-Lovelock gravity.

In the thermodynamic limit, there exists an entropy flux vector field s^a such that the entropy passing through the null hypersurface L can be written as

$$S_L = \int_L d^{D-2}x du\sqrt{\gamma}s \tag{22}$$

with the entropy density

$$s = -k_a s^a. \tag{23}$$

Analogies to the assumptions in Einstein gravity [10,11], Ref. [20] made two following assumptions in the modified gravitational theories,

$$\begin{aligned}&\text{(i)}\quad \partial_u s(x,u) \le 2\pi\mathcal{T}(x,u), \\ &\text{(ii)}\quad s(x,0) \le -\frac{1}{4}\Theta(x,0)\end{aligned} \tag{24}$$

on the null hypersurface L. The first assumption is from the requirement that the change rate of the entropy flux is not large than the energy flux and it can also be regarded as the consequence of the version of Bekenstein bound [20]. The second assumption is just an initial choice of the hypersurface such that the entropy bound is valid at the beginning of L.

With the above setups and assumptions, it is not hard to get

$$s(x,u) = s(x,0) + \int_0^u du\partial_u s(x,u) \\ \leq s(x,0) + 2\pi \int_0^u du\mathcal{T}(x,u), \tag{25}$$

in which we have used the first assumption at the last step. Then, using Eq. (20) and together with the second assumption in Eq. (24), we have

$$s(x,u) \leq s(x,0) - \frac{1}{4}\Theta(x,\lambda) + \frac{1}{4}\Theta(x,0) + \frac{1}{4}\int_0^u d\tilde{u}\mathcal{F}(x,\tilde{u}) \\ \leq -\frac{1}{4}\Theta(x,\lambda) + \frac{1}{4}\int_0^u d\tilde{u}\mathcal{F}(x,\tilde{u}). \tag{26}$$

Finally, after integrating the above identity over L, we have

$$S_L \leq S_{\text{JM}}(B_0) - S_{\text{JM}}(B_1) \\ + \frac{1}{4}\int_0^1 du \int_0^u d\tilde{u} \int d^{D-2}x\sqrt{\gamma(u)}\mathcal{F}(\tilde{u},x), \\ = S_{\text{JM}}(B_0) - S_{\text{JM}}(B_1) + \frac{1}{4}\int_0^1 du \int_0^u d\tilde{u}F(\tilde{u},u), \tag{27}$$

in which we have denoted

$$F(\tilde{u},u) = \int d^{D-2}x\sqrt{\gamma(u)}\mathcal{F}(\tilde{u},x). \tag{28}$$

From the above results, we can see that the key point to examining the generalized covariant entropy bound is to judge the sign of $F(\tilde{u},u)$. If we have $F(\tilde{u},u) \leq 0$, the inequality (27) reduces to

$$S_L \leq S_{\text{JM}}(B_0) - S_{\text{JM}}(B_1), \tag{29}$$

which is the entropy bound given by Eq. (14). For the Einstein gravity, Eq. (20) is just the Raychaudhuri equation and we have $\mathcal{F}(\tilde{u},x) \leq 0$, which gives the proof of the generalized covariant entropy bound in Einstein gravity. In the following, we would like to judge the sign of F in the Lanczos-Lovelock gravity.

IV. PROOF OF THE ENTROPY BOUND WITH HIGHER-CURVATURE CORRECTIONS

From the perspective of quantum corrections and string theory, it is natural to consider the models of gravity where the Einstein gravity is perturbed by higher curvature terms. Therefore, in the following, we consider the Lanczos-Lovelock gravity where the higher-order Lovelock terms are treated as small corrections to the Einstein gravity, i.e., we consider the Lovelock theory with $a_0 = -2\Lambda$, $a_1 = 1$ and $a_k = \lambda\alpha_k$ for $k \geq 2$, in which λ is a small quantity which describes the perturbation from the higher-curvature terms. From the perspective of effective field theory, the higher-order coupling constant α_k with $k \geq 3$ is also a small quantity to be proportional to λ^{k-2}. Since the order of α_k does not affect our following analysis, we will not express them concretely. Then, we have

$$H_a^b = G_a^b - \Lambda\delta_a^b - \lambda\sum_{k=2}^{k_{\max}} \frac{\alpha_k}{2^{k+1}} \delta^{ba_1b_1\cdots a_kb_k}_{ac_1d_1\cdots c_kd_k} R^{c_1d_1}_{a_1b_1}\cdots R^{c_kd_k}_{a_kb_k}, \tag{30}$$

and

$$\rho_{\text{JM}} = 1 + \lambda\rho \tag{31}$$

with

$$\rho = \sum_{k=2}^{k_{\max}} \frac{k\alpha_k}{2^{k-1}} \delta^{a_2b_2\cdots a_kb_k}_{c_2d_2\cdots c_kd_k} \hat{R}^{c_2d_2}_{a_2b_2}\cdots \hat{R}^{c_kd_k}_{a_kb_k}. \tag{32}$$

After considering the higher-curvature corrections, the solution in the theory will depend on the small parameter λ, i.e., $g_{ab}(\lambda)$, in which $\lambda = 0$ describes the solution of Einstein gravity.

To evaluate $\mathcal{F}(x,u,\lambda)$ on the null hypersurface L, we introduce the Gaussian null coordinate system $\{z,u,x\}$, in which the line element can be expressed as

$$ds^2(\lambda) = 2(dz + z^2\alpha du + z\beta_i dx^i)du + \gamma_{ij}dx^i dx^j, \tag{33}$$

in which the null hypersurface L is given by $z = 0$, and α, β_i, and γ_{ij} are the function of u, z, x, λ. Here the index i, j, k, l denotes the coordinate of the cross-section B. The null generator of L is given by $k^a = (\partial/\partial u)^a$. Using this line element, the nonvanishing component of the Christoffel symbol on L can be further obtained:

$$\Gamma^k{}_{ij} = \hat{\Gamma}^k{}_{ij}, \quad \Gamma^j{}_{ui} = K^j_i, \quad \Gamma^j{}_{zi} = \bar{K}^j_i, \quad \Gamma^1{}_{uz} = \frac{1}{2}\beta^i, \\ \Gamma^u{}_{ij} = -\bar{K}_{ij}, \quad \Gamma^u{}_{ui} = -\frac{1}{2}\beta_i, \quad \Gamma^z{}_{ij} = -\bar{K}_{ij}, \quad \Gamma^z{}_{zi} = \frac{1}{2}\beta_i, \tag{34}$$

in which $\hat{\Gamma}^k{}_{ij}$ is the Christoffel symbol of the induced metric γ_{ij}, and

$$K_{ij} = \frac{1}{2}\partial_u\gamma_{ij}, \qquad \bar{K}_{ij} = \frac{1}{2}\partial_z\gamma_{ij} \tag{35}$$

are the extrinsic curvature associated with the null vectors $(\partial/\partial u)^a$ and $(\partial/\partial z)^a$ separately. Further calculation gives

$$R^{kl}_{ij} = \hat{R}^{kl}_{ij} - 4K^{[k}_{[i}\bar{K}^{l]}_{j]}, \qquad R^{zj}_{ui} = -\partial_u K^j_i - K^k_i K^j_k,$$
$$R^{jk}_{ui} = -2D^{[j}K^{k]}_i + K^{[j}_i\beta^{k]}, \qquad R^{zi}_{jk} = -2D_{[j}K^i_{k]} + K^i_{[j}\beta_{k]} \tag{36}$$

on the hypersurface L. Using the above results and considering the symmetry of the generalized Kronecker tensor, it is not difficult to get

$$E_{ab}k^a k^b = R^z_u + \lambda \sum_{k=2}^{k_{\max}} \alpha_k [E^{(k)}]^z_u, \tag{37}$$

on the null hypersurface L. Considering the antisymmetry of the generalized Kronecker tensor and using Eq. (36), it is not hard to get

$$[E^{(k)}]^z_u = \frac{k}{2^{k-1}} R^{zj}_{ui}\delta^{ji_2j_2\cdots i_kj_k}_{il_2m_2\cdots l_km_k} R^{l_2m_2}_{i_2j_2}\cdots R^{l_km_k}_{i_kj_k} + \frac{k(k-1)}{2^{k-1}} R^{l_1m_1}_{uj_1} R^{zm_2}_{i_2j_2}\delta^{j_1i_2j_2i_3j_3\cdots i_kj_k}_{l_1m_1m_2l_3m_3\cdots l_km_k} R^{l_3m_3}_{i_3j_3}\cdots R^{l_km_k}_{i_kj_k} \tag{38}$$

for $k \geq 2$.

Then, using the result

$$\partial_u \hat{R}^{cd}_{ab} = K^{[c}_e R^{d]e}_{ab} - 2D_{[a}D^{[c}K^{d]}_{b]}, \tag{39}$$

and together with Eq. (19), we can further obtain

$$\partial_u\Theta = \partial_u\theta + \lambda\partial_u K^a_b \sum_{k=2}^{k_{\max}} \frac{k\alpha_k}{2^{k-1}} \delta^{ba_2b_2\cdots a_kb_k}_{ac_2d_2\cdots c_kd_k} \hat{R}^{c_2d_2}_{a_2b_2}\cdots \hat{R}^{c_kd_k}_{a_kb_k} + \lambda K^a_b \partial_u \hat{R}^{c_2d_2}_{a_2b_2} \sum_{k=2}^{k_{\max}} \frac{k(k-1)\alpha_k}{2^{k-1}} \times \delta^{ba_2b_2a_3b_3\cdots a_kb_k}_{ac_2d_2c_3d_3\cdots c_kd_k} \hat{R}^{c_3d_3}_{a_3b_3}\cdots \hat{R}^{c_kd_k}_{a_kb_k}. \tag{40}$$

Here we denote $\partial_u = \mathcal{L}_k$. Combing the above results, we have

$$\mathcal{F} = -K^b_a K^a_b + \lambda\mathcal{F}_2 \tag{41}$$

with

$$\mathcal{F}_2 = (\hat{H}^b_a - H^b_a)\partial_u K^a_b - K^c_b K^a_c H^b_a + (2D^d K^e_a - K^d_a\beta^e)(2D_b K^f_c - K^f_b\beta_c)P^{abc}_{def} + K^d_a(K^e_{\tilde{e}} R^{f\tilde{e}}_{bc} - 2D_b D^e K^f_c)\hat{P}^{abc}_{def}, \tag{42}$$

in which we have denoted

$$H^b_a = \sum_{k=2}^{k_{\max}} \frac{k\alpha_k}{2^{k-1}} \hat{\delta}^{ba_2b_2\cdots a_kb_k}_{ac_2d_2\cdots c_kd_k} R^{c_2d_2}_{a_2b_2}\cdots R^{c_kd_k}_{a_kb_k},$$
$$\hat{H}^b_a = \sum_{k=2}^{k_{\max}} \frac{k\alpha_k}{2^{k-1}} \hat{\delta}^{ba_2b_2\cdots a_kb_k}_{ac_2d_2\cdots c_kd_k} \hat{R}^{c_2d_2}_{a_2b_2}\cdots \hat{R}^{c_kd_k}_{a_kb_k},$$
$$P^{def}_{abc} = \sum_{k=2}^{k_{\max}} \frac{k(k-1)\alpha_k}{2^{k-1}} \hat{\delta}^{defa_cb_d\cdots a_kb_k}_{abcc_3d_3\cdots c_kd_k} R^{c_3d_3}_{a_3b_3}\cdots R^{c_kd_k}_{a_kb_k},$$
$$\hat{P}^{def}_{abc} = \sum_{k=2}^{k_{\max}} \frac{k(k-1)\alpha_k}{2^{k-1}} \hat{\delta}^{defa_cb_d\cdots a_kb_k}_{abcc_3d_3\cdots c_kd_k} \hat{R}^{c_3d_3}_{a_3b_3}\cdots \hat{R}^{c_kd_k}_{a_kb_k}, \tag{43}$$

in which

$$\hat{\delta}^{b_1\cdots b_i}_{a_1\cdots a_i} = i!\gamma^{b_1}_{[a_1}\cdots\gamma^{b_i}_{a_i]} \tag{44}$$

is the ith-order generalized Kronecker tensor on the cross-section B.

In the following, we would like to judge the sign of $F(\lambda)$ when the coupling constant λ is regarded as a small parameter. If we consider the solution $g_{ab}(\lambda)$ which is an analytic function of λ, then we can expand $\mathcal{F}(\lambda)$ by λ,

$$\mathcal{F}(\lambda) = \mathcal{F} + \lambda\delta\mathcal{F} + \frac{\lambda^2}{2}\delta^2\mathcal{F} + \cdots, \tag{45}$$

in which we have introduced the notation

$$\delta^i\eta(x) = \left.\frac{\partial^i\eta(x,\lambda)}{\partial\lambda^i}\right|_{\lambda=0} \tag{46}$$

to denote the ith-order variation of the quantity $\eta(x,\lambda)$, and the symbol without λ denotes its counterpart of $\lambda = 0$. In the following, we would like to analyze the sign of the integration of $\mathcal{F}(\lambda)$, i.e.,

$$F = \int_B dV\mathcal{A}\mathcal{F} \tag{47}$$

with $dV = d^{D-2}x\sqrt{\gamma}$ and

$$\mathcal{A} = \sqrt{\gamma(u)/\gamma(\tilde{u})} = \exp\left[\int_{\tilde{u}}^{u} \theta(\tilde{\tilde{u}})d\tilde{\tilde{u}}\right]. \tag{48}$$

From Eq. (27), we can see that the entropy bound is satisfied if $F \leq 0$. For the sake of simplicity, we define the operation "$\hat{=}$" as

$$Y \hat{=} \int_B dV\mathcal{A}Y. \tag{49}$$

In the following, we would like to analyze the sign of $F(\lambda)$ after assuming that the hypersurface L is smooth under the perturbation limit $\lambda \to 0$, i.e., we assume that K_i^j, θ and their derivatives along L are finite as $\lambda \to 0$.

A. Zeroth-order approximation

First, we consider the zeroth-order approximation of λ. From Eq. (41), we can obtain

$$\mathcal{F}(\lambda) = -K_b^a K_a^b + \mathcal{O}(\lambda). \tag{50}$$

Considering the fact that K_{ab} is a spatial tensor on B, we have $K_{ab}K^{ab} \geq 0$ and, therefore,

$$\mathcal{F}(\lambda) = -K_b^a K_a^b \leq 0 \tag{51}$$

under the zeroth-order approximation,[2] in which we neglect the first-order term of λ. This implies that the covariant entropy bound is satisfied under the zeroth-order approximation. This result is straightforward because the theory under the zeroth-order approximation is just the Einstein gravity. Assume

$$|K|\mathcal{A} \equiv \sqrt{\int_B dV \mathcal{A} K_a^b K_b^a} \propto \lambda^s \tag{52}$$

with $s \geq 0$. From Eq. (50), whether a first-order approximation of $F(\lambda)$ needs to be considered depends on the order of $|K|_\mathcal{A}$. When $s < 1/2$, the dominate term of $F(\lambda)$ is given by $-|K|_\mathcal{A}^2$ and therefore we have $F(\lambda) < 0$ even if $\mathcal{O}(\lambda)$ is taken into account. That is to say, only if $s \geq 1/2$ do we need to consider the first-order approximation.

Define the nth-order optimal condition by vanishing the nth-order approximation of $F(\lambda)$. Then, the second-order term needs to be taken into consideration only under the zeroth-order optimal condition

$$|K|_\mathcal{A} \propto \lambda^{s_0}, \tag{53}$$

with $s_0 \geq 1/2$ on the light sheet L. In this case, the extrinsic curvature of the light sheet is a small quantity. As the example of this light sheet, it can be chosen as a null hypersurface near the Killing horizons in a stationary black hole or an event horizon for a dynamical black hole which is a perturbation of a stationary one.

B. First-order approximation

In the following, we would like to discuss the first-order approximation of $F(\lambda)$ under the zeroth-order optimal condition

$$|K_\mathcal{A}| \propto \lambda^{s_0}. \tag{54}$$

[2] In this paper, we define the kth-order approximation as the result of ignoring the $(k+1)$th-order term $\mathcal{O}(\lambda^{k+1})$.

The first-order variation of $\mathcal{F}(\lambda)$ gives

$$\delta\mathcal{F} = -2K_b^a \delta K_a^b + \mathcal{F}_2. \tag{55}$$

From Eq. (42), $\mathcal{F}_2$ can be schematically expressed as

$$\mathcal{F}_2 = (\hat{H} - H)\partial_u K + C_1 KK + C_2 K\partial_x K + C_3 \partial_x K \partial_x K + C_4 K \partial_x^2 K, \tag{56}$$

in which we neglect the indexes on K_a^b, H_a^b, $\hat{H}_a^b$, C_1, C_2, and C_3, and $D_a^m K_b^c$ is denoted by $\partial_x^m K$. From Eq. (36), it is not hard to see

$$R_{ij}^{kl} - \hat{R}_{ij}^{kl} = \mathcal{O}(K) \tag{57}$$

under the first-order optimal condition. This implies that

$$H - \hat{H} = \mathcal{O}(K). \tag{58}$$

Then, under the zeroth-order optimal condition, we have

$$\mathcal{F}_2 = C_3 \partial_x K \partial_x K + \mathcal{O}(K). \tag{59}$$

Then, we have

$$\mathcal{F}(\lambda) = -K_a^b K_b^a + \lambda C_3 \partial_x K \partial_x K + \lambda \mathcal{O}(K). \tag{60}$$

From Eq. (48) and noting the assumption that θ is finite as $\lambda \to 0$, it is hard to see that the minimal value $\mathcal{A}_{\min}$ of $\mathcal{A}$ on B is a zeroth-order term of λ. Then, we have

$$|K|^2 = \int_B dV K_a^b K_b^a \leq \frac{1}{\mathcal{A}_{\min}} |K_\mathcal{A}|^2 \propto \lambda^{2s_0}, \tag{61}$$

which means

$$|K| = \mathcal{O}(\lambda^{s_0}). \tag{62}$$

Here we define the length of the spatial tensor $Y_{a_1\cdots a_m}$ by

$$|Y| = \sqrt{\int_B dV Y_{a_1\cdots a_m} Y^{a_1\cdots a_m}}. \tag{63}$$

Under the assumption that K_{ij} and its derivatives are all finite, it has been proven in the Appendix that

$$|\partial_x^m K| = \mathcal{O}(\lambda^{s_0}). \tag{64}$$

For the last term of Eq. (60) and using the Cauchy-Schwarz inequality (A4), we can further get

$$|\mathcal{O}(K)| \hat{=} \left| \int_B dV X_a^b K_a^b \right| \leq |X||K|. \tag{65}$$

Noting that $|X|$ is finite as $\lambda \to 0$, we have

$$\mathcal{O}(K) \hat{=} \int_B dV X_a^b K_a^b = \mathcal{O}(\lambda^{s_0}). \tag{66}$$

For the second term of Eq. (60), using the decomposition

$$\begin{aligned}\mathcal{A} C_3 \partial_x K \partial_x K &= X^{a_1 a_2 a_3 b_1 b_2 b_3} D_{a_1} K_{a_2 a_3} D_{b_1} K_{b_2 b_3} \\ &= X^{a_1 a_2 a_3 b_1 b_2 b_3} D^{c_1} K^{c_2 c_3} g_{a_1 c_1} g_{a_2 c_2} g_{a_3 c_3} D_{b_1} K_{b_2 b_3}\end{aligned} \tag{67}$$

and together with the Cauchy-Schwarz inequality (A4) as well as Eq. (64), we have

$$\begin{aligned}\left| \int_B dV \mathcal{A} C_3 \partial_x K \partial_x K \right| &\leq \sqrt{\int_B dV \kappa (D_a K_{bc})(D^a K^{bc})} \\ &\quad \times \sqrt{(D-2)^3} |\partial_x K| \\ &\leq \sqrt{(D-2)^3 \kappa_{\max}} |\partial_x K|^2 \\ &= \mathcal{O}(\lambda^{2s_0}),\end{aligned} \tag{68}$$

in which we denote $\kappa = X^{a_1 a_2 a_3 b_1 b_2 b_3} X_{a_1 a_2 a_3 b_1 b_2 b_3}$ and $\kappa_{\max}$ is the maximal value of κ. Then, we have

$$C_3 \partial_x K \partial_x K \hat{=} \int_B dV \mathcal{A} C_3 \partial_x K \partial_x K = \mathcal{O}(\lambda^{2s_0}). \tag{69}$$

With a similar calculation, it is not hard to show

$$C \partial_x^m K \partial_x^n K \hat{=} \mathcal{O}(\lambda^{2s_0}) \tag{70}$$

for any m and n.

Using the above results, we can further obtain

$$\mathcal{F}(\lambda) \hat{=} F(\lambda) = -|K|_{\mathcal{A}}^2 + \mathcal{O}(\lambda^{s_0+1}) \tag{71}$$

under the first-order approximation, in which we neglect the second-order term $\mathcal{O}(\lambda^2)$. Then, we can see that if $s_0 < 1$, the dominate term is given by $-|K|_{\mathcal{A}}^2 \propto \lambda^{2s_0}$ and we have $F(\lambda) < 0$ under the first-order approximation. When $s_0 \geq 1$, we have $F(\lambda) = 0$ under the first-order approximation. This indicates that the generalized covariant entropy bound is satisfied under the first-order approximation of λ.

For the case with $s_0 < 1$, we can see that the leading term of $F(\lambda)$ is always given by $-|K|_{\mathcal{A}}^2$ and therefore we have $F(\lambda) < 0$ even when the higher-order approximation is taken into account. Then, the second-order approximation is required only if the first-order optimal condition is satisfied, i.e.,

$$|K|_{\mathcal{A}} \propto \lambda^{s_1} \tag{72}$$

with $s_1 \geq 1$.

C. Second-order approximation

In this subsection, we evaluate the second-order approximation of $F(\lambda)$ under the first-order optimal condition (72).

For the first term of $\mathcal{F}(\lambda)$, we have

$$\begin{aligned}K_a^b(\lambda) K_b^a(\lambda) &= [K_a^b + \lambda \delta K_a^b + \mathcal{O}(\lambda^2)][K_a^b + \lambda \delta K_a^b + \mathcal{O}(\lambda^2)] \\ &= (K_a^b + \lambda \delta K_a^b)(K_a^b + \lambda \delta K_a^b) + \lambda^2 \mathcal{O}(K) + \mathcal{O}(\lambda^3).\end{aligned} \tag{73}$$

From Eq. (66), we can get

$$\lambda^2 \mathcal{O}(K) \hat{=} \mathcal{O}(\lambda^{2+s_1}). \tag{74}$$

Considering the first-order optimal condition $s_1 \geq 1$, we can further obtain

$$K_a^b(\lambda) K_b^a(\lambda) \hat{=} |K_a^b + \lambda \delta K_a^b|_{\mathcal{A}}^2 \tag{75}$$

under the second-order approximation of λ, i.e., we neglect the third-order term $\mathcal{O}(\lambda^3)$.

To simplify, we define

$$|K + \lambda \delta K|_{\mathcal{A}} \propto \lambda^{\tilde{s}_1}. \tag{76}$$

Considering the first-order optimal condition, we have $\tilde{s}_1 \geq 1$. When $\tilde{s}_1 \leq 2$, we have

$$|K_a^b(\lambda)|_{\mathcal{A}} \propto \lambda^{\tilde{s}_1}. \tag{77}$$

From Eq. (56), we have

$$\mathcal{F}_2(\lambda) = [\hat{H}(\lambda) - H(\lambda)] \partial_u K(\lambda) + C \partial_x^m K(\lambda) \partial_x^n K(\lambda). \tag{78}$$

Using Eq. (43), we have

$$(\hat{H} - H) \partial_u K = \partial_u (XKK) + \mathcal{O}(K^2) \tag{79}$$

Note that the final expression (27) in the entropy bound is the integration of F. Therefore, we consider an integral of the first term of the above equation,

$$\int_0^u \partial_{\tilde{u}} (XKK) = XKK|_0^u = \mathcal{O}(K^2) \hat{=} \mathcal{O}(\lambda^{2\tilde{s}_1}), \tag{80}$$

which means that the first term on the right-hand side of Eq. (78) only contributes a higher-order term under the second-order approximation. In Eqs. (79) and (80), we have neglected the parameter λ for simplify. For the second term of Eq. (78), using Eq. (70), we have

$$C\partial_x^m K(\lambda)\partial_x^n K(\lambda) \hat{=} \mathcal{O}(\lambda^{2\tilde{s}_1}). \tag{81}$$

Combining the above results, we have

$$F(\lambda) = -|K + \lambda\delta K|_{\mathcal{A}}^2 \leq 0 \tag{82}$$

under the second-order approximation. When $\tilde{s}_1 > 2$, we have $K(\lambda) = \mathcal{O}(\lambda^2)$. In this case, the calculation is same as $\tilde{s}_1 \leq 2$ and finally we can get $F(\lambda) = 0$ under the second-order approximation. These results show that the entropy bound is satisfied under the second-order approximation. Similarly, it is not hard to see that when $\tilde{s}_1 < 3/2$, the higher-order approximation of $F(\lambda)$ is also nonpositive, i.e., the third-order approximation only needs to be considered in the second-order optimal condition,

$$|K + \lambda\delta K|_{\mathcal{A}} \propto \lambda^{s_2}, \tag{83}$$

with $s_2 \geq 3/2$ such that the second-order approximation of $\mathcal{F}(\lambda)$ vanishes and we need to consider the third-order approximation of λ.

D. nth-order approximation

In the following, we would like to prove that $\mathcal{F}(\lambda)$ is always nonpositive under the nth-order approximation when the $(n-1)$th-order optimal conditions are based on the mathematical induction. By concluding the first two order results, it is equivalent to proving the following proposition: under the nth-order optimal condition

$$\left|\sum_{i=0}^{[n/2]} \frac{\lambda^i}{i!}\delta^i K\right|_{\mathcal{A}} \propto \lambda^{s_n} \tag{84}$$

with

$$s_n \geq \frac{n+1}{2}, \tag{85}$$

the $(n+1)$th-order approximation of $F(\lambda)$ is nonpositive, i.e.,

$$F(\lambda) \leq 0 \tag{86}$$

after neglecting $\mathcal{O}(\lambda^{n+2})$. Then, a saturation of this inequality demands the $(n+1)$th-order optimal condition and the $(m+1)$th-order optimal condition needs to be considered only when the $(n+1)$th-order optimal condition is satisfied.

Proof.—Obviously, the proposition is satisfied for $n = 0, 1, 2$.

(*Case of* $n = 2m$.) When $n = 2m$, the $(2m)$th-order optimal condition is given by

$$\left|\sum_{i=0}^{m} \frac{\lambda^i}{i!}\delta^i K\right|_{\mathcal{A}} \propto \lambda^{s_n} \tag{87}$$

with $s_n \geq m + 1/2$. Then, we have

$$K_a^b(\lambda)K_b^a(\lambda) = \left(\sum_{i=0}^{m}\frac{\lambda^i}{i!}\delta^i K_a^b\right)\left(\sum_{i=0}^{m}\frac{\lambda^i}{i!}\delta^i K_b^a\right) + \lambda^{m+1}\mathcal{O}(K) + \mathcal{O}(\lambda^{2m+2}). \tag{88}$$

From Eq. (66), we can get

$$\lambda^{m+1}\mathcal{O}(K) \hat{=} \mathcal{O}(\lambda^{s_n+m+1}). \tag{89}$$

Then, we have

$$|K(\lambda)|_{\mathcal{A}}^2 = \left|\sum_{i=0}^{m}\frac{\lambda^i}{i!}\delta^i K\right|_{\mathcal{A}}^2 \tag{90}$$

under the $(2m+1)$th-order approximation of λ.

With the same calculation as the second-order approximation, it is straightforward to show

$$\mathcal{F}_2(\lambda) \hat{=} \mathcal{O}(\lambda^{2s_n}) = \mathcal{O}(\lambda^{2m+1}), \tag{91}$$

which gives

$$F(\lambda) = -\left|\sum_{i=0}^{m}\frac{\lambda^i}{i!}\delta^i K\right|_{\mathcal{A}}^2 + \mathcal{O}(\lambda^{s_n+m+1}) \tag{92}$$

under the $(2m+1)$th-order approximation, in which the $O(\lambda^{2m+2})$ terms are neglected. When $s_n < m+1$, the leading term is given by

$$-\left|\sum_{i=0}^{m}\frac{\lambda^i}{i!}\delta^i K\right|_{\mathcal{A}}^2 \tag{93}$$

and thus we have $F(\lambda) < 0$ under the $(2m+1)$th-order approximation. For the case with $s_n \geq m+1$, we have $F(\lambda) = 0$ under the $(2m+1)$th-order approximation. These imply that $F(\lambda) \leq 0$ under the $(2m+1)$th-order approximation.

When $s_{2m+1} < m+1$, we can see that $F(\lambda) < 0$ even under the higher-order approximation of λ. That is to say, the $(2m+2)$th-order term needs to be taken into account only when the $(2m+1)$th-order optimal condition is satisfied, i.e.,

$$\left|\sum_{i=0}^{m}\frac{\lambda^i}{i!}\delta^i K\right|_{\mathcal{A}} \propto \lambda^{s_{2m+1}} \tag{94}$$

with $s_{2m+1} \geq m+1$. These show that the validity of the proposition with $n = 2m$.

(*Case of* $n = 2m+1$.) When $n = 2m+1$, we define

$$\left|\sum_{i=0}^{m}\frac{\lambda^i}{i!}\delta^i K\right|_{\mathcal{A}} \propto \lambda^{\tilde{s}_{2m+1}}. \tag{95}$$

When $s_{2m+1} \leq m+2$, the $(2m+1)$th-order optimal condition (94) implies $\tilde{s}_{2m+1} \geq m+1$. Then, with the same calculation as second-order approximation, it is not hard to obtain

$$F(\lambda) = -\left|\sum_{i=0}^{m+1} \frac{\lambda^i}{i!}\delta^i K\right|_{\mathcal{A}}^2 \leq 0 \tag{96}$$

under the $(2m+2)$th-order approximation. A saturation of this inequality demands the $(2m+2)$th-order optimal condition

$$\left|\sum_{i=0}^{m+1} \frac{\lambda^i}{i!}\delta^i K\right|_{\mathcal{A}} \propto \lambda^{s_{2m+2}} \tag{97}$$

with $s_{2m+2} \geq m+3/2$. This is actually the proposition with $n = 2m+1$, i.e., we have completed the proof. ∎

The above result shows that the generalized covariant entropy bound associated with the JM entropy is valid under any higher-order approximation of λ.

V. CONCLUSION AND DISCUSSION

In this paper, we consider the generalized covariant entropy bound for the theory in which the Einstein gravity is perturbed by the higher-order Lovelock terms and introduce a small parameter λ to characterize these perturbations. After considering the linearized second law of black holes in Lanczos-Lovelock gravity, the entropy bound in this theory is naturally proposed by replacing the Bekenstein-Hawking entropy with the JM entropy. Then, we showed that the key point to examine the validity of covariant entropy bound is to judge the sign of the quantity $F(\lambda)$, and the entropy bound is satisfied if $F(\lambda) \leq 0$. After assuming two physical assumptions and that the metric g_{ab} is an analytic function of λ, we illustrate that the dominant term of $F(\lambda)$ is always nonpositive based on the mathematical induction, i.e., the generalized covariant entropy bound is valid under any higher-order approximation of λ. This indicates that the entropy bound using the JM entropy is strictly satisfied under the perturbation level of the higher-order Lovelock terms. From the discussion in Sec. V D of Ref. [20], we can see that the above result also indicates the validity of the generalized second law under the higher-order approximation of λ for the theory where the Einstein gravity is perturbed by the higher-order Lovelock terms, this is a different result from the linearized second law of Lanczos-Lovelock gravity.

From the calculations presented in this paper, it is not hard to check that if we replace the Bekenstein-Hawking with the Wald entropy formula instead of the JM entropy formula in the entropy bound, we cannot show the nonpositive of $\mathcal{F}$ only using the assumptions given by the paper. This implies that the covariant entropy bound might be used to select the black hole entropy of the gravitational theory. Moreover, it is worth noting that our result is only suitable for the case where the higher-order Lovelock terms are regarded as some small corrections to Einstein gravity, and not for the nonperturbation cases. From the discussion in Sec. III, the key point to examine the entropy bound is also to check the sign of $\mathcal{F}$ given by Eq. (41). However, due to the complexity of the expression, it is difficult for us to judge its sign directly only based on the setups and assumptions in our paper. One of our future works is going to consider these cases. Furthermore, our strategy can also be applied in the theory in which the Einstein gravity is perturbed by other higher-curvature terms. For most higher-curvature theories, since there exist higher-order field equations and the entropy of black holes cannot be expressed as a purely spatial quantity on the cross-section B, it is hard to simplify the expression of $\mathcal{F}$ like the Lanczos-Lovelock gravity. We will leave that for future studies.

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APPENDIX: THE ORDER FOR THE DERIVATIVE OF EXTRINSIC CURVATURE

In this paper, we only consider the cases where the null hypersurface L is located in a finite region away from infinity and singularity. Assuming that the hypersurface L is smooth, K_i^j and its derivatives along L should be finite, i.e., we demand

$$\int_B dV (D_{a_1\cdots a_m} K_a^b)(D^{a_1\cdots a_m} K_b^a) \propto \lambda^{2\chi_m} \tag{A1}$$

with $\chi_m \geq 0$, in which we denote $dV = d^{D-2}x\sqrt{\gamma}$ and

$$D_{a_1\cdots a_n} = D_{a_1}\cdots D_{a_n}. \tag{A2}$$

Considering the assumption that B is compact and using the Gauss law, we can get

$$\int_B dV (D_{a_1\cdots a_m} K_a^b)(D^{a_1\cdots a_m} K_b^a) = \int_B dV (D^{a_1} D_{a_1 a_2\cdots a_m} K_a^b)(D^{a_2\cdots a_m} K_b^a). \tag{A3}$$

Using the Cauchy-Schwarz inequality

$$\int_B dV\eta^{a_1\cdots a_m}\zeta_{a_1\cdots a_m} \le \sqrt{\int_B dV\eta^{a_1\cdots a_m}\eta_{a_1\cdots a_m}\int_B dV\zeta^{a_1\cdots a_m}\zeta_{a_1\cdots a_m}} \quad (A4)$$

for any spatial tensor ξ and ζ, it is not hard to obtain

$$\int_B dV(D^{a_1}D_{a_1a_2\cdots a_m}K_a^b)(D^{a_2\cdots a_m}K_b^a) \le \sqrt{\int_B dV(D_{a_1\cdots a_{m+1}}K_a^b)(D^{a_1\cdots a_{m+1}}K_b^a)} \times \sqrt{(D-2)\int_B dV(D_{a_2\cdots a_m}K_a^b)(D^{a_2\cdots a_m}K_b^a)} \propto \lambda^{s_{m+1}+s_{m-1}}, \quad (A5)$$

which implies

$$\lambda^{2\chi_m} = \mathcal{O}(\lambda^{\chi_{m+1}+\chi_{m-1}}). \quad (A6)$$

Then, we have $2\chi_m \ge \chi_{m+1} + \chi_{m-1}$, i.e.,

$$\chi_m - \chi_{m-1} \ge \chi_{m+1} - \chi_m. \quad (A7)$$

This inequality implies that if there exists an "m" such that $\chi_{m+1} > \chi_m$, i.e., $\chi_{m+1} - \chi_m > 0$ we have

$$\chi_1 - \chi_0 \ge \chi_1 - \chi_2 \ge \chi_2 - \chi_3 \ge \cdots \ge \chi_{m+1} - \chi_m > 0, \quad (A8)$$

i.e., we have $\chi_1 > \chi_0$. Therefore, we only need to focus on the case in which

$$\chi_0 \ge \chi_1 \ge \chi_2 \ge \cdots \ge \chi_m \ge \cdots. \quad (A9)$$

Considering that $\chi_m \ge 0$ for any $m \ge 0$, the above inequality implies the existence of a limit on the sequence χ_m, i.e.,

$$\lim_{m\to\infty}\chi_m = \bar{\chi} \quad (A10)$$

with $\bar{\chi} \ge 0$. This result also implies that

$$\lim_{m\to\infty}(\chi_m - \chi_{m-1}) = 0. \quad (A11)$$

Using the inequality (A7), we can further obtain

$$\chi_1 - \chi_0 \ge \chi_2 - \chi_1 \ge \cdots \ge \lim_{m\to\infty}(\chi_m - \chi_{m-1}) = 0. \quad (A12)$$

Together with inequality (A9), this implies $\chi_1 = \chi_0$. Finally, we can summarize that $\chi_1 \ge \chi_0$. Similarly, we can obtain $\chi_m \ge \chi_{m-1}$ for any m.

A straightforward application of the above result is

$$\int_B dV(D_{a_1\cdots a_m}K_a^b)(D^{a_1\cdots a_m}K_b^a) = \mathcal{O}(\lambda^{2\chi_0}) \quad (A13)$$

for any m.

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