Universal treatment of the reduction for one-loop integrals in a projective space

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Recently, a nice work about the understanding of one-loop integrals has been given by Arkani-Hamed and Yuan [arXiv:1712.09991] using the language of the projective space associated to their Feynman parametrization. We find this language is also very suitable to deal with the reduction problem of one-loop integrals with general tensor structures as well as propagators having arbitrary higher powers. In this paper, we show how to combine Feynman parametrization and embedding formalism to give a universal treatment of reductions for general one-loop integrals, even including the degenerated cases, such as the vanishing Gram determinant. Results from this method can be written in a compact and symmetric form.

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I. INTRODUCTION

In recent years, we have witnessed enormous progress in computing and understanding the analytic structure of scattering amplitude. At the one-loop level, it is well-known that a general one-loop integral in the $D = 4 - 2\epsilon$ dimension can always be reduced to a linear combination of one-loop scalar integrals having no more than five propagators (*master integrals*) with reduction coefficients being rational functions of kinetic variables [1–17]. These master integrals at the one-loop level (i.e., tadpoles, bubbles, triangles, boxes, and pentagons) are well known. Therefore, the main problem of one-loop integrals is to calculate the reduction coefficients. There are a host of methods to deal with the reduction at the integrand level and integral level, such as Integration-By-Parts (IBP) [18,19], Passarino-Veltman(PV) reduction [3], Ossola-Papadopoulos-Pittau(OPP) reduction [20–22], and the unitarity cut method [14,17,23–29].

Although in practice, we will not meet many situations where propagators have higher powers, a complete reduction method should be able to deal with it. From this point of view, the IBP method is a complete method since it treats these complicated cases within the same framework as the ones without higher poles. Recently, combining the unitarity cut and the derivation over mass, the reduction coefficients for higher pole cases can be calculated [30], except for the tadpole coefficients.

Recently, we have proposed an improved PV-reduction method for one-loop integrals [31,32]. The reduction coefficients can be expressed with the cofactors of the Gram matrix and have some symmetries. Thus, it is useful to understand these symmetries appearing in our intermediate recursion relations and the final results. Notably, the analytical structure of one-loop integrals is studied by investigating Feynman parametrization in the projective space for its compactness and the close relation to geometry [33]. Inspired by the geometric angle, we find it could be convenient to do reduction for one-loop integrals in projective space. By our study in this paper, one can see that the symmetry and simplicity of reduction coefficients are illustrated clearly with the denotations in [33].

Motivated by the work [33], we will develop an alternative method to determine the reduction coefficients of one-loop integrals in the $D = 4 - 2\epsilon$ dimension. The general tensor integrals with higher poles are related to integrals $E_{n,k}[T]$ in projective space by

$$E_{n,k}[T] \equiv \int_\Delta \frac{\langle X \mathrm{d}^{n-1} X\rangle T[X^k]}{(XQX)^{\frac{n+k}{2}}}, \tag{1.1}$$

where Δ is a simplex in n-dimensional space, which is defined by $H_I X = X_I > 0$, $\forall I = 1, 2, \ldots, n$. The T is a general tensor, which is contracted with k X's. The homogeneous

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coordinate X_I is denoted by a square bracket $X = [x_1 : x_2 : \ldots : x_n]$, and two coordinates are equivalent to each other up to a scaling, i.e., $[x_1 : x_2 : \ldots : x_n] \sim [kx_1 : kx_2 : \ldots : kx_n]$ for any $k \neq 0$. The measure in the projective space is given by the differential form,

$$\langle X \mathrm{d}^{n-1} X \rangle = \frac{\epsilon^{I_1, I_2, \ldots, I_n}}{(n-1)!} X_{I_1} \mathrm{d}X_{I_2} \wedge \mathrm{d}X_{I_3} \wedge \ldots \wedge \mathrm{d}X_{I_n};$$
$$XQX = Q^{IJ} X_I X_J. \tag{1.2}$$

As pointed out in [33], the integral $E_{n,k}$ satisfies a nice recursion relation, which will be recalled in Appendix A. This property is the key to carry out reduction in this paper.

This paper is structured as follows. In Sec. II, we discuss how to write a general one-loop integral as a sum of integrals $E_{n,k}$ in projective space. In Sec. III, we derive recursion relations for $E_{n,k}[V^i \otimes L^{k-i}]$ and dimension recursion relations for nondegenerate Q, then apply them to the reduction of one-loop integrals. In Sec. IV, we discuss the reduction framework for degenerate Q. In Sec. V, we show how to obtain the general expression of reduction coefficient from n-gon tensor integrals to n-gon scalar integrals, while general expressions of reduction coefficients are given in Appendix C. More reduction results are listed in Appendix B.

II. ONE-LOOP INTEGRALS IN PROJECTIVE SPACE

In this paper, we will discuss the reduction of the most general one-loop integrals,

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{k^{\mu_1} k^{\mu_2} \ldots k^{\mu_r}}{\prod_{j=1}^n D_j^{v_j}} = \int \frac{d^D k}{i\pi^{D/2}} \frac{k^{\mu_1} k^{\mu_2} \ldots k^{\mu_r}}{\prod_{j=1}^n [(k - q_j)^2 - m_j^2]^{v_j}}, \tag{2.1}$$

where $q_j = \sum_{i=1}^{j-1} p_i$. As pointed out in [31], we can recover the tensor structure by multiplying each index with an auxiliary vector R_{i,μ_i}. Furthermore, we can combine these R_i to $R = \sum_{i=1}^r a_i R_i$ to simplify the expression (2.1) to a Lorentzian invariant form,

$$I^{(r)}_{\mathbf{v}_n;D} \equiv \int \frac{d^D k}{i\pi^{D/2}} \frac{(2R \cdot k)^r}{\prod_{j=1}^n ((k - q_j)^2 - m_j^2)^{v_j}}. \tag{2.2}$$

To recover the result of (2.1), one can expand R and extract the coefficient of $\prod_{i=1}^r a_i$ from the auxiliary formula (2.2). With the above explanation, we will focus on the form (2.2) and transform it into projective space as suggested in [33]. First, to make our formulas elegant, we denote $y^\mu \equiv k^\mu$, $y_i \equiv q_i$; thus, (2.2) becomes

$$I^{(r)}_{\mathbf{v}_n;D} = \int \frac{d^D y}{i\pi^{D/2}} \frac{(2R \cdot y)^r}{\prod_{j=1}^n ((y - y_i)^2 - m_j^2)^{v_j}}. \tag{2.3}$$

Then we put the whole formula into the embedding space with two higher dimensions by lifting

$$y^\mu \longmapsto Y^M = (Y^+, Y^-, Y^\mu) = (1, y^2, y^\mu), \tag{2.4}$$

where we use the light-cone coordinates, i.e., the metric $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$, $\eta_{\mu\nu} = \mathrm{diag}(+, - - - -)$ while all other entries vanish. For clarity, we will use the capital letters I, J to denote the components of vectors in the embedding space, greek letters μ, ν to denote the components of Lorentzian vectors, and lowercase letters i, j for the external legs. We will also use capital letters Y, X simultaneously to denote vectors in the embedding space and projective space without ambiguity. Therefore, we can simplify the denominator of (2.3) into the inner product of two vectors in the embedding space, and the quadratic expression has been somehow linearized, for example,

$$(y - y_i)^2 = -2Y \cdot Y_i. \tag{2.5}$$

After defining

$$\mathcal{Y}_i^M = (1, y_i^2 - m_i^2, y_i^\mu), \qquad Y_\infty^M = (0, 1, 0, \ldots, 0),$$
$$\mathcal{R}^M = (0, 0, R^\mu), \tag{2.6}$$

it is easy to check that (2.3) becomes

$$I^{(r)}_{\mathbf{v}_n;D} = \int [\mathrm{d}^D Y] \frac{(-2Y \cdot Y_\infty)^{v-D-r} (2Y \cdot \mathcal{R})^r}{\prod_{j=1}^n (-2Y \cdot \mathcal{Y}_j)^{v_j}}, \tag{2.7}$$

where $v = \sum_{j=1}^n v_j$. The projective space invariant measure is given by $\int [\mathrm{d}^D Y] \equiv \int \frac{\mathrm{d}^{D+2} Y \delta(Y^2)}{i\pi^{D/2} \mathrm{vol.GL}(1)}$ [GL(1) acts as an overall scaling of the Y coordinates] and the factor $(Y \cdot Y_\infty)^{v-D-r}$ is necessary for the last expression to be genuinely an integral over the projective light cone. Using the most general Feynman parametrization,

$$\frac{1}{A_1^{m_1} A_2^{m_2} \ldots A_n^{m_n}} = \frac{\Gamma(\sum_i m_i)}{\prod \Gamma(m_i)} \int_0^1 dx_1 \ldots dx_n \delta\left(\sum_i x_i - 1\right) \times \frac{\prod x_i^{m_i - 1}}{(\sum_i x_i A_i)^{\sum_i m_i}},$$

and putting the Feynman parameters into the projective space, (2.7) becomes

$$I^{(r)}_{\mathbf{v}_n;D} = \frac{\Gamma(v)}{\prod \Gamma(v_i)} \int_\Delta \langle X \mathrm{d}^{n-1} X \rangle X^{\mathbf{v}_n - 1} \times \int [\mathrm{d}^D Y] \frac{(-2Y \cdot Y_\infty)^{v-D-r} (2Y \cdot \mathcal{R})^r}{(-2Y \cdot W)^v}, \tag{2.8}$$

where $X = [x_1 : x_2 : \ldots : x_n]$, which is a vector in a different projective (Feynman parametrization) space, $W = \sum_j x_j \mathcal{Y}_j$, and there is an additional factor $X^{\mathbf{v}_n-1} \equiv \prod_{i=1}^n x_i^{v_i-1} = \prod_{i=1}^n (H_i X)^{v_i-1}$. Then the Feynman parametrization integral has been written into the compact form $\int_\Delta \langle X \mathrm{d}^{n-1} X\rangle$ (see [33] for more details). We can further simplify (2.8) using the common trick,

$$I^{(r)}_{\mathbf{v}_n;D} = \frac{\Gamma(v)}{\prod \Gamma(v_i)} \int_\Delta \langle X \mathrm{d}^{n-1} X\rangle X^{\mathbf{v}_n-1} \frac{(-)^{v+D+r}\Gamma(D)}{\Gamma(v)} \times \left(\mathcal{R}^M \frac{\partial}{\partial W^M}\right)^r \left(Y_\infty^M \frac{\partial}{\partial W^M}\right)^{v-D-r} \times \int [\mathrm{d}^D Y] \frac{1}{(-2Y\cdot W)^D}. \tag{2.9}$$

Up to now, the last integral in (2.9) can be done easily. One way to solve it is to translate it back to the form (2.3), which is[1]

$$\int \frac{d^D y}{i\pi^{D/2}} \frac{1}{\left(W_-\left(y - \frac{w}{W_-}\right)^2 + W_+ - \frac{w^2}{W_-}\right)^D} = \frac{\Gamma(D/2)}{(-)^D \Gamma(D)} (W\cdot W)^{-D/2}, \tag{2.10}$$

and we have

$$I^{(r)}_{\mathbf{v}_n;D} = \frac{(-)^{v+r}\Gamma(D/2)}{\prod \Gamma(v_i)} \int_\Delta \langle X \mathrm{d}^{n-1} X\rangle X^{\mathbf{v}_n-1} \left(\mathcal{R}^M \frac{\partial}{\partial W^M}\right)^r \times \left(Y_\infty^M \frac{\partial}{\partial W^M}\right)^{v-D-r} (W\cdot W)^{-D/2}. \tag{2.11}$$

The action of $(Y_\infty^M \frac{\partial}{\partial W^M})$ can be done easily after using the fact $Y_\infty \cdot Y_\infty = 0$, and we get

$$I^{(r)}_{\mathbf{v}_n;D} = \frac{(-)^{v+r}\Gamma(D/2)}{\prod \Gamma(v_i)} \int_\Delta \langle X \mathrm{d}^{n-1} X\rangle X^{\mathbf{v}_n-1} \frac{\Gamma(v-\frac{D}{2}-r)}{\Gamma(D/2)} \times (LX)^{v-D-r} \left(\mathcal{R}^M \frac{\partial}{\partial W^M}\right)^r (W\cdot W)^{-(v-D/2-r)}, \tag{2.12}$$

where we have written $(-2Y_\infty \cdot W) = \sum_i x_i \equiv L\cdot X$ with $L = [1{:}1{:}\ldots{:}1]$.[2] The action of $(\mathcal{R}^M \frac{\partial}{\partial W^M})$ is more complicated. By power counting, we have the general expansion,

$$(\mathcal{R}\cdot\partial_W)^r (W^2)^k = \sum_{i=0}^r \mathcal{C}^k_{r,i} (W^2)^{k-\frac{r+i}{2}} (R^2)^{\frac{r-i}{2}} (\mathcal{R}\cdot W)^i, \tag{2.13}$$

where k can be an arbitrary number and i has the same parity as r due to the power of R^2 must be an integer. The expansion coefficients $\mathcal{C}^k_{r,i}$ are determined by initial conditions $\mathcal{C}^k_{0,i} = \delta_{i,0}, \mathcal{C}^k_{1,i} = 2k\delta_{i,1}$ and the recursion relation,

$$\mathcal{C}^k_{r+1,i} = (i+1)\mathcal{C}^k_{r,i+1} + (2k-r-i+1)\mathcal{C}^k_{r,i-1}. \tag{2.14}$$

From the recursion relation, one can solve $\mathcal{C}^k_{r,i}$ for general r, i as

$$\mathcal{C}^k_{r,i} = \frac{2^r r! \Gamma(\frac{r-i+1}{2})}{\sqrt{\pi} i! (r-i)!} \prod_{j=1}^{\frac{r+i}{2}} (k+1-j) = \frac{2^r r! k! \Gamma(\frac{r-i+1}{2})}{\sqrt{\pi} i! (r-i)! (k-\frac{r+i}{2})!}, \qquad \frac{r-i}{2} \in \mathbb{N}. \tag{2.15}$$

Plugging (2.13) into (2.12), we get

$$I^{(r)}_{\mathbf{v}_n;D} = \frac{\Gamma(v-\frac{D}{2}-r)}{(-)^{v+r}\prod \Gamma(v_i)} \sum_{i=0}^r \mathcal{C}^{D/2+r-v}_{r,i} (R^2)^{\frac{r-i}{2}} \times \int_\Delta \frac{\langle X \mathrm{d}^{n-1} X\rangle X^{\mathbf{v}_n-1} (\mathcal{R}\cdot W)^i (LX)^{v-D-r}}{(W\cdot W)^{v-\frac{D+r-i}{2}}}. \tag{2.16}$$

Since the remaining integrals are in the projective space of Feynman parameters, we should rewrite $W\cdot W, \mathcal{R}\cdot W$ as

$$W\cdot W = \left(\sum_{a=1}^n x_a \mathcal{Y}_a\right)\cdot\left(\sum_{b=1}^n x_b \mathcal{Y}_b\right) = \sum_{a,b} x_a (\mathcal{Y}_a\cdot\mathcal{Y}_b) x_b = XQX,$$

$$\mathcal{R}\cdot W = \sum_{b=1}^n x_b \mathcal{R}\cdot\mathcal{Y}_b = V\cdot X, \qquad V = [R\cdot q_1 {:} R\cdot q_2 {:} \ldots {:} R\cdot q_n]. \tag{2.17}$$

Then, we get

$$I^{(r)}_{\mathbf{v}_n;D} = \frac{\Gamma(v-\frac{D}{2}-r)}{(-)^{v+r}\prod \Gamma(v_i)} \sum_{i=0}^r \mathcal{C}^{D/2+r-v}_{r,i} (R^2)^{\frac{r-i}{2}} \times \int_\Delta \frac{\langle X \mathrm{d}^{n-1} X\rangle X^{\mathbf{v}_n-1} (VX)^i (LX)^{v-D-r}}{(XQX)^{v-\frac{D+r-i}{2}}}. \tag{2.18}$$

For general one loop integrals (2.1), one can calculate Q as

$$Q_{ij} = \frac{1}{2}(m_i^2 + m_j^2 - q_{ij}^2), \qquad q_{ij} = q_i - q_j. \tag{2.19}$$

Now the expression (2.18) is written as the integration over the X-projective space. Using the result of [33]

$$E_{n,k}[T] \equiv \int_\Delta \frac{\langle X \mathrm{d}^{n-1} X\rangle T[X^k]}{(XQX)^{\frac{n+k}{2}}}, \tag{2.20}$$

[1] We have used the fact $W_- = 1$ by the Feynman parametrization. The integration result can be found in the formula (A.44) in the book of Peskin and Schroeder [34].

[2] The reason that we can ignore the action of $(\mathcal{R}^M \frac{\partial}{\partial W^M})$ on $(-2Y_\infty \cdot W)$ is because $Y_\infty \cdot \mathcal{R} = 0$.

where Δ is a simplex in n-dimensional space defined by $H_i X = X_i > 0$, $\forall\, i = 1, 2, \ldots, n$ and T is a kth tensor contracted with k X's, the general one loop integral in the projective space (2.18) can be written as

$$I^{(r)}_{\mathbf{v}_n;D} = \frac{\Gamma(v - D/2 - r)}{(-)^{v+r} \prod \Gamma(v_i)} \sum_{i=0}^{r} C_{r,i}^{D/2+r-v} (R^2)^{\frac{r-i}{2}} \times E_{n,2v-n-D-r+i}[\otimes_j H_j^{v_j-1} \otimes V^i], \quad (2.21)$$

where for simplicity, we write $E_{n,k}[V^a \otimes L^{k-a}] \equiv E_{n,k}[V^a]$ by neglecting the power of L.

For the later use, we need to do symmetrization for the tensor $\otimes_j H_j^{v_j-1} \otimes V^i$. To do so, we can use the same trick as used in (2.1) and (2.2), i.e., $Z \equiv \sum_{i=1}^{v-n} z_i H_i$ and $S = tV + Z$. Thus, (2.21) can be obtained from

$$\frac{\Gamma(v - D/2 - r)}{(-)^{v+r} \prod \Gamma(v_i)} \sum_{i=0}^{r} C_{r,i}^{D/2+r-v} (R^2)^{\frac{r-i}{2}} E_{n,2v-n-D-r+i}[S^{v-n+i}] \quad (2.22)$$

after taking the coefficients of $t^i z^{\mathbf{v}_n - 1} \equiv t^i \prod_{i=1}^{n} z_i^{v_i - 1}$. Taking care of numerical factors, the final expression is

$$I^{(r)}_{\mathbf{v}_n;D} = \sum_{i=0}^{r} \frac{i!\,\Gamma(v - D/2 - r)}{(-1)^{v+r} (v - n + i)!} C_{r,i}^{D/2+r-v} (R^2)^{\frac{r-i}{2}} \times E_{n,2v-n-D-r+i}[S^{v-n+i}]\Big|_{t^i z^{\mathbf{v}_n-1}}. \quad (2.23)$$

A special case of (2.23) is that for $\mathbf{v}_n = \mathbf{1}_n$ and $r = 0$, we have

$$I_{n;D} = (-1)^n \Gamma(n - D/2) E_{n,n-D}[L^{n-D}], \quad (2.24)$$

where $I_{n;D}$ is the scalar integral of n propagators in D dimension.

Before ending this section, we want to point out that in our discussion, for example in (2.8), the power $v - D - r$ could be positive or negative with the arbitrary choice of r. Since we have kept the dimension D arbitrary, we can take D properly (even a negative number) to make $v - D - r$ a positive integer to make later discussion legitimate. At the end of reduction, we can analytically continue D to the proper dimension. We have checked with several examples that such a continuation is allowed.

In this paper, we mainly discuss Feynman integrals in $D = 4 - 2\epsilon$-dimension space. At the one-loop level, the master integrals are related to $E_{n,n-D}[L^{n-D}]$, $n = 1, 2, \ldots, 5$. So the main task of one-loop integral reduction is to reduce general integral $E_{n,k}[S^a \otimes L^{k-a}]$ to the basis $E_{n\le 5, n-D}[L^{n-D}]$.

III. REDUCTION FOR NONDEGENERATE Q

Having transformed our problem (2.1) to the form (2.23), in this section, we will show how to use the tricks of integrals in projective space (see [33]) to generate recursion relations of $E_{n,k}[V^a \otimes L^{k-a}]$. By applying these recursion relations iteratively, one can reduce a general one-loop integral to the basis with coefficients written by elegant expressions. In other words, the reduction can be done universally in the new projective space form. As we will point out, the reduction coefficients will have an interesting pattern other than the obvious permutation symmetry.[3] Moreover, the reduction process can be carried out in *Mathematica* automatically.

A. Recursion relation

In this subsection, we derive the recursion relations of $E_{n,k}[V^a \otimes L^{k-a}]$. We first consider the case Q is nondegenerate. The key equation is the following [see Eq. (4.2) in [33]]:

$$\frac{\langle X \mathrm{d}^{n-1} X\rangle T[X^k]}{(XQX)^{\frac{n+k}{2}}} = \frac{1}{n+k-2} \mathrm{d}_X \left[\frac{\langle (Q^{-1}T)[X^{k-1}] X \mathrm{d}^{n-2} X\rangle}{(XQX)^{\frac{n+k-2}{2}}} \right] + \frac{k-1}{n+k-2} \frac{\langle X \mathrm{d}^{n-1} X\rangle (\mathrm{tr}_Q T)[X^{k-2}]}{(XQX)^{\frac{n+k-2}{2}}}, \quad (3.1)$$

where $\mathrm{d}_X = \mathrm{d}X^I \frac{\partial}{\partial X^I}$, $\mathrm{tr}_Q T = Q^{-1}_{I_1 I_2} T^{I_1 I_2 \ldots I_k}$. The proof of the formula can be found in Appendix A. By integrating (3.1), we get

$$E_{n,k}[T] = \alpha_{n,k} E^{(b)}_{n-1,k-1}[(H_b Q^{-1} T)] + \beta_{n,k} E_{n,k-2}[\mathrm{tr}_Q T], \quad (3.2)$$

where summing over b is implicit and to simplify our denotations, we have defined

$$\alpha_{n,k} \equiv \frac{1}{n+k-2}, \qquad \beta_{n,k} \equiv \frac{k-1}{n+k-2}. \quad (3.3)$$

Let us give a little explanation for the first term on the right-hand side of (3.2). When integrating a total derivative term, we should choose a patch. For simplicity, we assume $X_i = 1$. Then we get the contribution from the boundary $X_b = 0$ and $X_b = +\infty$ for $b \neq i$. By the dimensional regularization, the term with $X_b = +\infty$ gives zero. For the term $X_b = 0$, in $\langle (Q^{-1}T)[X^{k-1}] X \mathrm{d}^{n-2} X\rangle$, only when the first index of Q^{-1} takes the value b, the contribution is nonzero, which is equivalent to be written as $(H_b Q^{-1} T)$.

[3] In the work [31,32], one can see that the recursion relations, for example, the bubble tensors to bubble basis and the triangle tensor to triangle basis, are similar, except for the boundary conditions. This similarity has explained the interesting pattern we observe in this paper.

When we repeatedly do the recursion relation, there will be a set of X_b setting to zero. Writing the index set as $\mathbf{b}_j \equiv \{b_1, b_2, ..., b_j\}$ with $1 \leq b_i < b_{i+1} \leq n$, we define $X_{(\mathbf{b}_j)}$ to be the new vector in lower-dimensional projective space obtained by removing these components belonging to the set $\mathbf{b}_j$ from the original vector X. With this understanding, the meaning of

$$E^{(\mathbf{b}_j)}_{n-j,k}[T] \equiv \int_{\Delta_{(\mathbf{b}_j)}} \frac{T[X^k_{(\mathbf{b}_j)}]\langle X_{(\mathbf{b}_j)} \mathrm{d}^{n-1-j} X_{(\mathbf{b}_j)}\rangle}{(X_{(\mathbf{b}_j)} Q_{(\mathbf{b}_j)} X_{(\mathbf{b}_j)})^{\frac{n-j+k}{2}}} \tag{3.4}$$

is clear. Equation (3.4) represents the integral got by removing propagators belonging to $\mathbf{b}_j$.

Now we consider the reduction of $E_{n,k}[T]$ with $T = V^i \otimes L^{k-i}$ as given in (2.23). Since tensor T is contracted with k X's, we can symmetrize its last $k-1$ indices as below,

$$V^i \otimes L^{k-i} \to V \otimes \frac{1}{(k-1)!} \sum_{\sigma \in S_{k-1}} \sigma[V^{i-1} \otimes L^{k-i}], \tag{3.5}$$

where the σ is the permutation acting on the tensor $V^{i-1} \otimes L^{k-i}$. By applying (3.1), one has

$$E_{n,k}[V^{i+1}] = \alpha_{n,k}[(\overline{H_b V}) E^{(b)}_{n-1,k-1}[V^i] + i(\overline{VV}) E_{n,k-2}[V^{i-1}] + (k-i-1)(\overline{VL}) E_{n,k-2}[V^i]], \tag{3.6}$$

where for simplicity, the L tensor part has been omitted. To make our formula more compact, here we have defined $(\overline{AB}) \equiv AQ^{-1}B$. Here, we want to remark on a subtle point. In (3.6), the Q is $n \times n$ matrix as defined in $E_{n,k}$ and appears in front of $E^{(b)}_{n-1,k-1}$. When we try to iteratively use (3.6) for $E^{(b)}_{n-1,k-1}$, that Q will become the $(n-1) \times (n-1)$ matrix $Q_{(b)}$. Thus, for later convenience, we define $(\overline{AB})_{(\mathbf{a}_j)} = A_{(\mathbf{a}_j)} (Q_{(\mathbf{a}_j)})^{-1} B_{(\mathbf{a}_j)}$ where the matrix $(Q_{(\mathbf{a}_j)})^{-1}$ is the inverse of the matrix obtained by removing the rows and columns of the index set $\mathbf{a}_j$ from the original matrix Q.

As the main result of the whole paper, the recursion relation (3.6) plays a crucial role in the reduction of one-loop integrals. By comparing the power of V, one sees that it has been reduced from the lhs to rhs. Furthermore, from (2.23), one sees that the power is given by $v + i - n$, where v contains the contribution of higher power of propagators and i contains the contribution of the tensor numerator; thus, (3.6) provides the universal reduction of both cases. As shown in Fig. 1, after iteratively using (3.6), we get a linear combination of $E^{(\mathbf{a}_{n-n'})}_{n' \leq n, k' < k}$, i.e., the scalar integral $I_{n';D'}$ in dimension $D' = n' - k'$. So starting with a general one-loop integral, one can always reduce it to the scalar integrals in different dimensions with coefficients being rational functions of external momenta. If we prefer the scalar basis in a given D-dimensional space, we need to find the formula to shift the dimension of the scalar basis to a fixed D.

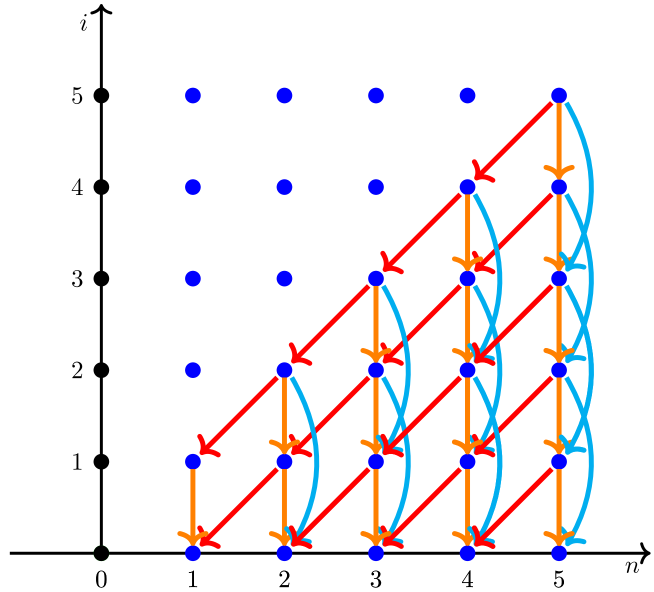

FIG. 1. The reduction process of $E_{n=5,k}[V^{i=5}]$, where the black points represent zero terms. The red, orange, and cyan arrows represent the first, second, and third terms, respectively in (3.6).

B. Dimension recursion

As we have seen in (2.24), the integral $E_{n,k}$ corresponds to the scalar n-gon diagram in $(n-k)$-dimensional space. To find dimension recursion relations, we set $V = L$ in (3.6) and get

$$E_{n,k} = \alpha_{n,k} (\overline{H_b L}) E^{(b)}_{n-1,k-1} + \beta_{n,k} (\overline{LL}) E_{n,k-2}. \tag{3.7}$$

To reduce $E_{n,k}$, where $n - k = D + 2s$, $s \in \mathbb{Z}, s \neq 0$, we can iteratively use (3.7), which is established for the scalar integrals already. Noticing that in the rhs of (3.7), the first term has the same dimension as the lhs with one propagator being removed, while the second term has two higher dimensions with the same number of propagators. Depending on the sign of s, we can take different manipulations.

(i) $s > 0$: For this case, we need to reduce an integral in a higher dimension to D dimension, so we solve the second term in the rhs of (3.7) and get

$$E_{n,k} = \frac{E_{n,k+2} - \alpha_{n,k+2} (\overline{H_j L}) E^{(j)}_{n-1,k+1}}{\beta_{n,k+2} (\overline{LL})}. \tag{3.8}$$

It is obvious that such a rewriting (3.8) is legitimate when and only when $(\overline{LL}) \neq 0$. For $(\overline{LL}) = 0$, we have

$$E_{n,k}=\alpha_{n,k}(\overline{H_jL})E^{(j)}_{n-1,k-1}. \tag{3.9}$$

Both sides have the same dimension, but the rhs of (3.9) has one less propagator. One well-known example of $(\overline{LL})=0$ is that the bubble with null external momentum is not a basis anymore, and it is reduced to two tadpoles. Having established (3.8) and (3.9), we can reduce $(D+2s)$-dimensional integrals to D dimensional iteratively using either (3.8) or (3.9) depending on if $(\overline{LL})$ is zero or not at that step.[4]

(ii) $s<0$: For this case, we can use (3.7) directly.

$$E_{n,k}=\alpha_{n,k}(\overline{H_jL})E^{(j)}_{n-1,k-1}+\beta_{n,k}(\overline{LL})E_{n,k-2}. \tag{3.10}$$

As pointed out before, the first term on the rhs corresponds to the scalar integrals in the same dimension but with the j-th propagator has been removed and the second term corresponds to the scalar integral in $D'=n-k+2$ dimension. For the boundary situation, i.e., $n=1$, the first term vanishes, and the second term gives a higher dimensional scalar basis. Repeating it, we can reduce $E_{n,k}$ to scalar integrals in D-dimensional space.

C. Examples

To illustrate our method and avoid complicated computation in general cases, we first consider the reduction of tensor bubbles,

$$I_2^{(r)}=\int\frac{d^D\ell}{i\pi^{D/2}}\frac{(2R\cdot\ell)^r}{D_1^{v_1}D_2^{v_2}}, \tag{3.11}$$

where

$$D_1=(\ell-q_1)^2-m_1^2,\qquad D_2=(\ell-q_2)^2-m_2^2. \tag{3.12}$$

We set $q_1=0$ for simplicity, then

$$L=\begin{bmatrix}1\\1\end{bmatrix},\qquad H_1=\begin{bmatrix}1\\0\end{bmatrix},\qquad H_2=\begin{bmatrix}0\\1\end{bmatrix},$$
$$Q=\begin{bmatrix}m_1^2 & \frac{1}{2}(m_1^2+m_2^2-q_2^2)\\ \frac{1}{2}(m_1^2+m_2^2-q_2^2) & m_2^2\end{bmatrix}. \tag{3.13}$$

Here, we give some results for different choices of ranks and powers to illustrate our idea.

(i) *Tensor bubble with primary propagator*

We first consider reducing a rank-1 bubble $I_2^{(1)}\equiv I^{(1)}_{\{1,1\}}$. Since there are no higher poles, we just choose $S=V, r=1, v=n=2$ in (2.23), and we have

$$\begin{aligned}I_2^{(1)}&=-\Gamma(1-D/2)\mathcal{C}_{1,1}^{D/2-1}E_{2,2-D}[V]\\&=-\Gamma(1-D/2)(D-2)E_{2,2-D}[V].\end{aligned} \tag{3.14}$$

First, we use (3.6) to reduce $E_{2,2-D}[V]$ and get

$$E_{2,2-D}[V]=\alpha_{2,2-D}(\overline{H_iV})E^{(i)}_{1,1-D}+\beta_{2,2-D}(\overline{VL})E_{2,-D}, \tag{3.15}$$

where the first term $E^{(i)}_{1,1-D}$ corresponds to D-dimensional scalar tadpoles generated by removing the ith propagator of bubble I_2, while the second term $E_{2,-D}$ corresponds to a $(D+2)$-dimensional bubble. We need to lower the dimension of the second term further. Here, we assume $(\overline{LL})\neq 0$, by using (3.8), we get

$$E_{2,-D}=\frac{E_{2,2-D}-\alpha_{2,2-D}(\overline{H_iL})E^{(i)}_{1,1-D}}{\beta_{2,2-D}(\overline{LL})}, \tag{3.16}$$

where the two terms in the numerator correspond to the D-dimensional bubble and two tadpoles. Plugging (3.16) and (3.15) to (3.14) and recognizing them as master integral according to (2.24), we have

$$I_2^{(1)}=\sum_i-\frac{(\overline{LL})(\overline{H_iV})-(\overline{H_iL})(\overline{VL})}{(\overline{LL})}I_{2;\hat{i}}+\frac{2(\overline{VL})}{(\overline{LL})}I_2. \tag{3.17}$$

So the reduction coefficients are

$$\begin{aligned}C^{(1)}_{2\to2;\hat{1}}&=-\frac{(\overline{LL})(\overline{H_1V})-(\overline{H_1L})(\overline{VL})}{(\overline{LL})}=\frac{R\cdot q_2}{q_2^2},\\C^{(1)}_{2\to2;\hat{2}}&=-\frac{(\overline{LL})(\overline{H_2V})-(\overline{H_2L})(\overline{VL})}{(\overline{LL})}=-\frac{R\cdot q_2}{q_2^2},\\C^{(1)}_{2\to2}&=\frac{2(\overline{VL})}{(\overline{LL})}=\frac{(m_1^2-m_2^2+q_2^2)R\cdot q_2}{q_2^2}.\end{aligned} \tag{3.18}$$

(ii) *Tensor bubble with massless legs*

One can notice there is a pole of q_2^2 in the reduction coefficients of $I_2^{(1)}$, which comes from $(\overline{LL})$,

$$(\overline{LL})=-\frac{4q_2^2}{-2(m_1^2+m_2^2)q_2^2+(m_1^2-m_2^2)^2+q_2^4}. \tag{3.19}$$

For $q_2^2=0$, we have $(\overline{LL})=0$, so we have

$$E_{2,2-D}[V]=\alpha_{2,2-D}(\overline{H_bV})E^{(b)}_{1,1-D}+\beta_{2,2-D}(\overline{VL})E_{2,-D}. \tag{3.20}$$

[4]Please remember that as emphasized under (3.6), at each step, Q is different.

Here we need to reduce $E_{2,-D}$,

$$E_{2,-D} = E^{(b)}_{1,-D-1} = \frac{\alpha_{2,-D}(\overline{H_b L})E^{(b)}_{1,1-D}}{\beta_{1,1-D}(\overline{LL})_{(b)}}, \quad (3.21)$$

where we have used (3.9) and

$$E^{(b)}_{1,-D-1} = \frac{E^{(b)}_{1,1-D}}{\beta_{1,1-D}(\overline{LL})_{(b)}}. \quad (3.22)$$

Using (2.24), we finally get

$$I_2^{(1);q_2^2=0} = \left[\frac{1-D}{D}\frac{(\overline{VL})(\overline{H_b L})}{(\overline{LL})_{(b)}} - (\overline{H_b V})\right] I_{2;\hat{b}}. \quad (3.23)$$

Explicitly, we have

$$I_2^{(1);q_2^2=0} = -\frac{2(Dm_1^2-(D-2)m_2^2)R\cdot q_2}{D(m_1^2-m_2^2)^2} I_{2;\hat{1}} + \frac{4m_1^2 R\cdot q_2}{D(m_1^2-m_2^2)^2} I_{2;\hat{2}}. \quad (3.24)$$

(iii) *Scalar bubble with higher poles*

Then, we consider reducing scalar bubbles $I_{\{v_1,v_2\}}$ with higher poles $v = v_1 + v_2 = 3$. Due to there being no tensor structure in the numerator, we just set $S = Z = z_1 H_1 + z_2 H_2$, $r = 0$ in (2.23),

$$I_{\mathbf{v}_2;v=3} = -\Gamma(3-D/2)E_{2,4-D}[Z]|_{z^{\mathbf{v}_2-1}}, \quad (3.25)$$

where $\mathbf{v}_2 = \{1,2\}, \{2,1\}$. First, we use (3.6) to reduce $E_{2,4-D}[Z]$ and get

$$E_{2,4-D}[Z] = \alpha_{2,4-D}(\overline{H_i Z})E^{(i)}_{1,3-D} + \beta_{2,4-D}(\overline{ZL})E_{2,2-D}, \quad (3.26)$$

where the first term $E^{(i)}_{1,3-D}$ corresponds to $(D-2)$-dimensional scalar tadpoles generated by removing the ith propagator of bubble I_2, while the second term $E_{2,2-D}$ corresponds to wanted D-dimensional bubble. We need to lift the dimension of the first term further. By using (3.10), we get

$$E^{(i)}_{1,3-D} = \beta_{1,3-D}(\overline{LL})E^{(i)}_{1,1-D}, \quad (3.27)$$

where we have used $E^{(ij)}_{0,2-D} = 0$, and the rhs corresponds to D-dimensional scalar tadpoles. Plugging (3.27) and (3.26) to (3.25) and recognizing them as master integrals according to (2.24), we have

$$I_{\mathbf{v}_2;v=3} = -\frac{1}{4}(D-2)(\overline{LL})_{(i)}(\overline{H_i Z})\Big|_{z^{\mathbf{v}_2-1}} I_{2;\hat{i}} + \frac{1}{2}(D-3)(\overline{ZL})\Big|_{z^{\mathbf{v}_2-1}} I_2. \quad (3.28)$$

There are two configurations,

$$\begin{aligned} I_{\{2,1\}} &= -\frac{1}{4}(D-2)(\overline{LL})_{(i)}(\overline{H_i Z})\Big|_{z_1} I_{2;\hat{i}} + \frac{1}{2}(D-3)(\overline{ZL})\Big|_{z_1} I_2 \\ &= -\frac{1}{4}(D-2)(\overline{LL})_{(i)}(\overline{H_i H_1})I_{2;\hat{i}} + \frac{1}{2}(D-3)(\overline{H_1 L})I_2, \\ I_{\{1,2\}} &= -\frac{1}{4}(D-2)(\overline{LL})_{(i)}(\overline{H_i Z})\Big|_{z_2} I_{2;\hat{i}} + \frac{1}{2}(D-3)(\overline{ZL})\Big|_{z_2} I_2 \\ &= -\frac{1}{4}(D-2)(\overline{LL})_{(i)}(\overline{H_i H_2})I_{2;\hat{i}} + \frac{1}{2}(D-3)(\overline{H_2 L})I_2. \end{aligned} \quad (3.29)$$

Explicitly, using (3.13), we find the reduction coefficients are

$$\begin{aligned} C_{\{2,1\}\to 2;\hat{1}} &= \frac{D-2}{((m_1-m_2)^2-q_2^2)((m_1+m_2)^2-q_2^2)}, \\ C_{\{2,1\}\to 2;\hat{2}} &= -\frac{(D-2)(m_1^2+m_2^2-q_2^2)}{2m_1^2((m_1-m_2)^2-q_2^2)((m_1+m_2)^2-q_2^2)}, \\ C_{\{2,1\}\to 2} &= \frac{(D-3)(m_1^2-m_2^2-q_2^2)}{((m_1-m_2)^2-q_2^2)((m_1+m_2)^2-q_2^2)}, \end{aligned} \quad (3.30)$$

$$\begin{aligned} C_{\{1,2\}\to 2;\hat{1}} &= -\frac{(D-2)(m_1^2+m_2^2-q_2^2)}{2m_2^2((m_1-m_2)^2-q_2^2)((m_1+m_2)^2-q_2^2)}, \\ C_{\{1,2\}\to 2;\hat{2}} &= \frac{D-2}{((m_1-m_2)^2-q_2^2)((m_1+m_2)^2-q_2^2)}, \\ C_{\{1,2\}\to 2} &= -\frac{(D-3)(m_1^2-m_2^2+q_2^2)}{((m_1-m_2)^2-q_2^2)((m_1+m_2)^2-q_2^2)}. \end{aligned} \quad (3.31)$$

(iv) *Tensor bubble with higher poles*

At last, we consider a combined case, $I^{(1)}_{\mathbf{v}_2;v=3}$. Here, we need to set $S = tV + Z$, $Z = z_1H_1 + z_2H_2$. Setting $v = 3$, $r = 1$, $n = 2$ in (2.23), we have

$$I^{(1)}_{\mathbf{v}_2;v=3} = -\Gamma(3 - D/2)E_{2,4-D}[S^2]|_{tz^{\mathbf{v}_2-1}}. \tag{3.32}$$

First, we use (3.6) iteratively to pull out all S's in the numerator,

$$\begin{aligned} E_{2,4-D}[S^2] &= \alpha_{2,4-D}[(\overline{H_iS})E^{(i)}_{1,3-D}[S] + (\overline{SS})E_{2,2-D} + (2-D)(\overline{SL})E_{2,2-D}[S]] \\ &= \alpha_{2,4-D}[\beta_{1,3-D}(\overline{H_iS})(\overline{SL})_{(i)}E^{(i)}_{1,1-D} + (\overline{SS})E_{2,2-D} \\ &\quad + (2-D)(\overline{SL})\alpha_{2,2-D}[(\overline{H_iS})E^{(i)}_{1,1-D} + (1-D)(\overline{SL})E_{2,-D}]]. \end{aligned} \tag{3.33}$$

Among the four terms above, we only need to deal with the last term $E_{2,-D}$ since it corresponds to a $(D+2)$-dimensional bubble, which has been discussed in (3.21). Finally, we have

$$\begin{aligned} I^{(1)}_{\mathbf{v}_2;v=3} = &-\frac{(D-2)((\overline{LL})(\overline{H_iZ})(\overline{ZL}) + (\overline{LL})(\overline{ZL})_{(i)}(\overline{H_iZ}) - (\overline{H_iL})(\overline{ZL})^2)}{4(\overline{LL})}\bigg|_{tz^{\mathbf{v}_2-1}} I_{2;\hat{i}} \\ &+\frac{1}{2}\left(\frac{(D-2)(\overline{ZL})^2}{(\overline{LL})} - (\overline{ZZ})\right)\bigg|_{tz^{\mathbf{v}_2-1}} I_2. \end{aligned} \tag{3.34}$$

One can get the reduction results for $I^{(1)}_{\{2,1\}}, I^{(1)}_{\{1,2\}}$. For example,

$$\begin{aligned} I^{(1)}_{\{2,1\}} = &-\frac{(D-2)}{4(\overline{LL})}[(\overline{LL})(\overline{H_iV})(\overline{H_1L}) + (\overline{LL})(\overline{H_iH_1})(\overline{VL}) + (\overline{LL})(\overline{VL})_{(i)}(\overline{H_iH_1}) \\ &+ (\overline{LL})(\overline{H_1L})_{(i)}(\overline{H_iV}) - 2(\overline{H_iL})(\overline{VL})(\overline{H_1L})]I_{2;\hat{i}} + \left[\frac{(D-2)(\overline{H_1L})(\overline{VL})}{(\overline{LL})} - (\overline{H_1V})\right]I_2, \end{aligned} \tag{3.35}$$

where for simplicity, we will not present explicit expressions for these coefficients.

Note that the reduction coefficients in these examples are rational functions. For some special masses and momenta configurations, denominators can become zero, which leads to several kinds of divergences. Since only $(\overline{LL}) \equiv LQ^{-1}L$ appears in the denominators, all divergences come from the Q matrix and its all submatrices, which have $\det Q = 0$ or $(\overline{LL}) = 0$. For example, the pole of q_2^2 in (3.18) comes from $LQ^{-1}L$ [see (3.19)]. The divergence of $C_{\{2,1\}\to 2;\hat{2}}$ is given by $m_1^2((m_1 - m_2)^2 - q_2^2)((m_1 + m_2)^2 - q_2^2) = 0$, which is corresponds to $\det Q_{(2)} = 0$ or $\det Q = 0$. One can find the pole $(\overline{LL}) = 0$ comes from the dimension shifting process (3.8), which can be addressed by employing (3.7) to reduce $E_{n,k}$ to lower topology. To deal with the divergences coming from $\det Q = 0$, we need to consider the reduction method for degenerate Q elaborated in the next section.

IV. REDUCTION FOR DEGENERATE Q

In this section, we generalize our reduction method to degenerate Q. The basic idea is to generalize the recursion relation (3.1) to the formula (A9). When Q is degenerate, the characteristic equation $Q\xi = 0$ always has solutions, and we denote the $\mathfrak{N}_Q$ as the null space spanned by linearly independent ξ's.

A. $\tilde{Q}L \neq 0$

When Q is degenerate, the recursion relation (3.1) in the last sections breaks down for $\det Q = 0$. Our idea is to consider the tensor structure with one L in the first place and make other $(k-1)$ indices completely symmetric by summing over all permutations between i V's and $(k-1-i)$ L's. Using (A9), we have

$$\begin{aligned} &E_{n,k}[(Q\tilde{Q}L)\otimes V^i \otimes L^{k-1-i}] \\ &= E_{n,k}\left[(Q\tilde{Q}L)\otimes\frac{1}{(k-1)!}\sum_{\sigma\in S_{k-1}}\sigma[V^i\otimes L^{k-1-i}]\right] \\ &= \alpha_{n,k}(H_b\tilde{Q}L)E^{(b)}_{n-1,k-1}[V^i] + \beta_{n,k}\left[\frac{i}{k-1}(V\tilde{Q}L)E_{n,k-2}[V^{i-1}]\right. \\ &\quad \left. + \frac{k-1-i}{k-1}(L\tilde{Q}L)E_{n,k-2}[V^i]\right]. \end{aligned} \tag{4.1}$$

For a degenerate Q, we can always find a matrix $\tilde{Q} = [\xi_1, \xi_2, \ldots, \xi_n]$, $\xi_i \in \mathfrak{N}_Q$ so that $Q\tilde{Q} = 0$. Then the lhs of (4.1) vanishes. If $Q^* L \neq 0$, where Q^* is the adjugate matrix of Q, we can take $\tilde{Q} = Q^*$. With the denotation,

$$(\dot{\overline{AB}}) = (A\tilde{Q}B), \tag{4.2}$$

(4.1) becomes

$$\alpha_{n,k}(\dot{\overline{H_b L}})E^{(b)}_{n-1,k+1}[V^i] + i\beta_{n,k}(\dot{\overline{VL}})E_{n,k}[V^{i-1}] + (k+1-i)\beta_{n,k}(\dot{\overline{LL}})E_{n,k}[V^i] = 0. \tag{4.3}$$

Depending on the value of $(\dot{\overline{LL}})$, we have following the two cases:

(1) When $(\dot{\overline{LL}}) \neq 0$, (4.3) can be rewritten as

$$E^{|Q|=0}_{n,k}[V^i] = \frac{1}{(i-k-1)(\dot{\overline{LL}})}[(\dot{\overline{H_b L}})E^{(b)}_{n-1,k+1}[V^i] + i(\dot{\overline{VL}})E_{n,k}[V^{i-1}]], \tag{4.4}$$

where the first term in rhs corresponds to the lower topologies and the second term has tensor rank reduced by one.

(2) When $(\dot{\overline{LL}}) = 0$, (4.3) can be rewritten as

$$E^{|Q|=0}_{n,k}[V^i] = \frac{-(\dot{\overline{H_b L}})}{(i+1)(\dot{\overline{VL}})}E^{(b)}_{n-1,k+1}[V^{i+1}], \tag{4.5}$$

where, although the tensor rank increased by one on the rhs, it belongs to the lower topologies. One particular thing of (4.5) is that since V depends on the auxiliary R, we will always have $(\dot{\overline{VL}}) \neq 0$. Another tricky point is that although R appears in the denominator in (4.5) in the intermediate steps, it will be canceled in the final reduction coefficients. Thus, they are a new type of spurious singularities, which is tightly related to our method.

B. $\tilde{Q}L = 0$

Since $\tilde{Q}L = 0$, every term in (4.3) vanishes. Now we can put V in the first place of the tensor structure and use (A9) to reach

$$E_{n,k}[(Q\tilde{Q}V)V^i] = A_{n,k}[(H_j\tilde{Q}V)E^{(j)}_{n-1,k-1}[V^i] + i(V\tilde{Q}V)E_{n,k-2}[V^{i-1}] + (k-i-1)(V\tilde{Q}L)E_{n,k-2}[V^i]]. \tag{4.6}$$

Using $Q\tilde{Q} = \tilde{Q}L = 0$, (4.6) becomes

$$0 = A_{n,k}[(H_j\tilde{Q}V)E^{(j)}_{n-1,k-1}[V^i] + i(V\tilde{Q}V)E_{n,k-2}[V^{i-1}]], \tag{4.7}$$

and we have

$$E_{n,k}[V^i] = \frac{-(\dot{\overline{H_b V}})}{(i+1)(\dot{\overline{VV}})}E^{(b)}_{n-1,k+1}[V^{i+1}]. \tag{4.8}$$

Again, although R appears in the denominator through $(\dot{\overline{VV}})$ in the intermediate steps, it will be canceled in the final reduction coefficients. Similarly, $(\dot{\overline{VV}})$ is another new type of spurious singularity in our method.

C. Dimension recursion

Having reduced to scalar integrals with different dimensions, we want to shift the dimension to a given D. Depending on various situations, we have

(i) For $\tilde{Q}L \neq 0$, we can always choose $Q\tilde{Q} = 0$, $(\dot{\overline{LL}}) \neq 0$. Then using (4.4) for the case $i = 0$, the second term vanishes, and we have

$$E_{n,k} = \frac{-(\dot{\overline{H_b L}})}{(1+k)(\dot{\overline{LL}})}E^{(b)}_{n-1,k+1}. \tag{4.9}$$

(ii) For $\tilde{Q}L = 0$, we use (4.8) with $i = 1$. Then, we shift $V \to L + \epsilon K$ with a reference K such that $K \not\subset \mathfrak{N}_Q$ and get

$$E_{n,k} + \epsilon E_{n,k}[K] = \frac{-(\dot{\overline{H_b K}})}{2\epsilon(\dot{\overline{KK}})}E^{(b)}_{n-1,k+1}[(L+\epsilon K)^2]. \tag{4.10}$$

Comparing both sides, especially the ϵ term, we have

$$E_{n,k} = \frac{-(\dot{\overline{H_b K}})}{(\dot{\overline{KK}})}E^{(b)}_{n-1,k+1}[K]. \tag{4.11}$$

We lower the topology in the rhs, so we can reduce it further by using the equation recursively. The dependence of the choice of K in the intermediate steps will vanish in the final reduction coefficients, as shown in the examples in the next section.

D. Examples

In this section, we illustrate our method for degenerate Q. To avoid unnecessary and complicated calculations and compare with the reduction procedures for nondegenerate Q discussed in the Sec. III C, we focus on bubbles with some special masses and momenta configurations.

(i) $\tilde{Q}L = 0$: *Massless scalar bubble with equal internal masses*

To check the validity of our method, we first consider a scalar bubble with $m_1 = m_2 = m$ and $q_2^2 = 0$, defined as

$$I_2^{m_1=m_2,q_2^2=0} = \int \frac{d^D\ell}{i\pi^{D/2}} \frac{1}{(\ell^2-m^2)[(\ell-q_2)^2-m^2]}, \tag{4.12}$$

which can be reduced to two tadpoles.[5]

We find Q defined in (3.13) degenerates to a corank-1 matrix,

$$Q = \begin{bmatrix} m^2 & m^2 \\ m^2 & m^2 \end{bmatrix}, \qquad \tilde{Q} = Q^* = \begin{bmatrix} m^2 & -m^2 \\ -m^2 & m^2 \end{bmatrix}. \tag{4.13}$$

Since $Q\tilde{Q} = 0$ and $Q^*L = 0$, using (2.24), we have

$$I_2 = \Gamma(2-D/2)E_{2,2-D}. \tag{4.14}$$

One can see the rhs is not irreducible anymore, by employing (4.11),

$$E_{2,2-D} = \frac{-(\overline{H_b K})^{\bullet}}{(\overline{KK})^{\bullet}} E^{(b)}_{1,3-D}[K], \tag{4.15}$$

where the reference vector K satisfies $Q^*K \neq 0$ and the rhs corresponds to integrals of tadpole topology acquired by removing one propagator from I_2. We then use (3.6) to pull out the K,

$$E^{(b)}_{1,3-D}[K] = \beta_{1,3-D}(\overline{KL})_{(b)}E^{(b)}_{1,1-D}. \tag{4.16}$$

Note that the Q matrix in (4.16) is just a number $Q_{(b)} = m^2$, so its inverse in $(\overline{KL})_{(b)}$ is just $Q^{-1}_{(b)} = \frac{1}{m^2}$. Combining (4.15), (4.16), and (4.14), we finally find

$$I_2^{m_1=m_2,q_2^2=0} = \frac{2-D}{2}\frac{(\overline{H_b K})^{\bullet}(\overline{KL})_{(b)}}{(\overline{KK})^{\bullet}} I_{2;\hat{b}} = \frac{D-2}{2m^2} I_{2;\hat{1}}. \tag{4.17}$$

One can check that the result above does not depend on the choice of $K = (a,b)$ as long as $a \neq b$.

(ii) $\tilde{Q}L = 0$: *Massless tensor bubble with equal internal masses*

Here, we consider the reduction of the tensor bubble $I_2^{(1);m_1=m_2,q_2^2=0}$. Recalling the first equation of (3.14),

$$I_2^{(1)} = -\Gamma(1-D/2)(D-2)E_{2,2-D}[V], \tag{4.18}$$

we just need to reduce $E_{2,2-D}[V]$. Due to $\tilde{Q}L = 0$, we use (4.8) to get

$$E_{2,2-D}[V] = \frac{-1}{2(\overline{VV})^{\bullet}}(\overline{H_b V})^{\bullet} E^{(b)}_{1,3-D}[V^2], \tag{4.19}$$

where the rhs corresponds to integrals of tadpoles with the nondegenerate Q matrix $Q_{(b)} = m^2$. We can use (3.6) to reduce it iteratively,

$$\begin{aligned} E^{(b)}_{1,3-D}[V^2] &= \alpha_{1,3-D}[(\overline{VV})_{(b)}E^{(b)}_{1,1-D} \\ &\quad + (1-D)(\overline{VL})_{(b)}E^{(b)}_{1,1-D}[V]] \\ &= \alpha_{1,3-D}[(\overline{VV})_{(b)}E^{(b)}_{1,1-D} \\ &\quad + (1-D)(\overline{VL})_{(b)}\beta_{1,1-D}(\overline{VL})_{(b)}E^{(b)}_{1,-1-D}]. \end{aligned} \tag{4.20}$$

There are two terms in the last line, we need to reduce the second one $E^{(b)}_{1,-1-D}$ since it corresponds to the $(D+2)$-dimensional tadpole. Using (3.8) to lower the dimension to D, we get

$$E^{(b)}_{1,-1-D} = \frac{E^{(b)}_{1,1-D}}{\beta_{1,1-D}(\overline{LL})_{(b)}}. \tag{4.21}$$

Plugging (4.20), (4.21), (4.19) into (4.18), we have

$$I_2^{(1);m_1=m_2,q_2^2=0} = \frac{(\overline{H_b V})^{\bullet}((1-D)(\overline{VL})^2_{(b)} + (\overline{LL})_{(b)}(\overline{VV})_{(b)})}{2(\overline{VV})^{\bullet}(\overline{LL})_{(b)}} I_{2;\hat{b}}. \tag{4.22}$$

Using

$$\begin{aligned} &(\overline{VV})^{\bullet} = m^2(R\cdot q_2)^2, \\ &(\overline{H_b V})^{\bullet} = \{-m^2 R\cdot q_2, m^2 R\cdot q_2\}, \\ &(\overline{VV})_{(b)} = \left\{\frac{(R\cdot q_2)^2}{m^2}, 0\right\}, \quad (\overline{VL})_{(b)} = \left\{\frac{R\cdot q_2}{m^2}, 0\right\}, \\ &(\overline{LL})_{(b)} = \frac{1}{m^2}, \end{aligned} \tag{4.23}$$

we find the reduction relation,

[5]One can check this by using FIRE or direct calculation.

$$I_2^{(1);m_1=m_2,q_2^2=0} = \frac{(D-2)R\cdot q_2}{2m^2} I_{2;\hat{1}}, \tag{4.24}$$

where $I_{2;\hat{1}}$ is just the tadpole $I_1[m]$ with mass m.

(iii) $\tilde{Q}L \neq 0$: *Scalar bubble*

We then consider the scalar bubble $I_2^{(1)}$ with degenerate Q but $m_1 \neq m_2$. Here, we will show that it can be reduced to two tadpoles using our method. The equation $\det Q = 0$ gives two solutions,

$$q_2^2 = (m_1 \pm m_2)^2, \tag{4.25}$$

which are just the poles in the reduction coefficients of bubbles [see (3.31)].[6] Here, we choose

$$\tilde{Q} = Q^* = \begin{pmatrix} m_2^2 & \pm m_1 m_2 \\ \pm m_1 m_2 & m_1^2 \end{pmatrix}. \tag{4.26}$$

One can check that $\tilde{Q}L \neq 0$ for $m_1 \neq m_2$. Due to $(\dot{\overline{LL}}) \neq 0$, we can use (4.4) to reduce $E_{2,2-D}$ in the expansion,

$$I_2 = \Gamma(2-D/2)E_{2,2-D}, \tag{4.27}$$

and we get

$$\begin{aligned} E_{2,2-D} &= \frac{-(\dot{\overline{H_b L}})}{(3-D)(\dot{\overline{LL}})} E^{(b)}_{1,3-D} \\ &= \frac{(\dot{\overline{H_b L}})\beta_{1,3-D}}{(D-3)(\dot{\overline{LL}})} (\overline{LL})_{(b)} E^{(b)}_{1,1-D}. \end{aligned} \tag{4.28}$$

Plugging (4.28) into (4.27), we have

$$I_2^{m_1=m_2,q_2^2=0} = \frac{(D-2)(\dot{\overline{H_b L}})(\overline{LL})_{(b)}}{2(D-3)(\dot{\overline{LL}})} I_{2;\hat{b}}. \tag{4.29}$$

One can find the explicit expression easily. For $q_2^2 = (m_1+m_2)^2$, the explicit expression is

$$\begin{aligned} I_2^{m_1=m_2,q_2^2=0} = &\frac{D-2}{2(D-3)m_2(m_1+m_2)} I_{2;\hat{1}} \\ &+ \frac{D-2}{2(D-3)m_1(m_1+m_2)} I_{2;\hat{2}}. \end{aligned} \tag{4.30}$$

(iv) $\tilde{Q}L \neq 0$: *Tensor bubble*

Here, we discuss the reduction of the rank-1 bubble with $q_2^2 = (m_1 \pm m_2)^2$. First, we expand it using (2.23)

$$I_2^{(1)} = -\Gamma(1-D/2)(D-2)E_{2,2-D}[V]. \tag{4.31}$$

Choosing $\tilde{Q} = Q^*$, due to $(\dot{\overline{LL}}) = LQ^*L \neq 0$, we can use (4.4) to reduce the rhs to

$$\begin{aligned} E_{2,2-D}[V] = &\frac{1}{(D-2)(\dot{\overline{LL}})} \\ &\times [(\dot{\overline{H_b L}})E^{(b)}_{1,3-D}[V] + (\dot{\overline{VL}})E_{2,2-D}], \end{aligned} \tag{4.32}$$

where the second term has been discussed in the last example. We just need to reduce the first term $E^{(b)}_{1,3-D}[V]$, which corresponds to a tadpole. Using (3.6), we find

$$E^{(b)}_{1,3-D}[V] = \beta_{1,3-D}(\overline{VL})_{(b)} E^{(b)}_{1,1-D}, \tag{4.33}$$

where the rhs is a D-dimensional scalar tadpole. Plugging (4.33), (4.32) into (4.31), we get

$$\begin{aligned} I_2^{(1);q_2^2=(m_1\pm m_2)^2} = &\frac{(\dot{\overline{H_b L}})(\overline{VL})_{(b)}}{(\dot{\overline{LL}})} I_{2;\hat{b}} \\ &+ \frac{2}{D-2}\frac{(\dot{\overline{VL}})}{(\dot{\overline{LL}})} I_2^{q_2^2=(m_1\pm m_2)^2}. \end{aligned} \tag{4.34}$$

Then, we refer to the result (4.29) and find

$$\begin{aligned} I_2^{(1);q_2^2=(m_1\pm m_2)^2} = &\frac{(\dot{\overline{H_b L}})(\overline{VL})_{(b)}}{(\dot{\overline{LL}})} I_{2;\hat{b}} \\ &+ \frac{(\dot{\overline{H_b L}})(\overline{LL})_{(b)}(\dot{\overline{VL}})}{(D-3)(\dot{\overline{LL}})^2} I_{2;\hat{b}}. \end{aligned} \tag{4.35}$$

For $q_2^2 = (m_1+m_2)^2$, the explicit expression is

$$\begin{aligned} I_2^{(1);q_2^2=(m_1+m_2)^2} = &\frac{((D-2)m_1+(D-3)m_2)R\cdot q_2}{(D-3)m_2(m_1+m_2)^2} I_{2;\hat{1}} \\ &+ \frac{R\cdot q_2}{(D-3)(m_1+m_2)^2} I_{2;\hat{2}}. \end{aligned} \tag{4.36}$$

V. GENERAL EXPRESSION OF $C^{(r)}_{n\to n}$

It seems hard to solve (3.6), but if we only care about the reduction coefficients to the same topology, we can ignore the first term for it contributes only to lower topologies. Then by iteratively using (3.6), keeping only the second and the third term, one finds that

[6] The condition is the threshold (or the Landau poles) of the integrals. It appears because the singularity of Landau poles becomes higher with higher power of propagators. Thus, it will appear in both master integrals and the reduction coefficients.

$$E_{n,k}[V^i] = \sum_j \frac{i!}{j!(i-j)!!}(\overline{VV})^{\frac{i-j}{2}}(\overline{VL})^j \frac{\prod_{l=1}^{j}(k-i-l+1)}{\prod_{l=1}^{\frac{i+j}{2}}(k+n-2l)} \times E_{n,k-i-j} + \text{Lower topology}. \quad (5.1)$$

Although the first term has the same topology, with different choices of i, j, the dimension is different, thus we need to reduce it further.

To simplify our denotation, we define

$$\mathcal{E}_{n,k,i;j} = \frac{i!}{j!(i-j)!!}\frac{(k-i)!(k+n-i-j-2)!!}{(k-i-j)!(k+n-2)!!}. \quad (5.2)$$

Thus, (5.1) becomes

$$E_{n,k}[V^i] = \sum_j \mathcal{E}_{n,k,i;j}(\overline{VV})^{\frac{i-j}{2}}(\overline{VL})^j E_{n,k-i-j} + \text{Lower topology}. \quad (5.3)$$

Here, we give the computation for tensor reduction with all propagators power one.[7] Thus, we need to reduce all $E_{n,n-D-(r-i)}[V^i]$ to $E_{n,n-D}$. After using (3.8) to repeatedly lift k, we have

$$E_{n,k-2s} = \frac{E_{n,k}}{(\overline{LL})^s \prod_{l=1}^{s}(\beta_{n,k-2s+2l})} + \sum_{l=1}^{s} \frac{-\alpha_{n,k-2s+2l}(\overline{H_j L})}{\prod_{p=1}^{l}\beta_{n,k-2s+2p}(\overline{LL})^l} E^{(j)}_{n-1,k-1-2s+2l} = \frac{\mathcal{K}^+_{n,k,s}}{(\overline{LL})^s}E_{n,k} + \sum_{j=1}^{s}\frac{\mathcal{K}^+_{n,k,s;j}}{(\overline{LL})^j}(\overline{H_b L})E^{(b)}_{n-1,k-1-2(s-j)}, \quad (5.4)$$

where we have defined

$$\mathcal{K}^+_{n,k,s} = \frac{1}{\prod_{l=1}^{s}(\beta_{n,k-2s+2l})}, \qquad \mathcal{K}^+_{n,k,s;j} = \frac{-\alpha_{n,k-2s+2j}}{\prod_{p=1}^{j}\beta_{n,k-2s+2p}}. \quad (5.5)$$

Since the second term in (5.4) belongs to lower topologies, which can be ignored, finally, we have

$$\begin{aligned} E_{n,n-D-(r-i)}[V^i] &= \sum_j \mathcal{E}_{n,n-D-(r-i),i;j}(\overline{VV})^{\frac{i-j}{2}}(\overline{VL})^j E_{n,n-D-(r+j)} + \text{Lower} \\ &= \sum_j \mathcal{E}_{n,n-D-(r-i),i;j}(\overline{VV})^{\frac{i-j}{2}}(\overline{VL})^j \frac{\mathcal{K}^+_{n,n-D,\frac{r+j}{2}}}{(\overline{LL})^{\frac{r+j}{2}}} E_{n,n-D} + \text{Lower} \\ &= \sum_j \mathcal{E}_{n,n-D-(r-i),i;j}(\overline{VV})^{\frac{i-j}{2}}(\overline{VL})^j \frac{\mathcal{K}^+_{n,n-D,\frac{r+j}{2}}}{(\overline{LL})^{\frac{r+j}{2}}} \frac{(-)^n}{\Gamma(n-D/2)} I_{n;D} + \text{Lower}. \end{aligned} \quad (5.6)$$

So

$$\begin{aligned} I^{(r)}_{n;D} &= \sum_{i=0}^{r} \frac{\Gamma(n-D/2-r)}{(-1)^{n+r}} \mathcal{C}^{D/2+r-n}_{r,i}(R^2)^{\frac{r-i}{2}} E_{n,n-D-(r-i)}[V^i] \\ &= \sum_{i=0}^{r} \frac{\Gamma(n-D/2-r)}{(-1)^{n+r}} \mathcal{C}^{D/2+r-n}_{r,i}(R^2)^{\frac{r-i}{2}} \sum_j \mathcal{E}_{n,n-D-(r-i),i;j}(\overline{VV})^{\frac{i-j}{2}}(\overline{VL})^j \frac{\mathcal{K}^+_{n,n-D,\frac{r+j}{2}}}{(\overline{LL})^{\frac{r+j}{2}}} \frac{(-)^n}{\Gamma(n-D/2)} I_{n;D} + \text{Lower} \\ &= \frac{(-)^r\Gamma(n-D/2-r)}{\Gamma(n-D/2)} \sum_{i=0}^{r}\sum_{j=0}^{i} \mathcal{C}^{D/2+r-n}_{r,i}\mathcal{E}_{n,n-D-(r-i),i;j}\mathcal{K}^+_{n,n-D,\frac{r+j}{2}}(R^2)^{\frac{r-i}{2}} \frac{(\overline{VV})^{\frac{i-j}{2}}(\overline{VL})^j}{(\overline{LL})^{\frac{r+j}{2}}} I_{n;D} + \text{Lower}. \end{aligned} \quad (5.7)$$

From it, we read out reduction coefficient,

$$C^{(r)}_{n\to n} = \sum_{i=0}^{r}\sum_{j=0}^{i} c^{(r)}_{n\to n;i,j}(R^2)^{\frac{r-i}{2}} \frac{(\overline{VV})^{\frac{i-j}{2}}(\overline{VL})^j}{(\overline{LL})^{\frac{r+j}{2}}} \quad (5.8)$$

with

$$c^{(n)}_{n\to n;i,j} = \frac{(-)^r\Gamma(n-D/2-r)}{\Gamma(n-D/2)} \times \mathcal{C}^{D/2+r-n}_{r,i}\mathcal{E}_{n,n-D-(r-i),i;j}\mathcal{K}^+_{n,n-D,\frac{r+j}{2}}, \quad (5.9)$$

where we require r, i, j having the same parity.

[7]The general results for arbitrary tensor structure, general powers as well as the coefficients for lower topologies are given in Appendix C.

VI. DISCUSSION

Although the main target of the paper [33] is to understand general one-loop integrals from a geometric point of view, it does contain many important and valuable results. In this paper, we have elaborated on the method to compute reduction coefficients for general one-loop integrals.

The essential idea of the method is to put the whole one-loop Feynman integrals in the projective space, in which integrals have compact forms and geometry properties. Using a vital recursion relation of $E_{n,k}[T]$, one achieves the wanted reduction. The advantage and the most promising point of this method are that we can solve any one-loop integrals with higher poles and tensor structures at the same time, which is demonstrated by some examples in Secs. III and IV. The language of projective space simplifies the reduction process a lot by keeping the elegant contractions like $(\overline{HL}), (\overline{LL})$ in these recursion relations without expansion, thus making the whole reduction process a symbolic calculation. Although results listed in the paper can be obtained by other methods, such as the PV-reduction and IBP method, the very compact and symmetric analytic form is the new feature of our method. It is also efficient for practical calculation.

For some programs based on the traditional IBP method, like FIRE, LiteRed, KIRA, etc. [35–41], they do reduction by solving linear equations where the determinant of the Gram matrix appears with the full-expanded form, which makes the final results too complicated to read. The appearance of (reduced) Gram matrix has also been observed in our recent work [31,32].

One obvious idea is to generalize the above method to the reduction of two-loops and higher-loops. Recently, using the improved PV-reduction method with auxiliary vectors, we have shown how to do the general tensor reduction for two-loop sunset integrals [42]. From the results in [31,32] and results in this paper, we see that these two methods have some correspondences. In other words, they treat the same thing from different but related angles. Thus, it is natural to ask if we can translate two-loop integrals to a form in the projective space and establish a similar recursion relation, possibly using the reduction results of the sub-one-loop integrals studied in this paper.

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APPENDIX A: RECURSION RELATIONS OF $E_{n,k}[T]$

In this section, we recall the proof of the relation (3.1) given in [33]. Let us calculate directly

$$\mathrm{d}_X\left[\frac{\langle Q^{-1}T[X^{k-1}]X\mathrm{d}^{n-2}X\rangle}{(XQX)^{\frac{n+k-2}{2}}}\right] = \frac{1}{(XQX)^{\frac{n+k-2}{2}}}\left[-(n+k-2)\frac{(XQ\mathrm{d}X)}{(XQX)}\langle Q^{-1}T[X^{k-1}]X\mathrm{d}^{n-2}X\rangle + (n-1)\langle Q^{-1}T[X^{k-1}]\mathrm{d}^{n-1}X\rangle\right.$$
$$\left. + (k-1)X_{I_1}\frac{1}{(n-2)!}Q^{-1}_{I_1 i_1}T^{i_1 i_2 i_3\ldots i_k}X_{i_2}X_{i_3}\cdots \mathrm{d}X_{i_k}X_{I_2}\wedge \mathrm{d}^{n-2}X\epsilon^{I_1 I_2\ldots I_k}\right], \tag{A1}$$

where we require T is completely symmetric of the last $(k-1)$ indices $i_2, i_3, \ldots i_k$. For simplicity, we denote

$$A_I = Q^{-1}_{Is}T^{s i_2 i_3\ldots i_k}X_{i_2}X_{i_3}\cdots X_{i_k}, \qquad B_{Ir} = (Q^{-1}T[X^{k-2}])_{Ir}. \tag{A2}$$

Thus, (A1) becomes

$$\mathrm{d}_X\left[\frac{\langle Q^{-1}T[X^{k-1}]X\mathrm{d}^{n-2}X\rangle}{(XQX)^{\frac{n+k-2}{2}}}\right] = \frac{1}{(XQX)^{\frac{n+k-2}{2}}}\left[-(n+k-2)\frac{(XQ\mathrm{d}X)}{(XQX)}\langle AX\mathrm{d}^{n-2}X\rangle + (n-1)\langle A\mathrm{d}^{n-1}X\rangle + (k-1)\mathrm{d}X_I\langle B^I X\mathrm{d}^{n-2}X\rangle\right]. \tag{A3}$$

Using the fact,

$$XQA = XQQ^{-1}T[X^{k-1}] = T[X^k], \qquad B_I{}^I = (\mathrm{tr}_Q T)[X^{k-2}], \tag{A4}$$

and the Schouten identity,

$$\mathrm{d}X^I\langle AX\mathrm{d}X^{n-2}\rangle + A^I\langle X\mathrm{d}^{n-1}X\rangle - X^I\langle A\mathrm{d}^{n-1}X\rangle = 0, \quad \forall A^I \tag{A5}$$

to simplify the first term in (A3) as

$$XQ\mathrm{d}X\langle AX\mathrm{d}^{n-2}X\rangle = QX\mathrm{d}X(-A\langle X\mathrm{d}^{n-1}X\rangle + X\langle A\mathrm{d}^{n-1}X\rangle)$$
$$= -(XQA)\langle X\mathrm{d}^{n-1}X\rangle + (QXX)\langle A\mathrm{d}^{n-1}X\rangle, \tag{A6}$$

and the third term in (A3) as

$$\mathrm{d}X_I\langle B^I X\mathrm{d}^{n-2}X\rangle = X^I\langle B_I\mathrm{d}^{n-1}X\rangle - B^I_I\langle X\mathrm{d}^{n-1}X\rangle, \tag{A7}$$

the (A3) becomes

$$\mathrm{d}_X\left[\frac{\langle Q^{-1}T[X^{k-1}]X\mathrm{d}^{n-2}X\rangle}{(XQX)^{\frac{n+k-2}{2}}}\right]=(n+k-2)\frac{T[X^k]\langle X\mathrm{d}^{n-1}X\rangle}{(XQX)^{\frac{n+k}{2}}}-(k-1)\mathrm{tr}_Q T[X^{k-2}]\langle X\mathrm{d}^{n-1}X\rangle, \tag{A8}$$

which is nothing but the wanted (3.1).

The derivation above can be generalized to the case $\det Q=0$, where the Q^{-1} does not exist. In this case, we can replace Q^{-1} by arbitrary matrix $\tilde{Q}$ in (A1) and repeat the same derivations to reach the similar expression likes (A8), except Q^{-1} replaced by $\tilde{Q}$. Rearranging (A8), we get

$$E_{n,k}[(Q\tilde{Q}T)]=\alpha_{n,k}E^{(b)}_{n-1,k-1}[(H_b\tilde{Q}T)]+\beta_{n,k}E_{n,k-2}[\mathrm{tr}_{\tilde{Q}}T], \tag{A9}$$

where $(Q\tilde{Q}T)$ has the same rank as T,

$$(Q\tilde{Q}T)^{I_1I_2,\ldots,I_k}=Q^{I_1J_1}\tilde{Q}_{J_1J_2}T^{J_2,I_2,I_3,\ldots,I_k},$$
$$(\mathrm{tr}_{\tilde{Q}}T)^{I_3,I_4,\ldots,I_k}=\tilde{Q}_{I_1I_2}T^{I_1I_2\ldots I_k}. \tag{A10}$$

In (A9), the lhs is the first term in rhs of (A8), while the first term on the rhs is the boundary contribution of lhs of (A8).

APPENDIX B: MORE RESULTS

1. Bubbles

Here list the results for rank from $r=1$ to $r=4$, where we set $q_1=0$.

(i) $r=1$

$$I_2^{(1)}=\sum_i-\frac{(\overline{LL})(\overline{H_iV})-(\overline{H_iL})(\overline{VL})}{(\overline{LL})}I_{2;\hat{i}}+\frac{2(\overline{VL})}{(\overline{LL})}I_2. \tag{B1}$$

So we have

$$C^{(1)}_{2\to2;\hat{2}}=-\frac{(\overline{LL})(\overline{H_2V})-(\overline{H_2L})(\overline{VL})}{(\overline{LL})}=-\frac{R\cdot q_2}{q_2^2},$$
$$C^{(1)}_{2\to2;\hat{1}}=-\frac{(\overline{LL})(\overline{H_1V})-(\overline{H_1L})(\overline{VL})}{(\overline{LL})}=\frac{R\cdot q_2}{q_2^2},$$
$$C^{(1)}_{2\to2}=\frac{2(VQ^{-1}L)}{(LQ^{-1}L)}=\frac{(m_1^2-m_2^2+q_2^2)R\cdot q_2}{q_2^2}. \tag{B2}$$

(ii) $r=2$

$$C^{(2)}_{2\to2;\hat{i}}=\frac{2R^2(\overline{H_iL})}{(D-1)(\overline{LL})}+\frac{2D(\overline{H_iL})(\overline{VL})^2}{(D-1)(\overline{LL})^2}-\frac{2(\overline{H_iV})((\overline{VL})+(\overline{VL})_{(i)})}{(\overline{LL})_{(i)}}-\frac{2(\overline{H_iL})(-D(\overline{VL})^2+(\overline{VL})^2+(\overline{LL})_{(i)}(\overline{VV}))}{(D-1)(\overline{LL})_{(i)}(\overline{LL})},$$
$$C^{(2)}_{2\to2}=\frac{4(R^2-(\overline{VV}))}{(D-1)(\overline{LL})}+\frac{4D(\overline{VL})^2}{(D-1)(\overline{LL})^2}. \tag{B3}$$

The exact form is

$$C^{(2)}_{2\to2;\hat{2}}=\frac{(m_1^2-m_2^2+q_2^2)R^2}{(D-1)q_2^2}-\frac{D(m_1^2-m_2^2+q_2^2)(R\cdot q_2)^2}{(D-1)q_2^4},$$
$$C^{(2)}_{2\to2;\hat{1}}=-\frac{R^2(m_1^2-m_2^2-q_2^2)}{(D-1)q_2^2}-\frac{(-Dm_1^2+Dm_2^2-3Dq_2^2+4q_2^2)(R\cdot q_2)^2}{(D-1)q_2^4},$$
$$C^{(2)}_{2\to2}=-\frac{R^2(-2m_1^2(m_2^2+q_2^2)+(m_2^2-q_2^2)^2+m_1^4)}{(D-1)q_2^2}+\frac{(-2m_1^2(Dm_2^2-(D-2)q_2^2)+D(m_2^2-q_2^2)^2+Dm_1^4)(R\cdot q_2)^2}{(D-1)q_2^4}.$$

(iii) $r=3$

$$\begin{aligned}C^{(3)}_{2\to2;\hat{i}}=&-\frac{4(D+1)(\overline{H_iV})((\overline{VL})_{(i)}(\overline{VL})+(\overline{VL})^2+(\overline{VL})^2_{(i)})}{D(\overline{LL})^2_{(i)}}-\frac{4(\overline{H_iV})(-2(\overline{VV})-(\overline{VV})_{(i)}+3R^2)}{D(\overline{LL})_{(i)}}\\&+\frac{4(D+2)(\overline{H_iL})(\overline{VL})^3}{(D-1)(\overline{LL})^3}+\frac{4(\overline{H_iL})(\overline{VL})(D(\overline{VL})^2+(\overline{VL})^2-3(\overline{LL})_{(i)}(\overline{VV})+3R^2(\overline{LL})_{(i)})}{D(\overline{LL})^2_{(i)}(\overline{LL})}\\&+\frac{4(\overline{H_iL})(\overline{VL})}{(D-1)D(\overline{LL})_{(i)}(\overline{LL})^2}(D^2(\overline{VL})^2+D(\overline{VL})^2-3D(\overline{LL})_{(i)}(\overline{VV})-2(\overline{VL})^2+3DR^2(\overline{LL})_{(i)}),\end{aligned}$$
$$C^{(3)}_{2\to2}=\frac{24R^2(\overline{VL})}{(D-1)(\overline{LL})^2}+\frac{8(D+2)(\overline{VL})^3}{(D-1)(\overline{LL})^3}-\frac{24(\overline{VV})(\overline{VL})}{(D-1)(\overline{LL})^2}. \tag{B4}$$

The exact form is

$$
\begin{aligned}
C^{(3)}_{2\to2;\hat{2}} =& -\frac{(D+2)(m_1^2-m_2^2)^2(R\cdot q_2)^3}{(D-1)q_2^6}+\frac{3R^2R\cdot q_2}{D-1}\\
&+\frac{(2D(D+2)m_2^2-2(D^2-2D+4)m_1^2)(R\cdot q_2)^3+3D(m_1^2-m_2^2)^2R^2R\cdot q_2}{(D-1)Dq_2^4}\\
&-\frac{R\cdot q_2(6R^2((D-2)m_1^2+Dm_2^2)+D(D+2)(R\cdot q_2)^2)}{(D-1)Dq_2^2},\\
C^{(3)}_{2\to2;\hat{1}} =& \frac{3R^2(4(D-1)m_2^2q_2^2-Dm_1^4+2Dm_2^2m_1^2-Dm_2^4+Dq_2^4)R\cdot q_2}{(D-1)Dq_2^4}\\
&\frac{(R\cdot q_2)^3}{(D-1)Dq_2^6}[-4(D^2+D-2)m_2^2q_2^2-2Dm_1^2((D+2)m_2^2-2(D-1)q_2^2)\\
&+D(D+2)m_1^4+D(D+2)m_2^4+D(7D-10)q_2^4],\\
C^{(3)}_{2\to2} =& \frac{(m_1^2-m_2^2+q_2^2)(R\cdot q_2)^3}{(D-1)q_2^6}[-2m_1^2((D+2)m_2^2-(D-4)q_2^2)+(D+2)(m_2^2-q_2^2)^2+(D+2)m_1^4]\\
&-\frac{3R^2(m_1^2-m_2^2+q_2^2)(-2m_1^2(m_2^2+q_2^2)+(m_2^2-q_2^2)^2+m_1^4)R\cdot q_2}{(D-1)q_2^4}. \qquad \text{(B5)}
\end{aligned}
$$

(iv) $r=4$

$$
\begin{aligned}
C^{(4)}_{2\to2;\hat{i}} =& \frac{8(D+2)(\overline{H_iL})(\overline{VL})^2((D^2+3D-4)(\overline{VL})^2+6D(\overline{LL})_{(i)}(R^2-(\overline{VV})))}{(D-1)D(D+1)(\overline{LL})_{(i)}(\overline{LL})^3}\\
&+\frac{8(\overline{H_iL})}{D(D+1)(\overline{LL})^3_{(i)}(\overline{LL})}((D^2+4D+3)(\overline{VL})^4+3(\overline{LL})^2_{(i)}(R^2-(\overline{VV}))^2+6(D+1)(\overline{LL})_{(i)}(\overline{VL})^2(R^2-(\overline{VV})))\\
&+\frac{8(\overline{H_iL})}{D(D^2-1)(\overline{LL})^2_{(i)}(\overline{LL})^2}((D^3+4D^2-D-4)(\overline{VL})^4+6(D^2+D-2)(\overline{LL})_{(i)}(\overline{VL})^2(R^2-(\overline{VV}))\\
&+3D(\overline{LL})^2_{(i)}(R^2-(\overline{VV}))^2)+\frac{8(D+2)(D+4)(\overline{H_iL})(\overline{VL})^4}{(D^2-1)(\overline{LL})^4}\\
&-\frac{8(\overline{H_iV})}{D(\overline{LL})^2_{(i)}}((\overline{VL})(-5(\overline{VV})-(\overline{VV})_{(i)}+6R^2)+(\overline{VL})_{(i)}(-3(\overline{VV})-3(\overline{VV})_{(i)}+6R^2))\\
&-\frac{8(D+3)((\overline{VL})+(\overline{VL})_{(i)})((\overline{VL})^2+(\overline{VL})^2_{(i)})(\overline{H_iV})}{D(\overline{LL})^3_{(i)}},\\
C^{(4)}_{2\to2} =& \frac{96(D+2)(\overline{VL})^2(R^2-(\overline{VV}))}{(D^2-1)(\overline{LL})^3}+\frac{48(R^2-(\overline{VV}))^2}{(D^2-1)(\overline{LL})^2}+\frac{16(D^2+6D+8)(\overline{VL})^4}{(D^2-1)(\overline{LL})^4}. \qquad \text{(B6)}
\end{aligned}
$$

2. Triangles

For triangle topology, we have presented results for scalar triangles with higher poles. Here, we present some examples, including the tensor triangles without higher poles and with higher poles.

For the tensor triangles without higher poles, we have

(i) $r=1$

$$
C^{(1)}_{3\to3;\hat{i}} = \frac{(\overline{H_iL})(\overline{VL})}{(\overline{LL})}-(\overline{H_iV}), \qquad C^{(1)}_{3\to3} = \frac{2(\overline{VL})}{(\overline{LL})}. \qquad \text{(B7)}
$$

The expression is the same as (B2) for bubble topology. This phenomenon is not an accident and will be persistent to other topologies.

(ii) $r=2$

$$\begin{aligned} C^{(2)}_{3\to 3;\widehat{ij}} &= \frac{(\overline{H_iL})(\overline{H_jL})_{(i)}(\overline{VL})^2}{(\overline{LL})_{(i)}(\overline{LL})} - \frac{(\overline{H_jL})_{(i)}(\overline{H_iV})((\overline{VL})_{(i)}+(\overline{VL}))}{(\overline{LL})_{(i)}} + (\overline{H_iV})(\overline{H_jV})_{(i)} + (i\leftrightarrow j),\\ C^{(2)}_{3\to 3;\hat{i}} &= \frac{2(\overline{H_iL})((D-2)(\overline{VL})^2+(\overline{LL})_{(i)}(R^2-(\overline{VV})))}{(D-2)(\overline{LL})_{(i)}(\overline{LL})} + \frac{2(D-1)(\overline{H_iL})(\overline{VL})^2}{(D-2)(\overline{LL})^2}\frac{2(\overline{H_iV})((\overline{VL})_{(i)}+(\overline{VL}))}{(\overline{LL})_{(i)}},\\ C^{(2)}_{3\to 3} &= \frac{4((D-1)(\overline{VL})^2+(\overline{LL})(R^2-(\overline{VV})))}{(D-2)(\overline{LL})^2}. \end{aligned} \tag{B8}$$

Again, the last two coefficients are very similar to these given in (B3).

For the tensor with higher poles, we have

(i) $v=4,\ r=1$

$$\begin{aligned} C^{(1)}_{\mathrm{v}_3\to 3;\widehat{ij}} &= \frac{1}{8}(D-2)(\overline{LL})_{(ij)}((\overline{H_iS})(\overline{H_jS})_{(i)} + (i\leftrightarrow j)),\\ C^{(1)}_{\mathrm{v}_3\to 3;\hat{i}} &= \frac{D-3}{4}\left(\frac{(\overline{H_iL})(\overline{SL})^2}{(\overline{LL})} - (\overline{H_iS})(\overline{SL})_{(i)} - (\overline{H_iS})(\overline{SL})\right),\\ C^{(1)}_{\mathrm{v}_3\to 3} &= \frac{(D-3)(\overline{SL})^2}{2(\overline{LL})} - \frac{1}{2}(\overline{SS}). \end{aligned} \tag{B9}$$

Here, we have suppressed the process to take the coefficient of tz^{v_3-1},i.e., $|_{tz^{\mathrm{v}_3-1}}$. The similarity of the above expression with the one given in (3.25) is obvious.

(ii) $v=4,\ r=2$

$$\begin{aligned} C^{(2)}_{\mathrm{v}_3\to 3;\widehat{ij}} &= \frac{(D-2)(\overline{H_iL})(\overline{H_jL})_{(i)}(\overline{SL})^3}{6(\overline{LL})_{(i)}(\overline{LL})} - \frac{(D-2)(\overline{H_jL})_{(i)}(\overline{H_iS})((\overline{SL})^2_{(i)}+(\overline{SL})(\overline{SL})_{(i)}+(\overline{SL})^2)}{6(\overline{LL})_{(i)}}\\ &\quad + \frac{1}{6}(D-2)(\overline{H_iS})(\overline{H_jS})_{(i)}((\overline{SL})_{(i)}+(\overline{SL})+(\overline{SL})_{(ij)}) + (i\leftrightarrow j),\\ C^{(2)}_{\mathrm{v}_3\to 3;\hat{i}} &= \frac{(\overline{H_iL})(\overline{SL})((D-2)(\overline{SL})^2+3(\overline{LL})_{(i)}(R^2-(\overline{SS})))}{3(\overline{LL})_{(i)}(\overline{LL})} + \frac{(D-1)(\overline{H_iL})(\overline{SL})^3}{3(\overline{LL})^2}\\ &\quad + \frac{(\overline{H_iS})((\overline{LL})_{(i)}((\overline{SS})_{(i)}+2(\overline{SS})-3R^2)-(D-2)((\overline{SL})^2_{(i)}+(\overline{SL})(\overline{SL})_{(i)}+(\overline{SL})^2))}{3(\overline{LL})_{(i)}},\\ C^{(2)}_{\mathrm{v}_3\to 3} &= \frac{2(D-1)(\overline{SL})^3}{3(\overline{LL})^2} + \frac{2(\overline{SL})(R^2-(\overline{SS}))}{(\overline{LL})}. \end{aligned} \tag{B10}$$

Here, when calculating the reduction coefficients, we take the coefficient of z^{v_3-1} for terms containing one R^2 and one S, while we take the coefficient of $t^2z^{\mathrm{v}_3-1}$ for terms containing three S's.

3. Boxes

For the box topology, we present three cases: (1) $r=1, v-n=0$, (2) $r=0, v-n=1$, (3) $r=1, v-n=1$. As pointed out in the triangle topology, the reduction coefficients have some similarities between different topologies. In fact, the similarity is classified by the pair $(r, v-n)$ as one can check by using the results listed in the Appendix and the main body of the paper.

(i) $r=1, v-n=0$

$$C^{(1)}_{4\to 4;\hat{i}} = \frac{(\overline{H_i L})(\overline{VL})}{(\overline{LL})} - (\overline{H_i V}), \qquad C^{(1)}_{4\to 4} = \frac{2(\overline{VL})}{(\overline{LL})}. \tag{B11}$$

(ii) $r=0, v-n=1$

$$\begin{aligned}
C_{\mathbf{v}_4\to 4;\widehat{ijk}} &= -\frac{1}{16}(D-2)(\overline{LL})_{(ijk)}((\overline{H_j L})_{(i)}(\overline{H_k L})_{(ij)}(\overline{H_i Z}) + \text{permutations of } (ijk)),\\
C_{\mathbf{v}_4\to 4;\widehat{ij}} &= \frac{1}{8}(D-3)(\overline{LL})_{(ij)}((\overline{H_j L})_{(i)}(\overline{H_i Z}) + (i\leftrightarrow j)),\\
C_{\mathbf{v}_4\to 4;\hat{i}} &= -\frac{1}{4}(D-4)(\overline{LL})_{(i)}(\overline{H_i Z}),\\
C_{\mathbf{v}_4\to 4} &= \frac{1}{2}(D-5)(\overline{ZL}).
\end{aligned} \tag{B12}$$

(iii) $r=1, v-n=1$

$$\begin{aligned}
C^{(1)}_{\mathbf{v}_4\to 4;\widehat{ijk}} &= -\frac{1}{16}(D-2)(\overline{LL})_{(ijk)}((\overline{H_k L})_{(ij)}(\overline{H_i S})(\overline{H_j S})_{(i)} + \text{permutations of } (ijk)),\\
C^{(1)}_{\mathbf{v}_4\to 4;\widehat{ij}} &= \frac{1}{8}(D-3)(\overline{LL})_{(ij)}((\overline{H_i S})(\overline{H_j S})_{(i)} + (i\leftrightarrow j)),\\
C^{(1)}_{\mathbf{v}_4\to 4;\hat{i}} &= -\frac{(D-4)((\overline{LL})(\overline{H_i S})((\overline{SL})_{(i)} + (\overline{SL})) - (\overline{H_i L})(\overline{SL})^2)}{4(\overline{LL})},\\
C^{(1)}_{\mathbf{v}_4\to 4} &= \frac{1}{2}\left(\frac{(D-4)(\overline{SL})^2}{(\overline{LL})} - (\overline{SS})\right).
\end{aligned} \tag{B13}$$

4. Pentagons

For pentagon topology, the reduction coefficients are similar, so we just present one example, i.e., scalar pentagon with higher poles.

(i) $v=6$

$$\begin{aligned}
C_{\mathbf{v}_5\to 5;\widehat{ijkl}} &= \frac{1}{32}(D-2)(\overline{LL})_{(ijkl)}((\overline{H_i L})_{(jk)}(\overline{H_j Z})(\overline{H_k L})_{(j)}(\overline{H_l L})_{(ijk)} + \text{permutations of } (ijkl)),\\
C_{\mathbf{v}_5\to 5;\widehat{ijk}} &= -\frac{1}{16}(D-3)(\overline{LL})_{(ijk)}((\overline{H_i L})_{(j)}(\overline{H_k L})_{(ij)}(\overline{H_j Z}) + \text{permutations of } (ijk)),\\
C^{(1)}_{5\to 5;\widehat{ij}} &= \frac{1}{8}(D-4)(\overline{LL})_{(ij)}((\overline{H_j L})_{(i)}(\overline{H_i Z}) + (i\leftrightarrow j)),\\
C_{\mathbf{v}_5\to 5;\hat{i}} &= -\frac{1}{4}(D-5)(\overline{LL})_{(i)}(\overline{H_i Z}),\\
C_{\mathbf{v}_5\to 5} &= \frac{1}{2}(D-6)(\overline{ZL}).
\end{aligned} \tag{B14}$$

One thing we want to point out is that these coefficients have a manifest permutation symmetry. Using these observations, the expression can be very compact, as shown above.

APPENDIX C: GENERAL EXPRESSION OF REDUCTION COEFFICIENTS

One can solve the recursion relations (3.6) in Sec. III iteratively[8] and get general expressions for the final reduction coefficients. Here, we present only the final results for nondegenerate Q without derivation details. We define

[8]For the degenerate case discussed in the Sec. IV, one can do similar computation, although it will be more complicated.

$$\mathcal{F}^{n,k,i}_{i-n_v,s} = \mathcal{P}^{k,i}_{i-n_v+1,s} \prod_{l=0}^{s} \alpha_{n,k-2l} \tag{C1}$$

with

$$\mathcal{P}^{k,i}_{i-d,s} = (i+1-d)\mathcal{P}^{k,i}_{i-d+2,s-1} + (k-i+1+d-2s)\mathcal{P}^{k,i}_{i-d+1,s-1}, \tag{C2}$$

where we set $\mathcal{P}^{k,i}_{i-d,s} = 0$ for either $d < 0$ or $s < 0$. With the initial condition $\mathcal{P}^{k,i}_{0,0} = 1$, (C2) can be solved. For simplicity, we define

$$\begin{aligned} E^{(\mathbf{b}_l)}_{n-l,k-l} &= E^{(b_1,b_2,b_3,\ldots,b_l)}_{n-l,k-l}, \qquad (\overline{WV})_{(\mathbf{b}_l)} = (\overline{WV})_{(b_1,b_2,b_3,\ldots,b_l)} \\ (\overline{HL})_{\hat{\mathbf{b}}_l} &= (\overline{H_{b_1}L})(\overline{H_{b_2}L})_{(b_1)}(\overline{H_{b_3}L})_{(b_1 b_2)}\cdots(\overline{H_{b_l}L})_{(\mathbf{b}_{l-1})} \\ (\overline{LL})^{(s;\mathbf{j}_l)}_{(\mathbf{b}_l)} &= (\overline{LL})^{j_1}(\overline{LL})^{j_2}_{(b_1)}(\overline{LL})^{j_3}_{(b_1,b_2)}\cdots(\overline{LL})^{j_l}_{(\mathbf{b}_{l-1})}(\overline{LL})^{s-j_1-j_2-j_3-\cdots-j_l}_{(\mathbf{b}_l)} \end{aligned} \tag{C3}$$

and

$$\begin{aligned} \sum_{\{\mathbf{l},\mathbf{j},\mathbf{s}\}_a}^{i} &= \sum_{l_1=0}^{i-1}\sum_{j_1=0}^{l_1}\sum_{s_1=\lceil\frac{i-l_1-1}{2}\rceil}^{i-l_1-1}\sum_{l_2=0}^{l_1-1}\sum_{j_2=0}^{l_2}\sum_{s_2=\lceil\frac{l_1-l_2-1}{2}\rceil}^{l_1-l_2-1}\cdots\sum_{l_a=0}^{l_{a-1}-1}\sum_{j_a=0}^{l_a}\sum_{s_a=\lceil\frac{l_{a-1}-l_a-1}{2}\rceil}^{l_{a-1}-l_a-1} \\ \mathcal{F}^{n,k,i}_{\{\mathbf{l},\mathbf{s}\}_a} &= \mathcal{F}^{n,k,i}_{l_1,s_1}\mathcal{F}^{n-1,k-1-2s_1,l_1}_{l_2,s_2}\mathcal{F}^{n-2,k-2-2s_1-2s_2,l_2}_{l_3,s_3}\cdots\mathcal{F}^{n-a,k-a-2s_1-2s_2-\cdots-2s_{a-1},l_{a-1}}_{l_a,s_a} \\ \mathcal{S}_{l_0,\{\mathbf{l},\mathbf{j},\mathbf{s}\}_a;\mathbf{b}_a} &= \prod_{p=1}^{a}(\overline{SS})^{-1+l_{p-1}-l_p-s_p}_{(\mathbf{b}_{p-1})}(\overline{SL})^{1-l_{p-1}+l_p+2s_p}_{(\mathbf{b}_{p-1})}(\overline{SS})^{\frac{l_a-j_a}{2}}_{(\mathbf{b}_a)}(\overline{SL})^{j_a}_{(\mathbf{b}_a)}, \end{aligned} \tag{C4}$$

where specially

$$\sum_{\{\mathbf{l},\mathbf{j},\mathbf{s}\}_0}^{i} = \sum_{j_0=0}^{i}, \qquad \mathcal{F}^{n,k,i}_{\{\mathbf{l},\mathbf{s}\}_0} = 1, \qquad (\overline{HS})_{\widehat{\mathbf{b}_0}} = 1, \qquad \mathcal{S}_{i,\{\mathbf{l},\mathbf{j},\mathbf{s}\}_0;\mathbf{b}_0} = (\overline{SS})^{\frac{i-j_0}{2}}(\overline{SL})^{j_0}, \qquad \mathcal{E}_{n,k-2|\mathbf{s}_0|,l_0;j_0} = \mathcal{E}_{n,k,i;j_0}. \tag{C5}$$

Then, we denote some notations used in dimension shifting as follows:

$$\mathcal{K}^{-}_{n,k,s} = \frac{1}{\prod_{l=1}^{s}(\beta_{n,k-2s+2l})}, \quad \mathcal{K}^{-}_{n,k,s;j} = \frac{-\alpha_{n,k-2s+2j}}{\prod_{p=1}^{j}\beta_{n,k-2s+2p}}, \quad \mathcal{K}^{+}_{n,k,s} = \prod_{p=1}^{s}\beta_{n,k+2s+2-2p}, \quad \mathcal{K}^{+}_{n,k,s;j} = \alpha_{n,k+2s-2j}\prod_{p=1}^{j}\beta_{n,k+2s+2-2p}, \tag{C6}$$

and

$$\begin{aligned} \mathcal{K}^{-;\mathbf{b}_l}_{n,k,2s;l} &= \sum_{j_1=1}^{s}\sum_{j_2=1}^{s-j_1}\sum_{j_3=1}^{s-j_1-j_2}\cdots\sum_{j_l=1}^{s-j_1-j_2-\cdots-j_{l-1}}\frac{1}{(\overline{LL})^{(s;\mathbf{j}_l)}_{(\mathbf{b}_l)}}\mathcal{K}^{-}_{n,k,s;j_1}\mathcal{K}^{-}_{n-1,k-1,s-j_1;j_2}\mathcal{K}^{-}_{n-2,k-2,s-j_1-j_2;j_3}\cdots\mathcal{K}^{-}_{n-l,k-l,s-j_1-j_2-j_3-\cdots-j_l} \\ \mathcal{K}^{+;\mathbf{b}_l}_{n,k,2s;l} &= \sum_{j_1=0}^{s}\sum_{j_2=0}^{s-j_1}\sum_{j_3=0}^{s-j_1-j_2}\cdots\sum_{j_l=0}^{s-j_1-j_2-\cdots-j_{l-1}}\mathcal{K}^{+}_{n,k,s;j_1}\mathcal{K}^{+}_{n-1,k-1,s-j_1;j_2}\mathcal{K}^{+}_{n-2,k-2,s-j_1-j_2;j_3}\cdots\mathcal{K}^{+}_{n-l,k-l,s-j_1-j_2-j_3-\cdots-j_l} \\ \mathcal{K}^{\mathbf{b}_l}_{n,k,2s;l} &= \mathcal{K}^{\mathrm{Sign}(s);\mathbf{b}_l}_{n,k,|2s|;l}. \end{aligned} \tag{C7}$$

The final expression for general reduction coefficients is

$$C^{(r)}_{\mathbf{v}_n\to n;\widehat{\mathbf{b}_a}} = \sum_{\substack{a_1+a_2=a,\\ \sigma\in S[\mathbf{b}_a]}}\sigma\left[\sum_{i=0}^{r}\sum_{\{\mathbf{l},\mathbf{j},\mathbf{s}\}_{a_1}}^{\xi(n,v,i)} c^{n,v,r}_{i,a_1,a_2,\{\mathbf{l},\mathbf{j},\mathbf{s}\}_{a_1}}(R^2)^{\frac{r-i}{2}}\mathcal{S}_{\xi,\{\mathbf{l},\mathbf{j},\mathbf{s}\}_{a_1};\mathbf{b}_{a_1}}(\overline{HS})_{\widehat{\mathbf{b}_{a_1}}}(\overline{HL})_{(\mathbf{b}_{a_1})\widehat{\mathbf{b}_{a_2}}}\mathcal{K}^{(\mathbf{b}_{a_1})\mathbf{b}_{a_2}}_{n-a_1,-\mu-2|\mathbf{s}_{a_1}|-l_{a_1}-j_{a_1};a_2}\right]\Bigg|_{t^i z^{\mathbf{v}_{n-1}}}, \tag{C8}$$

where we have defined $\mu(n,v,r,i) = r - i + 2n - 2v$, $\xi(n,v,i) = v - n + i$, and the dimensionless factor is

$$c^{n,v,r}_{i,a_1,a_2,\{\mathbf{l},\mathbf{j},\mathbf{s}\}_{a_1}} = \frac{(-)^{r+v+n+a_1+a_2}\Gamma(v-D/2-r)i!}{\Gamma(n-a_1-a_2-D/2)(v-n+i)!}\mathcal{F}^{n,n-D-\mu(n,v,r,i),\xi(n,v,i)}_{\{\mathbf{l},\mathbf{s}\}_{a_1}}\mathcal{E}_{n-a_1,n-a_1-D-\mu(n,v,r,i)-2|\mathbf{s}_{a_1}|,l_{a_1};j_{a_1}}. \tag{C9}$$

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