Scalar field damping at high temperatures

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The motion of a scalar field that interacts with a hot plasma, like the inflaton during reheating, is damped, which is a dissipative process. At high temperatures the damping can be described by a local term in the effective equation of motion. The damping coefficient is sensitive to multiple scattering. In the loop expansion its computation would require an all-order resummation. Instead we solve an effective Boltzmann equation, similarly to the computation of transport coefficients. For an interaction with another scalar field we obtain a simple relation between the damping coefficient and the bulk viscosity, so that one can make use of known results for the latter. The numerical prefactor of the damping coefficient turns out to be rather large, of order 10^4.

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I. INTRODUCTION

Scalar fields may play an important role in the early Universe. They can drive cosmic inflation, and their quantum fluctuations can provide the seed of galaxy formation, they can cause phase transitions [1] and generate the baryon asymmetry of the Universe [2]. They can also be part or all of dark matter [3], or be responsible for today's dark energy [4].

We consider a scalar field φ which is (approximately) constant in space and which evolves in time. Important examples are the inflaton field which drives inflation or the axion field which can be dark matter. To be specific we will consider φ being the inflaton, keeping in mind that our discussion applies to many other situations as well.

When inflation ends, the inflaton field φ starts oscillating coherently around the minimum of its potential. It interacts with other fields leading to an energy transfer thus creating a plasma and (re-)heating the Universe. At the same time, the motion of φ gets damped. The plasma makes up an increasing fraction of the total energy density. For sufficiently strong interaction the plasma thermalizes. Eventually this thermal plasma dominates the energy density; the corresponding temperature is called reheat temperature $T_{\rm RH}$. This is, however, not the largest temperature of the plasma, which rises early during the reheating process and then decreases before it reaches $T_{\rm RH}$ [5,6].

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An oscillating inflaton field with frequency $\omega = m_\varphi$, can be viewed as a state with high occupancy of inflaton particles with zero momentum and mass m_φ. When there are only few decay products present, the damping is dominated by inflaton decay into lighter particles [7]. If many particles have already been produced such that their occupation numbers are of order one or larger, other effects come into play. Parametric resonance can lead to very efficient particle production [8]. Then the decay products thermalize and acquire a thermal mass. At high temperature, thermal masses can become larger than the inflaton mass such that the decay of an inflaton into plasma particles is kinematically forbidden [9]. Then other processes that involve multiple scatterings become unsuppressed and open new channels for the energy transfer [10,11]. In this paper we consider the damping rate in the high-temperature regime where T is much larger than m_φ and the mass of the plasma particles. We assume that the characteristic frequency $\omega \sim \dot\varphi/\varphi$ and the damping rate γ of φ are small compared to the thermalization rate of the plasma. Then the inflaton interacts nearly adiabatically with an almost thermal plasma. In particular, there is no nonperturbative particle production through parametric resonance [8]. When the plasma is approximately thermal, its properties are fully specified by the temperature and by the instantaneous value of φ. Therefore the plasma "forgets" about its past, and its effect on the inflaton dynamics can be described by local terms. The effective equation of motion (without Hubble expansion) for the zero-momentum mode of φ takes the form [12]

$$\ddot\varphi + V'_{\rm eff} + \gamma\dot\varphi = 0, \qquad (1)$$

where the prime denotes a derivative with respect to φ. The effective potential $V_{\rm eff}$ and the damping coefficient γ only depend on the value of φ and the temperature. For sufficiently slow evolution, higher derivative terms in Eq. (1) can be neglected. Note that the form of Eq. (1) follows from the separation of timescales alone.

If ω and the damping rate γ are small compared to the thermalization rate, then γ can be obtained from a finite-temperature real-time correlation function, evaluated in the zero-momentum, zero-frequency limit [13,14]. E.g., the damping coefficient for the axion field is proportional to the Chern-Simons diffusion rate in QCD [13], the so-called strong sphaleron rate, which is nonperturbative and has been calculated on the lattice [15,16]. In many cases the required correlation functions can in principle be calculated perturbatively in thermal field theory. They are, however, sensitive to timescales much larger than the mean free time of the plasma particles, so that one has to take into account multiple scatterings. This requires the resummation of an infinite set of diagrams. Several authors have applied 1- or 2-loop approximations with resummed propagators containing a finite width (see e.g., [17–20]), which gives rise to a nonzero damping rate. However, proper treatment of the multiple interaction requires the resummation of a much larger class of diagrams [21,22].

Similar complications arise in the computation of transport coefficients, such as bulk viscosity, which can be written as the zero-momentum zero-frequency limit of a stress-tensor correlation function [23,24]. For viscosities, the required summation of diagrams has been performed. It was shown that this is equivalent to solving an effective Boltzmann equation [23].

The physics behind bulk viscosity ζ is closely related to the one of the damping coefficient γ. Both describe small deviations from thermal equilibrium. In the case of ζ it is due to a uniform expansion of the system. The deviation of the trace of the stress tensor from the ideal-fluid form is proportional to ζ. When the field φ changes with time and interacts with the plasma particles, it changes their parameters such as masses or couplings, driving the plasma out of equilibrium. Since φ is spatially constant, this deviation from equilibrium is homogeneous and isotropic as well.

For the model considered in Ref. [14] the damping coefficient could be related to a correlation function of the stress tensor. Thus there is a simple relation between γ and the bulk viscosity ζ of the thermal plasma, so that one can use the known result for ζ. In this work we consider interactions of φ with another scalar field χ through operators that cannot be expressed in terms the stress tensor, but which still allow for a perturbative treatment. We can proceed similarly to the computation of the bulk viscosity, for which the required resummation of diagrams is equivalent to solving an effective Boltzmann equation [23]. Damping coefficients have been computed from a Boltzmann equation long ago using a relaxation time approximation [12].[1] This approximation, however, does not give the correct result for the bulk viscosity in scalar theory [23,24]. Here we carefully treat the collision term as well as thermal effects by employing the effective Boltzmann equations which were used to perturbatively compute bulk viscosities in scalar theories [23,24], and in gauge theories[25,26]. This allows us to obtain the correct dependence on the coupling constants and explicitly compute γ at leading order in perturbation theory.

This paper is organized as follows. In Sec. II we obtain the effective equation of motion for φ and an expression for the damping coefficient in terms of the plasma-particle occupancy. The latter is computed in Sec. III from an effective Boltzmann equation. In Sec. IV the solution to the Boltzmann equation is inserted into the effective equation of motion for φ, and the damping coefficients is expressed in terms of the known bulk viscosity of the plasma. Section V contains conclusions and a brief outlook. Appendix A deals with the thermodynamics of the plasma particles, and Appendix B describes the solution of the Boltzmann equation.

II. EFFECTIVE EQUATION OF MOTION

In this section, largely following Ref. [12], we obtain the effective equation of motion (1) from quantum field theory and relate the coefficients therein to microscopic physics. We consider a scalar field Φ is coupled to another scalar field χ through the interaction

$$\mathcal{L}_{\Phi\chi} = -A(\Phi)\chi^2. \tag{2}$$

Restricting ourselves to renormalizable interactions we can have

$$A(\Phi) = \frac{\mu}{2}\Phi + \frac{\lambda}{4}\Phi^2 \tag{3}$$

with coupling constants μ and λ. Without Hubble expansion the equation of motion for Φ reads

$$\ddot{\Phi} - \Delta\Phi + V'(\Phi) + A'(\Phi)\chi^2 = 0, \tag{4}$$

where V is the part of the tree-level potential that depends only on Φ. Equation (4) is still an equation for field operators. We want to write an equation of motion for the zero-momentum mode φ of Φ, and we assume that φ can be approximated by a classical field. We write

$$\Phi = \varphi + \hat{\Phi}, \tag{5}$$

where $\hat{\Phi}$ contains the nonzero momentum modes of Φ. Through the interaction, χ particles are produced. Once the χ particles are created, they can also produce Φ particles

[1]The damping coefficients computed in Refs. [18–20] are of the same form as in Ref. [12].

which are represented by $\hat{\Phi}$. Thus the production of Φ particles also contributes to the damping of φ. This effect is discussed in [27], where it was found that this contribution is subdominant unless the energy density in φ is small compared to the energy density of the χ particles. In the context of reheating after inflation this would already be during radiation domination. We assume that φ still dominates the energy density and neglect this contribution. Then we can replace the forth term in Eq. (4) by $A'(\varphi)\chi^2$.

We assume that χ interacts rapidly with itself or other fields, so that it thermalizes on timescales which are short compared to the period of φ oscillations. Furthermore, we assume that the interactions of χ are weak enough, so that the typical mean free path of χ particles is much larger than their typical de Broglie wavelength. Then χ is made up of weakly interacting particles which can be described by their phase space density, or occupancy $f(t, \mathbf{p})$. Since we consider a homogeneous system, it only depends on time t and on the particle momentum $\mathbf{p}$. We may then replace χ^2 by its expectation value computed from the occupancy using the free-field expression

$$\langle\chi^2\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{f(t,\mathbf{p})}{E}, \tag{6}$$

where E is the one-particle energy (see below). Thus we arrive at the effective *classical* equation of motion

$$\ddot{\varphi} + V'(\varphi) + A'(\varphi)\langle\chi^2\rangle = 0. \tag{7}$$

The deviations from equilibrium are assumed to be small, so that the occupancy in Eq. (6) can be written as

$$f(t,\mathbf{p}) = f_{\mathrm{eq}}(t,\mathbf{p}) + \delta f(t,\mathbf{p}), \tag{8}$$

with the local equilibrium distribution

$$f_{\mathrm{eq}}(t,\mathbf{p}) = \frac{1}{\exp(E/T) - 1}. \tag{9}$$

and $\delta f \ll f_{\mathrm{eq}}$. The temperature T in Eq. (9) varies slowly with time. The mass of the χ particles depends on the value of φ,

$$m^2 = m_0^2 + 2A(\varphi) \tag{10}$$

where m_0 is the zero-temperature mass at vanishing φ. Throughout this paper we assume that m is small compared to the temperature.[2] The mass appearing in the one-particle energy E in Eqs. (6) and (9) also receives a thermal contribution $m_{\mathrm{th}}^2 \propto T^2$, so that $E = (\mathbf{p}^2 + m_{\mathrm{eff}}^2)^{1/2}$ with

$$m_{\mathrm{eff}}^2 = m^2 + m_{\mathrm{th}}^2. \tag{11}$$

To avoid a tachyonic instability [29], m_{eff}^2 must be positive.

With the help of Eq. (8), the expectation value in Eq. (6) becomes

$$\langle\chi^2\rangle = \langle\chi^2\rangle_{\mathrm{eq}} + \delta\langle\chi^2\rangle. \tag{12}$$

The first term in Eq. (12) is nondissipative. It gives a thermal correction in the effective potential in Eq. (1) [12],

$$V'_{\mathrm{eff}} = V' + A'\langle\chi^2\rangle_{\mathrm{eq}}, \tag{13}$$

which is precisely the leading term in the high-temperature limit of the 1-loop effective potential (see, e.g., [30]). The second term in Eq. (12) is dissipative and will give rise to the damping term in Eq. (1).

III. BOLTZMANN EQUATION

The occupancy of χ particles in Eq. (6) can be computed by solving a Boltzmann equation, because the timescale on which their mass changes is of order $1/\omega$ which is much larger than their typical de Broglie wavelength of order $1/T$. Due to the homogeneity, spatial momentum is conserved. Thus the Boltzmann equation takes the form

$$\partial_t f = C, \tag{14}$$

where C is the collision term. Now we insert Eq. (8) on the left-hand side of Eq. (14). We neglect $\partial_t \delta f$ because it is quadratic in small quantities, so that

$$\partial_t f \simeq -f_{\mathrm{eq}}(1 + f_{\mathrm{eq}})\partial_t(E/T). \tag{15}$$

The zero-momentum mode φ depends on time and changes the mass of the plasma particles through the interaction (2). If the oscillation is much slower than the thermalization of the plasma, this is an adiabatic process that changes the temperature in Eq. (9) at constant volume.[3] Thus the time dependence of the temperature is determined by

$$\partial_t T = \left(\frac{\partial T}{\partial m^2}\right)_{S,V} \partial_t m^2. \tag{16}$$

In the limit $T \gg m$ we obtain (see Appendix A)

$$\left(\frac{\partial T}{\partial m^2}\right)_{S,V} = \frac{T}{4\rho}\langle\chi^2\rangle_{\mathrm{eq}}, \tag{17}$$

where ρ is the energy density of the thermal plasma. The one-particle energy E depends on time through the effective mass. We thus have

[2]The opposite limit $m \gg T$ is considered in Ref. [28] with additional light degrees of freedom. Then χ can be integrated out giving rise to an effective interaction of φ with the light plasma particles.

[3]This does not apply to the case $\omega \gtrsim T$ which is considered in Refs. [31–33].

$$\partial_t(E/T) = \frac{1}{2TE}\left[1 - \frac{\langle\chi^2\rangle_{\rm eq}}{2\rho}\left(E^2 - T^2\frac{\partial m^2_{\rm eff}}{\partial T^2}\right)\right]\partial_t m^2. \quad (18)$$

The third term in the square bracket is small compared to the first, both for hard ($|\mathbf{p}| \sim T$) and for soft ($|\mathbf{p}| \sim m_{\rm eff}$) momenta, and can be neglected, so that

$$\partial_t f \simeq -f_{\rm eq}(1+f_{\rm eq})\frac{Q}{2T}\partial_t m^2 \quad (19)$$

with

$$Q(\mathbf{p}) \equiv \frac{1}{E} - \frac{\langle\chi^2\rangle_{\rm eq}}{2\rho}E. \quad (20)$$

Now we insert Eq. (8) into the collision term. Since C vanishes in equilibrium, its expansion in δf starts at linear order,

$$C \simeq \hat{C}\delta f. \quad (21)$$

Here we have neglected the contribution of Φ particles because the corresponding collision term is quadratic in the Φ-χ couplings which we assume to be much smaller than the self-coupling of χ entering $\hat{C}$.

It is convenient to write the deviation from equilibrium as

$$\delta f = -f_{\rm eq}(1+f_{\rm eq})X. \quad (22)$$

Similarly, we write the linearized collision term as

$$\hat{C}\delta f = f_{\rm eq}(1+f_{\rm eq})\tilde{C}X, \quad (23)$$

with the convolution

$$[\tilde{C}X](\mathbf{p}) \equiv \int\frac{d^3p'}{(2\pi)^3}\tilde{C}(\mathbf{p},\mathbf{p}')f_{\rm eq}(\mathbf{p}') \times (1+f_{\rm eq}(\mathbf{p}'))X(\mathbf{p}'). \quad (24)$$

Then the kernel $\tilde{C}$ is symmetric [34],

$$\tilde{C}(\mathbf{p},\mathbf{p}') = \tilde{C}(\mathbf{p}',\mathbf{p}). \quad (25)$$

The Boltzmann equation thus turns into an equation for X,

$$-\frac{\partial_t m^2}{2T}Q = \tilde{C}X. \quad (26)$$

Since the collision term vanishes in equilibrium for any temperature, the linearized collision term has a zero mode $X = X_1$ associated with a shift of the temperature, which is given by $X_1(\mathbf{p}) = E$. Due to the symmetry of $\tilde{C}$ the right-hand side of Eq. (26) is orthogonal to X_1. For Eq. (26) to be consistent, the left-hand side must be orthogonal to X_1 as well. This is indeed the case when the second term in Eq. (20) is taken into account, which can be easily checked.

Due to the zero mode the linear operator $\tilde{C}$ cannot be inverted. However, it can be inverted on the subspace orthogonal to the zero mode,[4] where orthogonality is defined with respect to the inner product

$$(X,X') \equiv \int\frac{d^3p}{(2\pi)^3}f_{\rm eq}(1+f_{\rm eq})X(\mathbf{p})X'(\mathbf{p}). \quad (27)$$

We can then write the solution as

$$X = -\frac{\partial_t m^2}{2T}\tilde{C}^{-1}Q, \quad (28)$$

and we finally obtain

$$\delta f = f_{\rm eq}(1+f_{\rm eq})\frac{\partial_t m^2}{2T}\tilde{C}^{-1}Q. \quad (29)$$

IV. DAMPING COEFFICIENT AND BULK VISCOSITY

Coming back to the effective equation of motion (7), we insert the solution (29) into Eq. (6) to obtain the second term in Eq. (12) as

$$\delta\langle\chi^2\rangle = \frac{1}{2T}(Q',\tilde{C}^{-1}Q)\partial_t m^2. \quad (30)$$

Here we have introduced $Q'(\mathbf{p}) \equiv 1/E$. The factor $\partial_t m^2$ is proportional to $\dot{\varphi}$. Comparing Eqs. (1) and (7) we see that the second term in Eq. (12) is indeed responsible for the damping,

$$\gamma\dot{\varphi} = A'\delta\langle\chi^2\rangle. \quad (31)$$

Inserting A from Eq. (3) we obtain

$$\gamma = \frac{1}{4T}(Q',\tilde{C}^{-1}Q)(\mu+\lambda\varphi)^2, \quad (32)$$

which is our main result.

The computation of $\tilde{C}^{-1}Q$ is described in Appendix B. However, at this point we do not need it explicitly,[5] because, as we shall see in a moment, the coefficient $(Q',\tilde{C}^{-1}Q)$ also appears in the computation of the bulk viscosity ζ of the χ plasma. Therefore it can be read off directly from known results for ζ. To see this, we first recall that $\tilde{C}^{-1}Q$ is orthogonal to $X_1 = E$, i.e., $(E,\tilde{C}^{-1}Q) = 0$.

[4]This is equivalent to imposing the Landau-Lifshitz condition on the energy density $\delta\rho = (2\pi)^{-3}\int d^3pE\,\delta f = 0$.

[5]It will be useful later, when we estimate the size of δf in order to check the accuracy of our approximations.

In Eq. (32) we may therefore replace $Q' = Q + (\langle\chi^2\rangle_{\rm eq}/2\rho)E$ by Q without changing our result for γ, which then reads

$$\gamma = \frac{1}{4T}(Q, \tilde{C}^{-1}Q)(\mu + \lambda\varphi)^2. \tag{33}$$

Let us now recall some properties of the bulk viscosity, as described, e.g., in Ref. [25]. When a plasma is uniformly compressed or rarified it leaves equilibrium, unless this happens infinitely slowly. The pressure of the plasma then differs from the value it would have in the equilibrium state with the same energy density. This deviation of the pressure from equilibrium is proportional to the bulk viscosity.

In a plasma with scale invariance the bulk viscosity vanishes, for two different reasons. The first one is that a uniform expansion or rarefaction is a dilatation which is a symmetry transformation in a scale invariant theory. Therefore such a transformation does not take the system out of equilibrium. The second is that in a scale invariant theory the trace of the energy-momentum tensor $T^{\mu\nu}$ always vanishes. Therefore the pressure $P = T^{mm}/3$ equals $\rho/3$ even out of equilibrium.

Scale invariance is broken by zero-temperature masses and by the trace anomaly, i.e., by quantum effects. The bulk viscosity is then quadratic in the measure which controls the breaking of scale invariance.

The bulk viscosity ζ of the thermal plasma of scalar particles with mass m was computed for scalar theory in Refs. [23,24]. Like in QCD [25] it can be written as

$$\zeta = \frac{1}{T}(q, \tilde{C}^{-1}q), \tag{34}$$

with

$$q(\mathbf{p}) = -\frac{1}{E}\left[\left(c_{\rm s}^2 - \frac{1}{3}\right)\mathbf{p}^2 + c_{\rm s}^2 m_{\rm sub}^2\right]. \tag{35}$$

Here $c_{\rm s}$ with $c_{\rm s}^2 = \partial P/\partial\rho$ is the speed of sound. In a scale invariant theory $c_{\rm s}^2$ equals $1/3$, so that the first term in the square bracket in Eq. (35) vanishes. Furthermore, $m_{\rm sub}$ with

$$m_{\rm sub}^2 \equiv m_{\rm eff}^2 - T^2\frac{\partial m_{\rm eff}^2}{\partial T^2} \tag{36}$$

is the so-called subtracted mass. In the massless limit $m = 0$, $m_{\rm eff}^2$ equals T^2 times a function of the coupling constants [see Eqs. (10), (11)]. Then the only contribution to the subtracted mass is from the running of the couplings renormalized at the scale T. The subtracted mass is thus a measure of the deviation from scale invariance as well, because it vanishes when $m = 0$ and the couplings do not run. Since q appears twice in Eq. (34), the bulk viscosity is indeed quadratic in the measure of scale-invariance violation.

Now we replace $\mathbf{p}^2$ by $E^2 - m_{\rm eff}^2$ in Eq. (35) which turns it into

$$q(\mathbf{p}) = \left[\left(c_{\rm s}^2 - \frac{1}{3}\right)m_{\rm eff}^2 - c_{\rm s}^2 m_{\rm sub}^2\right]\frac{1}{E} - \left(c_{\rm s}^2 - \frac{1}{3}\right)E. \tag{37}$$

Comparing Eqs. (20) and (37) we see that both Q and q consist of a term proportional to $1/E$, and one proportional to E. Furthermore, q appears on the left-hand side of a Boltzmann equation precisely like Q in Eq. (19),[6] and is thus orthogonal to X_1 as well. Therefore q must be proportional to Q. Here we are interested in the limit $T \gg m$ in which [23]

$$|c_{\rm s}^2 - 1/3| = O(m_{\rm sub}^2/T^2) \ll 1 \qquad (T \gg m), \tag{38}$$

and also $m_{\rm eff}^2 \ll T^2$. Therefore we can approximate the square bracket in (37) by $-m_{\rm sub}^2/3$. This gives us the approximate factor of proportionality, so that $q \simeq -(m_{\rm sub}^2/3)Q$. Then we obtain the following simple relation

$$\gamma(\varphi, T) = \frac{9}{4}\frac{\zeta}{m_{\rm sub}^4}(\mu + \lambda\varphi)^2 \tag{39}$$

of the damping coefficient in the effective equation of motion (1) and the bulk viscosity of the χ plasma. Like in Ref. [14] the nontrivial dependence on the interaction of the plasma particles, on thermal masses, etc., is precisely the same for both quantities. Note that $m_{\rm sub}^4$ in the denominator of Eq. (39) removes the factors related to the breaking of scale invariance from ζ, which can also be seen explicitly in Eqs. (41) and (43) below. Thus, despite its similarity to the bulk viscosity, the damping coefficient is not related to the breaking of scale invariance.

The bulk viscosity for a self-interacting scalar field was computed in Ref. [23]. For the quartic self-interaction

$$\mathcal{L}_{\chi\chi} = -\frac{g^2}{4!}\chi^4 \tag{40}$$

and $T \gg m$ the leading-order result reads

$$\zeta = \frac{b}{4}\frac{m_{\rm sub}^4 m_{\rm eff}^2}{g^8 T^3}\ln^2\left(\frac{\kappa^2 m_{\rm eff}^2}{T^2}\right), \tag{41}$$

with $b = 5.5 \times 10^4$ and $\kappa = 1.25$. The effective mass for the χ particles is given by

$$m_{\rm eff}^2 = m^2 + \frac{g^2}{24}T^2, \qquad m^2 = m_0^2 + \frac{\lambda}{2}\varphi^2. \tag{42}$$

Inserting Eq. (41) into Eq. (39) we obtain the damping coefficient

[6] See Eq. (3.7) of Ref. [25].

$$\gamma(\varphi, T) = a\frac{m_{\rm eff}^2}{g^8 T^3}\ln^2\left(\frac{\kappa^2 m_{\rm eff}^2}{T^2}\right)(\mu+\lambda\varphi)^2 \tag{43}$$

with the remarkably large numerical prefactor

$$a = 3.1\times 10^4. \tag{44}$$

In the temperature range $T \gg m/g^2$ the form (41) and thus Eq. (43) remain valid when a cubic self-interaction is included in (40), while in the intermediate regime $m \ll T \ll m/g^2$ the bulk viscosity depends nontrivially on the relative strengths of cubic and quartic χ self-couplings [23]. It is obvious from the dependence on the coupling constant g that the result (43) cannot be obtained from a one-loop approximation to a correlation function, as anticipated in Refs. [22,28]. Instead, by solving the Boltzmann equation we have summed an infinite set of diagrams which all contribute at leading order in g.

We can compare our result with the one obtained in Ref. [12] for a single scalar field by putting $\chi = \varphi$, $\mu = 0$, and, up to an $O(1)$ factor, $g^2 = \lambda$. In Ref. [12] the Boltzmann equation was solved in the collision-time approximation, i.e., by replacing the linearized collision term on the right-hand side of Eq. (21) by a constant times δf. Such an approximation does not take into account the zero eigenvalues and the hierarchy of nonzero eigenvalues of $\tilde{C}$. In Ref. [12] the collision time is determined by $2\to 2$ scattering which changes momenta but not particle numbers. Bulk viscosity and the damping coefficient γ are, however, determined by the slowest equilibration process, corresponding to the smallest eigenvalue of the linear collision operator, since it is the inverse of the collision operator that appears in Eqs. (32) and (34). In scalar field theory the slowest process is particle number equilibration. Therefore the computation of Ref. [12] does not give the correct dependence on the coupling constant and underestimates the values of γ and ζ. Similarly, in Ref. [35] the rate for elastic scattering was used to estimate the damping coefficient.

The importance of particle number changing processes for the bulk viscosity is well known. The reason why they are also important for the damping coefficient is the following. When φ evolves in time, it changes the mass of the χ particles, but not their momenta. This, in turn, changes the energy density of χ particles but leaves their number density unaffected. In order to relax to equilibrium, the χ particle number has to adjust to the equilibrium value corresponding to their new energy density.

Let as finally discuss the range of validity of the effective equation of motion (1). There the damping term is linear in $\dot{\varphi}$. This is related to the linearization of the Boltzmann equation, which requires that $\delta f \ll f_{\rm eq}$. In Appendix B we show that

$$\delta f/f_{\rm eq} \sim \frac{m_{\rm eff}}{g^8 T^4}\partial_t m^2 \tag{45}$$

for the interaction (40). When φ oscillates with frequency ω and amplitude $\tilde{\varphi}$, the time derivative $\partial_t m^2$ can be estimated as $\lambda\omega\tilde{\varphi}^2$. For $gT \gg m$ we thus need

$$\lambda\omega\tilde{\varphi}^2 \ll g^7 T^3 \tag{46}$$

in order to be able to linearize the Boltzmann equation.

We may also apply the condition (46) to a model with a single scalar field φ which was considered in Ref. [12] by putting $\mu = 0$ and $\lambda \sim g^2$. Then (46) turns into $\omega\tilde{\varphi}^2 \ll g^5 T^3$. The energy density in φ would be $\rho_\varphi \sim \omega^2\tilde{\varphi}^2 \ll (\omega/T) g^5 T^4$. Due to $\omega \ll T$, the energy carried by φ would be only a tiny fraction of the plasma energy density $\rho \sim T^4$. For the more interesting case that we have several fields, φ can give the dominant contribution to the total energy without violating the condition (46).

V. CONCLUSION

A slowly moving homogeneous scalar field φ interacting with a thermal plasma drives it slightly out of equilibrium, giving rise to dissipation and damping. In the high-temperature regime the damping coefficient in the effective equation of motion for φ can be efficiently computed by solving an appropriate Boltzmann equation, see Eq. (32). We have considered a plasma made of a single species of scalar particles. In this case we obtained a simple relation of the damping coefficient to the bulk viscosity of the plasma, Eq. (39). This extends a result [14] which was obtained for a scalar field with derivative interaction. Like in the computation of viscosity, the solution of the Boltzmann equation is dominated by the slowest process required for equilibration. This can be easily generalized to multicomponent plasmas, where again one has to identify the slowest process to solve the Boltzmann equation and then use the resulting phase space density to compute the dissipative terms in the effective equation of motion for the scalar field.

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APPENDIX A: MASS DEPENDENCE OF THE TEMPERATURE

The free energy of an ideal gas has the high-temperature expansion

$$F(T,V,m^2) = V(-aT^4 + bT^2m^2 + \cdots) \qquad \text{(A1)}$$

with positive constants a and b; m is the mass of one particle species. The coefficient a can also contain the contributions from other light species. At leading order our expansion is related to the energy density by $\rho = 3aT^4$. For a scalar

$$\frac{\partial F}{\partial m^2} = \frac{V}{2}\langle\chi^2\rangle_{\rm eq}, \qquad \text{(A2)}$$

which gives $b = \langle\chi^2\rangle_{\rm eq}/(2T^2)$. The entropy is

$$S = -\frac{\partial F}{\partial T} = V(4aT^3 - 2bTm^2 + \cdots). \qquad \text{(A3)}$$

This can be inverted to obtain the expansion for the temperature, $T = T_0 + T_2 + \cdots$, for which we obtain

$$T_0 = \left(\frac{S}{4aV}\right)^{1/3}, \qquad \text{(A4)}$$

$$T_2 = \frac{b}{6a}\frac{m^2}{T_0}. \qquad \text{(A5)}$$

Differentiating T_2 with respect to m^2 then gives Eq. (17).

APPENDIX B: SOLVING THE BOLTZMANN EQUATION AND ESTIMATING δf

The linearization of the Boltzmann equation is only possible if the deviation from equilibrium is small, $\delta f \ll f_{\rm eq}$. This condition restricts the allowed values of the couplings and the amplitude of the zero-momentum mode φ.

To estimate the size of δf in Eq. (29) we derive its explicit form, closely following Ref. [24]. For a self-interacting scalar field one has to include two contributions in the collision term,

$$\tilde{C} = \tilde{C}_{\rm el} + \tilde{C}_{\rm inel}. \qquad \text{(B1)}$$

$\tilde{C}_{\rm el}$ describes elastic $2 \to 2$ scattering which conserves particle number. Therefore it has the additional zero mode $X_0 = 1$, associated with a shift of the chemical potential, and cannot be inverted on the subspace orthogonal to $X_1 = E$. One also has to include an inelastic contribution $\tilde{C}_{\rm inel}$ describing particle number changing processes, even though its matrix element is higher order. $\tilde{C}$ has a single small eigenvalue c on the subspace orthogonal to X_1, with the approximate eigenvector

$$X_{0\perp} = X_0 - \alpha X_1, \qquad \text{(B2)}$$

where $\alpha = (X_1, X_0)/(X_1, X_1)$. The small eigenvalue is approximately

$$c = \frac{(X_{0\perp}, \tilde{C}_{\rm inel}X_{0\perp})}{(X_{0\perp}, X_{0\perp})}, \qquad \text{(B3)}$$

while the other nonvanishing eigenvalues are of order $\tilde{C}_{\rm el}$. In the numerator of Eq. (B3) we may replace $X_{0\perp}$ by X_0 because $\tilde{C}_{\rm inel}X_1$ vanishes. This eigenvalue gives the leading contribution to $\tilde{C}^{-1}$, so that

$$\tilde{C}^{-1}Q \simeq \frac{(X_{0\perp}, Q)}{(X_0, \tilde{C}_{\rm inel}X_0)}X_{0\perp}. \qquad \text{(B4)}$$

In the numerator of Eq. (B4) we can replace $X_{0\perp}$ by X_0, because X_1 is orthogonal to Q. We insert this into Eq. (29), which finally gives

$$\delta f(\mathbf{p}) = f_{\rm eq}(\mathbf{p})[1 + f_{\rm eq}(\mathbf{p})] \times \frac{\partial_t m^2}{2T}\frac{(X_0, Q)}{(X_0, \tilde{C}_{\rm inel}X_0)}X_{0\perp}(\mathbf{p}). \qquad \text{(B5)}$$

We now use this to estimate the size of δf. We will encounter the integrals

$$I_n \equiv \int \frac{d^3p}{(2\pi)^3} f_{\rm eq}(1 + f_{\rm eq})E^n, \qquad \text{(B6)}$$

for $n = -1$, 0, and 1. For $n \geq 0$ these are saturated at $|\mathbf{p}| \sim T$, giving $I_n \sim T^{3+n}$. Since $f_{\rm eq} \simeq T/E$ for $E \ll T$, the integral I_{-1}, is logarithmically infrared divergent in the massless limit and is cut off by $m_{\rm eff}$. Thus I_{-1} receives leading order contributions both from $|\mathbf{p}| \sim T$ and from $|\mathbf{p}| \sim m_{\rm eff} \ll T$, with the result

$$I_{-1} = \frac{T^2}{2\pi^2}\ln\left(\frac{2T}{m_{\rm eff}}\right). \qquad \text{(B7)}$$

The factor (X_0, Q) in the numerator of Eq. (B5) contains I_{-1} and I_1 and is of order T^2 modulo logarithms, because $\langle\chi^2\rangle_{\rm eq}/\rho \sim T^{-2}$. The denominator depends on the type of interaction (see below).

Since the size of $\delta f(\mathbf{p})$ depends on $|\mathbf{p}|$, we need to know which values of $|\mathbf{p}|$ give the dominant contributions to $(Q', \tilde{C}^{-1}Q) \propto (Q', X_{0\perp})$, which enters the damping coefficient in Eq. (32). Using Eq. (B2) we find that the integrals (B6) appear in the combination $I_{-1} - \alpha I_0$. The factor α is of order $1/T$. Thus $|\mathbf{p}| \sim T$ and $|\mathbf{p}| \sim m_{\rm eff}$ are equally important. In both regions $X_{0\perp} \sim 1$. Due to the Bose factors in Eq. (22) the ratio $\delta f/f_{\rm eq}$ increases with decreasing $|\mathbf{p}|$, so that it takes its largest value when $|\mathbf{p}| \sim m_{\rm eff}$. Putting $|\mathbf{p}| \sim m_{\rm eff}$, collecting all factors and ignoring logarithms we obtain

$$\delta f/f_{\rm eq} \sim \frac{T^2\partial_t m^2}{m_{\rm eff}(X_0, \tilde{C}_{\rm inel}X_0)}. \tag{B8}$$

For the χ self-interaction (40) and the χ-φ interaction (2) with $\mu = 0$, $\tilde{C}_{\rm inel}$ describes scattering involving 6 particles. The corresponding squared matrix is proportional to g^8. The momentum integral which enters the denominator in Eq. (B4) is saturated by soft momenta ($T \sim m_{\rm eff}$)) [23]. It contains up to six Bose distributions, which for soft momenta satisfy $f_{\rm eq}(\mathbf{p}) \simeq T/E$, giving rise to a factor T^6. By dimensional analysis one then finds $(X_0, \tilde{C}_{\rm inel}X_0) \sim g^8T^6/m_{\rm eff}^2$, which yields the estimate (45).

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