Nonlocal gluon condensates in QCD sum rules

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Nonlocal gluon condensates are vacuum expectations of the product of gluon field strength tensors. Short-distance expansions of two-, three-, and four-gluon condensates are presented up to dimension-eight local operators. We propose a method for calculating the Wilson coefficients based on the presented expansions and the Feynman diagram technique in the background field approach. The method is demonstrated using the glueball current correlators as examples. Methodological aspects of the background field approach are discussed in relation to glueball studies within QCD sum rules. We confirm the results for operator product expansion of the two-gluon $0^{\pm+}$ glueball current correlators and calculate additional contributions coming from dimension-six four-quark condensate and dimension-eight mixed quark-gluon condensates. The operator product expansions used in QCD sum rules for three-gluon $0^{\pm+}$ glueballs are revisited up to dimension-six order.

I. INTRODUCTION

A practical method to evaluate quantitatively the physical characteristics of hadrons from the QCD is provided by an approach called the QCD sum rules (SRs) [1–3]. This approach gives a the direct correspondence between the hadron parameters and Wilson's operator-product expansion (OPE) [4] of correlation functions. The correspondence arises from the dispersion relations. Various aspects and reviews of QCD SRs applications can be found in Refs. [5–27], for recent development of QCD SR in medium, see [28–30]. Sum rules are useful beyond the Standard Model in application for bounds on new physics effects [31–33].

The OPE terms are given as a product of the coefficient function and the vacuum condensate that reflect the separation of short and large distance effects, respectively. The approaches to calculate the coefficient functions include the method of projectors [34,35], Schwinger's operator method [36] and its branch based on the Fock-Schwinger gauge, and the background field approach, see references in [9].

The technical details of calculating the coefficient function are usually omitted due to complexity. A consistent procedure for such calculations was elaborated in [9] and further developed in [37]. The presented study can be considered as a

continuation of [9,37]. We propose a method for calculating the coefficient function based on the nonlocal condensates (NLCs) [38–40] and Feynman diagram technique in the framework of the background field approach. Nonlocal gluon condensates are vacuum expectations of the product of gluon field strength tensors. For example, the two-gluon nonlocal condensate has a dimension of four in energy units and can be defined as a "physical vacuum" expectation of the normal product $\langle 0|{:}G_{\mu\nu}(x)[x;y]G_{\alpha\beta}(y){:}|0\rangle$, where $G_{\mu\nu}$ is the gluon field strength tensor and $[x;y]$ is a Wilson line, see details in Sec. II A.

The methodological aspects of QCD SR are often considered for the case of quark current correlators, see [41], while the glueball currents correlators have particularities left unattended. Therefore, applications of the proposed approach are given by the examples of the glueball currents correlators used in QCD SRs studies [42–45].

Glueballs are an exotic state that contains only gluons and no valence quarks. This type of state has candidates among observations [46–51] and is included in the running and projected large-scale experiments: Belle (Japan), BESIII (Beijing, China), LHC (CERN), GlueX (JLAB, USA), NICA (Dubna, Russia), HIAF (China), and FAIR (GSI, Germany). Electron ion colliders have potential for glueball state observations [52]. There is an open discussion, whether the C-odd glueball bound state was seen in the observation of the Odderon [50,53,54] by TOTEM and D0 [55,56]. The reviews of glueball physics can be found in [47,57,58]. There are many applications of QCD SR to glueball states [7,42–45,59–86]. Here we shortly discuss some of them. The first QCD SR study [42] of glueballs considered a pseudoscalar 0^{-+} state with an obtained mass

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of ~1 GeV, where a two-gluon current was used. Later QCD SR was applied [43] to a scalar 0^{++} glueball state, where the glueball mass was estimated to be ~0.7 GeV. The OPE used in SRs for the scalar and pseudoscalar glueballs was extended [73–75,87] by including the direct instanton contribution, the two loop radiative corrections [88,89] to the perturbative term, and the one loop radiative correction [90] to dimension-four term (dimension of the considered condensate is four in energy units). Three-gluon glueballs were first studied in [44] for a 0^{++} state. Then, QCD SRs for three-gluon glueballs was extended [45] to the 0^{-+} scalar, vector, and tensor states [70].

Here we study the technical aspects of the calculation of OPE coefficients up to dimension-eight order and suggest a new way to organize and perform calculations using NLCs [38–40]. Applying the algorithms formulated in [37] for calculating higher power corrections, we extend the results of [37] to a full set of gluon NLCs needed for calculations of OPE up to dimension-eight terms without considering the radiative corrections to the coefficient functions, see Sec. II. The three-loop coefficient functions of two-gluon NLC OPE was obtained in [91] for the leading dimension-four term. We develop and obtain expansions of NLCs for two- and three-gluon condensate up to dimension-eight, where terms of expansions are given employing local condensates. Vacuum expectation of normal product of local operators are usually called condensate but here we use phrase "local condensate" to distinguish between local and nonlocal condensate. The result for four-gluon condensate expansion includes only the leading dimension-eight term. Coordinate dependence of higher dimension terms is also discussed using permutation symmetries of gluon fields strength tensor in condensate. Our result is in agreement with the two-gluon condensate expansions that were first obtained in [92] in dimension-six order and then in [37] up to dimension-eight term.

The number of terms that need to be evaluated is growing with the mass dimension (dimension in energy units) of OPE order. We propose [79–81] to use the so-called nonlocal condensates, which help to systematize the contributions and simplify calculations. We consider NLCs in the form of their truncated expansions given in terms of local condensates. As a result, each NLC-based OPE contribution is represented in this work through a finite set of the OPE terms given by the local condensates of various dimensions starting from the mass dimension of the original NLC. In other words, we use NLCs only as intermediate while the final results for OPEs are given through local condensates. Originally, NLCs were used differently in QCD SR—for resumming an infinite series of local condensates by modeling of the long-range dependence [93] of NLCs. One of the models for the two-gluon NLC could be found in [94]. The NLCs were successfully applied in studies of the hadron structures (distribution amplitudes, form factors) [40,92,93,95–103].

The idea of using a truncated series of NLCs for OPE was first applied in [79–81] for studying C-odd $0^{\pm-}$ three-gluon glueball states in QCD SR. Although, we focus here on the gluon condensates and glueball state, the considered ideas can be applied to other states within QCD SR and beyond. Recent applications of gluon condensates beyond QCD SR include studies of heavy quarkonium within potential nonrelativistic QCD [104,105], the rapidity anomalous dimension or Collins-Soper kernel [106], see references in [91].

The suggested method to calculate OPE using a truncated series of NLCs is demonstrated for correlators that represent the two-gluon and three-gluon $0^{\pm+}$ glueball states in QCD SRs. The demonstrations lead to the following results: (i) for the two-gluon current cases we obtained additional dimension-eight terms that were not considered in [42,43]; (ii) for the three-gluon current cases [44,45] the OPE for correlators are revisited in dimension-six order and compared with [44,45].

The paper is organized as follows. In the next section, the explicit expressions for tensor and Taylor expansions of gluon NLCs are presented in terms of local condensates. Then, in Sec. III, we apply these expansions to correlators of glueball currents. In Sec. III, we also discuss aspects of the background field approach, especially those that are relevant to glueball studies within QCD SR. In Sec. IV, we summarize our observations and results. In the Appendices, we provide the Feynman rules and a detailed example of their application.

II. GLUON CONDENSATES

In this section, we present the expansion of two-, three-, and four-gluon NLCs in terms of local condensates. The expansions are obtained by the method formulated in [37]. The coefficient functions and operators in OPE are gauge invariant; therefore, the choice of gauge is a matter of convenience. The expansion of NLCs is performed in the FS (Fock-Schwinger) gauge [107]:

$$(x_\mu - z_\mu)A_\mu^a(x) = 0, \tag{1}$$

with the gauge-fixing point $z_\mu = 0$. In the FS gauge the Taylor expansion for the gluon field strength tensor $G_\mu^a(x)$ can be written in gauge-covariant form [9]:

$$G_{\mu\nu}^a(x) = G_{\mu\nu}^a(0) + \frac{1}{1!}x_\alpha D_\alpha G_{\mu\nu}^a(0)$$
$$+ \frac{1}{2!}x_\beta x_\alpha D_\beta D_\alpha G_{\mu\nu}^a(0) + O(x^3). \tag{2}$$

Here we use the covariant derivative $D_\mu G_{\rho\sigma}^a = (\partial_\mu \delta^{ab} + gf^{acb}A_\mu^c)G_{\rho\sigma}^b$ in the adjoint representation. The advantage of FS gauge is that, due to Eq. (1), the gluon field can be expressed via the field strength tensor:

$$A_\nu^a(x) = z_\mu \int_0^1 dt\, t\, G_{\mu\nu}^a(tx). \quad (3)$$

Taylor's expansion of the gluon field can be obtained from Eqs. (3) and (2). Using the above Eqs. (2) and (3), one can expand gluon NLCs in terms of local condensates.

The reader may be familiar with the following notation for local condensates: [108] one dimension-four condensate $\langle \alpha_S G^2 \rangle \equiv \langle \alpha_S G_{\mu\nu}^a G_{\mu\nu}^a \rangle$, two dimension-six condensates $\langle gG^3 \rangle \equiv \langle g f^{abc} G_{\mu\nu}^a G_{\nu\rho}^b G_{\rho\mu}^c \rangle$, $\langle J^2 \rangle \equiv \langle J_\mu^a J_\mu^a \rangle$, and two dimension-eight condensates $\langle (f^{abc} G_{\mu\nu}^b G_{\alpha\beta}^c)^2 \rangle$, $\langle (f^{abc} G_{\mu\nu}^b G_{\nu\rho}^c)^2 \rangle$, where g is the coupling constant of strong interaction with $\alpha_S = g^2/(4\pi)$ and the quark current $J_\mu^a = \sum \bar{q} \gamma_\mu t^a q$. We use a different notation for local gluon condensates suggested in [37,109] that includes more condensates at dimension-eight order. Gluon condensate of dimension-four and condensates of dimension-six are defined as follows:

$$G^4 = \langle \text{Tr}\, G_{\mu\nu} G_{\mu\nu} \rangle, \qquad G_1^6 = i \langle \text{Tr}\, G_{\lambda\mu} G_{\mu\nu} G_{\nu\lambda} \rangle,$$
$$G_2^6 = \langle \text{Tr}\, J_\mu J_\mu \rangle, \quad (4)$$

where we use compact matrix notation $G_{\mu\nu} \equiv g G_{\mu\nu}^a(0) t^a$ for the gluon field strength tensor and $J_\mu \equiv g J_\mu^a t^a$ for the quark current. A set of independent dimension-eight gluon condensates was found in [110]. Here we use the notation introduced in [109,111]:

$$G_1^8 = \langle \text{Tr}\, G_{\mu\nu} G_{\mu\nu} G_{\alpha\beta} G_{\alpha\beta} \rangle,$$
$$G_2^8 = \langle \text{Tr}\, G_{\mu\nu} G_{\alpha\beta} G_{\mu\nu} G_{\alpha\beta} \rangle,$$
$$G_3^8 = \langle \text{Tr}\, G_{\mu\alpha} G_{\alpha\nu} G_{\nu\beta} G_{\beta\mu} \rangle,$$
$$G_4^8 = \langle \text{Tr}\, G_{\mu\alpha} G_{\alpha\nu} G_{\mu\beta} G_{\beta\nu} \rangle,$$
$$G_5^8 = i \langle \text{Tr}\, J_\mu G_{\mu\nu} J_\nu \rangle,$$
$$G_6^8 = i \langle \text{Tr}\, J_\lambda [D_\lambda G_{\mu\nu}, G_{\mu\nu}] \rangle,$$
$$G_7^8 = \langle \text{Tr}\, J_\mu D^2 J_\mu \rangle, \quad (5)$$

where the condensate G_7^8 is defined in a different way as suggested in [37]. The notation G_j^i specifies the local condensate of dimension i. The trace is taken in the fundamental representation with the covariant derivative $D_\mu = \partial_\mu - iA_\mu$ and $A_\mu = gA_\mu^a t^a$.

Applying the expansion Eq. (2) in calculation of OPE truncated at dimension-eight order, one may encounter 70 tensor condensates formed by the gluon field strength tensor and its derivatives, e.g., a rank-eight tensor $\langle \text{Tr}\, G_{\mu_1\mu_2} D_{\mu_3} G_{\mu_4\mu_5} D_{\mu_6} G_{\mu_7\mu_8} \rangle$. These tensor condensates can be expressed by 11 scalar local condensates given in Eqs. (4) and (5) and the various combinations of the metric tensors. Instead of considering an expansions for various tensor local condensates, we expand two-, three-, and four-gluon nonlocal condensates and observe that the 15 out of

21 dimension-eight coefficients, gathered in Tables III–V, are expressed only by four linear combinations of seven condensates G_i^8 ($i = 1, \ldots, 7$):

$$G_{12}^8 = G_1^8 - G_2^8, \qquad G_{34}^8 = G_3^8 - G_4^8,$$
$$G_{56}^8 = 4G_5^8 - G_6^8, \qquad G_{67}^8 = 6G_7^8 - G_6^8. \quad (6)$$

The remaining dimension-eight terms of gluon NLC expansions depend on the same seven condensates, see Eq. (5), in linear combinations different from those given in Eq. (6).

There are the following relations with common notations:

$$G_1^4 = 2\pi \langle \alpha_S G^2 \rangle,$$
$$G_1^6 = -g^2 \langle gG^3 \rangle / 4, \qquad G_2^6 = g^2 \langle J^2 \rangle / 2,$$
$$G_{12}^8 = g^4 \langle (f^{abc} G_{\mu\nu}^b G_{\alpha\beta}^c)^2 \rangle / 4,$$
$$G_{34}^8 = g^4 \langle (f^{abc} G_{\mu\nu}^b G_{\nu\rho}^c)^2 \rangle / 4. \quad (7)$$

The combinations G_{56}^8, G_{67}^8 have no commonly used notation.

As we will see in the next subsection, the expansion of gluon fields provides for NLCs up to dimension eight not only the gluon condensates G^4, G_1^8, G_{12}^8, G_{34}^8 but also the four-quark condensate G_2^6 and mixed quark-gluon condensates G_5^8, G_6^8, G_7^8. The quark and mixed quark-gluon condensates have not been considered in glueball studies, including [42,43]. In instanton models, these condensates are equal to zero due to the self-duality of the vacuum gluon field strength tensor. One may expect that the condensates G_2^6, G_5^8, G_6^8, G_7^8 could give a numerically minor correction compared to the pure gluon condensates G_1^6, G_{12}^8, G_{34}^8. In the general case, these condensates should be included in the OPE as they could be enhanced by the coefficient function.

In the next subsection, we present expansions of nonlocal gluon condensates in terms of local condensates. Expanding NLCs at dimension-eight order requires tedious calculation described in [37]. The obtained expansions are one of the important results of this work.

A. Two-gluon NLC

The two-gluon NLC expansion was first presented in dimension-six order [92], where only the collinear part of the condensate was considered. Later, expansion in the noncollinear case was obtained in [37] up to dimension-eight term. The result obtained here is in agreement with the expansion given in [37]. We suggest using the following form for two-gluon NLCs:

$$\langle \text{Tr}\, G_{\mu_1\nu_1}(x)[x;y] G_{\mu_2\nu_2}(y)[y;x] \rangle$$
$$= \frac{1}{d(d-1)} \mathbb{A}(\mu_1,\nu_1) \mathbb{A}(\mu_2,\nu_2) \sum_{k=0}^{8} \Gamma_k(x,y) M_k(x,y), \quad (8)$$

where d is the space dimension, and Wilson's line $[x; y]$ ensures gauge invariance and is defined by the P-ordered exponent:

$$[x; y] = \mathbb{P} \exp\left\{ ig \int_{P(x,y)} d\omega_\mu A_\mu(\omega) \right\}.$$

Here and below we use the operator $\mathbb{A}(\mu, \nu)$ for anti-symmetrization: $\mathbb{A}(\mu, \nu)t_{\mu\nu} \equiv t_{\mu\nu} - t_{\nu\mu}$. The suggested form of expansion, Eq. (8), is more appropriate for modeling scalar functions than the one given in [37] and explicitly displays the symmetries of the condensate with respect to the Lorentz indices of strength tensors and with respect to the transformation $\mu_1 \leftrightarrow \mu_2$, $\nu_1 \leftrightarrow \nu_2$, and $x \leftrightarrow y$ related to the symmetry of gluons field strength tensor permutations in the condensate.

The new basis leads to simpler expansions of the scalar functions M_k that are easier for modeling long distance behavior of NLC. The path $P(x, y)$ is chosen to be a broken line with apexes at the points $x, 0, y$; therefore, the links can be omitted $[x; y] = [x; 0][0; y] = 1$ due to the gauge condition Eq. (1). The same path will be applied to three-gluon and four-gluon condensates. The operators of asymmetrizaton $\mathbb{A}(\mu_1, \nu_1)$ allow one to introduce a brief notation for master tensors:

$$\Gamma_0(x, y) = g_{\mu_1\mu_2}g_{\nu_1\nu_2}/2,$$
$$\Gamma_1(x, y) = (x - y)_{\mu_1}(x - y)_{\mu_2}g_{\nu_1\nu_2}(d + 4)^{-1},$$
$$\Gamma_2(x, y) = (x_{\mu_1}y_{\mu_2} - y_{\mu_1}x_{\mu_2})g_{\nu_1\nu_2},$$
$$\Gamma_3(x, y) = (x_{\mu_1}y_{\mu_2} + y_{\mu_1}x_{\mu_2})g_{\nu_1\nu_2},$$
$$\Gamma_4(x, y) = \Delta \cdot \Gamma_0(x, y), \quad \Delta = x^2y^2 - (xy)^2,$$
$$\Gamma_5(x, y) = (x^2y_{\mu_1}y_{\mu_2} + y^2x_{\mu_1}x_{\mu_2}$$
$$\quad - xy(x_{\mu_1}y_{\mu_2} + y_{\mu_1}x_{\mu_2}))g_{\nu_1\nu_2},$$
$$\Gamma_6(x, y) = x_{\mu_1}y_{\nu_1}x_{\mu_2}y_{\nu_2},$$
$$\Gamma_7(x, y) = (y^2)^2x_{\mu_1}x_{\mu_2}g_{\nu_1\nu_2},$$
$$\Gamma_8(x, y) = (x^2)^2y_{\mu_1}y_{\mu_2}g_{\nu_1\nu_2}. \tag{9}$$

The tensor basis, Eq. (9), is sufficient in all orders of expansion. It has been constructed to have simple expansions for the scalar functions $M_i(x, y)$. The properties of the introduced tensors Γ_n are collected in Table I. The table includes the leading dimension of the corresponding condensate and the specification of which of the tensors contributes when one of the gluon coordinates coincides with the gauge fixing point, $x = 0$ or $y = 0$, and when the coordinates x and y lie on one line with the gauge fixing point $x = ty$ (collinearity condition). The three tensors Γ_0, Γ_1, and Γ_6 are the same as in [37]. At dimension-six order, the tensor Γ_2 is the addition to the Γ_0, Γ_1 tensors considered in [92]. The tensor Γ_4 is introduced to separate the Δ term. The remaining tensors Γ_3 and Γ_5 are chosen so that an

TABLE I. The properties of the tensors $\Gamma_n(x, y)$ for the expansion of the two-gluon NLC. The first row specifies the tensor. The second row gives the leading dimension of the condensate that contributes to the tensor. The third row tells us which of the tensors contribute in the case when one of the gluon coordinates coincides with the gauge fixing point, e.g., $x = 0$ or $y = 0$. The fourth row demonstrates which of the tensors is nonzero in the collinear case when the coordinates x and y lying on one line with the gauge fixing point $x = ty$.

n	0	1	2	3	4	5	6	7
Dimension	4	6	6	8	8	8	8	8
$x = 0$	$\neq 0$	$\neq 0$	0	0	0	0	0	0
$x = ty$	$\neq 0$	$\neq 0$	0	$\neq 0$	0	0	0	0

expansion of the corresponding scalar functions start from dimension-eight condensates. The special form of the tensor Γ_5 comes from the gluon field strength tensor exchange symmetry in the condensate.

The expansions for the introduced scalar functions are defined in the following way:

$$M_0(x, y) = G_1^4 + \frac{(x - y)^2}{(d - 2)(d + 2)}\left(k_0^{6,2} + \frac{k_1^{6,2}}{d + 4}\right.$$
$$\left. + \left(k_{0,1}^{8,2} + \frac{k_{1,1}^{8,2}}{2(d + 4)}\right)\frac{(x - y)^2}{4!}\right) + \dots, \tag{10}$$

$$M_i(x, y) = \frac{1}{(d - 2)(d + 2)}\left(k_i^{6,2} + \frac{k_{i,1}^{8,2}(x - y)^2 + 2k_{i,2}^{8,2}xy}{4!}\right)$$
$$+ \dots, \quad \text{for } i = 1, 2,$$

$$M_3(x, y) = \frac{k_{3,1}^{8,2}(x - y)^2}{4!(d - 2)(d + 2)} + \dots,$$

$$M_i(x, y) = \frac{2k_{i,1}^{8,2}}{4!(d - 2)(d + 2)(d - 3)(d + 1)} + \dots,$$
$$\text{for } i = 4, 5, 6, \tag{11}$$

where the dimension-six coefficients $k_i^{6,2}$ and dimension-eight coefficients $k_{i,j}^{8,2}$ are given in Tables II and III, respectively. The expansion is given for space dimension d.

TABLE II. The dimension-six coefficients $k_i^{6,2}$ of two-gluon scalar function expansions, Eqs. (10) and (11), are given in the second column with the subscripts i defined in the first column. The third column presents the coefficients in the case of the space dimension $d = 4$.

i	$k_i^{6,2}$	$d = 4$
0	$-(d + 2)G_1^6$	$-6G_1^6$
1	$(d + 4)[-(d - 4)G_1^6 - (d - 2)G_2^6]$	$-16G_2^6$
2	$-(d + 2)G_1^6$	$-6G_1^6$

TABLE III. Expressions for the dimension-eight coefficients $k_{i,j}^{8,2}$ of two-gluon condensate expansions are given in the second column with the subscripts i, j given in the first column. In the third column, the coefficients in the case of gluodynamics in space dimension $d = 4$ are given. The last column presents the coefficients when the vacuum gluon field strength tensor is (anti-)self-dual $\tilde{G}_{\mu\nu}^a \equiv i\epsilon_{\mu\nu\alpha\beta}G^{a\alpha\beta}/2 = \pm G_{\mu\nu}^a$.

i, j	$k_{i,j}^{8,2}$	$d = 4, J = 0$	Self-dual
0,1	$2G_{12}^8 + 11G_{34}^8 - 3G_{56}^8$	$2G_{12}^8 + 11G_{34}^8$	$15G_{34}^8$
1,1	$2((d-6)G_{12}^8 + (13d-48)G_{34}^8 - 6(d-3)G_{56}^8 - (d-2)G_{67}^8)$	$-4(G_{12}^8 - 2G_{34}^8)$	0
1,2	$(d+4)(G_{12}^8 - 2G_{34}^8 + G_{56}^8)$	$8(G_{12}^8 - 2G_{34}^8)$	0
2,1	$3G_{12}^8 + 24G_{34}^8 - 7G_{56}^8$	$3(G_{12}^8 + 8G_{34}^8)$	$30G_{34}^8$
2,2	$3G_{34}^8 - G_{56}^8$	$3G_{34}^8$	$3G_{34}^8$
3,1	$-G_{12}^8 + 2G_{34}^8 - G_{56}^8$	$2G_{34}^8 - G_{12}^8$	0
4,1	$2(d-3)(d-2)G_6^8 + (7-3d)dG_{12}^8 + 4(d+3)G_{34}^8$	$-4(5G_{12}^8 - 7G_{34}^8)$	$-12G_{34}^8$
5,1	$2(d-3)(d-2)G_6^8 - ((d-6)d+3)G_{12}^8 - (d(d+6)-3)G_{34}^8$	$5G_{12}^8 - 37G_{34}^8$	$-27G_{34}^8$
6,1	$6((d-1)^2G_{34}^8 + (d-3)(d-2)G_6^8 + (d-4)G_{12}^8)$	$54G_{34}^8$	$54G_{34}^8$

The expansions for $M_7(x, y)$, $M_8(x, y)$ start from the dimension-10 condensates. Due to the symmetry concerning gluon field strength tensor exchange, the following scalar functions are related $M_8(x, y) = M_7(y, x)$.

Note that the two-gluon NLC expansion violates translational invariance in the FS gauge, since the expansion depends on the coordinate of the gauge fixing point. The contribution of the tensor Γ_2 starts in dimension-six order that is the leading order (LO) where violation of translational invariance occurs. The translational invariance of correlators is restored when all contributions to the coefficient function of a given dimension are taken into account [92,112].

The main result of Sec. II is NLC expansions. The expansions are defined by the Lorentz tensors and scalar functions. The latter are presented in the form of expansion whose coefficients are linear combinations of the local condensates. For clarity of expansions, we accumulated the coefficients in Tables II–V.

B. Three-gluon NLC

The result of the three-gluon nonlocal condensate expansion up to dimension-eight can be presented by the rank-six Lorentz tensor and depends on three coordinates:

$$i\langle \text{Tr}\, G_{\mu_1\nu_1}(x_1)[x_1; x_2]G_{\mu_2\nu_2}(x_2)[x_2; x_3]G_{\mu_3\nu_3}(x_3)[x_3; x_1]\rangle$$
$$= \frac{\Gamma(d-2)}{\Gamma(d+1)}\left(\prod_j^3 \mathbb{A}(\mu_j, \nu_j)\right)\frac{1}{2}\sum_{i=0}^7 \mathbb{A}\Gamma_i^{(abc)}M_i(x_a, x_b, x_c)$$
$$+ \dots, \tag{12}$$

where the index (abc) could be one of the six permutations of (123) and is used to denote permutations of three sets. Each of the sets includes two Lorentz indices and one coordinate: (μ_1, ν_1, x_1), (μ_2, ν_2, x_2), and (μ_3, ν_3, x_3). The scalar functions are denoted by M_i. The dependence of the rank-six master Lorentz tensor $\Gamma_i^{(abc)}$ on three coordinates x_1, x_2, x_3 is implied:

$$\Gamma_i^{(abc)}M_i(x_a, x_b, x_c) = T_{\mu_a\nu_a\mu_b\nu_b\mu_c\nu_c}(x_a, x_b, x_c) = T_{(abc)},$$

where $T_{(abc)}$ is short for the rank-six tensor. The operator $\mathbb{A}$ is introduced to shorten the expression and restore asymmetry of the condensate with respect to permutations of the gluon field strength tensors. The operator is defined by the following antisymmetrization with respect to permutations of three sets:

$$\mathbb{A}T_{(abc)} = T_{(123)} - T_{(132)} + T_{(231)} - T_{(213)} + T_{(312)} - T_{(321)},$$

To shorten the expression, we also define the operator of symmetrization $\mathbb{S}$:

$$\mathbb{S}T_{(abc)} = T_{(123)} + T_{(132)} + T_{(231)} + T_{(213)} + T_{(312)} + T_{(321)},$$

Using the operators $\mathbb{S}$ and $\mathbb{A}$ makes the expressions eye-readable and simplifies calculations of OPE. The master tensors $\Gamma_0^{(abc)}$ are given as follows: [113]

$$\Gamma_0^{(abc)} = g_{\nu_c\mu_a}g_{\nu_a\mu_b}g_{\nu_b\mu_c}/3,$$
$$\Gamma_1^{(abc)} = \frac{1}{d+1}[(\mathbb{S}r_A^{(abc)}) + (\mathbb{S}r_B^{(abc)}/2)](x_c)_\rho(x_a)_\sigma,$$
$$\Gamma_2^{(abc)} = [(x_c)_\rho(x_a)_\sigma - (x_c)_\sigma(x_a)_\rho]r_A^{(abc)},$$
$$\Gamma_3^{(abc)} = (x_c)_\rho(x_a)_\sigma r_B^{(abc)},$$
$$\Gamma_4^{(abc)} = [(\mathbb{A}r_A^{(abc)}) - (\mathbb{S}r_B^{(abc)}/2)](x_c)_\rho(x_a)_\sigma,$$
$$\Gamma_5^{(abc)} = [(x_c)_\rho(x_a)_\sigma + (x_c)_\sigma(x_a)_\rho]r_A^{(abc)},$$
$$\Gamma_6^{(abc)} = (x_b)_\rho(x_b)_\sigma r_A^{(abc)},$$
$$\Gamma_7^{(abc)} = [(x_a)_\rho(x_a)_\sigma + (x_b)_\rho(x_b)_\sigma + (x_c)_\rho(x_c)_\sigma]r_A^{(abc)},$$
$$r_A^{(abc)} = g_{\nu_c\rho}g_{\sigma\mu_a}g_{\nu_a\mu_b}g_{\nu_b\mu_c},$$
$$r_B^{(abc)} = g_{\mu_a\mu_c}g_{\nu_a\nu_c}g_{\mu_b\rho}g_{\nu_b\sigma}. \tag{13}$$

The choice for the tensors has been motivated by the simplicity of the scalar function expansions considered up to dimension-eight order:

$$M_0(x, y, z) = G_1^6 + \frac{k_0^{8,3}}{8(d-3)(d+2)}$$
$$\cdot \frac{(y-x)^2 + (x-z)^2 + (z-y)^2}{4} + \ldots,$$

$$M_i(x, y, z) = \frac{k_i^{8,3}}{8(d-3)(d+2)} + \ldots, \quad \text{for } i \geq 1, \quad (14)$$

where the coefficients $k_i^{8,3}$ are collected in Table IV. The same as in the two-gluon case, the gauge link path is a broken line with apex at the gauge fixing point; therefore, the link can be omitted $[x; y] = [x; 0][0; y] = 1$ due to the gauge condition, Eq. (1). Note that only the C-even part contributes in Eq. (12):

$$\langle \operatorname{Tr} G_{\mu_1\nu_1}(x) G_{\mu_2\nu_2}(y) G_{\mu_3\nu_3}(z) \rangle$$
$$= \frac{1}{2} \langle \operatorname{Tr} G_{\mu_1\nu_1}(x)[G_{\mu_2\nu_2}(y), G_{\mu_3\nu_3}(z)] \rangle,$$

while the C-odd contribution is equal to zero

$$\langle \operatorname{Tr} G_{\mu_1\nu_1}(x)\{G_{\mu_2\nu_2}(y), G_{\mu_3\nu_3}(z)\} \rangle = 0,$$

which is explicitly confirmed by the obtained expansion, Eq. (12), up to dimension-eight order. The C-parity causes antisymmetrization of the condensate with respect to permutations of the gluon field strength tensors in three-gluon NLC.

C. Four-gluon NLC

We present the four-gluon condensate in the following form [114]:

$$\langle \operatorname{Tr} (G_{\mu_1\nu_1}(x_1) G_{\mu_2\nu_2}(x_2) G_{\mu_3\nu_3}(x_3) G_{\mu_4\nu_4}(x_4)) \rangle$$
$$= \frac{\Gamma(d-3)}{\Gamma(d+3)} \left(\prod_{n=1}^{4} \mathbb{A}(\mu_n, \nu_n) \right)$$
$$\times \sum_{m=1}^{4} \Gamma_m M_m(x_1, x_2, x_3, x_4) + \ldots, \quad (15)$$

where M_m are the scalar functions and Γ_m are the Lorentz tensors that contribute starting with the leading dimension-eight order. The four-gluon condensate is symmetric with respect to cyclic permutations of the gluon field strength tensors and the reflection of their order in the trace. To respect these symmetries, the scalar functions M_m should be invariant to cyclic permutations of the arguments (x_1, x_2, x_3, x_4) and the reflection of their order:

$$M_m(x_1, x_2, x_3, x_4) = k_m^{8,4} + a_m(x_1^2 + x_2^2 + x_3^2 + x_4^2)$$
$$+ b_m(x_1 \cdot x_2 + x_2 \cdot x_3 + x_3 \cdot x_4 + x_4 \cdot x_1)$$
$$+ c_m(x_1 \cdot x_3 + x_2 \cdot x_4) + \cdots.$$

The leading order terms $k_m^{8,4}$ are given in Table V in terms of the dimension-eight local condensates. The dimension-10 order coefficients a_m, b_m, and c_m are not considered here, as we work in dimension-eight order. The Lorentz tensors respect the discussed symmetry due to antisymmetrization of four pairs of indices $\mu_n\nu_n$, $n = 1, 2, 3, 4$:

$$\Gamma_1 = r_A^{(1234)}, \qquad \Gamma_2 = r_A^{(1234)} + r_A^{(2314)} + r_A^{(3124)},$$
$$\Gamma_3 = r_B^{(1234)}, \qquad \Gamma_4 = r_B^{(1234)} + r_B^{(2314)} + r_B^{(3124)},$$
$$r_A^{(ijkl)} = g_{\nu_k\mu_l} g_{\nu_l\mu_i} g_{\nu_i\mu_j} g_{\nu_j\mu_k},$$
$$r_B^{(ijkl)} = g_{\mu_i\mu_k} g_{\nu_i\nu_k} g_{\mu_j\mu_l} g_{\nu_j\nu_l}/4. \quad (16)$$

The considered vacuum expectation corresponds to the general case $\langle G_{\mu_1\nu_1}^{a_1} G_{\mu_2\nu_2}^{a_2} G_{\mu_3\nu_3}^{a_3} G_{\mu_4\nu_4}^{a_4} \rangle$ condensate that

TABLE IV. Expressions for the dimension-eight coefficients $k_i^{8,3}$ of the three-gluon condensate expansions are given in the second column with the subscript i given in the first column. In the third column, the coefficients in the case of gluodynamics in space dimension $d = 4$ are given. The last column presents the coefficients when the vacuum gluon field strength tensor is (anti-)self-dual $\tilde{G}_{\mu\nu}^a \equiv i\epsilon_{\mu\nu\alpha\beta}G^{a\alpha\beta}/2 = \pm G_{\mu\nu}^a$.

i	$k_i^{8,3}$	$d = 4, J = 0$	Self-dual
0	$4(G_{12}^8 - 4(d-2)G_{34}^8 + (d-3)G_{56}^8)$	$4(G_{12}^8 - 8G_{34}^8)$	$-24G_{34}^8$
1	$-2(d-3)(d-2)G_6^8 + (d-4)(d-1)G_{12}^8 + 4(d-1)G_{34}^8$	$12G_{34}^8$	$12G_{34}^8$
2	$4G_{12}^8 - 4(d+1)G_{34}^8$	$4(G_{12}^8 - 5G_{34}^8)$	$-12G_{34}^8$
3	$2(d-2)G_{12}^8 - 8G_{34}^8$	$4(G_{12}^8 - 2G_{34}^8)$	0
4	$(d-2)G_{12}^8 - 4G_{34}^8$	$2(G_{12}^8 - 2G_{34}^8)$	0
5	$-2(d-3)(G_{12}^8 - 2G_{34}^8 + G_{56}^8)$	$-2(G_{12}^8 - 2G_{34}^8)$	0
6	$-2(d-3)(G_{12}^8 - 2G_{34}^8 + G_{56}^8)$	$-2(G_{12}^8 - 2G_{34}^8)$	0
7	$-2(2(d-5)G_{34}^8 - (d-3)G_{56}^8 + G_{12}^8)$	$-2(G_{12}^8 - 2G_{34}^8)$	0

TABLE V. Expressions for the dimension-eight coefficients $k_i^{8,4}$ of the four-gluon condensate expansion are given in the second column with the subscript i given in the first column. In the third column, the coefficients in the case of gluodynamics in space dimension $d = 4$ are given. The last column presents the coefficients when the vacuum gluon field strength tensor is (anti-)self-dual $\tilde{G}_{\mu\nu}^a \equiv i\epsilon_{\mu\nu\alpha\beta}G^{a\alpha\beta}/2 = \pm G_{\mu\nu}^a$.

i	$k_i^{8,4}$	$d = 4, J = 0$	Self-dual
1	$(d+1)[(d+1)G_{34}^8 - G_{12}^8]$	$-5(G_{12}^8 - 5G_{34}^8)$	$15G_{34}^8$
2	$(d^2+3)G_4^8 + (1-d)G_1^8 - dG_2^8 + (1-d)G_3^8$	$-3G_1^8 - 4G_2^8 - 3G_3^8 + 19G_4^8$	$-3(G_1^8 + G_{34}^8)$
3	$-(d+1)[(d-2)G_{12}^8 - 4G_{34}^8]$	$-10(G_{12}^8 - 2G_{34}^8)$	0
4	$(d^2-d+2)G_1^8 - 4dG_3^8 - 4(d-1)G_4^8 + 2G_2^8$	$2(7G_1^8 + G_2^8 - 8G_3^8 - 6G_4^8)$	$9G_1^8 - 6G_{34}^8$

is contracted with the color tensor $\mathrm{Tr}\, t^{a_1}t^{a_2}t^{a_3}t^{a_4}$, where a_i are the color labels (i.e., group index) of the gluon field strength tensors. This expansion can be used for calculating OPE of the quark fields correlators. From the obtained expression (15) we can extract the expansion of the four-gluon condensate $\langle G_{\mu_1\nu_1}^{a_1} G_{\mu_2\nu_2}^{a_2} G_{\mu_3\nu_3}^{a_3} G_{\mu_4\nu_4}^{a_4} \rangle$ contracted with the color tensor $f^{ba_1a_2}f^{ba_3a_4}$, which is useful for OPE calculations related to glueball states:

$$\langle \mathrm{Tr}\,([G_{\mu_1\nu_1}, G_{\mu_2\nu_2}][G_{\mu_3\nu_3}, G_{\mu_4\nu_4}])\rangle$$
$$= \frac{\Gamma(d-3)}{\Gamma(d+3)}\left(\prod_{j=1}^4 \mathbb{A}(\mu_j, \nu_j)\right)\sum_{i=1}^2 \tilde{\Gamma}_i\tilde{M}_i,$$
$$\tilde{\Gamma}_1 = (r_A^{(1234)} - r_A^{(3124)}), \qquad \tilde{\Gamma}_2 = (r_B^{(1234)} - r_B^{(3124)}),$$
$$\tilde{M}_1 = 2k_1^{8,4}, \qquad \tilde{M}_2 = 2k_3^{8,4}, \tag{17}$$

where $G_{\mu\nu} \equiv gG_{\mu\nu}^a(0)t^a$, the tensors r_i^A, r_i^B are defined in (16), and the coefficients $k_1^{8,4}$ and $k_3^{8,4}$ are given in Table V. The expansion (17) coincides with the one given in [45].

III. USAGE OF GLUON NLC EXPANSIONS

In the previous section, we obtained two-, three-, and four-gluon NLC expansions in the FS gauge. This section is dedicated to a discussion of the practical importance of the obtained expansions. The usual way of OPE calculations includes applying the Taylor expansion of vacuum fields, see Eq. (2). In the case of gluon condensates, the Taylor expansion in mass dimension-D order leads to intermediate expressions in the form of the rang-D tensor. In dimension-eight order of OPE, there are 70 tensor condensates formed by the gluon field strength tensor and its derivatives, e.g., $\langle \mathrm{Tr}\, G_{\mu_1\mu_2}D_{\mu_3}G_{\mu_4\mu_5}D_{\mu_6}G_{\mu_7\mu_8}\rangle$. The tensor expansion of these condensates leads to an expression defined by 11 scalar local condensates given in Eqs. (4) and (5).

Using NLC expansions causes a significant reduction of computational work by jumping over the Taylor and tensor expansions. As we will see in the next subsection, the application of NLC expansions is especially efficient in the OPE of a vacuum correlator for the currents with the gluon field strength tensor, such as currents of glueball and hybrid states [54,115–132].

In the QCD SR approach, the correlator OPE serves as a source of information on hadron parameters. The OPE of vacuum correlator Π based on the dimension-D truncated NLC expansions is the same as OPE defined by the local condensates up to dimension-D order:

$$\Pi = \Pi_{\mathrm{pert}} + \sum_{i=1}^{N_D} \Pi_{\mathrm{NLC}-i} = \Pi_{\mathrm{pert}} + \sum_{j=3}^D \Pi_j, \tag{18}$$

where Π_{pert} is the perturbative contribution, Π_j is the nonperturbative contribution [133] of dimension j, and $\Pi_{\mathrm{NLC}-i}$ is the ith NLC contribution. The number of NLC-based diagrams that contribute to dimension D is denoted by N_D. Equation (18) reflects the idea of the OPE rearrangement in terms of NLCs.

The NLC expansion can be widely applied due to its universality. The usage of NLC expansions is demonstrated by the vacuum correlators of two-gluon and three-gluon glueball currents with quantum numbers $0^{\pm+}$. The discussion below is given in general terms, while the technical details are placed in the Appendices. In particular, the full set of Feynman rules needed for such calculations is presented in Appendix A, while Appendix B provides the detailed calculation of one of the contributions to OPE of the glueball current correlators.

Before considering these applications, we want to refresh some aspects of OPE calculations [9,37] in the next subsection. We cover aspects of the background approach [134–136] related to glueball studies and the gluon propagator and its precalculated expansions in the background gluon field.

A. Background field approach

In the background field approach, the total gluon field is considered as a compound of two fields $\bar{A}_\mu^a = A_\mu^a + a_\mu^a$: the perturbative quantum gluon field a_μ^a and background field A_μ^a. To keep the gluon propagator of the quantum field a_μ^a in Feynman gauge form, one should add the generalization gauge fixing term $(D_\mu a_\mu^a)^2/(-2)$ to the Lagrangian that causes modification of the interaction between background and quantum fields. In Fig. 7, we provide the Feynman rules for vertices of quantum gluon field interaction with

background fields for the case of three- and four-gluon vertices where two fields are quantum and the rest are background fields. Using these rules, one obtains expansion of the gluon propagator D in the external field up to dimension-four order:

$$
\begin{aligned}
-iD(p) = &-iD_0 + (-i)^2 D_0 V_1 D_0 \\
&+ \frac{2}{2!}(-i)^3 D_0 V_1 D_0 V_1 D_0 \\
&+ (-i)^2 D_0 V_2 D_0 + \dots,
\end{aligned}
$$

where D_0 is the free propagator, V_1 and V_2 are the vertices of interaction between quantum field and background field. Definitions and graphical notations of vertices are given in Fig. 7. The gluon propagator can be expressed by four terms given in graphical form in Fig. 1:

$$
\begin{aligned}
D_{\alpha\beta}(p) &= i \int d^4 x\, e^{ipx} \langle T\{a_\alpha(x) a_\beta(0)\}\rangle, \\
&= D_{0\alpha\beta} + D_{1\alpha\beta} + D_{21\alpha\beta} + D_{22\alpha\beta} + \dots \quad (19)
\end{aligned}
$$

Note that there is an additional diagram for the third term D_{21} that gives combinatoric factor of 2. The gluon propagator expansion was obtained in [137] (see also [9,37]). For reader's convenience, we present the result for each term

$$
\begin{aligned}
D_{0\alpha\beta} &= \frac{1}{p^2}\delta_{\alpha\beta}, \\
D_{1\alpha\beta} &= \frac{2G_{\alpha\beta}}{p^4} + \frac{1}{p^6}\left(\frac{2}{3}ip_\mu J_\mu \delta_{\alpha\beta} + 4ip_\lambda D_\lambda G_{\alpha\beta}\right) \\
&\quad + \frac{1}{p^8}[-2p_\lambda D_\lambda p_\mu J_\mu \delta_{\alpha\beta} - 2(4(p_\lambda D_\lambda)^2 - p^2 D^2)G_{\alpha\beta}] \\
&\quad + \Delta D_{1\alpha\beta}, \\
D_{21\alpha\beta} &= \frac{4}{p^6}G_{\alpha\lambda}G_{\lambda\beta}, \\
D_{22\alpha\beta} &= \frac{\delta_{\alpha\beta}}{2p^8}(p^2 G_{\mu\nu}G_{\mu\nu} + 4p_\mu G_{\mu\lambda}G_{\lambda\nu}p_\nu) - \Delta D_{1\alpha\beta}, \\
\Delta D_{1\alpha\beta} &= \frac{1}{p^8}(p^2[G_{\alpha\mu}, G_{\mu\beta}] + 2p_\mu p_\nu[G_{\alpha\mu}, G_{\beta\nu}]), \quad (20)
\end{aligned}
$$

where we use compact matrix notation for the field strength tensor $G = G^{ac} = gf^{abc}G^b$ and the current $J = J^{ac} = gf^{abc}J^b$.

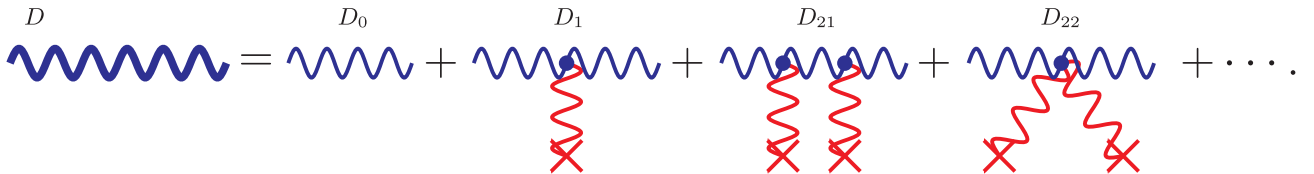The glueball and hybrid state currents include the gluon field strength tensor. For such currents, the correlator OPE has terms that depend on the modified propagator with fields derivatives:

$$
\begin{aligned}
D_{\alpha\beta,\mu\nu}(p) &= i \int d^4 x\, e^{ipx} \langle T\{\partial_\mu a_\alpha(x) \partial_\nu a_\beta(0)\}\rangle, \\
&= \frac{g_{\alpha\beta}q_\mu q_\nu}{q^2} + \frac{2q_\mu q_\nu G_{\alpha\beta}}{q^4} - \frac{g_{\alpha\beta}q_\mu q_\rho G_{\nu\rho}}{q^4} + \dots \quad (21)
\end{aligned}
$$

The first two terms can be easily obtained from propagator's expansion given in Eq. (19), while the third term is additional and related to the derivative of the quantum field $a_\beta(0)$. Therefore, in glueball and hybrid-related studies within QCD SR, one needs to calculate additional background field corrections compared to those given in Eq. (19).

The usage of NLCs truncated series allows the calculation of all background field corrections to the final expression for the correlator OPE without applying the gluon propagator expansion. This approach to perform OPE appears especially useful for C-odd glueball studies [79–81], where the glueball currents include high derivatives of the gluon field strength tensor. The suggested approach can be widely applied due to the universality of NLC expansions. In the next subsections, we demonstrate the application of the approach to OPE of glueball current correlators.

B. Two-gluon $0^{\pm +}$ glueballs

QCD SR for glueballs were first considered in [42,43] by evaluating the correlator

$$
\Pi^P(q) = i \int d^4 x\, e^{iqx} \langle T\{J_2^P(0) J_2^{P\dagger}(x)\}\rangle \quad (22)
$$

of the two-gluon current for the scalar 0^{++} and pseudoscalar 0^{-+} glueball states:

$$
J_2^P(x) = \alpha_S \delta^{a_1 a_2} T^P_{\mu_1\nu_1\mu_2\nu_2} G^{a_1}_{\mu_1\nu_1}(x) G^{a_2}_{\mu_2\nu_2}(x), \quad (23)
$$

where rank-4 Lorentz tensors $T^+_{\mu_1\nu_1\mu_2\nu_2} \equiv g_{\mu_1\mu_2}g_{\nu_1\nu_2}$ and $T^-_{\mu_1\nu_1\mu_2\nu_2} \equiv i\epsilon_{\mu_1\nu_1\mu_2\nu_2}/2$ specify the parity $P = \pm$. The OPE of the correlator can be presented as

$$
\Pi^P(q) = \Pi^P_{\text{LO}} + \Pi^P_4 + \Pi^P_6 + \Pi^P_8 + \dots, \quad (24)
$$

FIG. 1. Contributions to the gluon propagator in the external fields, Eq. (19).

where Π_{LO}^P is the leading order perturbative contibution, and the nonperturbative corrections Π_n^P include only condensates with even dimension if we do not consider a radiative correction. This OPE was calculated in [42,43] up to dimension-eight where only gluon condensates were taken into account. The radiative correction to the dimension-four and dimension-six gluon condensate terms was obtained in [88–90].

We rearrange the contributions to OPE in Eq. (24), by collecting terms arising from one NLC and a specific hard part of the diagram

$$\Pi^P(q) = \Pi_{\mathrm{LO}}^P + \sum_{i=0}^{5} \Pi_{\mathrm{NLC}-i}^P + \cdots, \qquad (25)$$

where $\Pi_{\mathrm{NLC}-i}^P$ is one of the six NLC groups of OPE contributions to $\Pi^P(q)$ that provide contribution to the dimension-eight condensates or lower dimension terms. Each group $\Pi_{\mathrm{NLC}-i}^P$ is depicted by one diagram in Figs. 2 and 3. The NLCs that have dimensions higher than eight are not considered in our work. The crosses on the figures specify the background field and the blob around the crosses denotes NLC. There are two types of lines: the single line represents gluon fields; the double line depicts the gluon field strength tensor. The red color and the cross at the end of the line denote the soft part that forms a vacuum condensate. The blue color of the line denotes the hard part of the diagram. For condensates with a gluon fields (single red line with the cross at the end), we apply Eq. (3) to express the term through obtained NLCs expansions. The Feynman rules for the diagrams are given in Appendix A.

The first diagram in Fig. 2 is the LO perturbative contribution, while the second diagram could contribute from dimension-four to higher dimensions. The OPE for the third and fourth diagrams in Fig. 2 starts from dimension-six terms. The diagrams that contribute starting from dimension-eight condensates are presented in Fig. 3. Since we do not consider higher dimension, the diagrams in Fig. 3 could be considered in both forms: either as local condensates or as NLC where only the leading term is taken into account. In any case, using the leading terms of the expansion, Eq. (15), is useful, as it is presented in the form of the product of local scalar condensates and Lorentz tensors.

We use the NLCs and their expansions to obtain nonperturbative contributions to OPE:

$$\Pi_4^{\pm} = \pm 4\alpha_S \langle \alpha_S G^2 \rangle,$$

$$\Pi_6^{\pm} = \frac{8\alpha_S^2}{Q^2}\left(\pm\langle gG^3 \rangle + \frac{1}{3}\langle J^2 \rangle\right),$$

$$\Pi_8^{\pm} = \frac{2\alpha_S}{\pi Q^4}(2G_{34}^8 - G_{12}^8 \pm 12G_{34}^8 \pm G_{56}^8), \qquad (26)$$

(a) $\Pi_{\mathrm{LO}}^{\pm}$ \qquad (b) $\Pi_{\mathrm{NLC}\text{-}0}^{\pm}$ \qquad (c) $\Pi_{\mathrm{NLC}\text{-}1}^{\pm}$ \qquad (d) $\Pi_{\mathrm{NLC}\text{-}2}^{\pm}$
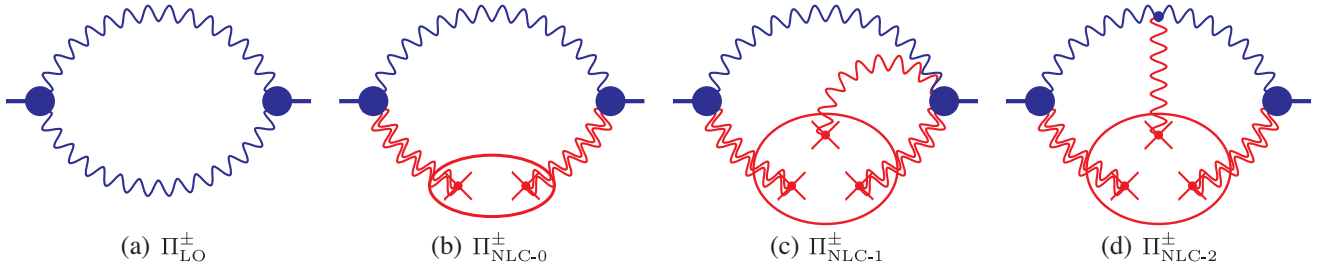
FIG. 2. Diagrams for the LO perturbative term (the first diagram) and nonperturbative contributions to the correlator. The expansions of the depicted terms start from dimension-four condensate ($\langle G^2 \rangle$ for the second diagram) and dimension-six ($\langle G^3 \rangle$ for the third and the fourth diagrams). NLC is represented by crosses that denote vacuum gluon fields (single line) or vacuum gluon field strength tensors (double line). The Feynman rules for the elements of the diagrams are given in Appendix A.

(e) $\Pi_{\mathrm{NLC}\text{-}3}^{\pm}$ \qquad (f) $\Pi_{\mathrm{NLC}\text{-}4}^{\pm}$ \qquad (g) $\Pi_{\mathrm{NLC}\text{-}5}^{\pm}$
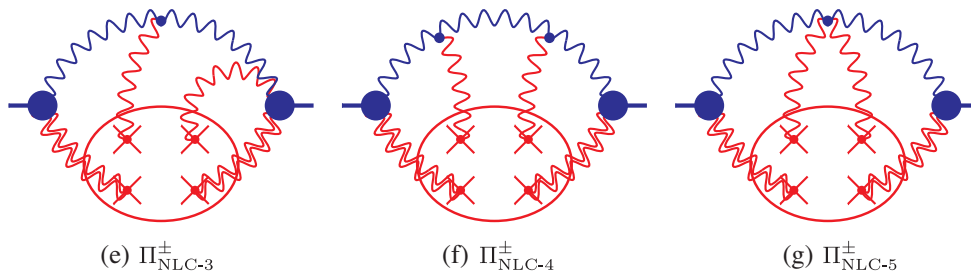
FIG. 3. Diagrammatic representation of three groups of nonperturbative contributions to the correlator $\Pi^P(q)$ OPE that starts from dimension-eight order. The Feynman rules for the elements of the diagrams are given in Appendix A.

where the notation for dimension-eight condensates G_i^8 can be found in Eqs. (5) and (6), and $Q^2 = -q^2$. Our result agrees with [42,43] and provides additional contributions: dimension-six four-quark condensate $\langle J^2 \rangle$ and dimension-eight mixed quark-gluon condensates G_{56}^8. They were omitted in earlier studies [42,43] as they were expected to give a minor contribution. The details and procedures of calculation can be found in Appendices A and B. There, the contributions from each diagram are given for dimension-six and dimension-eight orders. The case of applying the NLC expansion is discussed in detail in Appendix B using the example of Fig. 2(d).

C. Three-gluon $0^{\pm +}$ glueballs

Here we revisit OPE of the correlator used for QCD SR for the three-gluon $0^{\pm +}$ glueballs [44,45]:

$$\tilde{\Pi}^\pm(q) = i \int d^4 x \, e^{iqx} \langle T\{J_3^\pm(x) J_3^{\dagger\pm}(0)\} \rangle,$$

$$= \tilde{\Pi}_{\text{LO}}(Q^2) + \sum_{k=1}^{5} \tilde{\Pi}_k^\pm(Q^2) + \cdots, \qquad (27)$$

where the subscript k numerates the NLC-based terms $\tilde{\Pi}_k^\pm$ and the currents are defined as follows:

$$J_3^+(x) = g_s^3 f^{abc} G_{\mu\nu}^a(x) G_{\nu\rho}^b(x) G_{\rho\mu}^c(x),$$
$$J_3^-(x) = g_s^3 f^{abc} \tilde{G}_{\mu\nu}^a(x) \tilde{G}_{\nu\rho}^b(x) \tilde{G}_{\rho\mu}^c(x), \qquad (28)$$

where the dual tensor $\tilde{G}_{\mu\nu}^a = i\epsilon_{\mu\nu\alpha\beta} G_{\alpha\beta}^a / 2$. We suggest another identical expression for a negative parity current that is easier in use:

$$J_3^-(x) = g_s^3 f^{abc} G_{\mu\nu}^a(x) G_{\nu\rho}^b(x) \tilde{G}_{\rho\mu}^c(x). \qquad (29)$$

Note that any other similar construction of gluon field strength tensors and dual tensors will be identical to $J_3^\pm$, see discussion in [79], where the currents were constructed using helicity formalism. We recalculated the leading perturbative term, the dimension-four and the dimension-six terms of OPE for two current correlators. The leading perturbative term $\tilde{\Pi}_{\text{LO}}(Q^2)$ is depicted by the first diagram

in Fig. 4, the dimension-four term comes only from the second diagram in Fig. 4. Apart from the first diagram, all diagrams presented in Fig. 4 contribute to the dimension-six terms and terms of higher dimensions. Using expansions of NLCs, we got the following result for contributions to OPE in dimension-six order:

$$\tilde{\Pi}_{\text{LO}}(Q^2) = -\frac{N_c^2 C_F}{5 \cdot 8} \alpha_s^3 Q^8 \ln \frac{Q^2}{\mu^2},$$

$$\tilde{\Pi}_1^\pm(Q^2) = \pm 6\pi N_c \alpha_s^2 Q^4 \langle \alpha_s G^2 \rangle$$
$$\qquad - 6\pi N_f C_F \alpha_s^3 \langle \bar{q}q \rangle^2 Q^2 \ln \frac{Q^2}{\mu^2},$$

$$\tilde{\Pi}_2^\pm(Q^2) = \mp (9/4) N_c \alpha_s^2 \langle g^3 G^3 \rangle Q^2,$$

$$\tilde{\Pi}_3^\pm(Q^2) = \pm 5(9/4) N_c \alpha_s^2 \langle g^3 G^3 \rangle Q^2,$$

$$\tilde{\Pi}_4^\pm(Q^2) = \mp 9 N_c \alpha_s^2 \langle g^3 G^3 \rangle Q^2 \ln \frac{Q^2}{\mu^2},$$

$$\tilde{\Pi}_5^\pm(Q^2) = -\tilde{\Pi}_4^\pm(Q^2), \qquad (30)$$

where μ is the renormalization scale, N_c is the number of colors, N_f is the number of flavors, and C_F is the Casimir operator in the fundamental representation. Our result has the same properties as the results in [42,43,81]. Namely, the dimension-four and dimension-six gluon condensate terms have the same absolute value for both parities but different signs. The dimension-eight terms are expected to have the same property but only for the case of (anti-)self-dual gluon fields that can also be observed in the results for other glueball states [42,43,79].

For the three-gluon 0^{++} glueball, our expression for the dimension-four term $\langle \alpha_s G^2 \rangle$ agrees with [44]; the dimension-six term, however, differs from [44]. In the case of the 0^{-+} glueball, our result for the dimension-four contribution is the same in absolute value but has the opposite sign compared to [45]. The dimension-six terms have partial correspondence and partial agreement with the terms given in [45]. Note that we take into account not only the first term of expansion but all necessary terms, Eq. (2), for the gluon field and its strength tensor; this causes a difference for the $\tilde{\Pi}_1^-$ contribution. The difference for the $\tilde{\Pi}_3^-$ term, denoted by diagram (d) in Fig. 4, arises from the
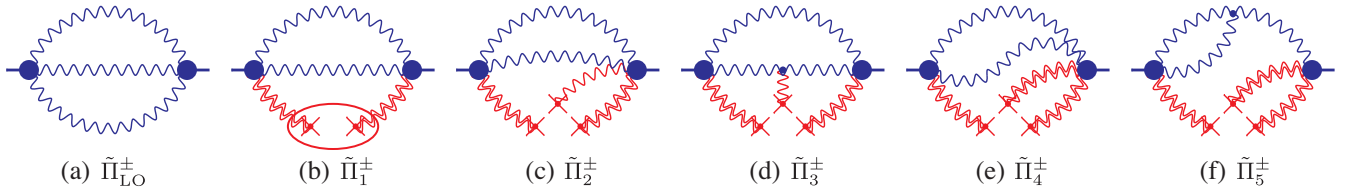

FIG. 4. The diagrams representing the LO perturbative term, dimension-four and dimension-six contributions to the correlator OPE of three-gluon currents. Each term is given by $\tilde{\Pi}_k^\pm$ where k is the number given below the diagram. The Feynman rules for the elements of the diagrams are given in Appendix A.

gauge-fixing term that makes the background field vertex different from the quantum three-gluon vertex, see Fig. 7. The diagram (f) term $\tilde{\Pi}_5^-$ seems to be omitted in [45]. The $\tilde{\Pi}_2^-$ and $\tilde{\Pi}_4^-$ contributions are the same as in [45].

IV. CONCLUSIONS

In this work, we have developed a new scheme to calculate OPE of vacuum correlators. We suggest using precalculated expansions for NLC as intermediate to simplify calculations, where the final results for OPE are given by truncated series that include only local condensates up to dimension eight. This way of using NLCs is different from the standard applications [38–40], where condensates are considered in nonlocal form and correlator's OPE is partially resummed. Using the procedures elaborated in [9,37], we have obtained the full set of NLC expansions needed for calculations of gluon condensate contributions to correlators OPE up to dimension-eight order.

The background field approach is revisited for OPE of glueball correlators. We discuss some calculation issues which are crucial for hadron parameter evaluation within QCD SR: (i) one of the issues is the misusing of perturbative gluon vertices that differ from the vertices for interaction of the vacuum gluon with the quantum gluon field due to the gauge fixing term; (ii) another problem could be caused by using the incomplete Taylor expansion, Eq. (2)—all vacuum gluon fields and vacuum gluon field strength tensors should be expanded up to the order that could contribute to the final expression of OPE in goal dimension order; (iii) the third issue is related to the gluon propagator expansion in the background field. The precalculated expressions for propagator in momentum space, Eq. (19), could not give a full answer when the calculation of the coefficient function requires a derivative of the propagator in the configuration space, as in the case of glueball current correlators [9]. In the light of discussed issues and obtained results, the glueball QCD SRs need reconsideration.

The usage of NLC expansions avoids the above problems and can be a good alternative to using precalculated expansion of the propagator in the background field. The derived NLC expansions are universal and can be used in various applications. Although, only gluon condensates and their applications are considered here, the same scheme is applicable for quark and quark-gluon NLCs. We provide the scheme for applying the NLC expansions and demonstrate it on four correlators used in glueball state studies. Using NLC expansions, we confirm OPEs [42,43] for two-gluon $0^{\pm+}$ glueball currents and calculate additional contributions coming from the dimension-six four-quark condensate $\langle J^2 \rangle$ and dimension-eight mixed quark-gluon condensates $\langle \mathrm{Tr}\, J_\mu G_{\mu\nu} J_\nu \rangle$ and $\langle \mathrm{Tr}\, J_\lambda [D_\lambda G_{\mu\nu}, G_{\mu\nu}] \rangle$. The OPE used for the three-gluon $0^{\pm+}$ glueballs [44,45] are

revisited up to dimension-6 order, and the corrected expressions are provided.

APPENDIX A: FEYNMAN RULES

In this appendix, the mathematical notation for the diagrams depicted in Sec. III are given. The right and left shaded blobs in the diagrams shown in Figs. 2–4 represent the glueball currents with the Feynman rules provided in Fig. 6. The gluon field strength tensor $\bar{G}_{\mu\nu}^a$ in the glueball currents has contributions from two components of the gluon field $\bar{A}_\mu^a = A_\mu^a + a_\mu^a$, the perturbative quantum gluon field a_μ^a, and the background field A_μ^a:

$$\bar{G}_{\mu\nu}^a(x) = G_{\mu\nu}^a(x) + g f^{abc}(A_\mu^b(x)a_\nu^c(x) - A_\nu^b(x)a_\mu^c(x))$$
$$+ g f^{abc} a_\mu^b(x) a_\nu^c(x) + (\partial_\mu a_\nu^a(x) - \partial_\nu a_\mu^a(x)),$$

where $G_{\mu\nu}^a(x)$ is the vacuum gluon field strength tensor. The graphical notation for these contributions is shown in Fig. 5. There are two types of lines: the single line represents gluon fields; the double line depicts the gluon field strength tensor. The red color and the cross at the end of the line denote the soft part that forms a vacuum condensate. The blue color of the line denotes the hard part of the diagram. The non-Abelian part of the gluon field strength tensor contributes to Figs. 2(c) and 3(e), where one of the gluon fields goes to the vacuum and the other field becomes part of the gluon propagator.

The graphic notation and definitions for the vertices of the quantum field and background field interactions are provided in Fig. 7. The vertices are given in terms of the operators of the vacuum gluon field and its strength tensor at the point of interaction. These operators become part of NLCs. Otherwise, these operators need to be expanded to the Taylor series, Eq. (2).

Calculation in the configuration space requires integration over the coordinates of interaction points, see, for example, Figs. 3(e)–3(g). We consider differentiating easier than integrating; therefore, we work in the momentum

FIG. 5. The diagrammatic representation for the terms of the gluon field strength tensor in the glueball current. The double red line is used for the vacuum gluon field strength tensor $G_{\mu\nu}^a$. The single red line with the cross at the end denotes the vacuum gluon field. The single blue line denotes the quantum gluon field. The blue lines are part of the coefficient function and the red lines form a condensate.
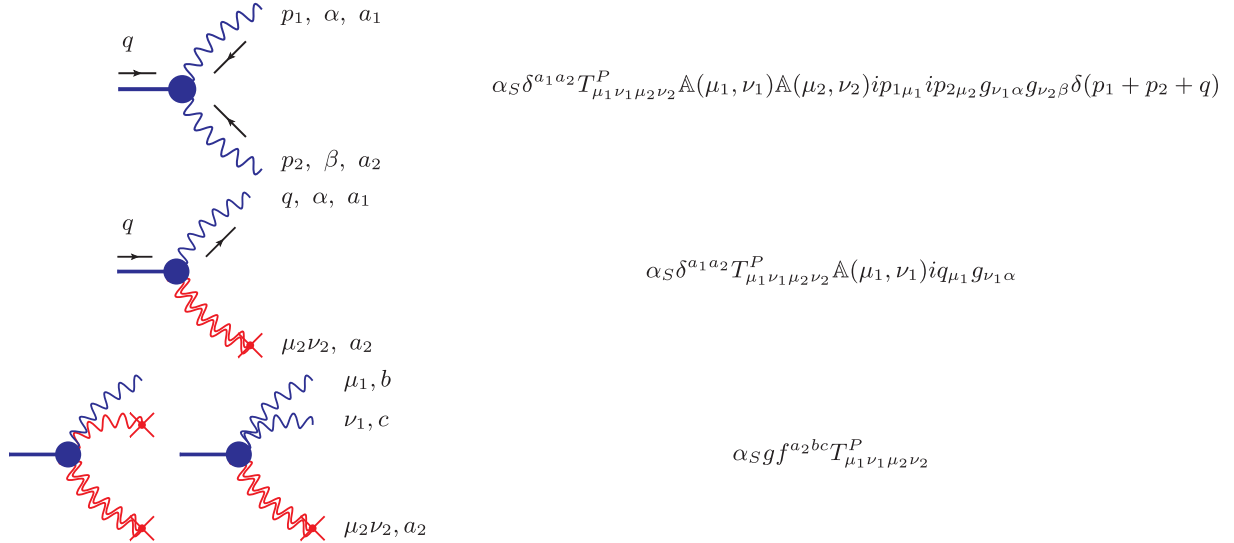
$$\alpha_S \delta^{a_1 a_2} T^P_{\mu_1 \nu_1 \mu_2 \nu_2} \mathbb{A}(\mu_1, \nu_1) \mathbb{A}(\mu_2, \nu_2) i p_{1\mu_1} i p_{2\mu_2} g_{\nu_1 \alpha} g_{\nu_2 \beta} \delta(p_1 + p_2 + q)$$

$$\alpha_S \delta^{a_1 a_2} T^P_{\mu_1 \nu_1 \mu_2 \nu_2} \mathbb{A}(\mu_1, \nu_1) i q_{\mu_1} g_{\nu_1 \alpha}$$

$$\alpha_S g f^{a_2 bc} T^P_{\mu_1 \nu_1 \mu_2 \nu_2}$$

FIG. 6.   The Feynman rules for contributions to glueball current vertices in the momentum representation. The single line represents the gluon field, while the double line depicts the gluon field strength tensor. The cross at the end of the single or double line tells that the given gluon field goes to vacuum and the field is part of a condensate. The explicit expression for short notation of rank-four Lorentz tensor $T^P_{\mu_1 \nu_1 \mu_2 \nu_2}$ can be found below in the text, see also discussion in Sec. III B.
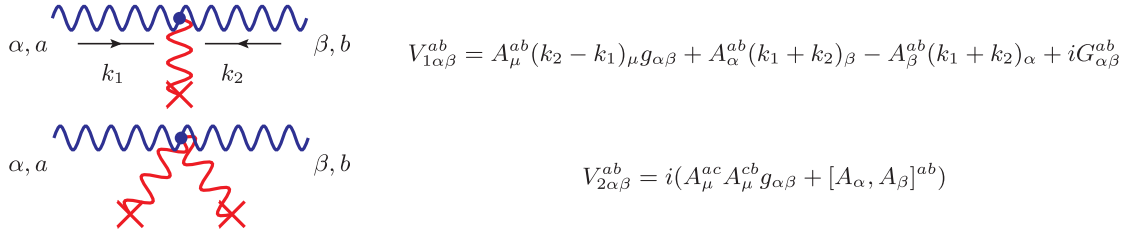

$$V^{ab}_{1\alpha\beta} = A^{ab}_\mu (k_2 - k_1)_\mu g_{\alpha\beta} + A^{ab}_\alpha (k_1 + k_2)_\beta - A^{ab}_\beta (k_1 + k_2)_\alpha + i G^{ab}_{\alpha\beta}$$

$$V^{ab}_{2\alpha\beta} = i(A^{ac}_\mu A^{cb}_\mu g_{\alpha\beta} + [A_\alpha, A_\beta]^{ab})$$

FIG. 7.   The Feynman rules for the background field vertices. We use the following matrix notation for the vacuum gluon field $A^{ab}_\mu = g f^{acb} A^c_\mu(y)$ and its strength tensor $G^{ab}_{\mu\nu} = g f^{acb} G^c_{\mu\nu}(y)$, where y is the interaction point.

representation with the following definition for the coordinates that appear in the obtained expansion of nonlocal condensates. For the coordinate x, where one of the currents in the correlator is located, we have

$$x_{\rho_1} x_{\rho_2} \ldots x_{\rho_n} \rightarrow \frac{\partial}{i\partial q_{\rho_1}} \frac{\partial}{i\partial q_{\rho_2}} \cdots \frac{\partial}{i\partial q_{\rho_n}}. \qquad (A1)$$

For any arbitrary coordinate y in NLC expansion differing from x we apply

$$y_{\rho_1} y_{\rho_2} \ldots y_{\rho_k} \rightarrow \int d^4 l \, e^{-ily} \left(\frac{\partial}{i\partial l_{\rho_1}} \frac{\partial}{i\partial l_{\rho_2}} \cdots \frac{\partial}{i\partial l_{\rho_k}} \delta^4(l) \right), \quad (A2)$$

where additional coordinate y is a point of interaction with the background field and l is auxiliary momentum running from point 0 to y.

The general procedure for OPE calculation in our approach is the following. Using Wick's theorem, we obtain all possible contributions and depict the corresponding Feynman diagrams. We do not perform Taylor and tensor

expansions of the vacuum fields, but use NLC expansions to get a truncated OPE. As the next step, we apply the following Feynman rules and expressions to the background field vertices, Fig. 7; current vertices for the two-gluon glueball, Fig. 6; two-gluon NLC expansion, Eq. (8); three-gluon NLC expansion, Eq. (12); four-gluon NLC expansion, Esq. (15) and (17); coordinates from NLC expansions, Eqs. (A1) and (A2); and free gluon propagator (without background field corrections).

APPENDIX B: EXAMPLE OF NLC-BASED CALCULATION

In this section, we provide the technical details of the OPE calculation of the glueball current correlator $\Pi^P(q)$, Eq. (22), based on grouping the contributions in NLCs and using precalculated expressions for NLC expansions. As an example, the $\Pi^\pm_{\text{NLC}-2}$ term depicted in Fig. 2(d) is considered. Applying the Feynman rules given in Appendix A, we have

$$\Pi^\pm_{\text{NLC}-2} = i4 \langle G^{a_2}_{\mu_2\nu_2}(0) V^{c_1c_2}_{1\rho_1\rho_2}(y) G^{b_2}_{\alpha_2\beta_2}(x) \rangle$$

$$\cdot \frac{g_{\bar\nu_1\rho_2}\delta^{a_1c_2}}{i(q-l)^2} \cdot \frac{g_{\bar\beta_1\rho_1}\delta^{c_1b_1}}{iq^2}$$

$$\cdot \alpha_S \delta^{a_1a_2} T^\pm_{\mu\nu} \mathbb{A}(\mu_1,\nu_1) i(q-l)_{\mu_1} g_{\nu_1\bar\nu_1}$$

$$\cdot \alpha_S \delta^{b_1b_2} T^\pm_{\alpha\beta} \mathbb{A}(\alpha_1,\beta_1) i(-q_{\alpha_1}) g_{\beta_1\bar\beta_1}, \qquad \text{(B1)}$$

where expression is given in the form of the product of NLC, two propagators and two glueball vertices. The tensors $T^+_{\mu\nu} \equiv g_{\mu_1\mu_2}g_{\nu_1\nu_2}$, and $T^-_{\mu\nu} \equiv i\epsilon_{\mu_1\nu_1\mu_2\nu_2}/2$ define the glueball current.

The expression for the correlator is given in the momentum space where the coordinates x and y are considered as differential operators defined in Eqs. (A1) and (A2). Next, we consider the example of the application of these operators. The condensate part is given by

$$\langle G^a_{\mu_2\nu_2}(0) A^b_{\rho_3}(y) G^c_{\alpha_2\beta_2}(x) \rangle$$

$$= y_{\rho_0} \int_0^1 dt\, t \langle G^a_{\mu_2\nu_2}(0) G^b_{\rho_0\rho_3}(ty) G^c_{\alpha_2\beta_2}(x) \rangle,$$

$$= \frac{y_{\rho_0}}{2} \langle G^a_{\mu_2\nu_2}(0) G^b_{\rho_0\rho_3}(0) G^c_{\alpha_2\beta_2}(0) \rangle + \dots, \qquad \text{(B2)}$$

where, in the third line, we keep only the leading term in the expansion of the gluon field to demonstrate the calculation of the dimension-six term. Then, for leading order, we replace the condensate by its expansion, Eq. (12), and replace the coordinates by their representations in the momentum space, Eq. (A2), which takes a shorter form here

$$y_\rho \to \int d^4 l\, e^{-ily} \frac{\partial}{i\partial l_\rho} \delta^4(l). \qquad \text{(B3)}$$

To consider the next term that has dimension-8, we apply the three-gluon NLC expansion given in Eq. (12). One could choose not to use the NLC expansion, then one should take the first three terms of Taylor expansion, see Eq. (2), for all gluon field strength tensors in Eq. (B2).

Applying Eq. (3) and the expansion of the three-gluon NLC given in Eq. (12), one gets

$$\Pi^\pm_{\text{NLC}-2} = \frac{-\alpha_S}{6\pi} T^\pm_{\mu_1\nu_1\mu_2\nu_2} T^\pm_{\alpha_1\beta_1\alpha_2\beta_2} \mathbb{A}(\mu_1,\nu_1)\mathbb{A}(\mu_2,\nu_2)$$

$$\times \mathbb{A}(\alpha_1,\beta_1)\mathbb{A}(\alpha_2,\beta_2)\mathbb{A}(\rho_0,\rho_3)$$

$$\times \int_0^1 dt \left(\frac{1}{2}\mathbb{A} \sum_{i=0}^7 \Gamma_i^{(abc)} M_i(x_a,x_b,x_c) \right)$$

$$\cdot (-ity_{\rho_0} V_{1\beta_1\nu_1\rho_3}(-q, q-l, l) - g_{\rho_0\beta_1}g_{\rho_3\nu_1}\delta(t-1))$$

$$\cdot \frac{q_{\alpha_1}(q-l)_{\mu_1}}{q^2(q-l)^2}, \qquad \text{(B4)}$$

where part of the background field vertex is given by

$$V_{1\alpha_1\alpha_2\alpha_3}(k_1,k_2,k_3) = g_{\alpha_1\alpha_2}(k_1-k_2)_{\alpha_3} - g_{\alpha_1\alpha_3}(k_1-k_2)_{\alpha_2}$$

$$+ g_{\alpha_2\alpha_3}(k_1-k_2)_{\alpha_1}, \qquad \text{(B5)}$$

and the coordinates of the scalar functions and tensors are $x_1 = 0$, $x_2 = yt$, $x_3 = x$. The master tensors Γ_i carry the Lorentz indices $\mu_2\nu_2\alpha_2\beta_2\rho_0\rho_3$, see Eq. (13). From Eq. (B4) one gets

$$\Pi^\pm_{\text{NLC}-2} = \pm\frac{12\alpha_S^2\langle gG^3\rangle}{Q^2}$$

$$+ \frac{2\alpha_S}{3\pi Q^4}(\mp 12G^8_{34} \pm 3G^8_{56} + 2G^8_{34} - G^8_{12}). \qquad \text{(B6)}$$

The OPE of the scalar and pseudoscalar glueballs current correlator $\Pi^\pm$, defined in Eq. (22), is given by the sum of NLC-based terms $\Pi^\pm_{\text{NLC}-k}$ depicted in Figs. 2 and 3. We denote the dimension-n contribution of the given NLC-based diagram by $\Pi^\pm_{nk}$ and consider the contributions up to dimension-eight order:

$$\Pi^\pm_{\text{NLC}-k} = \sum_{n=d_k}^8 \Pi^\pm_{nk}, \qquad \text{(B7)}$$

where d_k is the dimension of the kth NLC-based term. Only diagram (b) in Fig. 2 contributes to leading dimension four:

$$\Pi^\pm_4 = \Pi^\pm_{40} = \pm 4\alpha_S\langle \alpha_S G^2\rangle. \qquad \text{(B8)}$$

There are three diagrams that contribute to dimension-six order:

TABLE VI. The dimension-eight contribution to each diagram $\Pi^\pm_{8k}$, where k is number of the diagram, as depicted in Figs. 2 and 3, see Eq. (B10).

k	Π^+_{8k}	Π^-_{8k}
0	$O(d-4)$	$O(d-4)$
1	$O(d-4)$	$O(d-4)$
2	$-\frac{1}{3}(G^8_{12} + 10G^8_{34} - 3G^8_{56})$	$-\frac{1}{3}(G^8_{12} - 14G^8_{34} + 3G^8_{56})$
3	$O(d-4)$	$O(d-4)$
4	$18G^8_{34}$	$-G^8_{12} - 16G^8_{34}$
5	$-2(G^8_{12} + G^8_{34})/3$	$(G^8_{12} + 4G^8_{34})/3$
Sum	$14G^8_{34} - G^8_{12} + G^8_{56}$	$-(10G^8_{34} + G^8_{12} + G^8_{56})$

$$\Pi_6^\pm = \sum_{k=0}^2 \Pi_{6k}^\pm = \frac{8\alpha_S^2}{Q^2}\left(\pm\langle gG^3\rangle + \frac{1}{3}\langle J^2\rangle\right),$$

$$\Pi_{60}^\pm = \frac{8\alpha_S^2}{3Q^2}\langle J^2\rangle, \qquad \Pi_{61}^\pm = \mp\frac{4\alpha_S^2\langle gG^3\rangle}{Q^2},$$

$$\Pi_{62}^\pm = \pm\frac{12\alpha_S^2\langle gG^3\rangle}{Q^2}. \tag{B9}$$

All considered diagrams can contribute to dimension-eight order:

$$\Pi_8^\pm = \frac{2\alpha_S}{\pi Q^4}\sum_{k=0}^5 \Pi_{8k}^\pm,$$

$$= \frac{2\alpha_S}{\pi Q^4}(2G_{34}^8 - G_{12}^8 \pm 12G_{34}^8 \pm G_{56}^8), \tag{B10}$$

where the kth diagram contribution is denoted by $\Pi_{8k}^\pm$ and given in Table VI by local condensates. Diagrams (b), (c), and (e) in Fig. 2 give a zero contribution for the space dimension $d = 4$.

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