Editors' Suggestion

Waveforms from amplitudes

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We show how to compute classical wave observables using quantum scattering amplitudes. We discuss observables both with incoming and with outgoing waves. The required classical limits are naturally described by coherent states of massless bosons. We recompute the classic gravitational deflection of light, and also show how to rederive Thomson scattering. We introduce a new class of local observables, which includes the asymptotic electromagnetic and gravitational Newman-Penrose scalars. As an example, we compute a simple radiated waveform: the expectation of the electromagnetic field in charged-particle scattering. At leading order, the waveform is trivially related to the five-point scattering amplitude.

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I. INTRODUCTION

Theoretical waveforms play an important role in the LIGO/Virgo Collaboration's observational program of gravitational-wave events from binary mergers [1,2]. These waveforms provide templates that enable the detection of events against otherwise overwhelming noise backgrounds. They also allow observers to extract the masses and spins of the binaries' constituents [3]. To date, theorists have computed waveforms (or equivalently, spectral functions for decaying binaries) using long-established effective-one-body methods [4] and numerical-relativity approaches [5], in addition to methods based on the "traditional" Arnowitt-Deser-Misner Hamiltonian formalism [6], direct post-Newtonian solutions in harmonic gauge [7], and computations in the effective-field theory approach pioneered by Goldberger and Rothstein [8,9].

The start of the gravitational-wave observational era has spurred theorists to explore new approaches to computing classical observables for the two-body problem in gravity, in particular using quantum scattering amplitudes. The connection between the quantum S-matrix and observables in classical general relativity (GR) was first explored nearly 50 years ago by Iwasaki [10]. More recently, renewed interest has been driven by modern on shell techniques for computing amplitudes and the double-copy relation between Yang-Mills and gravitational amplitudes [11–39], as well as the bounty of observations. Earlier investigations included extraction of the two-body potential from amplitudes and the study of quantum corrections to gravity [40,41].

An important step was taken by Cheung, Solon, and Rothstein [42], building on earlier work by Neill and Rothstein [43]. Cheung, Solon, and Rothstein showed how to match effective field theories (EFTs) efficiently to scattering amplitudes above threshold in order to extract a classical potential. The classical potential can then be used in the effective-one-body or other frameworks to make predictions for bound-state quantities. Bern, Cheung, Roiban, Shen, Solon, and Zeng used [44] this approach to compute the third-order corrections (G^3) to the conservative potential. This milestone computation went beyond what had been known from direct classical GR calculations and provided the first concrete fulfillment of the promise of the scattering-amplitudes class of methods. It used a two-loop scattering amplitude for massive particles and was followed by many new calculations using amplitude methods [45–66]. New EFT-based results have also emerged [67–94]. In this context, Kälin and Porto have pointed out an interesting analytic continuation from scattering to bound-state observables [84,89]. Several groups have pursued an eikonal approach [95–101] and connections to it [102]. Another approach which has seen

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recent attention is the worldline formalism [8,103,104]. In the context of EFT, this worldline approach is particularly important since it makes immediate sense classically. Treated as an effective quantum field theory, this means that it organizes quantum corrections particularly simply. Finally, two of the present authors have examined light-ray operators [105] and shock waves [106]. Researchers working within a traditional GR framework have also continued to produce new results [107–121].

In a previous paper [122], two of the present authors and Maybee outlined an observables-based approach to computing classical quantities. It starts with an observable in the quantum theory, expressing it in terms of scattering amplitudes; and then uses an efficient and controlled method for taking the classical limit. In this approach, rather than trying to compute intermediate quantities such as the conservative potential, we write down a formal expression for an observable of interest—for example, the total change in the momentum of one of two scattered particles, also known as its impulse—in the quantum theory. With an appropriate wave function for the initial state, we can express the chosen observable in terms of quantum scattering amplitudes. We further restore powers of $\hbar$ via dimensional analysis. At this stage, the $\hbar$ scaling is naively bad, as the observable may be seemingly divergent in the classical, $\hbar \to 0$ limit, and loop corrections appear to be increasingly divergent with increasing order.

The original paper [122] focused on scattering two massive particles. Appropriate wave functions were necessary to localize each incoming particle. This localization will sharpen in the classical limit, when we are focusing on point particles. The localization will in turn lead us to retain momenta for the scattering particles in the expression for the observable, but to use *wave numbers* for exchanged, emitted, or virtual massless particles (photons or gravitons). The change of variables from momenta to wave numbers for the latter reveals additional powers of $\hbar$ that then yield a finite classical limit at each perturbative order. Herrmann, Parra-Martinez, Ruf, and Zeng [48,49], and separately Bautista and Guevara [62] have applied this approach in their calculations.

Reference [122] did not discuss massless bosonic particles, in particular in the initial state. We remedy that in this article. Furthermore, Ref. [122] focused only on global observables, which require surrounding an event with a detector of 4π coverage. We remedy this as well with a discussion of local observables, such as electromagnetic and gravitational waveforms. Newman-Penrose [123] scalars provide a natural language for these quantities. We will introduce these two principal topics of our article in the remainder of this introduction.

Let us begin with the question of initial-state massless bosons. In the classical limit, one describes *massive* particles as superpositions of single-particle states. They ultimately appear as pointlike particles or extended bodies. In contrast, *massless* bosons appear as waves or wave packets. It is no longer possible to describe them as superpositions of single-particle states. Instead, we shall see that they emerge most naturally from coherent states of the corresponding quantum fields. Such states are inherently superpositions of multiparticle states.

The significance of coherent states was emphasized by Glauber from 1963 on. He proved that every quantum state of radiation—that is, every density matrix—can be described as a suitable superposition of coherent states [124,125]. In particular, in the classical limit one can describe these density matrices using the so-called Glauber-Sudarshan P representation [125,126]. In this representation, there is a classical probability distribution in the space of coherent states. The application of coherent states to the classical limit of quantum scattering amplitudes started soon afterwards in the work of Frantz, Kibble, and Brown [127–129], but a systematic analysis of the question was still lacking [130]. Most calculations were limited to the solvable model of the linear interactions of a current (or a stress tensor) with the associated field [131–133]: in this case the S matrix is solvable to all orders in perturbation theory, and its structure is exactly equivalent to a coherent state. Yaffe later showed [134] that coherent states are very convenient for understanding the emergence of the classical approximation from quantum physics quite generally. Concrete applications are nonetheless rare in the literature, especially outside the case of a single particle interacting with a fixed coherent background (see Ref. [135] and references therein).[1] Coherent states have a close connection to soft limits and infrared divergences, which provide a natural arena for their emergence in the late-time dynamics of QED and linearized gravity [137–142].

Let us turn next to the question of local observables. In Ref. [122], the authors studied time-integrated observables, in the context of scalar electrodynamics, and validated the amplitude-based approach through comparisons with direct calculations in classical electromagnetism. What is of more direct interest to observers, however, are time-dependent observables such as radiation waveforms. These are examples of a class of observables which are *local* in the sense that they describe a measurement at one spacetime point (or in a small region of spacetime). The time-integrated observables of Ref. [122] in principle require an apparatus which completely surrounds a scattering event, so that (for example) the impulse of any incoming particle can be measured. We describe this class of observables as *global* as a result.

In this article, we establish a direct connection between local observables, such as waveforms, and scattering amplitudes. We validate our approach with a calculation

[1] Reference [136] offers a notable exception in the context of the superradiance problem.

of a simple waveform, arising from the scattering of two charged particles in scalar QED. We will see that waveforms are effectively amplitudes for detecting massless particles, or waves in the classical limit. We show how to write appropriate quantum observables, and how to take their limits. Finally, we provide a direct connection between the celebrated Newman-Penrose formalism [123] and scattering amplitudes.

As our work has progressed, we have become aware of a parallel line of investigation by Bautista, Guevara, Kavanagh, and Vines [143]. Their work is broadly complementary to ours, but touches on some of the same themes: the connection between the Compton amplitude and classical wave scattering, for example, and the close connection between the Newman-Penrose scalars and helicity amplitudes.

Our article is organized as follows. We begin in the next section with a review of the formalism of Ref. [122]. In Sec. III, we review coherent states for the electromagnetic field, show how they correspond to classical fields, and give a simple example of a light beam built from them. In Sec. IV, we discuss global observables with massless waves in the initial state, concentrating on the impulse in this context. As examples, we discuss Thomson scattering and its relation to the Compton amplitude, and we examine the calculation of the gravitational deflection of light within our formalism. We turn to the second major topic of our article in Sec. V with a discussion of the general form of local observables far from some event. Section VI follows with an introduction to spectral aspects of local observables, leading to the Newman-Penrose projection formalism. In Sec. VII, we pause the general development to give example of a local observable: the scattered radiation field in Thomson scattering. In Sec. VIII, we present the general form of the emission waveform when two massive particles scatter, and in Sec. IX we give explicit results for electromagnetic emission in charged-particle scattering to leading order. We discuss the connection between the waveform and the total radiated momentum in Sec. X, and end with concluding remarks in Sec. XI.

II. REVIEW OF FORMALISM

We use relativistic units, retaining $c = 1$, even as we restore $\hbar$ explicitly. This means that we must distinguish units of energy and length, which we denote by $[M]$ and $[L]$, respectively. In this article, we will use a different normalization than the conventions of Ref. [122] (which are also the conventions of Peskin and Schroeder [144]). Here, we normalize the annihilation and creation operators such that

$$[a_p, a^\dagger_{p'}] = (2\pi)^3 2E_p \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}'). \tag{2.1}$$

(Bold symbols denote spatial three-vectors.) Accordingly, n-point scattering amplitudes continue to have dimension $[M]^{4-n}$.

We keep $[M]^{-1}$ as the dimension of single-particle states $|p\rangle$,

$$|p\rangle \equiv a^\dagger_p|0\rangle, \tag{2.2}$$

with the vacuum state being dimensionless. We define n-particle plane-wave states as simply the tensor product of normalized single-particle states. (The normalization of the single-particle states is the same as in Ref. [122].) The state $|p\rangle$ represents a particle of momentum p and positive energy, while $\langle p| = \langle 0|a_p$ is the conjugate state.

We find it convenient to define an n-fold Dirac δ distribution with normalization absorbing 2πs,

$$\hat{\delta}^{(n)}(p) \equiv (2\pi)^n \delta^{(n)}(p). \tag{2.3}$$

The scattering matrix S and the transition matrix T are both dimensionless. Scattering amplitudes are matrix elements of the latter between plane-wave states,

$$\langle p'_1 \cdots p'_m|T|p_1 \cdots p_n\rangle = \mathcal{A}(p_1 \cdots p_n \to p'_1 \cdots p'_m) \times \hat{\delta}^{(4)}(p_1 + \cdots + p_n - p'_1 - \cdots - p'_m). \tag{2.4}$$

As our formalism encompasses both QED and gravity, as well as other theories with massless force carriers, we denote the coupling by g. In electrodynamics, it corresponds to e, while in gravity to $\kappa = \sqrt{32\pi G}$. It is not dimensionless once we have restored the factors of $\hbar$; rather, it is $g/\sqrt{\hbar}$ that is the dimensionless coupling.

We start by taking the momenta of all particles as the primary variables; but as explained in the introduction, for most massless momenta, wave numbers are the variables of interest. We introduce a notation for the wave number $\bar{p}$ associated to the momentum p,

$$\bar{p} \equiv p/\hbar. \tag{2.5}$$

We use the notation of Ref. [122] for the on shell phase-space measure,

$$d\Phi(p_i) \equiv \hat{d}^4 p_i\, \hat{\delta}^{(+)}(p_i^2 - m_i^2). \tag{2.6}$$

We will leave the mass implicit, along with the designation of the integration variable as the first summand when the argument is a sum. The notation for the measure again absorbs factors of 2π,

$$\hat{d}^4 p \equiv \frac{d^4 p}{(2\pi)^4}, \tag{2.7}$$

and, as usual,

$$\delta^{(+)}(p^2 - m^2) = \Theta(p^t)\delta(p^2 - m^2), \quad (2.8)$$

so that

$$\hat{\delta}^{(+)}(p^2 - m^2) = 2\pi\Theta(p^t)\delta(p^2 - m^2). \quad (2.9)$$

(p^t is the energy component of the four-vector.)

Given our convention for normalizing single-particle states, their inner product is

$$\langle p'|p\rangle = 2E_p\hat{\delta}^{(3)}(\boldsymbol{p} - \boldsymbol{p}'). \quad (2.10)$$

The expression on the right-hand side is the appropriately normalized delta function for the on shell measure, which is convenient to express in more compact notation,

$$\hat{\delta}_\Phi(p_1 - p_1') \equiv 2E_{p_1}\hat{\delta}^{(3)}(\boldsymbol{p}_1 - \boldsymbol{p}_1'). \quad (2.11)$$

We should understand the argument on the left-hand side as a function of four-vectors. In this notation, Eq. (2.10) is simply

$$\langle p'|p\rangle = \hat{\delta}_\Phi(p - p'). \quad (2.12)$$

With this notation, we can also rewrite the normalization of creation and annihilation operations (2.1) in a natural form:

$$[a_p, a^\dagger_{p'}] = \hat{\delta}_\Phi(p - p'). \quad (2.13)$$

We will also employ the notation $a(k) \equiv a_k$ and $a^\dagger(k) \equiv a_k^\dagger$ to allow for additional indices.

Reference [122] exclusively considers the scattering of two massive pointlike particles. In this article we go beyond that discussion to consider initial states which may involve massless radiation. However, when appropriate we will continue to use the notation of Ref. [122] for initial states involving only massive particles: we take the initial momenta to be p_1 and p_2, initially separated by a transverse impact parameter b. The latter is transverse in that $p_1 \cdot b = 0 = p_2 \cdot b$.

In the quantum theory, the system of massive particles is described by wave functions, which we build out of plane waves. In the classical limit, these wave functions must localize the two pointlike particles and must separate them clearly. We describe the incoming particles in the far past by wave functions $\phi_i(p_i)$, which we take to have reasonably well-defined positions and momenta. We will review the requirements on the wave packets, discussed in detail in Sec. 4 of Ref. [122].

We express the initial state in terms of plane waves $|p_1 p_2\rangle_{\rm in}$,

$$\begin{aligned}|\psi\rangle_{\rm in} &= \int \hat{d}^4p_1\hat{d}^4p_2\hat{\delta}^{(+)}(p_1^2 - m_1^2)\hat{\delta}^{(+)}(p_2^2 - m_2^2) \\ &\quad\times \phi_1(p_1)\phi_2(p_2)e^{ib\cdot p_1/\hbar}|p_1p_2\rangle_{\rm in} \\ &= \int d\Phi(p_1)d\Phi(p_2)\phi_1(p_1)\phi_2(p_2)e^{ib\cdot p_1/\hbar}|p_1p_2\rangle_{\rm in}. \end{aligned} \quad (2.14)$$

We require each wave function ϕ_i to be normalized to unity,

$$\int d\Phi(p_1)|\phi_1(p_1)|^2 = 1, \quad (2.15)$$

so that our incoming state is also normalized to unity,

$$\begin{aligned}{}_{\rm in}\langle\psi|\psi\rangle_{\rm in} &= \int d\Phi(p_1)d\Phi(p_2)d\Phi(p_1')d\Phi(p_2')e^{ib\cdot(p_1-p_1')/\hbar} \\ &\quad\times \phi_1(p_1)\phi_1^*(p_1')\phi_2(p_2)\phi_2^*(p_2')\hat{\delta}_\Phi(p_1 - p_1') \\ &\quad\times \hat{\delta}_\Phi(p_2 - p_2') \\ &= \int d\Phi(p_1)d\Phi(p_2)|\phi_1(p_1)|^2|\phi_2(p_2)|^2 \\ &= 1. \end{aligned} \quad (2.16)$$

Finally, we turn to a review of the classical limit. As discussed in Ref. [122], there are three scales we must consider in the context of massive particle scattering: the Compton wavelengths of the particles, $\ell_c^{(i)} \equiv \hbar/m_i$; the intrinsic spread of the two particles' wave packets, given by ℓ_w; and the scattering length, ℓ_s. Taking the classical limit requires that we impose the "Goldilocks" conditions,

$$\ell_c^{(i)} \ll \ell_w \ll \ell_s. \quad (2.17)$$

The calculation of the scattering reveals that $\ell_s \sim \sqrt{-b^2}$.

In order to expand in the $\hbar \to 0$ limit and extract the leading, classical, term for any observable, as mentioned above we must make the powers of $\hbar$ explicit. These arise from two sources: powers ordinarily hidden inside electromagnetic or gravitational couplings and powers arising from keeping the wave numbers of massless particles fixed rather than their momenta. This is true both for emitted and virtual particles, when considering quantities such as the total emitted radiation.

III. CLASSICAL LIMIT FOR MASSLESS PARTICLES

We are now ready to address the first major topic of this article: how to include initial-state massless classical waves in the formalism of Ref. [122]. A naive extension of the considerations of Ref. [122] to massless particles is clearly impossible. A particle's Compton wavelength diverges when its mass goes to zero, making it impossible to satisfy the required conditions (2.17). It does not make sense to treat messengers (photons or gravitons) as pointlike particles. Indeed, Newton and Wigner [145] and Wightman [146] proved rigorously long ago that a strict localization of known massless particles in position space is impossible.[2] A proper treatment instead relies on coherent states. We begin such a treatment in the following subsection by discussing general aspects of coherent states, focusing on the electromagnetic case. We then describe the kind of coherent states of interest to us.

A. Coherent states of the electromagnetic field

We can write the electromagnetic field operator as

$$\mathbb{A}_\mu(x) = \frac{1}{\sqrt{\hbar}} \sum_\eta \int d\Phi(k) [a_{(\eta)}(k) \varepsilon_\mu^{(\eta)*}(k) e^{-ik\cdot x/\hbar} + a^\dagger_{(\eta)}(k) \varepsilon_\mu^{(\eta)}(k) e^{+ik\cdot x/\hbar}], \tag{3.1}$$

where $\eta = \pm$ labels the helicity, and the polarization vectors satisfy

$$[\varepsilon_\mu^{(\eta)}(k)]^* = \varepsilon_\mu^{(-\eta)}(k). \tag{3.2}$$

We follow the usual amplitudes convention of representing an outgoing positive-helicity photon of momentum k by $\varepsilon_\mu^{(+)}(k)$, which also corresponds to an incoming negative-helicity photon of the opposite momentum. To understand the helicity flip for an incoming state, note that we can analytically continue an incoming momentum k to an outgoing momentum $k' = -k$. The energy component k'^t of the outgoing momentum is now negative. Thus, in an all-outgoing convention, positive-helicity photons of momentum k with $k^t > 0$ are represented by the polarization vector $\varepsilon_\mu^{(+)}(k)$, while positive-helicity photons of momentum k with $k^t < 0$ are represented by the polarization vector $\varepsilon_\mu^{(-)}(k)$.

More generally, $a^\dagger_{(\eta)}(k)$ creates a single-messenger state of momentum k and helicity η, while $a_{(\eta)}(k)$ destroys such a state. Equivalently, the latter operator creates a conjugate state of momentum k and helicity η.

The commutation relations are

$$[a_{(\eta)}(k), a^\dagger_{(\eta')}(k')] = \delta_{\eta,\eta'} \hat{\delta}_\Phi(k - k'). \tag{3.3}$$

For example, a single-particle positive-helicity state is

$$|k^+\rangle \equiv a^\dagger_{(+)}(k)|0\rangle = [a_{(+)}(k)]^\dagger |0\rangle. \tag{3.4}$$

The conjugate state is $\langle k^+|$.

[2]The proof holds for vector bosons and gravitons.

Using the form of the electromagnetic field in Eq. (3.1), the electromagnetic field strength operator is

$$\mathbb{F}_{\mu\nu}(x) = -\frac{2i}{\hbar^{3/2}} \sum_\eta \int d\Phi(k) \Big[a_{(\eta)}(k) k_{[\mu} \varepsilon_{\nu]}^{(\eta)*}(k) e^{-ik\cdot x/\hbar} - a^\dagger_{(\eta)}(k) k_{[\mu} \varepsilon_{\nu]}^{(\eta)}(k) e^{+ik\cdot x/\hbar} \Big], \tag{3.5}$$

where as usual the subscripted brackets denote antisymmetrization:

$$A_{[\mu} B_{\nu]} = \frac{1}{2}(A_\mu B_\nu - A_\nu B_\mu). \tag{3.6}$$

We introduce the coherent-state operator,

$$\mathbb{C}_{\alpha,(\eta)} \equiv \mathcal{N}_\alpha \exp\left[\int d\Phi(k) \alpha(k) a^\dagger_{(\eta)}(k) \right], \tag{3.7}$$

where the normalization $\mathcal{N}_\alpha$ will be given below. We can build coherent states of the electromagnetic field using this operator, such as a positive-helicity one,

$$|\alpha^+\rangle = \mathbb{C}_{\alpha,(+)}|0\rangle. \tag{3.8}$$

More generally, we could consider coherent states containing both helicities. As coherent-state operators for different helicities commute and every polarization vector can be decomposed in the helicity basis, there is no loss of generality in making a specific helicity choice for the coherent states we consider. The coherent state operators are unitary,

$$(\mathbb{C}_{\alpha,(\eta)})^\dagger = (\mathbb{C}_{\alpha,(\eta)})^{-1}. \tag{3.9}$$

The normalization factor $\mathcal{N}_\alpha$ is determined by the condition $\langle \alpha^+|\alpha^+\rangle = 1$, that is,

$$\mathcal{N}_\alpha = \exp\left[-\frac{1}{2} \int d\Phi(k) |\alpha(k)|^2 \right], \tag{3.10}$$

as can be seen by using the Baker-Campbell-Hausdorff formula.

At this stage, the function $\alpha(k)$ is quite general, however in specific examples, we may take it to be real. We will see below that it is subject to certain restrictions in the classical limit. We will also see that its functional form will determine the physical shape of the corresponding state, so we will call it the "wave shape" function.

The coherent-state creation operator acting on the vacuum can be rewritten using the Baker-Campbell-Hausdorff identity as a displacement operator [127,128] yielding

$$\mathbb{C}_{\alpha,(\eta)}|0\rangle = \exp\left[\int d\Phi(k)(\alpha(k)a^{\dagger}_{(\eta)}(k) - \alpha^*(k)a_{(\eta)}(k))\right]|0\rangle. \tag{3.11}$$

Its action on creation and annihilation operators is given by

$$\begin{aligned}\mathbb{C}^{\dagger}_{\alpha,(\eta)}a_{(\rho)}(k)\mathbb{C}_{\alpha,(\eta)} &= a_{(\rho)}(k) + \delta_{\eta\rho}\alpha(k),\\ \mathbb{C}^{\dagger}_{\alpha,(\eta)}a^{\dagger}_{(\rho)}(k)\mathbb{C}_{\alpha,(\eta)} &= a^{\dagger}_{(\rho)}(k) + \delta_{\eta\rho}\alpha^*(k).\end{aligned} \tag{3.12}$$

To interpret the state, let us compute $\langle\alpha^+|\mathbb{A}^\mu(x)|\alpha^+\rangle$. It is useful to note that

$$\begin{aligned}a_{(+)}(k)|\alpha^+\rangle &= \alpha(k)|\alpha^+\rangle,\\ a_{(-)}(k)|\alpha^+\rangle &= 0,\\ \langle\alpha^+|a^{\dagger}_{(+)}(k) &= \langle\alpha^+|\alpha^*(k),\\ \langle\alpha^+|a^{\dagger}_{(-)}(k) &= 0,\end{aligned} \tag{3.13}$$

which incidentally imply that the dimension of $\alpha(k)$ is the same as the dimension of the annihilation operator: inverse mass. It is then easy to see that

$$\begin{aligned}\langle\alpha^+|\mathbb{A}_\mu(x)|\alpha^+\rangle &= \frac{1}{\sqrt{\hbar}}\int d\Phi(k)\left[\alpha(k)\varepsilon^{(+)*}_\mu(k)e^{-ik\cdot x/\hbar} + \alpha^*(k)\varepsilon^{(+)}_\mu(k)e^{+ik\cdot x/\hbar}\right]\\ &= \int d\Phi(\bar{k})\left[\bar{\alpha}(\bar{k})\varepsilon^{(+)*}_\mu(\bar{k})e^{-i\bar{k}\cdot x} + \bar{\alpha}^*(\bar{k})\varepsilon^{(+)}_\mu(\bar{k})e^{+i\bar{k}\cdot x}\right]\\ &\equiv A_{\mathrm{cl}\,\mu}(x),\end{aligned} \tag{3.14}$$

where we define

$$\bar{\alpha}(\bar{k}) \equiv \hbar^{3/2}\alpha(k). \tag{3.15}$$

Additional constraints on $\bar{\alpha}$ will emerge below from the consideration of correlators in the classical limit. Note that the polarization vector is invariant under the rescaling from a momentum to a wave vector: $\varepsilon^{(\eta)}(\bar{k}) = \varepsilon^{(\eta)}(k)$ is independent of $\hbar$.

Now, the most general solution of the classical Maxwell equation in empty space is

$$\sum_\eta A^{(\eta)\mu}_{\mathrm{cl}}(x) = \sum_\eta\int d\Phi(\bar{k})[\tilde{A}_\eta(\bar{k})\varepsilon^{(\eta)*\mu}(\bar{k})e^{-i\bar{k}\cdot x} + \tilde{A}^*_\eta(\bar{k})\varepsilon^{(\eta)\mu}(\bar{k})e^{+i\bar{k}\cdot x}] \tag{3.16}$$

in terms of Fourier coefficients $\tilde{A}_\eta(\bar{k})$, which we can identify as $\bar{\alpha}(\bar{k})$. Evidently our state $|\alpha^+\rangle$ contributes only the terms of positive helicity ($\eta = +$). A general coherent state is created by a product of coherent-state operators,

$$|\alpha_{(+)}, \alpha_{(-)}\rangle = \mathbb{C}_{\alpha_{(+)},(+)}\mathbb{C}_{\alpha_{(-)},(-)}|0\rangle, \tag{3.17}$$

where the different helicities have different wave shapes $\alpha_{(\pm)}(k)$. The expectation value of this state will correspond to any given classical wave as a linear superposition of the two circular polarizations via suitable choice of $\alpha_{(\pm)}(k)$. In examples we will consider, the simpler state $|\alpha^+\rangle$ will suffice.

To further illuminate the meaning of coherent states, we may consider scattering amplitudes in the presence of a nontrivial background field $A_{\mathrm{cl}}(x)$. The scattering matrix in the presence of this background field depends on it. We denote this dependence by $S(A_{\mathrm{cl}})$. Using the properties of the coherent state operator it can be shown that

$$\mathbb{C}^{\dagger}_{\alpha,(\eta)}S(A)\mathbb{C}_{\alpha,(\eta)} = S(A + A^{(\eta)}_{\mathrm{cl}}). \tag{3.18}$$

Here, $S(A)$ denotes the S matrix in the presence of a nontrivial quantum background field A. Coherent states thus allow us to capture the physics of a specific background field based on vacuum scattering amplitudes:

$$\mathbb{C}^{\dagger}_{\alpha,(\eta)}S(0)\mathbb{C}_{\alpha,(\eta)} = S(A^{(\eta)}_{\mathrm{cl}}). \tag{3.19}$$

The formulation of the perturbation theory in a fixed background is particularly convenient when the Feynman rules—or the scattering amplitudes—in the background are known exactly [147].

B. Classical coherent states

The coherence of a state does not suffice for it to behave classically. We must also require factorization of expectation values,

$$\langle\alpha^+|\mathbb{A}^\mu(x)\mathbb{A}^\nu(y)|\alpha^+\rangle \simeq \langle\alpha^+|\mathbb{A}^\mu(x)|\alpha^+\rangle\langle\alpha^+|\mathbb{A}^\nu(y)|\alpha^+\rangle. \tag{3.20}$$

A straightforward calculation in a light cone gauge defined by a lightlike vector q shows that

$$\langle\alpha^+|\mathbb{A}^\mu(x)\mathbb{A}^\nu(y)|\alpha^+\rangle = \langle\alpha^+|\mathbb{A}^\mu(x)|\alpha^+\rangle\langle\alpha^+|\mathbb{A}^\nu(y)|\alpha^+\rangle + \frac{1}{\hbar}\int d\Phi(k)\left[\eta^{\mu\nu} - \frac{k^\mu q^\nu + k^\nu q^\mu}{k\cdot q + i\delta}\right]e^{-ik\cdot(x-y)/\hbar},$$
$$= \langle\alpha^+|\mathbb{A}^\mu(x)|\alpha^+\rangle\langle\alpha^+|\mathbb{A}^\nu(y)|\alpha^+\rangle + \hbar\int d\Phi(\bar{k})\left[\eta^{\mu\nu} - \frac{\bar{k}^\mu q^\nu + \bar{k}^\nu q^\mu}{\bar{k}\cdot q + i\delta}\right]e^{-i\bar{k}\cdot(x-y)}. \tag{3.21}$$

Similarly for the field strengths, in a gauge independent way using Eq. (3.5), we obtain

$$\langle\alpha^+|\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\rho\sigma}(y)|\alpha^+\rangle = \langle\alpha^+|\mathbb{F}^{\mu\nu}(x)|\alpha^+\rangle\langle\alpha^+|\mathbb{F}^{\rho\sigma}(y)|\alpha^+\rangle + 4\hbar\partial^{[\mu}\eta^{\nu][\sigma}\partial^{\rho]}\int d\Phi(\bar{k})e^{-i\bar{k}\cdot(x-y)}. \tag{3.22}$$

For classical behavior, the second term on the right-hand side of Eq. (3.22) must be negligible compared to the first term. Writing $F^{\mu\nu}_{\mathrm{cl}}(x) \equiv \langle\alpha^+|\mathbb{F}^{\mu\nu}(x)|\alpha^+\rangle$, the right-hand side becomes

$$F^{\mu\nu}_{\mathrm{cl}}(x)F^{\rho\sigma}_{\mathrm{cl}}(y) + \frac{\hbar}{\pi^2}\partial^{[\mu}\eta^{\nu][\sigma}\partial^{\rho]}\frac{1}{(\mathbf{x}-\mathbf{y})^2 - (x^0 - y^0 - i\delta)^2}. \tag{3.23}$$

The first term has a nontrivial limit as $\hbar \to 0$, whereas the second term goes to zero in the limit, consistent with our expectations. For $\hbar \neq 0$, it is not possible to satisfy the inequality in the full spacetime region due to the divergence on the light cone $(x^0 - y^0)^2 = |\boldsymbol{x} - \boldsymbol{y}|^2$ of the massless photon propagator: causally connected measurements cannot be disentangled. We expect these contributions to fade away in the classical limit of a physical observable [128]. The factorization condition, which is trivial in the classical limit, has been dubbed the "complete coherence condition" in the literature,[3] a term coined by Glauber [125].

As usual, we define the operator measuring the number of photons to be

$$\mathbb{N}_\gamma = \sum_\eta \int d\Phi(k) a^\dagger_{(\eta)}(k) a_{(\eta)}(k). \tag{3.24}$$

A short computation shows that the expectation number N_γ of photons in our coherent state is

$$N_\gamma = \langle\alpha^+|\mathbb{N}_\gamma|\alpha^+\rangle$$
$$= \int d\Phi(k)|\alpha(k)|^2,$$
$$= \frac{1}{\hbar}\int d\Phi(\bar{k})|\bar{\alpha}(\bar{k})|^2. \tag{3.25}$$

[3]In the quantum optics literature the normal-ordered correlator of the electric field at different spatial locations can have various degrees of coherence [148].

The classical limit $\hbar \to 0$ thus corresponds to the limit of a large number of photons, which is a limit of a large occupation number [134]. The desired factorization property Eq. (3.20) will thus hold when

$$N_\gamma \gg 1. \tag{3.26}$$

We must choose the wave shape α such that the integral in the last line of Eq. (3.25) is not parametrically small as $\hbar \to 0$. A simple way to do so is to chose $\bar{\alpha}$ independent of $\hbar$.

Similarly, the momentum carried by the coherent state is

$$K^\mu_\odot = \langle\alpha^+|\mathbb{K}^\mu|\alpha^+\rangle,$$
$$= \int d\Phi(k)|\alpha(k)|^2 k^\mu,$$
$$= \int d\Phi(\bar{k})|\bar{\alpha}(\bar{k})|^2\bar{k}^\mu. \tag{3.27}$$

This quantity ("K beam") is finite in the classical limit, as expected.

We emphasize that this coherent-state construction and its connection to classical states generalizes to any massless particle, including gravitons. Finally, it is worth remarking on the important and familiar case of geometric optics. This is a purely classical approximation to wave phenomena, valid in situations where the wavelength is negligible in comparison to other physical scales. An important example, which we discuss below, is of the gravitational bending of light.

C. Localized beams of light

In this paper, one of our foci will be on phenomena associated with scattering light from a pointlike object. For problems of this type to be well defined, the incoming wave must be spatially separated from the incoming particle in the far past. Consequently, we need to understand how to describe a localized incoming beam of light. We can choose the beam to be moving in the z direction, localized around the origin of the $x - y$ plane. To see how to do this, let us consider some examples.

The simplest option for the wave shape is

$$\alpha(k) = \alpha_\odot\hat{\delta}_\Phi(k - \hbar\bar{k}_\odot), \tag{3.28}$$

where $\bar{k}_\odot$ ("k-bar beam") is the overall wave vector of the wave, and $\alpha_\odot$ ("α beam") is a constant which scales like $\sqrt{\hbar}$. Defining $\bar{\alpha}_\odot = \hbar^{-1/2}\alpha_\odot$, this choice implies that

$$\bar{\alpha}(\bar{k}) = \bar{\alpha}_\odot \hat{\delta}_\Phi(\bar{k} - \bar{k}_\odot), \tag{3.29}$$

and that the classical field takes the form

$$A^\mu_{\rm cl}(x) = 2{\rm Re}\, \bar{\alpha}_\odot \varepsilon^{*\mu}_\odot(\bar{k}_\odot) e^{-i\bar{k}_\odot\cdot x}. \tag{3.30}$$

It is worth pointing out here that the expectation value of the gauge potential between coherent states is always a real quantity: a physical field which can be measured. We can choose

$$\begin{aligned} \bar{k}^\mu_\odot &= (\omega, 0, 0, \omega), \\ \varepsilon^\mu_\odot &= \frac{1}{\sqrt{2}}(0, 1, i, 0), \end{aligned} \tag{3.31}$$

to provide an explicit example. If we pick the normalization of $\bar{\alpha}$ to be given by $\bar{\alpha}_\odot = A_\odot/\sqrt{2}$ with $A_\odot$ real, then the classical field for this example is

$$A^\mu_{\rm cl}(x) = A_\odot(0, \cos\omega(t-z), -\sin\omega(t-z), 0), \tag{3.32}$$

which is a plane wave of circular polarization[4] moving in the z direction with angular frequency ω. This wave is completely delocalized, which is a disadvantage for our purposes: we wish to have a clean separation between the incoming wave and pointlike particle states.

To localize the wave, we may "broaden" the delta function in Eq. (3.28). We define

$$\delta_\sigma(\bar{k}) \equiv \frac{1}{\sigma\sqrt{\pi}} \exp\left[-\frac{\bar{k}^2}{\sigma^2}\right], \tag{3.33}$$

which is normalized so that

$$\int_{-\infty}^{\infty} d\bar{k}\delta_\sigma(\bar{k}) = 1. \tag{3.34}$$

The peak width is controlled by the parameter σ. As $\bar{k}$ is a wave number, σ has dimensions of inverse length. We may choose our incoming wave, moving along the z axis, to be symmetric under a rotation about that axis. Consider the choice,

$$\begin{aligned} \alpha(k) = \frac{1}{\hbar^3}|\boldsymbol{k}|(2\pi)^3 A_\odot \sqrt{2\hbar}\delta_{\sigma_\parallel}(\omega - k^z/\hbar) \\ \times \delta_{\sigma_\perp}(k^x/\hbar)\delta_{\sigma_\perp}(k^y/\hbar), \end{aligned} \tag{3.35}$$

or equivalently,

$$\bar{\alpha}(\bar{k}) = \sqrt{2}|\bar{\boldsymbol{k}}|(2\pi)^3 A_\odot \delta_{\sigma_\parallel}(\omega - \bar{k}^z)\delta_{\sigma_\perp}(\bar{k}^x)\delta_{\sigma_\perp}(\bar{k}^y), \tag{3.36}$$

with $A_\odot$ being real. (We use the superscripts t, x, y, and z to denote the corresponding components of $\bar{k}$.) We have introduced two measures of beam spread, $\sigma_\parallel$ and $\sigma_\perp$, along and transverse to the wave direction, respectively. The corresponding classical field is

$$\begin{aligned} A^\mu_{\rm cl}(x) = \sqrt{2}A_\odot {\rm Re}\int d^3\bar{k}\varepsilon^{*\mu}_\odot(\bar{k})\delta_{\sigma_\parallel}(\omega - \bar{k}^z) \\ \times \delta_{\sigma_\perp}(\bar{k}^x)\delta_{\sigma_\perp}(\bar{k}^y)e^{-i\bar{k}\cdot x}\bigg|_{\bar{k}^t = \sqrt{(\bar{k}^x)^2+(\bar{k}^y)^2+(\bar{k}^z)^2}}. \end{aligned} \tag{3.37}$$

We emphasize that other choices of wave shape are available in the classical theory: the only constraint is that N_γ must be large.

Let us further refine our example by taking $\sigma_\parallel$ to be very small compared to the other two scales, $\sigma_\perp$ and $\omega = \bar{k}^t_\odot$. We are thus considering a monochromatic beam, for which we can replace $\delta_{\sigma_\parallel}$ by a Dirac delta distribution. Doing so, we obtain

$$\begin{aligned} A^\mu_{\rm cl}(x) = \sqrt{2}A_\odot {\rm Re}\int d^2\bar{k}_\perp \varepsilon^{*\mu}_\odot(\bar{k})\delta_{\sigma_\perp}(\bar{k}^x) \\ \times \delta_{\sigma_\perp}(\bar{k}^y)e^{-it\sqrt{\omega^2+(\bar{k}^x)^2+(\bar{k}^y)^2}}e^{i\omega z}e^{i\bar{k}^x x}e^{i\bar{k}^y y}. \end{aligned} \tag{3.38}$$

We can simplify this expression with the following considerations. For the beam to be moving in the z direction, the photons in the beam should dominantly have their momenta, or equivalently their wave numbers, aligned in the z direction. However, the broadened distribution $\delta_{\sigma_\perp}$ does allow small components of momentum in the x and y directions. These components should be subdominant. The corresponding x and y wave numbers are of order $\sigma_\perp$, while the wave number in the z direction is of order ω. Let us define the (reduced) wavelength $\lambda \equiv \omega^{-1}$. We must thus require

$$\lambda^{-1} \gg \sigma_\perp. \tag{3.39}$$

We can also define a transverse size of the beam,

$$\ell_\perp = \sigma_\perp^{-1}, \tag{3.40}$$

along with a "pulse length,"

$$\ell_\parallel = \sigma_\parallel^{-1}. \tag{3.41}$$

We see that we must require

$$\lambda \ll \ell_\perp. \tag{3.42}$$

[4]The wave $\langle\alpha^-|\mathbb{A}^\mu|\alpha^-\rangle$ is circularly polarized in the opposite sense.

In other words, a collimated monochromatic beam must have a transverse size which is large in units of the beam's wavelength. The requirement (3.42) is in some respects analogous to the first part of the "Goldilocks" condition (2.17). However, we emphasize that Eq. (3.42) arises from our desire to localize the wave in the far past. In particular, waves violating the requirement (3.42) may still be classical.

Turning back to Eq. (3.38), we may now simplify the time-dependent exponential factor. The broadened delta distribution $\delta_{\sigma_\perp}$ forces

$$(\bar{k}^x)^2 + (\bar{k}^y)^2 \lesssim \sigma_\perp^2 = \ell_\perp^{-2}, \tag{3.43}$$

so that

$$\sqrt{\omega^2 + (\bar{k}^x)^2 + (\bar{k}^y)^2} \lesssim \sqrt{\omega^2 + \ell_\perp^{-2}} \simeq \omega + \mathcal{O}(\ell_\perp^{-2}\omega^{-2}) \simeq \omega. \tag{3.44}$$

For the wave's field, we obtain, in this approximation,

$$\begin{aligned} A^\mu_{\mathrm{cl}}(x) &= \sqrt{2}A_\odot \mathrm{Re}\left\{ e^{-i\omega(t-z)} \int d^2\bar{k}_\perp \varepsilon^{*\mu}_\odot(\bar{k}) \delta_{\sigma_\perp}(\bar{k}^x)\delta_{\sigma_\perp}(\bar{k}^y) e^{i\bar{k}^x x} e^{i\bar{k}^y y} \right\} \\ &= \sqrt{2}A_\odot \mathrm{Re}\left\{ e^{-i\omega(t-z)} \varepsilon^{*\mu}_\odot(\bar{k}_\odot) \int d^2\bar{k}_\perp \delta_{\sigma_\perp}(\bar{k}^x)\delta_{\sigma_\perp}(\bar{k}^y) e^{i\bar{k}^x x} e^{i\bar{k}^y y} \right\}, \end{aligned} \tag{3.45}$$

where we can replace $\varepsilon^\mu_\odot(\bar{k})$ by $\varepsilon^\mu_\odot(\bar{k}_\odot)$ because of the smallness of the transverse components of $\bar{k}$. [Recall that $\bar{k}^\mu_\odot = (\omega, 0, 0, \omega)$.] To continue, we may note that the integral

$$\int_{-\infty}^{\infty} d\bar{q} e^{i\bar{q}x} \delta_\sigma(\bar{q}) = e^{-x^2\sigma^2/4}, \tag{3.46}$$

so that we finally obtain

$$A^\mu_{\mathrm{cl}}(x) = \sqrt{2}A_\odot \mathrm{Re}[e^{-i\omega(t-z)} \varepsilon^{*\mu}_\odot(\bar{k}_\odot) e^{-(x^2+y^2)/(4\ell_\perp^2)}]. \tag{3.47}$$

This is indeed a wave of circular polarization along the z axis, with finite size in the $x-y$ plane.

Our approximation that $\sigma_\parallel$ is negligible gives us a beam of *infinite* spatial extent along the direction of propagation (here, the z axis). Were we to stop short of the $\sigma_\parallel \to 0$ limit, we would find a finite size in this direction too. The occupation number, which is divergent for infinite extent in the z direction, would also become finite for nonvanishing $\sigma_\parallel$.

The classical field in Eq. (3.47) describes a beam of light that does not spread in the transverse direction, in apparent contradiction to the nonzero transverse momenta the integral contains. This seeming contradiction is lifted when we compute the field of Eq. (3.38) to the next order in $1/(\omega\ell_\perp)$ and $t/\ell_\perp$, as described in Appendix A. The result for short enough times is

$$\begin{aligned} A^\mu_{\mathrm{cl}}(x) &= \sqrt{2}A_\odot \mathrm{Re}\left\{ \frac{\exp[-i\omega(t-z)]}{1 + i\frac{t}{2\omega\ell_\perp^2}} \varepsilon^{*\mu}_\odot(\bar{k}_\odot) \exp\left[-\frac{(x^2+y^2)}{4\ell_\perp^2[1 + it/(2\omega\ell_\perp^2)]} \right] \right\} \\ &\quad + \frac{A_\odot}{\sqrt{2}} \mathrm{Re}\left\{ \exp[-i\omega(t-z)] \left[i\frac{x}{\ell_\perp^2} \partial_{\bar{k}^x} \varepsilon^{*\mu}_\odot(\bar{k}) \Big|_{\bar{k}=\bar{k}_\odot} + i\frac{y}{\ell_\perp^2} \partial_{\bar{k}^y} \varepsilon^{*\mu}_\odot(\bar{k}) \Big|_{\bar{k}=\bar{k}_\odot} \right] \exp\left[-\frac{(x^2+y^2)}{4\ell_\perp^2} \right] \right\} + \cdots. \end{aligned} \tag{3.48}$$

IV. GLOBAL OBSERVABLES WITH INCOMING RADIATION

In the previous section, we examined the use of coherent states to describe waves built up of massless messengers (photons or gravitons), and understood that the classical limit emerges in the limit of large occupation number. In this section, we turn to dynamics: we will consider the scattering of a messenger wave and a scalar point particle. Real-life examples are the classical scattering of a light beam off a charged point particle, a light beam scattering gravitationally off a point particle, or a gravitational wave scattering off a point particle.

Our focus in this section will be on *global* observables, obtained by surrounding the scattering event with a distant sphere of detectors. These detectors can register the total change in momentum (or impulse) of the particle, or of the wave, during scattering. These are the same kinds of observables considered in Ref. [122]. The main novelty in this section will be the computation of global observables for scattering with incoming classical radiation, which we will describe using the coherent states discussed in the previous section. In the following sections we will discuss local observables.

Two examples will allow us to explore different aspects of the dynamics: the electromagnetic impulse on a charge in a spatially localized beam of light (Thomson scattering); and the general-relativistic deflection of light in the geometric-optics limit. We begin by discussing the details of the requirements imposed by the dynamics in the classical limit and the nature of the initial state.

A. Setup

In the classical limit, the Compton wavelength ℓ_c of a pointlike particle must be unobservably small. However, there is (in general) no need for the wavelength of massless waves to be small. On the contrary, finite-wavelength classical waves are quotidian phenomena, and propagate along the pages of many classical-physics textbooks.

In the scattering of two pointlike particles, this requirement on ℓ_c would be violated if the particles approach at distances smaller than (or of order of) their Compton wavelength, because then the underlying wave nature of the particles becomes important. Thus we arrive at the conclusion that classical scattering of two particles obtains only when the impact parameter $b \neq 0$.

In contrast, for a wave of wavelength λ interacting with a particle, we simply require that λ be much larger than the Compton wavelength ℓ_c of the particle. When this is the case, the messengers comprising the wave cannot resolve the quantum structure of the particle. For the classical point-particle approximation to be valid, we further require that λ should be large compared to the finite size ℓ_w of the particle's wave packet. We thus have the requirement

$$\ell_c \ll \ell_w \ll \lambda \tag{4.1}$$

for classical interactions of a wave with a particle of Compton wavelength ℓ_c. There is no *a priori* constraint on the impact parameter b.

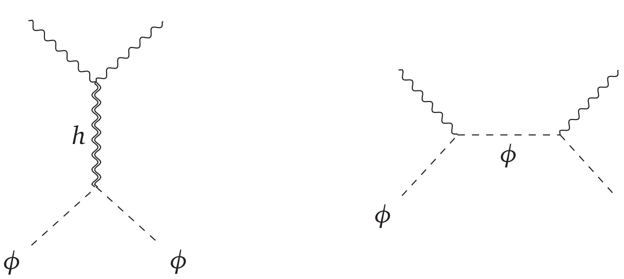

FIG. 1. While the t-channel graviton exchange contribution exists for a photon interacting gravitationally with a scalar, this is not true in electromagnetic case.

As exemplified in Fig. 1, in the electromagnetic scattering of a photon off a charged particle, there is no t-channel contribution. Correspondingly we are primarily interested in the $b \simeq 0$ case. (More precisely, we are interested in b smaller than the transverse size of the beam.) We will explore this in more detail below. In contrast, in the gravitational scattering of a photon off a neutral particle, there are both s- and t-channel contributions. In this case, we are interested in general b.

The interaction between our particle and our wave introduces another length scale to consider, namely the scattering length ℓ_s. Let $q = \hbar \bar{q}$ be a characteristic momentum exchange associated with the interaction; then the scattering length is defined to be

$$\ell_s = \frac{1}{\sqrt{|\bar{q}^2|}}. \tag{4.2}$$

The value of the scattering length depends on the details of the scattering process. In the case where two pointlike particles scatter, for instance, one finds that $\ell_s \sim b$. In the case at hand where a particle interacts with a wave this need not be the case. Indeed for an s channel processes it is more natural to expect ℓ_s to be determined by the off-shellness of intermediate propagators such as $s - m^2$. For definiteness let us take the momentum of the incoming particle to be $p_1 = m_1 u_1$ while the incoming wave has a characteristic wave number $\bar{k}_\odot$. Then $s - m_1^2 = 2\hbar \bar{k}_\odot \cdot p_1$, so that the scattering length should be

$$\ell_s \sim \frac{1}{\bar{k}_\odot \cdot u_1}. \tag{4.3}$$

This is simply of the order of the wavelength of the incoming wave.

We turn next to the construction of the incoming state. As in Ref. [122] and in Eq. (2.14), we write the point particle as a superposition of plane-wave states weighted by a wave function $\phi(p)$. Following the discussion in the previous section, we write the messenger wave as a coherent state of helicity η characterized by the wave shape $\alpha(k)$. We start with a basis of states constructed out of coherent states (3.8) of definite helicity $|\alpha^\eta\rangle$ for the messenger and plane-wave states for the massive particle

$$|p_1 \alpha_2^\eta\rangle_{\text{in}} = |p_1\rangle |\alpha_2^\eta\rangle. \tag{4.4}$$

Our initial state then takes the form

$$|\psi_w\rangle_{\text{in}} = \int d\Phi(p_1) \phi_1(p_1) e^{ib\cdot p_1/\hbar} |p_1 \alpha_2^\eta\rangle_{\text{in}}. \tag{4.5}$$

The impact parameter b now separates the particle from the center of the beam in the far past. As in the earlier discussion, the state is normalized to unity,

${}_{\rm in}\langle\psi_w|\psi_w\rangle_{\rm in} = 1$. (We will leave the "in" subscript implicit going forward.)

Information about the classical four-velocity of the point particle is hidden inside $\phi(p)$. The explicit example studied in Ref. [122] made use of a linear exponential (which slightly counterintuitively reduces to a Gaussian in the nonrelativistic limit). In the same way, the information about the overall momentum $K_\odot$ of the messenger wave is hidden inside $\alpha(k)$.

In the following, we will make use of the coherent wave shape $\alpha(k)$ chosen in Eq. (3.35) corresponding to the choice of $\bar{\alpha}(k)$ of Eq. (3.36), independent of $\hbar$ as desired. We will elucidate inequalities between the various parameters defining the beam below, where relevant.

B. General expression for the impulse

Before we discuss the details of specific examples, let us investigate the general structure of the impulse, $\langle\Delta p_1\rangle$, on a massive particle during a scattering event with a classical wave. We can carry over the expression from Ref. [122],

$$\begin{aligned}\langle\Delta p_1^\mu\rangle &= \langle\psi_w|i[\mathbb{P}_1^\mu, T]|\psi_w\rangle + \langle\psi_w|T^\dagger[\mathbb{P}_1^\mu, T]|\psi_w\rangle\\ &= I_{w(1)}^\mu + I_{w(2)}^\mu.\end{aligned}\tag{4.6}$$

Compared to Ref. [122], only the initial state is different.

Before studying the expansion of this expression, we remark that there is an equivalent formulation in terms of the background field,

$$\begin{aligned}\langle\Delta p_1^\mu\rangle &= \int d\Phi(p_1)d\Phi(p_1')\phi_1(p_1)\phi_1^*(p_1')e^{-ib\cdot(p_1'-p_1)/\hbar}\langle p_1'|\mathbb{C}_{\alpha,(\eta)}^\dagger i[\mathbb{P}_1^\mu, T]\mathbb{C}_{\alpha,(\eta)}|p_1\rangle\\ &\quad+\int d\Phi(p_1)d\Phi(p_1')\phi_1(p_1)\phi_1^*(p_1')e^{-ib\cdot(p_1'-p_1)/\hbar}\langle p_1'|\mathbb{C}_{\alpha,(\eta)}^\dagger T^\dagger[\mathbb{P}_1^\mu, T]\mathbb{C}_{\alpha,(\eta)}|p_1\rangle\\ &= \int d\Phi(p_1)d\Phi(p_1')\phi_1(p_1)\phi_1^*(p_1')e^{-ib\cdot(p_1'-p_1)/\hbar}\langle p_1'|i[\mathbb{P}_1^\mu, T(A_{\rm cl}^{(\eta)})]|p_1\rangle\\ &\quad+\int d\Phi(p_1)d\Phi(p_1')\phi_1(p_1)\phi_1^*(p_1')e^{-ib\cdot(p_1'-p_1)/\hbar}\langle p_1'|T^\dagger(A_{\rm cl}^{(\eta)})[\mathbb{P}_1^\mu, T(A_{\rm cl}^{(\eta)})]|p_1\rangle,\end{aligned}\tag{4.7}$$

where the scattering matrix computed from the background $A_{\rm cl}^{(\eta)}$ is denoted by $T(A_{\rm cl}^{(\eta)})$, and we have used the relation $\mathbb{C}_{\alpha,(\eta)}^\dagger\mathbb{C}_{\alpha,(\eta)} = \mathbb{1}$. While we will focus on the formulation (4.6), it is intriguing to notice the linear term of the impulse $I_{w(1)}^\mu$ is closely related to the two-point function of the massive scalar field in the coherent state background. As a consequence, we should expect a resummation of all higher-order results.

Returning to Eq. (4.6), we note that—just as in the scattering of two massive particles—only the first term contributes at leading order (LO) in the generic coupling g. This LO contribution arises at $\mathcal{O}(g^2)$; the second term only contributes starting at $\mathcal{O}(g^4)$. Let us focus on the $I_{w(1)}^\mu$ term and write out the details of the wave function (4.5),

$$I_{w(1)}^\mu = \int d\Phi(p_1)d\Phi(p_1')e^{-ib\cdot(p_1'-p_1)/\hbar}\phi_1(p_1)\phi_1^*(p_1')i(p_1'-p_1)^\mu\langle p_1'\alpha_2^\eta|T|p_1\alpha_2^\eta\rangle.\tag{4.8}$$

The matrix elements of coherent states are not of definite order in perturbation theory. In order to isolate the contributions at each order, one would ordinarily introduce a complete set of states of definite particle number on each side of the T matrix,

$$\begin{aligned}I_{w(1)}^\mu = \sum_{X,X'}\sum_{\zeta,\zeta'=\pm}\int & d\Phi(p_1)d\Phi(p_1')d\Phi(r_1)d\Phi(r_1')d\Phi(k_2)d\Phi(k_2')e^{-ib\cdot(p_1'-p_1)/\hbar}\phi_1(p_1)\phi_1^*(p_1')i(p_1'-p_1)^\mu\\ &\times\langle p_1'\alpha_2^\eta|r_1'k_2'^{\zeta'}X'\rangle\langle r_1'k_2'^{\zeta'}X'|T|r_1k_2^\zeta X\rangle\langle r_1k_2^\zeta X|p_1\alpha_2^\eta\rangle.\end{aligned}\tag{4.9}$$

The sums over X and X' are over different numbers of messengers, including none, and include the phase-space integrals over their momenta. Charge conservation implies that each intermediate state must contain one net massive-particle number; we drop additional particle-antiparticle pairs as their effects will disappear in the classical limit, and we denote the

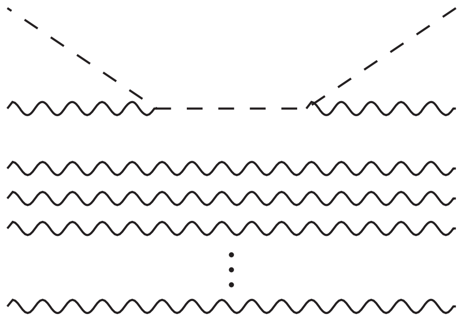

FIG. 2. S-matrix elements relevant to the scattering of a pointlike particle and a coherent state of radiation. It is important to include disconnected diagrams containing messengers trivially connected from the ket state on the left to the bra state on the right.

massive-particle momenta by r_1 and r_1'. Moreover, in order to satisfy on shell conditions of the T matrix element, each intermediate state must contain at least one messenger, whose momenta are denoted by k_2 and k_2'.

The LO contribution to $I^{\mu}_{w(1)}$ is the simplest. One may be tempted to believe that it arises from terms with $X = X' = \varnothing$, but this is not quite right: that would omit relevant disconnected parts of the S matrix, depicted in Fig. 2. In the situation at hand, a great many messengers are present in the initial state; the dominant contribution to the interaction occurs when most messengers connect directly from the ket state to the bra state. Thus rather than taking $X = X' = \varnothing$, we instead need to sum over additional messengers in the coherent states. These sums over noninteracting messengers, contributing disconnected S-matrix terms, are necessary to recover the correct normalization. Disconnected parts containing interaction vertices are higher-order in perturbation theory.

One can carry out these sums explicitly, but it is convenient instead to introduce an alternate representation for the T matrix in terms of creation and annihilation operators. As the incoming state $|\psi_w\rangle$ given in Eq. (4.5) contains one massive particle and an arbitrary number of messengers, we must consider terms with a pair of massive-particle annihilation and creation operators, and an arbitrary nonzero number of messenger annihilation and creation operators (not necessarily paired). That representation [following from Eq. (4.2.8) in the first volume of Weinberg's quantum field theory textbook [149]] has the form

$$T = \sum_{\tilde\eta,\tilde\eta'} \int d\Phi(\tilde r_1, \tilde r_1', \tilde k_2, \tilde k_2') \langle \tilde r_1' \tilde k_2'^{\tilde\eta'} | T | \tilde r_1 \tilde k_2^{\tilde\eta} \rangle \times a^{\dagger}_{(\tilde\eta')}(\tilde k_2') a^{\dagger}(\tilde r_1') a(\tilde r_1) a_{(\tilde\eta)}(\tilde k_2) + \cdots, \qquad (4.10)$$

where the ellipsis indicates higher order terms in the coupling g as well as amplitudes which do not contribute in the classical limit. We will summarily drop all these terms in the following, retaining only the explicit $\mathcal{O}(g^2)$ term. The measure here is a shorthand

$$d\Phi(\tilde r_1, \tilde r_1', \tilde k_2, \tilde k_2') = d\Phi(\tilde r_1) d\Phi(\tilde r_1') d\Phi(\tilde k_2) d\Phi(\tilde k_2'). \qquad (4.11)$$

The advantage of the representation (4.10) is that the creation and annihilation operators act simply on coherent states, yielding factors of $\alpha(k_2)$ and $\alpha^*(k_2')$, and taking care of the normalization for us. Each term within this representation contains an ordinary (connected) amplitude with a definite number of external messengers. We have carried out the expansions and sums explicitly in the standard representation, and find that they reproduce the results from this alternate representation in a far more laborious way. One encounters no extra divergences when doing so.

The required matrix element for the integrand term in Eq. (4.10) can be computed easily,

$$\begin{aligned} \langle p_1' \alpha_2^{\eta} | T | p_1 \alpha_2^{\eta} \rangle &= \langle \tilde r_1' \tilde k_2'^{\tilde\eta'} | T | \tilde r_1 \tilde k_2^{\tilde\eta} \rangle \langle p_1' \alpha_2^{\eta} | a^{\dagger}_{(\tilde\eta')}(\tilde k_2') a^{\dagger}(p_1') a_{(\tilde\eta)}(\tilde k_2) a(p_1) | p_1 \alpha_2^{\eta} \rangle \\ &= \hat\delta_{\Phi}(\tilde r_1 - p_1) \hat\delta_{\Phi}(\tilde r_1' - p_1') \delta_{\tilde\eta,\eta} \delta_{\tilde\eta',\eta} \alpha_2(\tilde k_2) \alpha_2^*(\tilde k_2') \langle \tilde r_1' \tilde k_2'^{\tilde\eta'} | T | \tilde r_1 \tilde k_2^{\tilde\eta} \rangle, \end{aligned} \qquad (4.12)$$

where we neglected all the terms in the ellipsis of Eq. (4.10). Notice that we encountered the matrix element $\langle \alpha_2^{\eta} | \alpha_2^{\eta} \rangle = 1$: this conveniently takes care of all the disconnected diagrams. The remaining matrix element introduces the desired scattering amplitude,

$$\langle \tilde r_1' \tilde k_2'^{\eta'} | T | \tilde r_1 \tilde k_2^{\tilde\eta} \rangle = \mathcal{A}(\tilde r_1 \tilde k_2^{\tilde\eta} \to \tilde r_1' k_2'^{\tilde\eta'}) \hat\delta^{(4)}(\tilde r_1 + \tilde k_2 - \tilde r_1' - \tilde k_2'). \qquad (4.13)$$

As usual, the superscripts on the messenger momenta denote the corresponding physical helicity. To write it in the usual amplitudes convention, $A(0 \to p_1, p_2, \ldots)$, we must cross the momenta to the other side. This flips the helicity of incoming messengers.

Using the results of Eqs. (4.12) and (4.13) in Eq. (4.8) and carrying out the sums over $\tilde\eta, \tilde\eta'$, we obtain

$$I^{\mu}_{w(1)} = \int d\Phi(p_1)d\Phi(p_1')d\Phi(k_2)d\Phi(k_2')\phi_1(p_1)\phi_1^*(p_1')\alpha_2(k_2)\alpha_2^*(k_2')$$
$$\times e^{-ib\cdot(p_1'-p_1)/\hbar} i(p_1'-p_1)^{\mu}\mathcal{A}(p_1 k_2^{\eta} \to p_1' k_2'^{\eta})\hat{\delta}^{(4)}(p_1+k_2-p_1'-k_2'), \tag{4.14}$$

where we have dropped the tildes on k_2 and k_2'.

We make the usual change of variables to the momentum mismatches $q_{1,2}$,

$$\begin{aligned} q_1 &= p_1' - p_1, \\ q_2 &= k_2' - k_2, \end{aligned} \tag{4.15}$$

then we use the delta function to integrate over q_2; and, dropping the subscript on q_1, we find

$$I^{\mu}_{w(1)} = \int d\Phi(p_1)d\Phi(k_2)\hat{d}^4q\hat{\delta}(2q\cdot p_1+q^2)\hat{\delta}(2q\cdot k_2-q^2)\Theta(p_1^t+q^t)\Theta(k_2^t-q^t)$$
$$\times \phi_1(p_1)\phi_1^*(p_1+q)\alpha_2^*(k_2-q)\alpha_2(k_2)e^{-ib\cdot q/\hbar} iq^{\mu}\mathcal{A}(p_1k_2^{\eta}\to p_1+q,(k_2-q)^{\eta}). \tag{4.16}$$

The analysis of the classical limit as far as the $\phi_1(p_1)\phi_1^*(p_1+q)$ factor is concerned is the same as in Ref. [122]. It requires us to take the wave number mismatch as our integration variable in lieu of the momentum mismatch. At leading order, we do not have to worry about terms singular in $\hbar$, so the evaluation as far as the massive particle is concerned will take

$$\begin{aligned} \hat{\delta}(2q\cdot p_1+q^2) &\to \hbar^{-1}\hat{\delta}(2\bar{q}\cdot p_1), \\ \phi(p_1+q) &\to \phi(p_1). \end{aligned} \tag{4.17}$$

Removing the coupling from inside the scattering amplitude (as in Ref. [122], the reduced amplitude is denoted by $\bar{\mathcal{A}}$), we find for the classical limit,

$$\begin{aligned} I^{\mu,\mathrm{cl}}_{w(1)} = g^2 \Big\langle\!\!\Big\langle \int & d\Phi(\bar{k}_2)\hat{d}^4\bar{q}\,\hat{\delta}(2\bar{q}\cdot p_1)\hat{\delta}(2\bar{q}\cdot\bar{k}_2-\bar{q}^2) \\ &\times \Theta(\bar{k}_2^t-\bar{q}^t)\bar{\alpha}_2^*(\bar{k}_2-\bar{q})\bar{\alpha}_2(\bar{k}_2) \\ &\times e^{-ib\cdot\bar{q}} i\bar{q}^{\mu}\bar{\mathcal{A}}(p_1\hbar\bar{k}_2^{\eta}\to p_1+\hbar\bar{q},\hbar(\bar{k}_2-\bar{q})^{\eta})\Big\rangle\!\!\Big\rangle. \end{aligned} \tag{4.18}$$

As in Ref. [122], the double-angle brackets indicate an average over the wave function of the pointlike particle. Classically, this is a function of the momentum p_1 with a very sharp peak at $p_1 = m_1u_1$, where u_1 is the classical (proper) velocity and m_1 is the particle's mass.

We can now apply this general result in a variety of specific cases. We shall describe two examples in detail: Thomson scattering of a charge by a wave, with $b \simeq 0$, and gravitational scattering of light by a mass in the geometric-optics limit.

C. Impulse in Thomson scattering

Our first application is to Thomson scattering, of a particle of charge Qe and mass m, by a collimated beam of light as shown in Fig. 3. We take the light beam to have positive helicity, corresponding to the coherent state $|\alpha^+\rangle$. We need the four-point tree Compton amplitude in scalar QED,

$$\begin{aligned} \bar{\mathcal{A}}(p_1,k_2^{\eta}\to p_1',k_2'^{\eta'}) &= 2Q^2\varepsilon^{(\eta)*}(k_2)\cdot\varepsilon^{(\eta')}(k_2') \\ &= 2Q^2\varepsilon^{(-\eta)}(k_2)\cdot\varepsilon^{(\eta')}(k_2'), \end{aligned} \tag{4.19}$$

where we have chosen the gauge,

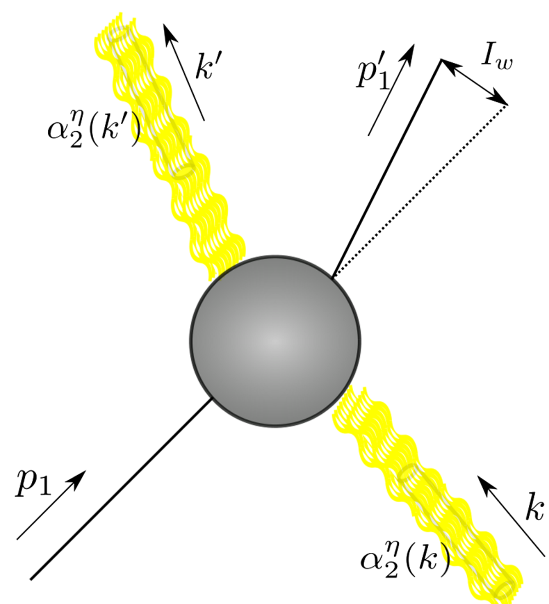

FIG. 3. Impulse in scattering of a massive object off a coherent state background.

$$\varepsilon^{(\eta)} \cdot p_1 = 0, \tag{4.20}$$

for both photons. Alternatively, in spinor variables, we have a gauge-invariant expression for the helicity amplitude, namely

$$\bar{\mathcal{A}}(p_1, k_2^+ \to p_1', k_2'^+) = -\frac{Q^2}{2}\frac{\langle k_2|p_1|k_2']^2}{k_2 \cdot p_1 k_2' \cdot p_1}. \tag{4.21}$$

This form of the amplitude is manifestly gauge independent, but it depends explicitly on spinors $|k_2'\rangle$ and $|k_2]$ associated with photon momenta. As usual, in the classical limit we prefer to work with photon wave numbers. We therefore introduce rescaled spinors,

$$\begin{aligned} |\bar{k}_2'\rangle &\equiv \hbar^{-1/2}|k_2'\rangle, \\ |\bar{k}_2] &\equiv \hbar^{-1/2}|k_2], \end{aligned} \tag{4.22}$$

which are directly associated with the photon wave numbers. The amplitude then has the expression

$$\bar{\mathcal{A}}(p_1, k_2^+ \to p_1', k_2'^+) = -\frac{Q^2}{2}\frac{\langle \bar{k}_2|p_1|\bar{k}_2']^2}{\bar{k}_2 \cdot p_1 \bar{k}_2' \cdot p_1}. \tag{4.23}$$

Choosing $b = 0$, and for a more symmetric presentation, writing $k = k_2$ and $k' = k_2 - q$, the impulse Eq. (4.18) takes the form

$$\begin{aligned} \langle \Delta p^\mu \rangle = \frac{Q^2 e^2}{2} \int d\Phi(\bar{k}) d\Phi(\bar{k}') \hat{\delta}(2p \cdot (\bar{k} - \bar{k}')) \bar{\alpha}^*(\bar{k}') \\ \times \bar{\alpha}(\bar{k})\, i(\bar{k}' - \bar{k})^\mu \frac{\langle \bar{k}|p|\bar{k}']^2}{(\bar{k} \cdot p)^2}. \end{aligned} \tag{4.24}$$

This expression may be compared with the classical electromagnetic result, obtained by iterating the classical Lorentz force twice. Thus we see in an explicit example that a vanishing impact parameter is perfectly acceptable in the classical scattering of waves off matter, in contrast to the situation for two massive particles scattering.

It is interesting that the Compton amplitude appears at tree level in the classical physics of wave scattering off massive particles. This amplitude is also relevant [150] for purely massive particle scattering, though at one loop order. While the amplitude is very simple for spinless particles, it is considerably more complicated [151] for particles with large spins. Currently we do not have a clear understanding of the appropriate Compton amplitude for the Kerr black hole, or of what principle we could use to determine it. This is an important area for further research. Our work suggests one angle of attack: information about the classical part of the Compton amplitude could be extracted by a purely classical analysis of the impulse on a massive spinning object in scattering off a messenger wave. This is one topic under independent study in Ref. [143].

D. Light deflection in gravitational scattering

A second interesting application of the formulas derived in the previous section is to the gravitational deflection of light by a massive object. We may access this observable by computing the change in momentum of a narrow (small $\ell_\perp$) beam of light passing with nonzero impact parameter b past a massive pointlike particle. At leading order, there is no radiation of momentum, so the change in momentum of the wave is simply the negative of the change in momentum of the massive point source: our starting point is once again Eq. (4.18).

Before we discuss the details of the calculation, it is worth dwelling for a moment on our setup. Eddington's famous observations demonstrated that starlight is deflected by the sun in accordance with general relativity. Near the sun, light emitted by a distant star is essentially a spherical wave, and so the incoming wave is extremely delocalized. In contrast, we have chosen to study a collimated, narrow beam of light. Nevertheless, the difference between our setup and Eddington's case is immaterial. We work in the situation where the wavelength λ of the light is very small compared to the impact parameter: this is the domain of geometric optics and also applies to Eddington's case. It is in the context of geometric optics that the bending is well defined; the geometric bending does not depend on the details of the wave.

For our purposes the setup of a narrow beam in the far past is just a simpler place to start. The reason is that we can then determine the bending of light by computing the impulse on the beam: this impulse is directly the change in direction of the wave. By contrast the impulse on starlight due to the sun involves integrating over the whole incoming spherical wave front: this is not related in a simple manner to the bending of light.

In the geometric-optics regime, we need the wavelength of the light λ to be small. At the same time we must suppress all quantum effects, so we choose λ to be large compared to the Compton wavelength ℓ_c of our point source. To keep our beam collimated, Eq. (3.42) requires that $\ell_\perp \gg \lambda$. The requirement that our beam is narrow is $\ell_\perp \ll b$. Thus there is a series of inequalities:

$$\ell_c \ll \lambda \ll \ell_\perp \ll \ell_s \sim b. \tag{4.25}$$

Note that the scattering length ℓ_s is expected to be of order of the impact parameter in this case, as we are considering a t channel process. For simplicity, we consider a monochromatic beam with $\sigma_\parallel \to 0$. The final length scale to consider is the size ℓ_w of the point-particle's wave packet. As usual we require $\ell_c \ll \ell_w \ll \ell_s$. Once these conditions are met, there will be little overlap between the beam and the wave packet, so we do not anticipate that the values of the ratios λ/ℓ_w or $\ell_\perp/\ell_w$ will be important.

The impulse given in Eq. (4.18) simplifies due to the constraints of Eq. (4.25). Note that the quantity

$|\bar{q}\cdot\bar{k}_2| \gg |\bar{q}^2|$ in the second delta function, as $\bar{k}_2 \sim 1/\lambda$ while $\bar{q} \sim 1/\ell_s$. The wave number $\bar{q}$ is then dominantly in the plane of scattering. In this plane, the coherent wave shape $\bar{\alpha}_2$ is of width $1/\ell_\perp$ so that we may approximate $\bar{\alpha}_2^*(\bar{k}_2 - \bar{q}) \simeq \bar{\alpha}_2^*(\bar{k}_2)$. For the same reason, the explicit theta function in the impulse simplifies: $\Theta(\bar{k}_2^t - \bar{q}^t) = 1$. Taking into account the sign demanded by momentum balance, the impulse on the wave is

$$\langle \Delta p_2^\mu \rangle = -g^2 \left\langle\!\!\left\langle \int d\Phi(\bar{k}_2) \hat{d}^4\bar{q}\, \hat{\delta}(2\bar{q}\cdot p_1)\hat{\delta}(2\bar{q}\cdot\bar{k}_2)|\bar{\alpha}_2(\bar{k}_2)|^2 \times e^{-ib\cdot\bar{q}}\, i\bar{q}^\mu \bar{\mathcal{A}}(p_1 \hbar\bar{k}_2^\eta \to p_1 + \hbar\bar{q}, \hbar(\bar{k}_2 - \bar{q})^\eta) \right\rangle\!\!\right\rangle. \tag{4.26}$$

The integral over $\bar{k}_2$ is now in a great many respects analogous to the integral over the massive particle wave function which is hidden in our double-angle brackets. In the geometric optics limit, $\bar{\alpha}_2(\bar{k}_2)$ is a steeply-peaked function of the wave number peaked at $\bar{k}_2 = \bar{k}_\odot$; in view of Eq. (3.25), its normalization is related to the number of photons in the beam. The amplitude, meanwhile, is a smooth function in this region. The $\bar{k}_2$ integral then has the structure

$$\int d\Phi(\bar{k}_2)\hat{\delta}(2\bar{q}\cdot\bar{k}_2)|\bar{\alpha}_2(\bar{k}_2)|^2 f(\bar{k}_2) \simeq f(\bar{k}_\odot) \int d\Phi(\bar{k}_2)\hat{\delta}(2\bar{q}\cdot\bar{k}_2)|\bar{\alpha}_2(\bar{k}_2)|^2, \tag{4.27}$$

where f is a slowly varying function. We thus encounter the convolution of a delta function and the sharply peaked $|\alpha_2(k)|^2$. The result of the convolution is a broadened delta function centered at $\bar{k}_2 = \bar{k}_\odot$. Neglecting the width (of order $\sigma_\perp$) of this function we have

$$\int d\Phi(\bar{k}_2)\hat{\delta}(2\bar{q}\cdot\bar{k}_2)|\bar{\alpha}_2(\bar{k}_2)|^2 f(\bar{k}_2) \simeq f(\bar{k}_\odot) N_\gamma \hbar \hat{\delta}(2\bar{q}\cdot\bar{k}_\odot). \tag{4.28}$$

Notice the appearance of the number of photons N_γ in the beam: this normalization constant emerges from the integral over $|\alpha_2(k)|^2$. The classical geometric optics approximation does not have access to this number of photons, and correspondingly it will cancel in our expression for the deflection angle below. Certain other physical quantities do involve this number of photons: for example, the total momentum of the beam is

$$K_\odot^\mu = \int d\Phi(\bar{k})|\bar{\alpha}(\bar{k})|^2 \bar{k}^\mu, \simeq N_\gamma \hbar \bar{k}_\odot^\mu. \tag{4.29}$$

Returning to the impulse on the beam, use of Eq. (4.28) leads to the expression

$$\langle \Delta p_{\mathrm{geom}}^\mu \rangle = -N_\gamma \hbar g^2 \left\langle\!\!\left\langle \int \hat{d}^4\bar{q}\, \hat{\delta}(2\bar{q}\cdot p_1)\hat{\delta}(2\bar{q}\cdot\bar{k}_\odot) \times e^{-ib\cdot\bar{q}}\, i\bar{q}^\mu \bar{\mathcal{A}}(p_1 \hbar\bar{k}_\odot^\eta \to p_1 + \hbar\bar{q}, \hbar(\bar{k}_\odot - \bar{q})^\eta) \right\rangle\!\!\right\rangle. \tag{4.30}$$

The subscript reminds us that the approximation is valid in the geometric-optics limit.

At leading order, we only need the four-point tree-level amplitude. As there are no contributions singular in $\hbar$ at this order, we can simply retain only the terms that survive in the classical limit:

$$\begin{aligned}\bar{\mathcal{A}}(p_1 k_2^\eta \to p_1', k_2'^\eta) &= \frac{p_1\cdot k_2 p_1\cdot k_2'}{q^2}\varepsilon^{(\eta)*}(k_2)\cdot\varepsilon^{(\eta)}(k_2') + \cdots \\ &= \frac{p_1\cdot \bar{k}_2 p_1\cdot \bar{k}_2'}{\bar{q}^2}\varepsilon^{(\eta)*}(\bar{k}_2)\cdot\varepsilon^{(\eta)}(\bar{k}_2') + \cdots,\end{aligned} \tag{4.31}$$

where we have chosen the gauge $p_1\cdot\varepsilon^{(\eta)}(k) = 0$ for each polarization vector, and the ellipsis indicates terms which are suppressed by powers of $\hbar$.

This amplitude simplifies further in the geometric-optics limit. The inequalities Eq. (4.25) require in particular that the wave number $\bar{q} \sim 1/b \ll \bar{k}_2$. We may therefore replace the scalar product $p\cdot\bar{k}_2'$ with $p\cdot\bar{k}_2$ in Eq. (4.31), up to terms which are neglected in the geometric-optics limit. At the same time, we may replace the polarization vector $\varepsilon^{(\eta)}(\bar{k}_2')$ with $\varepsilon^{(\eta)}(\bar{k}_2)$ to the same order of approximation. The amplitude is then simply

$$\bar{\mathcal{A}}(p_1 k_2^\eta \to p_1', k_2'^\eta) = -\frac{(p_1\cdot\bar{k}_2)^2}{\bar{q}^2} + \cdots. \tag{4.32}$$

We note that the geometric-optics limit of the amplitude for the scattering of a photon off a massive scalar is helicity independent. Up to constant factors, it reduces to the amplitude between one massless and one massive scalar.[5] This is as expected from the equivalence principle: if the classical limit were not universal, then the impulse and hence the scattering angle would have helicity-dependent contributions.

In order to the evaluate the impulse, we insert the geometric-optics amplitude (4.32) into the expression (4.30) for the impulse in the geometric-optics limit. We obtain

$$\begin{aligned}\langle \Delta p_{\mathrm{geom}}^\mu \rangle &= i\kappa^2 N_\gamma \hbar (p_1\cdot\bar{k}_\odot)^2 \int \hat{d}^4\bar{q}\, \hat{\delta}(2\bar{q}\cdot p_1)\hat{\delta}(2\bar{q}\cdot\bar{k}_\odot) e^{-ib\cdot\bar{q}} \frac{\bar{q}^\mu}{\bar{q}^2} \\ &= i\kappa^2 (p_1\cdot K_\odot)^2 \int \hat{d}^4\bar{q}\, \hat{\delta}(2\bar{q}\cdot p_1)\hat{\delta}(2\bar{q}\cdot K_\odot) e^{-ib\cdot\bar{q}} \frac{\bar{q}^\mu}{\bar{q}^2}.\end{aligned} \tag{4.33}$$

[5] See the beautiful and pedagogical discussion in Ref. [152] for more details.

Here, we have replaced the general coupling g by the appropriate gravitational coupling κ, and the wave number $\bar{k}_\odot$ by the total beam momentum $K_\odot$. The second line of this equation is strikingly similar to the impulse in a scattering process between two *massive* classical objects. Indeed, the integral remaining in Eq. (4.33) is essentially the same as the integral appearing in the LO impulse in Ref. [122]. It can easily be performed by taking the light beam in the z direction, $K^\mu_\odot = (E, 0, 0, E)$. The result is

$$\langle \Delta p^\mu_{\rm geom} \rangle = -\kappa^2 \frac{p_1 \cdot K_\odot}{8\pi b^2} b^\mu. \tag{4.34}$$

The impact parameter b^μ is directed from the massive particle towards the wave, so the sign above indicates that the interaction is attractive.

The scattering angle θ is then determined geometrically in terms of the impulse,

$$\sin\theta = \frac{|b \cdot \Delta p|}{|\boldsymbol{b}| E}, \tag{4.35}$$

once we have fixed a frame. We have taken the absolute value to drop the sign of the angle, understanding that the bending is towards the scatterer. Working in the rest frame of the massive scalar, and using $\kappa^2 = 32\pi G_N$, we reproduce the well-known value for the gravitational bending of light,

$$\theta = \frac{4G_N m}{|\boldsymbol{b}|} + \cdots. \tag{4.36}$$

As a final comment, it is satisfying that the impulse we have obtained in Eq. (4.33) is essentially the same as the impulse on massive point particles as discussed in Ref. [122]. This occurred as the inequalities Eq. (4.25) greatly simplified the impulse. These inequalities themselves are very similar to the Goldilocks conditions Eq. (2.17) for classical pointlike particles. The fact that the dynamics of massive particles is so similar to the behavior of waves in the geometric-optics regime was a celebrated aspect of 19th- and early 20th-century physics, known as the Hamiltonian analogy. This analogy was highlighted by Schrödinger [153] and others as an important consideration in the early days of quantum mechanics.

E. Higher orders

Although in Secs. IV C and IV D we focused on leading-order applications, our formalism is completely general and Eq. (4.6) holds to all perturbative orders. As we have seen, the leading-order contribution arises at $\mathcal{O}(g^2)$. The second term, $I^\mu_{w(2)}$, in the impulse of Eq. (4.6) involves one-loop amplitudes, and therefore contributes only starting at $\mathcal{O}(g^4)$. Consequently, we can identify a further contribution, at $\mathcal{O}(g^3)$, which receives no contribution from $I^\mu_{w(2)}$ but only from $I^\mu_{w(1)}$. It arises from the leading corrections to Eq. (4.10),

$$\begin{aligned}\delta T_3 \equiv \sum_{\tilde\eta,\tilde\eta',\tilde\eta''} \int d\Phi(\tilde r_1, \tilde r_1', \tilde k_2, \tilde k_2', \tilde k_3)[&\langle \tilde r_1' \tilde k_2'^{\tilde\eta'} | T | \tilde r_1 \tilde k_2^{\tilde\eta} \tilde k_3^{\tilde\eta''}\rangle a^\dagger_{(\tilde\eta')}(\tilde k_2') a^\dagger(\tilde r_1') a(\tilde r_1) a_{(\tilde\eta)}(\tilde k_2) a_{(\tilde\eta'')}(\tilde k_3) \\ &+ \langle \tilde r_1' \tilde k_2'^{\tilde\eta'} \tilde k_3^{\tilde\eta''} | T | \tilde r_1 \tilde k_2^{\tilde\eta}\rangle a^\dagger_{(\tilde\eta'')}(\tilde k_3) a^\dagger_{(\tilde\eta')}(\tilde k_2') a^\dagger(\tilde r_1') a(\tilde r_1) a_{(\tilde\eta)}(\tilde k_2)],\end{aligned} \tag{4.37}$$

where the additional argument in the measure corresponds to a factor of $d\Phi(\tilde k_3)$.

Inserting the integrand of δT_3 into the matrix element in Eq. (4.8), we obtain [analogously to Eq. (4.12)],

$$\begin{aligned}\langle p_1' \alpha_2^\eta | \delta T_3 | p_1 \alpha_2^\eta\rangle &= [\langle \tilde r_1' \tilde k_2'^{\tilde\eta'} | T | \tilde r_1 \tilde k_2^{\tilde\eta} \tilde k_3^{\tilde\eta''}\rangle \langle p_1' \alpha_2^\eta | a^\dagger_{(\tilde\eta')}(\tilde k_2') a^\dagger(\tilde r_1') a(\tilde r_1) a_{(\tilde\eta)}(\tilde k_2) a_{(\tilde\eta'')}(\tilde k_3) | p_1 \alpha_2^\eta\rangle \\ &\quad + \langle \tilde r_1' \tilde k_2'^{\tilde\eta'} \tilde k_3^{\tilde\eta''} | T | \tilde r_1 \tilde k_2^{\tilde\eta}\rangle \langle p_1' \alpha_2^\eta | a^\dagger_{(\tilde\eta'')}(\tilde k_3) a^\dagger_{(\tilde\eta')}(\tilde k_2') a^\dagger(\tilde r_1') a(\tilde r_1) a_{(\tilde\eta)}(\tilde k_2) | p_1 \alpha_2^\eta\rangle] \\ &= \hat\delta_\Phi(\tilde r_1 - p_1) \hat\delta_\Phi(\tilde r_1' - p_1') \delta_{\tilde\eta,\eta} \delta_{\tilde\eta',\eta} \alpha_2(\tilde k_2) \alpha_2^*(\tilde k_2') \\ &\quad \times [\delta_{\tilde\eta'',\eta} \alpha_2(k_3) \langle \tilde r_1' \tilde k_2'^{\tilde\eta'} | T | \tilde r_1 \tilde k_2^{\tilde\eta} \tilde k_3^{\tilde\eta''}\rangle + \delta_{\tilde\eta'',\eta} \alpha_2^*(k_3) \langle \tilde r_1' \tilde k_2'^{\tilde\eta'} \tilde k_3^{\tilde\eta''} | T | \tilde r_1 \tilde k_2^{\tilde\eta}\rangle].\end{aligned} \tag{4.38}$$

The scattering matrix elements in this expression introduce five-point amplitudes,

$$\begin{aligned}\langle \tilde r_1' \tilde k_2'^{\tilde\eta'} | T | \tilde r_1 \tilde k_2^{\tilde\eta} \tilde k_3^{\tilde\eta''}\rangle &= \mathcal{A}(\tilde r_1 \tilde k_2^{\tilde\eta} \tilde k_3^{\tilde\eta''} \to \tilde r_1' \tilde k_2'^{\tilde\eta'}) \hat\delta^{(4)}(\tilde r_1 + \tilde k_2 + \tilde k_3 - \tilde r_1' - \tilde k_2'), \\ \langle \tilde r_1' \tilde k_2'^{\tilde\eta'} \tilde k_3^{\tilde\eta''} | T | \tilde r_1 \tilde k_2^{\tilde\eta}\rangle &= \mathcal{A}(\tilde r_1 k_2^{\tilde\eta} \to \tilde r_1' \tilde k_2'^{\tilde\eta'} \tilde k_3^{\tilde\eta''}) \hat\delta^{(4)}(\tilde r_1 + \tilde k_2 - \tilde r_1' - \tilde k_2' - \tilde k_3).\end{aligned} \tag{4.39}$$

By crossing, we could choose to identify

$$\mathcal{A}(\tilde{r}_1\tilde{k}_2^{\tilde{\eta}}\tilde{k}_3^{\tilde{\eta}''} \to \tilde{r}'_1\tilde{k}_2'^{\tilde{\eta}'}) = \mathcal{A}(\tilde{r}_1\tilde{k}_2^{\tilde{\eta}} \to \tilde{r}'_1, \tilde{k}_2'^{\tilde{\eta}'}, (-\tilde{k}_3)^{-\tilde{\eta}''}). \tag{4.40}$$

Substituting these expressions into Eq. (4.8) and dropping tildes, we obtain

$$\begin{aligned} I^\mu_{w(1)}|_{g^3} = \int & d\Phi(p_1)d\Phi(p'_1)d\Phi(k_2)d\Phi(k'_2)d\Phi(k_3)\,\alpha_2^*(k'_2)\alpha_2(k_2)e^{-ib\cdot(p'_1-p_1)/\hbar}\phi_1(p_1)\phi_1^*(p'_1)i(p'_1-p_1)^\mu \\ & \times [\alpha_2(k_3)\mathcal{A}(p_1k_2^\eta k_3^\eta \to p'_1k_2'^\eta)\hat{\delta}^{(4)}(p_1+k_2+k_3-p'_1-k'_2) \\ & + \alpha_2^*(k_3)\mathcal{A}(p_1k_2^\eta \to p'_1k_2'^\eta k_3^\eta)\hat{\delta}^{(4)}(p_1+k_2-p'_1-k'_2-k_3)]. \end{aligned} \tag{4.41}$$

This $\mathcal{O}(g^3)$ term is interesting as it differs in structure from contributions to the impulse for massive-particle scattering studied in Ref. [122]. In that case, the first corrections arise at $\mathcal{O}(g^4)$, from one-loop amplitudes in $I^\mu_{(1)}$ and cut one-loop amplitudes in $I^\mu_{(2)}$. We leave an investigation of the new contributions (4.41) to future work.

Another difference between purely massive scattering and particle-on-wave scattering relates to the radiation reaction. In the massive case [122], radiation reaction first occurs at next-to-next-to-leading order, that is at $\mathcal{O}(g^6)$. In contrast, radiation reaction arises at $\mathcal{O}(g^4)$ in wave-particle scattering. This radiation reaction must contain contributions from the second term in the impulse, $I^\mu_{w(2)}$, which contributes at that order.

V. POINTLIKE OBSERVABLES

In the previous section, we built on Ref. [122] to analyze what we may call *global* observables, requiring an array of detectors covering the celestial sphere at infinity in order to measure the quantity. This is most manifest for the total radiated momentum, defined by Eq. (3.33) of Ref. [122],

$$R^\mu \equiv \langle k^\mu\rangle = {}_{\text{in}}\langle\psi|S^\dagger\mathbb{K}^\mu S|\psi\rangle_{\text{in}} = {}_{\text{in}}\langle\psi|T^\dagger\mathbb{K}^\mu T|\psi\rangle_{\text{in}}. \tag{5.1}$$

Even in electromagnetic scattering, achieving 4π coverage would make this a challenging measurement. In the gravitational context, where we would be looking to detect emission from scattering of distant black holes, such a measurement would be hopelessly impractical. Instead, for the remainder of this article, we turn to what we may call *local* observables, which can be measured with a localized detector, albeit still sitting somewhere on the celestial sphere, say at x. The paradigm for such a measurement is that of the waveform $W(t,\hat{\mathbf{n}};x)$ of radiation emitted during a scattering event in direction $\hat{\mathbf{n}}$ from an event at the coordinate origin. (That is, we adopt the convention that $-\hat{\mathbf{n}}$ points back from the observer towards the scattering event.) We will focus on electromagnetic radiation here, but much of the formalism will carry over to the gravitational case. Let us keep in mind that we will be interested in several detectors, all nearby x, though with separations that are completely negligible compared to the distance from the origin.

Local observables have a general structure which, as we will see, is determined by some source (the scattering event) and the propagation of messengers over very large distances. In fact it is convenient to break up our discussion of these observables along these lines. In the present section we will discuss this overall structure in more detail, with a focus on the crucial aspect of propagation. In the following sections, we will extract general expressions for local observables from quantum field theory, and connect to the Newman-Penrose formalism. Then we will examine global observables in cases where a classical wave scatters off a massive particle before turning to the physically important case where two massive particles scatter and radiate.

It will be easier to discuss and manipulate the Fourier transform of the waveform with respect to time. We will refer to this as the spectral waveform $f(\omega,\hat{\mathbf{n}};x)$:

$$f(\omega,\hat{\mathbf{n}};x) = \int_{-\infty}^{+\infty} dt\, W(t,\hat{\mathbf{n}};x)e^{i\omega t}. \tag{5.2}$$

Given a result for the spectral waveform, we can of course recover the time-dependent waveform via an inverse Fourier transform. Because we are interested in radiation produced by long-range forces, the idealized waveforms for the scattering processes we will consider stretch infinitely far back and forward in time. The idealization is implicit in the infinite limits for the integral in Eq. (5.2). In an actual measurement, however, the waveform would be below the noise floor of the detector for all times before a "signal start time" preceding the moment of closest approach, and likewise for all times after a "signal end time" following that moment. We can then take the theoretical waveforms to be approximations to actual ones cut off at the start and end times. Label the interval between the two by Δt_s.

Let us imagine that the point of closest approach during the scattering event is at the coordinate origin, $(t,\mathbf{x}) = (0,\mathbf{0})$. When a massless wave scatters off a point particle, the wave may overlap the particle; we take a suitable event of maximum overlap as the origin. We can treat the scattering as occurring in a box of temporal length

Δt_s, and of spatial size Δx_s. Radiation is emitted inside the box during the scattering event, and then spreads out. We will take an (idealized) measurement of the radiation in some direction $\hat{\mathbf{n}}$, at a much later time and at a point very far away in that direction. The details of the scattering—the particles' interaction and spins—will determine the radiation emitted inside the box. Modifying those details could radically change the emission. Those details, however, will have no effect on the propagation of the radiation out to the distant measuring apparatus. Only the spin of the radiated field can have any effect. We thus expect the form of the result to be a Green's function convoluted with a source. More precisely, given that we have only outgoing radiation, we expect a retarded Green's function $G_{\rm ret}$. We can then expand the Green's function in the large-distance limit to obtain the connection between the observable and the emitted radiation inside the box.

The details of the scattering inside the box around $(0, \mathbf{0})$ define a current for our radiation. In a real-world context, we are interested in electromagnetic or gravitational radiation, but we can equally well treat the case of (massless) scalar radiation as well. The details of the scattering inside the box give rise to a wave-number-space field-strength current, $\tilde{J}_{\vec{\mu}}(\bar{k})$, where the notation $\vec{\mu}$ denotes a number of indices appropriate to the radiated messenger: none for a scalar, two for a photon, and four for a graviton,

$$\begin{aligned} \tilde{J}(\bar{k})&: \text{ scalar}, \\ \tilde{J}_{\mu\nu}(\bar{k})&: \text{ electromagnetism}, \\ \tilde{J}_{\mu\nu\rho\sigma}(\bar{k})&: \text{ gravity}. \end{aligned} \tag{5.3}$$

In a slight abuse of language, we will refer to these quantities simply as currents. They will satisfy appropriate conservation conditions. We will later obtain an expression for such a current in terms of scattering amplitudes.

Given this current, the usual position-space current can of course be obtained by taking a Fourier transform,

$$J_{\vec{\mu}}(x) = \int \hat{d}^4\bar{k}\tilde{J}_{\vec{\mu}}(\bar{k})e^{-i\bar{k}\cdot x}. \tag{5.4}$$

Clearly we can also write $\tilde{J}_{\vec{\mu}}(\bar{k})$ in terms of $J_{\vec{\mu}}(x)$ via an inverse transform,

$$\tilde{J}_{\vec{\mu}}(\bar{k}) = \int d^4x J_{\vec{\mu}}(x)e^{i\bar{k}\cdot x}. \tag{5.5}$$

Both of these forms of the current will be helpful for us below.

As we will show in detail in the next section, we obtain an x-dependent radiation observable in the general form,

$$R_{\vec{\mu}}(x) = i\int d\Phi(\bar{k})[\tilde{J}_{\vec{\mu}}(\bar{k})e^{-i\bar{k}\cdot x} - \tilde{J}^*_{\vec{\mu}}(\bar{k})e^{+i\bar{k}\cdot x}], \tag{5.6}$$

that is, as an integral of the source $\tilde{J}_{\vec{\mu}}(\bar{k})$ over the on shell massless phase space for the radiated messenger. Examples will include expectations of Hermitian operators, such as the field-strength operator in electromagnetism or the Riemann tensor in gravity.

The Hermiticity properties of our radiation observables is manifest in Eq. (5.6). But notice that the observables are defined as integrals over positive frequencies $\bar{k}^t \geq 0$. Yet in writing the innocuous-seeming Fourier transform in Eq. (5.4), we have assumed knowledge of the current for both positive *and* negative frequency. So we must fill a gap: what do we mean by the current for negative frequency? In fact, the reality condition provides the necessary information. Our currents are real in position space, and we may note that

$$J_{\vec{\mu}}(x) = \int \hat{d}^4\bar{k}\theta(\bar{k}^t)[\tilde{J}_{\vec{\mu}}(\bar{k})e^{-i\bar{k}\cdot x} + \tilde{J}_{\vec{\mu}}(-\bar{k})e^{i\bar{k}\cdot x}]. \tag{5.7}$$

The reality condition then leads to the relation

$$\tilde{J}_{\vec{\mu}}(-\bar{k}) = \tilde{J}^*_{\vec{\mu}}(\bar{k}). \tag{5.8}$$

We use this relation to define the current for negative frequency.

A key simplification arises because the source event, occurring in our box, is sourced in a comparatively localized region compared to the very large propagation distance of the outgoing radiation. To access this simplification, we follow a well-trodden path [154] by rewriting our radiation observables as integrals over the spatial extent of the source. Thus, we express the observable of Eq. (5.6) in terms of the spatial current $J_{\vec{\mu}}(x)$, yielding

$$R_{\vec{\mu}}(x) = i\int d\Phi(\bar{k})d^4yJ_{\vec{\mu}}(y)[e^{-i\bar{k}\cdot(x-y)} - e^{+i\bar{k}\cdot(x-y)}]. \tag{5.9}$$

Next, we interchange orders of integration. Judicious forethought reveals the combination of phase space integrals to be a difference of retarded and advanced Green's functions,

$$R_{\vec{\mu}}(x) = \int d^4yJ_{\vec{\mu}}(y)[G_{\rm ret}(x-y) - G_{\rm adv}(x-y)]. \tag{5.10}$$

In the far future, where the observer measures the wave train emitted from the scattering event, $G_{\rm adv}$ will vanish. Put in an explicit form for $G_{\rm ret}$, and switch back to the wave-number-space current in order to make the complete dependence of the integrand on x and y manifest. The result is

$$R_{\vec{\mu}}(x) = \int \hat{d}\omega \hat{d}^3\bar{\boldsymbol{k}} d^4y\, \tilde{J}_{\vec{\mu}}(\bar{k}) e^{-i\bar{k}\cdot y} \frac{\delta(x^0 - y^0 - |\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$$
$$= \int \hat{d}\omega \hat{d}^3\bar{\boldsymbol{k}} d^3\boldsymbol{y}\, \tilde{J}_{\vec{\mu}}(\bar{k}) \frac{e^{-i\omega x^0} e^{+i\omega|\boldsymbol{x}-\boldsymbol{y}|} e^{+i\bar{\boldsymbol{k}}\cdot\boldsymbol{y}}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}. \quad (5.11)$$

Notice that the integral is now over *all* wave numbers. We have split the four-dimensional momentum integration into integrals over spatial and frequency components for later convenience.

From the earlier discussion, we know that $J_{\vec{\mu}}(y)$ is concentrated around $y \simeq 0$, whereas x is far away ($x \gg y$). Accordingly we can expand the integrand there, using

$$|\boldsymbol{x}-\boldsymbol{y}| \sim [\boldsymbol{x}^2 - 2\boldsymbol{x}\cdot\boldsymbol{y}]^{1/2}$$
$$\sim |\boldsymbol{x}|\left(1 - \frac{\hat{\boldsymbol{n}}\cdot\boldsymbol{y}}{|\boldsymbol{x}|}\right). \quad (5.12)$$

We must be careful in performing this expansion: while it is sufficient to retain the leading term in the denominator, we must retain formally subleading terms that contribute to nontrivial phases. Even in those exponents, we can of course still drop terms beyond the subleading, as they give rise to no nontrivial phases.

Substituting the expansion (5.12) into Eq. (5.11), we obtain

$$R_{\vec{\mu}}(x) = \int \hat{d}\omega \hat{d}^3\bar{\boldsymbol{k}} d^3\boldsymbol{y}\, \tilde{J}_{\vec{\mu}}(\bar{k}) \frac{e^{-i\omega x^0} e^{+i\omega|\boldsymbol{x}|} e^{-i\omega\hat{\mathbf{n}}\cdot\boldsymbol{y}} e^{+i\bar{\boldsymbol{k}}\cdot\boldsymbol{y}}}{4\pi|\boldsymbol{x}|}; \quad (5.13)$$

performing in turn the $\boldsymbol{y}$ and $\bar{\boldsymbol{k}}$ integrals, we finally obtain

$$R_{\vec{\mu}}(x) = \frac{(2\pi)^3}{4\pi|\boldsymbol{x}|} \int \hat{d}\omega \hat{d}^3\bar{\boldsymbol{k}}\, \tilde{J}_{\vec{\mu}}(\bar{k}) e^{-i\omega x^0} e^{+i\omega|\boldsymbol{x}|} \delta^3(\bar{\boldsymbol{k}} - \omega\hat{\mathbf{n}})$$
$$= \frac{1}{4\pi|\boldsymbol{x}|} \int \hat{d}\omega\, \tilde{J}_{\vec{\mu}}(\omega, \omega\hat{\mathbf{n}}) e^{-i\omega(x^0 - |\boldsymbol{x}|)}. \quad (5.14)$$

We can thus identify the waveform with the coefficient of the leading-power term $|\boldsymbol{x}|^{-1}$,

$$W_{\vec{\mu}}(t, \hat{\mathbf{n}}; x) = \frac{1}{4\pi} \int \hat{d}\omega \tilde{J}_{\vec{\mu}}(\omega, \omega\hat{\mathbf{n}}) e^{-i\omega(x^0 - |\boldsymbol{x}|)}. \quad (5.15)$$

In this equation, t represents the observer's clock time. We could take it to be x^0, or $x^0 - |\boldsymbol{x}|$, or some other convenient time. We must nonetheless retain the separate dependence on x^0 and $|\boldsymbol{x}|$, because these quantities will differ between the cluster of nearby observers in which we are interested. The relative phases between nearby observers are measurable.

Choosing $t = x^0 - |\boldsymbol{x}|$, the corresponding spectral waveform is then simply

$$f_{\vec{\mu}}(\omega, \hat{\mathbf{n}}) = \frac{1}{4\pi} \tilde{J}_{\vec{\mu}}(\omega, \omega\hat{\mathbf{n}}). \quad (5.16)$$

More precisely, Eq. (5.16) is the waveform for positive frequencies. For negative frequencies, the waveform follows from Eq. (5.8),

$$f_{\vec{\mu}}(\omega, \hat{\mathbf{n}}) = \frac{1}{4\pi} \tilde{J}^*_{\vec{\mu}}(-\omega, -\omega\hat{\mathbf{n}}). \quad (5.17)$$

We notice that $-\omega$ is now positive. In both cases, once we know the current $\tilde{J}_{\vec{\mu}}(\bar{k})$, we can immediately write down the spectral waveform.

VI. SPECTRAL WAVEFORMS

As we have seen, the waveform is directly related to the current $\tilde{J}_{\vec{\mu}}(\bar{k})$ generated by the scattering event. We must choose a specific local radiation observable to determine this current using its definition, Eq. (5.6). In this section we will study examples in both electrodynamics and gravity. Let us begin with a simple case: the field-strength tensor (3.5) in electrodynamics.

We choose an observer at x, in the far future of the event, equipped to measure the expectation value of the electric and magnetic field at the point x. The observable is therefore

$$\langle F^{\text{out}}_{\mu\nu}(x)\rangle \equiv {}_{\text{out}}\langle\psi|\mathbb{F}_{\mu\nu}(x)|\psi\rangle_{\text{out}}. \quad (6.1)$$

We can rewrite the outgoing state in terms of the incoming state using the time-evolution operator or S matrix,

$$\langle F^{\text{out}}_{\mu\nu}(x)\rangle = {}_{\text{in}}\langle\psi|S^\dagger\mathbb{F}_{\mu\nu}(x)S|\psi\rangle_{\text{in}}, \quad (6.2)$$

where (as usual) $|\psi\rangle_{\text{in}}$ is the incoming state in the far past. This state could contain, for example, two isolated massive pointlike particles, or a single isolated massive particle and a coherent state describing incoming radiation. A state of the former type would be appropriate to study radiation emitted as two particles scatter, while a state of the latter type can be used to study the scattered radiation field in a Thomson scattering process. We will study both of these examples in detail later in this article.

Inserting the expression for the field-strength tensor (3.5) into this expectation value, and converting to integrals over wave numbers, we learn that

$$\langle F^{\text{out}}_{\mu\nu}(x)\rangle$$
$$= -2i\hbar^{3/2}\sum_\eta \int d\Phi(\bar{k})[\langle\psi|S^\dagger a_{(\eta)}(\bar{k})S|\psi\rangle \bar{k}_{[\mu}\varepsilon^{(\eta)*}_{\nu]}(\bar{k})e^{-i\bar{k}\cdot x}$$
$$- \langle\psi|S^\dagger a^\dagger_{(\eta)}(\bar{k})S|\psi\rangle \bar{k}_{[\mu}\varepsilon^{(\eta)}_{\nu]}(\bar{k})e^{+i\bar{k}\cdot x}], \quad (6.3)$$

where we have again dropped the "in" subscript, leaving it implicit in the rest of our discussion. (Recall that k is just a label for the creation and annihilation operators, and we can use $\bar{k}$ interchangeably for this purpose.)

We now see the virtue of our definition of the general class of radiation observables in Eq. (5.6). Evidently the expectation value $\langle F^{\rm out}_{\mu\nu}(x)\rangle$ is of precisely this form, and we can read off the current $\tilde{J}_{\vec{\mu}}(\bar{k})$ as

$$\tilde{J}_{\mu\nu}(\bar{k}) = -2\hbar^{3/2}\sum_\eta \langle\psi|S^\dagger a_{(\eta)}(\bar{k})S|\psi\rangle \bar{k}_{[\mu}\varepsilon^{(\eta)*}_{\nu]}(\bar{k}). \quad (6.4)$$

The discussion of the previous section therefore applies, and we see from Eq. (5.16) that the corresponding spectral waveform is

$$f_{\mu\nu}(\omega,\hat{\mathbf{n}}) = -\frac{1}{2\pi}\hbar^{3/2}\sum_\eta \langle\psi|S^\dagger a_{(\eta)}(\bar{k})S|\psi\rangle \bar{k}_{[\mu}\varepsilon^{(\eta)*}_{\nu]}(\bar{k})\Big|_{\bar{k}=(\omega,\omega\hat{\mathbf{n}})}, \quad (6.5)$$

for positive frequency ($\omega > 0$). For negative frequency ($\omega < 0$) the waveform is

$$f_{\mu\nu}(\omega,\hat{\mathbf{n}}) = -\frac{1}{2\pi}\hbar^{3/2}\sum_\eta \langle\psi|S^\dagger a^\dagger_{(\eta)}(\bar{k})S|\psi\rangle \bar{k}_{[\mu}\varepsilon^{(\eta)}_{\nu]}(\bar{k})\Big|_{\bar{k}=-(\omega,\omega\hat{\mathbf{n}})}. \quad (6.6)$$

This result holds to all orders in perturbation theory.

It is straightforward to extend this result to gravity. We work in Einstein gravity, and assume that the spacetime is asymptotically Minkowskian. In this case our observer at x is very far from the source of gravitational waves, and is equipped to measure the expectation value of the local spacetime curvature $\langle R^{\rm out}_{\mu\nu\rho\sigma}(x)\rangle$. The corresponding spectral waveform is nothing but the double copy of Eq. (6.5),

$$f_{\mu\nu\rho\sigma}(\omega,\hat{\mathbf{n}}) = \frac{i\kappa}{2\pi}\hbar^{3/2}\sum_\eta \langle\psi|S^\dagger a_{(\eta)}(\bar{k})S|\psi\rangle \bar{k}_{[\mu}\varepsilon^{(\eta)*}_{\nu]}(\bar{k})\bar{k}_{[\rho}\varepsilon^{(\eta)*}_{\sigma]}(\bar{k})\Big|_{\bar{k}=(\omega,\omega\hat{\mathbf{n}})}, \quad (6.7)$$

for $\omega > 0$. In this equation, the operator $a_{(\eta)}(k)$ annihilates perturbative gravitational states. We have included a factor $\kappa/2$ so that the Riemann tensor has the conventional normalization. Noting that the metric perturbation falls off as inverse distance, it follows that nonlinear terms in the Riemann tensor produce corrections which fall off faster than inverse distance. Consequently, we have neglected them. Notice that all possible traces of Eq. (6.7) vanish, consistent with the fact that the Riemann tensor in vacuum equals the Weyl tensor. The waveform for negative frequency is

$$f_{\mu\nu\rho\sigma}(\omega,\hat{\mathbf{n}}) = -\frac{i\kappa}{2\pi}\hbar^{3/2}\sum_\eta \langle\psi|S^\dagger a^\dagger_{(\eta)}(\bar{k})S|\psi\rangle \bar{k}_{[\mu}\varepsilon^{(\eta)}_{\nu]}(\bar{k})\bar{k}_{[\rho}\varepsilon^{(\eta)}_{\sigma]}(\bar{k})\Big|_{\bar{k}=-(\omega,\omega\hat{\mathbf{n}})}. \quad (6.8)$$

The Lorentz indices on these observables reflects the tensor structure of electrodynamics and gravity. In both cases, however, there are only two possible polarizations of the outgoing radiation. It is helpful to project the waveform onto one of these polarizations. Classically, a convenient way to do so is to use the Newman-Penrose (NP) [123] formalism, which is intimately connected to the spinor-helicity method of scattering amplitudes [38,39,143]. We can adopt the same idea in the present context. For us, a simple route to the NP formalism is to pick a complex basis of vectors which is aligned with our setup. We choose the vectors[6]

$$\begin{aligned} &L^\mu = \bar{k}^\mu/\omega = (1,\hat{\mathbf{n}})^\mu, \qquad N^\mu = \zeta^\mu, \\ &M^\mu = \varepsilon^{(+)\mu}, \qquad M^{*\mu} = \varepsilon^{(-)\mu}. \end{aligned} \quad (6.9)$$

The null vector ζ is simply a gauge choice, satisfying $\zeta\cdot\varepsilon^{(\pm)} = 0$ and $L\cdot N = L\cdot\zeta = 1$. Furthermore note that $M\cdot M^* = -1$. The scaling of the NP vector L ensures that it does not depend on frequency ω, and is dimensionless. Indeed the polarization vectors $\varepsilon^{(\pm)}$ do not depend on the scaling of $\bar{k}$ so they are also independent of frequency. These vectors therefore make sense as a spacetime basis, not merely as a basis in Fourier space.

It is easy to check that the only nonzero components of $f_{\mu\nu}$ in the NP basis are $f_{\mu\nu}M^{*\mu}N^\nu$ and $f_{\mu\nu}M^\mu N^\nu$. These are the leading radiative NP scalar, traditionally [155] denoted Φ^0_2, and its conjugate. We can write these NP scalars as Fourier transforms:

$$\Phi^0_2(t,\hat{\mathbf{n}}) = \int \hat{d}\omega\, e^{-i\omega t}\tilde{\Phi}^0_2(\omega,\hat{\mathbf{n}}). \quad (6.10)$$

Notice that we commuted the NP basis vectors through the frequency integration sign. This is permissible as the basis

[6] We use capital letters to denote the elements of our NP basis rather than the more traditional lowercase symbols in order to distinguish the vectors from loop momenta, masses, et cetera.

vectors are independent of frequency. For positive frequency ω, we find

$$\tilde{\Phi}_2^0(\omega,\hat{\mathbf{n}}) = -\frac{\omega}{4\pi}\hbar^{3/2}\langle\psi|S^\dagger a_{(-)}(\bar{k})S|\psi\rangle\Big|_{\bar{k}=(\omega,\omega\hat{\mathbf{n}})}, \quad (6.11)$$

while for negative frequency, the corresponding expression reads

$$\tilde{\Phi}_2^0(\omega,\hat{\mathbf{n}}) = +\frac{\omega}{4\pi}\hbar^{3/2}\langle\psi|S^\dagger a^\dagger_{(+)}(\bar{k})S|\psi\rangle\Big|_{\bar{k}=-(\omega,\omega\hat{\mathbf{n}})}. \quad (6.12)$$

Combining these results, we find that the time-domain NP scalar is

$$\begin{aligned}\Phi_2^0(t,\hat{\mathbf{n}}) = &-\frac{\hbar^{3/2}}{4\pi}\int \hat{d}\omega\,\Theta(\omega)\\ &\times\omega[e^{-i\omega t}\langle\psi|S^\dagger a_{(-)}(\bar{k})S|\psi\rangle\\ &+e^{+i\omega t}\langle\psi|S^\dagger a^\dagger_{(+)}(-\bar{k})S|\psi\rangle]\Big|_{\bar{k}=(\omega,\omega\hat{\mathbf{n}})}.\end{aligned} \quad (6.13)$$

In gravity, the corresponding radiative NP scalar is defined by

$$\Psi_4(x) = -N_\mu M^*_\nu N_\rho M^*_\sigma\langle W^{\mu\nu\rho\sigma}(x)\rangle, \quad (6.14)$$

where $W^{\mu\nu\rho\sigma}(x)$ is the Weyl tensor, equal to the Riemann tensor in our case. Expanded at large distances, the leading term in the NP scalar is Ψ_4^0:

$$\Psi_4(x) = \frac{1}{|\boldsymbol{x}|}\Psi_4^0 + \cdots. \quad (6.15)$$

This object is directly relevant to gravitational waveforms [5,156]. We find that the spectral version of the NP scalar is

$$\tilde{\Psi}_4^0(\omega,\hat{\mathbf{n}}) = -i\frac{\kappa\omega^2}{8\pi}\hbar^{3/2}\langle\psi|S^\dagger a_{(--)}(\bar{k})S|\psi\rangle\Big|_{\bar{k}=(\omega,\omega\hat{\mathbf{n}})} \quad (6.16)$$

for positive ω. Let us emphasize once again that these results hold to all orders of perturbation theory.

NP scalars are particularly well-suited for comparison with helicity amplitudes in quantum field theory. However, they may be slightly less familiar than the more elementary field strengths; field strengths also have the virtue of being Hermitian quantities. Therefore, in the remainder of this article, we will also study the expectation of the radiative field-strength tensor in perturbation theory. This entails rewriting the scattering matrix in terms of the transition matrix T, $S = 1 + iT$,

$$\begin{aligned}\langle F^{\text{out}}_{\mu\nu}(x)\rangle &= \langle\psi|(1-iT^\dagger)\mathbb{F}_{\mu\nu}(x)(1+iT)|\psi\rangle\\ &= \langle\psi|\mathbb{F}_{\mu\nu}(x)|\psi\rangle + 2\text{Re}\,i\langle\psi|\mathbb{F}_{\mu\nu}(x)T|\psi\rangle\\ &\quad+\langle\psi|T^\dagger\mathbb{F}_{\mu\nu}(x)T|\psi\rangle.\end{aligned} \quad (6.17)$$

The first term in Eq. (6.17) is the expectation value of the field strength due to any incoming radiation which may be present in $|\psi\rangle_{\text{in}}$; the following term is linear in amplitudes, and thus of $\mathcal{O}(g^3)$ (or higher); the last term is quadratic in amplitudes (or equivalently, linear in a cut amplitude), and contains terms of $\mathcal{O}(g^5)$ and higher.

Using unitarity, we can rewrite Eq. (6.17),

$$\begin{aligned}\langle F^{\text{out}}_{\mu\nu}(x)\rangle(x) = &\langle\psi|\mathbb{F}_{\mu\nu}(x)|\psi\rangle + i\langle\psi|[\mathbb{F}_{\mu\nu}(x),T]|\psi\rangle\\ &+\langle\psi|T^\dagger[\mathbb{F}_{\mu\nu}(x),T]|\psi\rangle.\end{aligned} \quad (6.18)$$

The commutator in the second term of this expression is reminiscent of the form of the impulse Δp (although in case of the field strength, the first term above need not vanish). This second form of the field strength can be both instructive and useful, but it has a slight disadvantage that reality properties are somewhat obscured compared to Eq. (6.17). When taking the classical limit, we are interested in the leading term in the large-distance expansion as well; for such radiation observables, we will understand the $\langle\!\langle\cdots\rangle\!\rangle$ notation to impose that expansion as well.

We will use this observable to analyze emitted radiation in the scattering of two charged particles in Sec. VIII. We first continue our analysis of Thomson scattering in the next section.

VII. FROM COMPTON SCATTERS TO THOMSON SCATTERING

In Sec. IV C, we considered the Thomson scattering process: electromagnetic scattering of a classical beam off of a massive point charge. In our earlier discussion we studied the impulse suffered by the massive particle during the process. We are now equipped to deepen our analysis by determining the scattered light generated during Thomson scattering as shown schematically in Fig. 4. We will do so by using the results of the previous section to compute the NP scalar Φ_2^0, which describes that scattered light at very large distances.

In this situation, our initial state Eq. (4.5) describes an isolated massive particle, and a localized beam of incoming classical radiation described as in Sec. III C by a coherent state with an appropriate wave shape function. Correspondingly, the incoming state generates a nonvanishing expectation value for the electromagnetic field strength tensor. This is the incoming classical radiation $\langle F^{\text{in}}_{\mu\nu}(x)\rangle$:

$$\langle F^{\text{in}}_{\mu\nu}(x)\rangle = \langle\psi_w|\mathbb{F}_{\mu\nu}|\psi_w\rangle. \quad (7.1)$$

In particular, there is a nonvanishing NP scalar Φ_2^0 in the far past.

To focus attention on the scattered light, it is convenient to study the overall change in the NP scalar during the process,

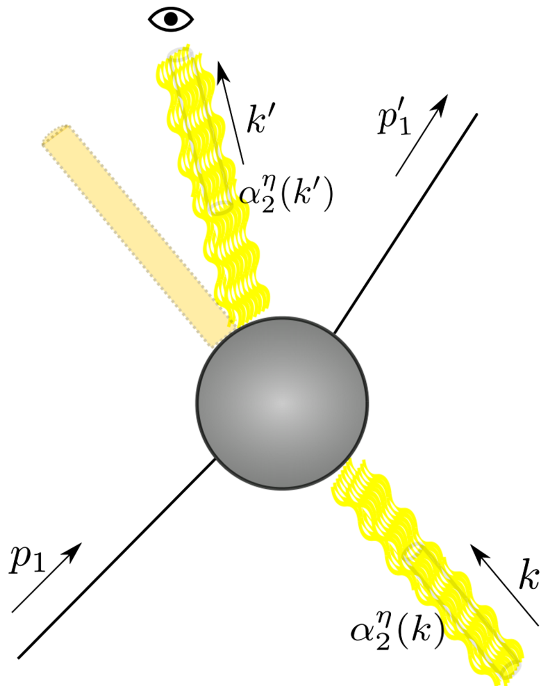

FIG. 4. The observer measures the field strength of the outgoing wave.

$$\Delta\Phi_2^0(\omega,\hat{\mathbf{n}}) = -\frac{\omega}{4\pi}\hbar^{3/2}[\langle\psi_w|S^\dagger a_{(-)}(\bar{k})S|\psi_w\rangle - \langle\psi_w|a_{(-)}(\bar{k})|\psi_w\rangle]\Big|_{\bar{k}=(\omega,\omega\hat{\mathbf{n}})}. \tag{7.2}$$

This simply subtracts the contribution of the incoming beam to the radiation field in the future. We will compute this quantity at leading order, focusing on the positive-frequency part throughout.

Using unitarity of the S matrix, we may write $\Delta\Phi_2^0$ in terms of a commutator,

$$\Delta\Phi_2^0(\omega,\hat{\mathbf{n}}) = -\frac{i}{4\pi}\omega\hbar^{3/2}\langle\psi_w|[a_{(-)}(\bar{k}_2'),T]|\psi_w\rangle\Big|_{\bar{k}_2'=(\omega,\omega\hat{\mathbf{n}})}. \tag{7.3}$$

We relabeled the quantity $\bar{k}$ appearing in Eq. (7.2) as $\bar{k}_2'$ because, as we will see below, it has the interpretation of the wave vector associated with the outgoing wave which was denoted $\bar{k}_2'$ in Sec. IV.

To compute the commutator $[a_{(-)}(k_2'),T]$, we make use of Eq. (4.10) to expand the T matrix in terms of creation and annihilation operators. Dropping the terms in the ellipsis of Eq. (4.10), the commutator is easily computed to be

$$[a_{(-)}(k_2'),T] = \sum_{\tilde{\eta}}\int d\Phi(\tilde{r}_1,\tilde{r}_1',\tilde{k}_2)\langle\tilde{r}_1'k_2'^{-}|T|\tilde{r}_1\tilde{k}_2^{\tilde{\eta}}\rangle \times a^\dagger(\tilde{r}_1')a(\tilde{r}_1)a_{(\tilde{\eta})}(\tilde{k}_2). \tag{7.4}$$

Inserting this result in Eq. (7.3), and expanding the state $|\psi_w\rangle$ using its definition (4.5) specialized to the case $b=0$ we easily find that

$$\begin{aligned}\Delta\Phi_2^0(\omega,\hat{\mathbf{n}}) &= -\frac{i}{4\pi}\omega\hbar^{3/2}\sum_\eta\int d\Phi(p_1,p_1',k_2)\phi^*(p_1')\phi(p_1)\langle p_1'k_2'^{-}|T|p_1k_2^\eta\rangle\langle\alpha^+|a_{(\eta)}(k_2)|\alpha^+\rangle \\ &= -\frac{i}{4\pi}\omega\hbar^{3/2}\int d\Phi(p_1,p_1',k_2)\phi^*(p_1')\phi(p_1)\langle p_1'k_2'^{-}|T|p_1k_2^+\rangle\alpha(k_2).\end{aligned} \tag{7.5}$$

The matrix element of the transition operator yields the Compton amplitude, as well as the usual delta function enforcing overall momentum conservation. We may perform the p_1' integral using this delta function to find that

$$\Delta\Phi_2^0(\omega,\hat{\mathbf{n}}) = -\frac{i}{4\pi}\omega\hbar^{3/2}\int d\Phi(p_1,k_2)\,\hat{\delta}(2p_1\cdot(k_2'-k_2))|\phi(p_1)|^2\alpha(k_2)\mathcal{A}(p_1k_2^+\to p_1'k_2'^{-}). \tag{7.6}$$

We replaced the (conjugated) wave function $\phi^*(p_1'+k_2-k_2')$ by $\phi^*(p_1')$ because the difference $(k_2-k_2')/\hbar = \bar{k}_2-\bar{k}_2'$ is small (of order $1/\lambda$) compared to the width of the wave function (which is of order $1/\ell_w$). The integral over the wave function is now precisely of the form required for the double-angle-bracket notation of Ref. [122] so that we arrive at

$$\Delta\Phi_2^0(\omega,\hat{\mathbf{n}}) = -\frac{i}{4\pi}\left\langle\!\!\left\langle\omega\hbar^{3/2}\int d\Phi(k_2)\,\hat{\delta}(2p_1\cdot(k_2'-k_2))\alpha(k_2)\mathcal{A}(p_1k_2^+\to p_1'k_2'^{-})\right\rangle\!\!\right\rangle. \tag{7.7}$$

Finally, we insert the explicit Compton amplitude of Eq. (4.19), and replace the remaining integral over k_2 with an integral over the associated wave number $\bar{k}_2$ to learn that the LO NP scalar due to the scattering process is

$$\Delta\Phi_2^0(\omega,\hat{\mathbf{n}}) = i\frac{Q^2e^2}{16\pi}\left\langle\!\!\left\langle \omega\int d\Phi(\bar{k}_2)\,\hat{\delta}(2p_1\cdot(\bar{k}_2'-\bar{k}_2))\bar{\alpha}(\bar{k}_2)m^2\frac{\langle\bar{k}_2\bar{k}_2'\rangle}{[\bar{k}_2\bar{k}_2']\bar{k}_2\cdot p_1}\right\rangle\!\!\right\rangle. \tag{7.8}$$

The same result would also be obtained from a classical analysis of the leading order radiation field of a point charge moving under the influence of an incoming classical wave.

Alternatively, it is possible to compute the expectation value of the field strength in the very far future. Focusing again on the change in the field strength,

$$\langle\Delta F_{\mu\nu}(x)\rangle \equiv \langle F_{\mu\nu}^{\text{out}}(x)\rangle - \langle F_{\mu\nu}^{\text{in}}(x)\rangle, \tag{7.9}$$

it is straightforward to use Eq. (6.17) and find that

$$\langle\Delta F_{\mu\nu}(x)\rangle = i\langle\psi_w|[\mathbb{F}_{\mu\nu}(x),T]|\psi_w\rangle + \cdots. \tag{7.10}$$

We have indicated higher order terms are present in the ellipsis. It may be worth emphasizing once again that this result is the same as one would find to be a direct computation using background field methods:

$$\begin{aligned}\langle\Delta F^{\mu\nu}(x)\rangle &= \langle\psi_w|S^\dagger\mathbb{F}^{\mu\nu}(x)S|\psi_w\rangle - \langle\psi_w|\mathbb{F}^{\mu\nu}(x)|\psi_w\rangle\\ &= \int d\Phi(p_1)d\Phi(p_1)\phi(p_1)\phi^*(p_1')e^{-ib\cdot(p_1'-p_1)/\hbar}\{i\langle p_1'|[\mathbb{F}^{\mu\nu}(x),T(A_{\text{cl}}^{(\eta)})]|p_1\rangle + \langle p_1'|T^\dagger(A_{\text{cl}}^{(\eta)})\mathbb{F}^{\mu\nu}(x)T(A_{\text{cl}}^{(\eta)})|p_1\rangle\},\end{aligned} \tag{7.11}$$

where $A_{\text{cl}}^{(\eta)}(x)$ denotes the classical background field corresponding to our coherent state, and we once again used the relation $\mathbb{C}_{\alpha,(\eta)}^\dagger\mathbb{C}_{\alpha,(\eta)} = \mathbb{1}$.

Returning to the LO computation of the scattered field strength, by inserting the definition of the field strength operator, we now encounter two commutators:

$$\langle\Delta F_{\mu\nu}(x)\rangle = \frac{2}{\hbar^{3/2}}\sum_{\eta'}\int d\Phi(k')[\langle\psi_w|[a_{(\eta')}(\bar{k}'),T]|\psi_w\rangle\bar{k}'_{[\mu}\varepsilon_{\nu]}^{(\eta')*}(k')e^{-ik'\cdot x/\hbar} - \langle\psi_w|[a_{(\eta')}^\dagger(\bar{k}'),T]|\psi_w\rangle\bar{k}'_{[\mu}\varepsilon_{\nu]}^{(\eta')}(k')e^{+ik'\cdot x/\hbar}]. \tag{7.12}$$

The first of these was computed explicitly above; the second is very similar. After a short computation, the field strength can be expressed as

$$\langle\Delta F_{\mu\nu}(x)\rangle = \text{Re}\left\langle\!\!\left\langle 4g^2\sum_{\eta'}\int d\Phi(\bar{k}_2,\bar{k}_2')\,\hat{\delta}(2p_1\cdot(\bar{k}_2'-\bar{k}_2))\bar{\alpha}(\bar{k}_2)\bar{\mathcal{A}}(p_1k_2^+\to p_1'k'^{\eta'})\bar{k}'_{2[\mu}\varepsilon_{\nu]}^{(\eta')*}(\bar{k}_2')e^{-i\bar{k}_2'\cdot x}\right\rangle\!\!\right\rangle. \tag{7.13}$$

Comparison with the NP scalar is facilitated by performing the $\bar{k}_2'$ integral using the methods of Sec. V. Indeed, the field strength change of Eq. (7.13) is of the general form of the radiation observable Eq. (5.6). The corresponding current is

$$\tilde{J}_{\mu\nu}(\bar{k}_2) = -4i\left\langle\!\!\left\langle\sum_{\eta'}\int d\Phi(\bar{k}_2')\,\hat{\delta}(2p_1\cdot(\bar{k}_2'-\bar{k}_2))\bar{\alpha}(\bar{k}_2)\bar{\mathcal{A}}(p_1k_2^+\to p_1'k_2'^{\eta'})\bar{k}'_{2[\mu}\varepsilon_{\nu]}^{(\eta')*}(\bar{k}_2')\right\rangle\!\!\right\rangle. \tag{7.14}$$

The NP scalar can be obtained directly from this current as

$$\Delta\Phi_2^0(\omega,\hat{\mathbf{n}}) = \frac{1}{4\pi}\tilde{J}_{\mu\nu}(\bar{k})M^{*\mu}N^\nu. \tag{7.15}$$

Performing the dot products, we recover our earlier result, Eq. (7.8).

Earlier, we identified incoming classical radiation with coherent states. The reader may wonder then about the nature of outgoing radiation. A necessary condition for the outgoing radiation to be represented by a coherent state is that expectation values of observables, such as the field strength, should factorize. We have proved this explicitly earlier, see Eq. (3.22).

Perhaps surprisingly, it turns out that this is also a sufficient condition. Indeed, one can work out the constraints on the probability density of the outgoing (pure) radiation: in the coherent state space (also called the Glauber-Sudarshan representation), the classical factorization of observables implies that the distribution has zero variance. In turn, this makes the distribution degenerate, i.e., supported on isolated points. But as shown by Hillery [157], the normalization condition together with the purity constraint suffices to reduce the sum of delta functions in the coherent state space to just a single delta function. That is, we have only a single outgoing coherent state in the classical limit. In Appendix B, we prove that the factorization condition holds at the lowest order in the coupling constant, which makes the outgoing radiation state of the Thomson scattering coherent up to order g^2. A more detailed discussion on this point will appear in forthcoming work [158].

VIII. EMISSION WAVEFORM

We turn now to photon emission in the scattering of two charged point particles. At leading order in perturbation theory, only the second term in Eq. (6.17) [or similarly, in Eq. (6.18)] contributes. It will be of order $\mathcal{O}(g^3)$, whereas the second term will be of $\mathcal{O}(g^5)$.

If we now substitute the expression (3.5), along with that (2.14) for the initial-state wave function for the scattering particles into the first term of Eq. (6.17), then we obtain

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle_1 &= \frac{4}{\hbar^{3/2}}\mathrm{Re}\sum_\eta\int d\Phi(p_1)d\Phi(p_2)d\Phi(p_1')d\Phi(p_2')d\Phi(k)\\&\quad\times e^{-ib\cdot(p_1'-p_1)/\hbar}\phi(p_1)\phi^*(p_1')\phi(p_2)\phi^*(p_2')k^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-ik\cdot x/\hbar}\langle p_1'p_2'|a_{(\eta)}(k)T|p_1p_2\rangle\\&= \frac{4}{\hbar^{3/2}}\mathrm{Re}\sum_\eta\int d\Phi(p_1)d\Phi(p_2)d\Phi(p_1')d\Phi(p_2')d\Phi(k)\\&\quad\times e^{-ib\cdot(p_1'-p_1)/\hbar}\phi(p_1)\phi^*(p_1')\phi(p_2)\phi^*(p_2')k^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-ik\cdot x/\hbar}\langle p_1'p_2'k^\eta|T|p_1p_2\rangle. \end{aligned}\tag{8.1}$$

We can identify the matrix element as a five-point amplitude,

$$\langle p_1'p_2'k^\eta|T|p_1p_2\rangle = \mathcal{A}(p_1,p_2\to p_1',p_2',k^\eta)\hat\delta^{(4)}(p_1+p_2-p_1'-p_2'-k). \tag{8.2}$$

At leading order, we replace the amplitude by its LO contribution, given by a tree-level expression. To compute the required waveform, we must identify the expectation of $F^{\mu\nu}(x)$ as the spatial current $J_{\vec\mu}(x)$ in Eqs. (5.4) and (5.5), and via Eq. (5.5) in Eq. (5.15).

Beyond leading order, the expectation of $F^{\mu\nu}(x)$ will receive higher-order contributions to the amplitudes in Eq. (8.2), alongside contributions from the last term in Eq. (6.18),

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle_2 &= -\frac{2i}{\hbar^{3/2}}\sum_\eta\int d\Phi(p_1)d\Phi(p_2)d\Phi(p_1')d\Phi(p_2')d\Phi(k)\,e^{-ib\cdot(p_1'-p_1)/\hbar}\phi(p_1)\phi^*(p_1')\phi(p_2)\phi^*(p_2')\\&\quad\times[k^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-ik\cdot x/\hbar}\langle p_1'p_2'|T^\dagger a_{(\eta)}(k)T|p_1p_2\rangle - k^{[\mu}\varepsilon^{(\eta)\nu]}e^{+ik\cdot x/\hbar}\langle p_1'p_2'|T^\dagger a^\dagger_{(\eta)}(k)T|p_1p_2\rangle].\end{aligned}\tag{8.3}$$

We insert a complete set of states to the right of each $T^\dagger$,

$$\langle\psi|T^\dagger\mathbb{F}^{\mu\nu}T|\psi\rangle = \sum_X\int d\Phi(r_1)d\Phi(r_2)\langle\psi|T^\dagger|r_1r_2X\rangle\langle r_1r_2X|\mathbb{F}^{\mu\nu}T|\psi\rangle, \tag{8.4}$$

where the sum over X is over all states, including no additional particles, and includes an implicit integral over momenta of any particles in X and a sum over any other quantum numbers. As in Ref. [122], we assume that each of the incoming massive particles carries a separately conserved global charge, so that each intermediate state has one net particle of each type. We can ignore additional particle-antiparticle pairs of the massive particles, as these contributions will disappear in the classical limit. As there are no messengers in the initial state, and hence no coherent states, there is no need to sum over arbitrary numbers of messengers. Accordingly, we do not need to switch to a coherent-friendly representation (4.10) of the T matrix. We obtain

$$
\begin{aligned}
\langle F^{\mu\nu}(x)\rangle_2 = &-\frac{2i}{\hbar^{3/2}}\sum_X\sum_\eta\int d\Phi(r_1)d\Phi(r_2)d\Phi(p_1)d\Phi(p_2)d\Phi(p'_1)d\Phi(p'_2)d\Phi(k)e^{-ib\cdot(p'_1-p_1)/\hbar}\phi(p_1)\phi^*(p'_1)\phi(p_2)\phi^*(p'_2)\\
&\times[k^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-ik\cdot x/\hbar}\langle p'_1,p'_2|T^\dagger|r_1r_2X\rangle\langle r_1r_2X|a_{(\eta)}(k)T|p_1p_2\rangle\\
&-k^{[\mu}\varepsilon^{(\eta)\nu]}e^{+ik\cdot x/\hbar}\langle p'_1,p'_2|T^\dagger|r_1r_2X\rangle\langle r_1r_2X|a^\dagger_{(\eta)}(k)T|p_1p_2\rangle]\\
=&-\frac{2i}{\hbar^{3/2}}\sum_X\sum_\eta\int d\Phi(r_1)d\Phi(r_2)d\Phi(p_1)d\Phi(p_2)d\Phi(p'_1)d\Phi(p'_2)d\Phi(k)e^{-ib\cdot(p'_1-p_1)/\hbar}\phi(p_1)\phi^*(p'_1)\phi(p_2)\phi^*(p'_2)\\
&\times[k^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-ik\cdot x/\hbar}\langle p'_1,p'_2|T^\dagger|r_1r_2X\rangle\langle r_1r_2k^\eta X|T|p_1p_2\rangle\\
&-k^{[\mu}\varepsilon^{(\eta)\nu]}e^{+ik\cdot x/\hbar}\langle p'_1p'_2|T^\dagger|r_1r_2k^\eta X\rangle\langle r_1r_2X|T|p_1p_2\rangle].
\end{aligned}
\tag{8.5}
$$

In the second term within brackets, the creation operator requires a photon in the intermediate state and eliminates it from the bra. We then relabeled X to exclude it. Note as well that at next-to-next-leading order and beyond, we necessarily require amplitudes with three incoming particles. These can just as easily be obtained by crossing. The term (8.5) has the interpretation of a cut of an amplitude, just as for the second term in the impulse in Ref. [122], as seen in Eqs. (3.26)–(3.31) therein.

The contribution of Eq. (8.5) first appears at next-to-leading order. At this order, we are interested in contributions with $X=\varnothing$, and we can identify the required matrix elements as a combination of four- and five-point amplitudes:

$$
\begin{aligned}
\langle r_1r_2|T|p_1p_2\rangle &= \mathcal{A}(p_1p_2\to r_1r_2)\hat{\delta}^{(4)}(p_1+p_2-r_1-r_2),\\
\langle p'_1,p'_2|T^\dagger|r_1r_2\rangle &= \mathcal{A}^*(p'_1,p'_2\to r_1,r_2)\hat{\delta}^{(4)}(p'_1+p'_2-r_1-r_2),\\
\langle r_1r_2k^\eta|T|p_1p_2\rangle &= \mathcal{A}(p_1,p_2\to r_1,r_2,k^\eta)\hat{\delta}^{(4)}(p_1+p_2-r_1-r_2-k),\\
\langle p'_1p'_2|T^\dagger|r_1r_2k^\eta\rangle &= \mathcal{A}^*(p'_1,p'_2\to r_1,r_2,k^\eta)\hat{\delta}^{(4)}(p'_1+p'_2-r_1-r_2-k).
\end{aligned}
\tag{8.6}
$$

For the next-to-leading order contribution to $\langle F^{\mu\nu}(x)\rangle$, we use tree-level amplitudes in Eq. (8.6).

IX. THE DETECTED WAVE AT LEADING ORDER

The leading-order contribution to the waveform will arise at $\mathcal{O}(g^3)$, as described in the previous section. We apply the approach of Ref. [122] to Eq. (8.1). Similar to that reference, and to Sec. IV, we define the momentum mismatches

$$
\begin{aligned}
q_1 &= p'_1-p_1,\\
q_2 &= p'_2-p_2,
\end{aligned}
\tag{9.1}
$$

and trade the integrals over the p'_i for integrals over the q_i,

$$
\begin{aligned}
\langle F^{\mu\nu}(x)\rangle_1 = &\frac{4}{\hbar^{3/2}}\mathrm{Re}\sum_\eta\int d\Phi(p_1)d\Phi(p_2)\hat{d}^4q_1\hat{d}^4q_2d\Phi(k)\,\hat{\delta}(2p_1\cdot q_1+q_1^2)\,\hat{\delta}(2p_2\cdot q_2+q_2^2)\\
&\times e^{-ib\cdot q_1/\hbar}\Theta(p_1^t+q_1^t)\Theta(p_2^t+q_2^t)\phi(p_1)\phi^*(p_1+q_1)\phi(p_2)\phi^*(p_2+q_2)\\
&\times k^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-ik\cdot x/\hbar}\mathcal{A}(p_1,p_2\to p_1+q_1,p_2+q_2,k^\eta)\hat{\delta}^{(4)}(q_1+q_2+k).
\end{aligned}
\tag{9.2}
$$

We can take the classical limit, and change to the required wave number variables for the q_i and k,

$$
\begin{aligned}
\langle F^{\mu\nu}(x)\rangle_{1,\mathrm{cl}} = g^3\bigg\langle\!\!\bigg\langle &\hbar^2\mathrm{Re}\sum_\eta\int d\Phi(\bar{k})\bar{k}^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-i\bar{k}\cdot x}\\
&\times\int\prod_{i=1,2}\hat{d}^4\bar{q}_i\hat{\delta}(p_i\cdot\bar{q}_i)e^{-ib\cdot\bar{q}_1}\hat{\delta}^{(4)}(\bar{q}_1+\bar{q}_2+\bar{k})\bar{\mathcal{A}}(p_1,p_2\to p_1+\hbar\bar{q}_1,p_2+\hbar\bar{q}_2,\hbar\bar{k}^\eta)\bigg\rangle\!\!\bigg\rangle.
\end{aligned}
\tag{9.3}
$$

We have also extracted powers of $\hbar$ from the coupling, and dropped the $\hbar$-suppressed terms inside the on shell delta functions as well as the positive-energy theta functions. We recognize the inner integral in the second term as the radiation kernel defined in Eq. (4.42) of Ref. [122] (after changing variables there $p_i\to p_i-\hbar\bar{w}_i$ and $\bar{w}_i\to-\bar{q}_i$),

$$\mathcal{R}^{(0)}(\bar{k}^\eta;b) \equiv \hbar^2 \int \prod_{i=1,2} \hat{d}^4\bar{q}_i\,\hat{\delta}(p_i\cdot\bar{q}_i)e^{-ib\cdot\bar{q}_1}\hat{\delta}^{(4)}(\bar{q}_1+\bar{q}_2+\bar{k})\bar{\mathcal{A}}(p_1,p_2\to p_1+\hbar\bar{q}_1,p_2+\hbar\bar{q}_2,\hbar\bar{k}^\eta). \tag{9.4}$$

We have made the impact parameter an explicit argument here. At LO, we can then write

$$\langle F^{\mu\nu}(x)\rangle_{1,\mathrm{cl}} = g^3\left\langle\!\!\left\langle \mathrm{Re}\sum_\eta \int d\Phi(\bar{k})\bar{k}^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-i\bar{k}\cdot x}\mathcal{R}^{(0)}(\bar{k}^\eta;b)\right\rangle\!\!\right\rangle. \tag{9.5}$$

The integrand has the form of the radiation observables introduced in Sec. VI. The spectral waveform is then

$$f_{\mu\nu}(\omega,\hat{\mathbf{n}}) = -\frac{ig^3}{8\pi}\sum_\eta[\Theta(\omega)\bar{k}^{[\mu}\varepsilon^{(\eta)\nu]*}\mathcal{R}^{(0)}(\bar{k}^\eta;b)|_{\bar{k}=\omega(1,\hat{\mathbf{n}})} - \Theta(-\omega)\bar{k}^{[\mu}\varepsilon^{(\eta)\nu]}\mathcal{R}^{(0)*}(\bar{k}^\eta;b)|_{\bar{k}=-\omega(1,\hat{\mathbf{n}})}]. \tag{9.6}$$

The corresponding result for the Fourier-space NP scalar is

$$\tilde{\Phi}_2^0(\omega,\hat{\mathbf{n}}) = -\frac{ig^3\omega}{16\pi}\left\langle\!\!\left\langle\Theta(\omega)\mathcal{R}^{(0)}(\omega(1,\hat{\mathbf{n}})^-;b)+\Theta(-\omega)\mathcal{R}^{(0)*}(-\omega(1,\hat{\mathbf{n}})^+;b)\right\rangle\!\!\right\rangle. \tag{9.7}$$

Equivalently, we may write

$$\Phi_2^0(t,\hat{\mathbf{n}}) = -\frac{ig^3}{16\pi}\left\langle\!\!\left\langle\int \hat{d}\omega\,\Theta(\omega)\omega[e^{-i\omega\cdot t}\mathcal{R}^{(0)}(\omega(1,\hat{\mathbf{n}})^-;b) - e^{+i\omega\cdot t}\mathcal{R}^{(0)*}(\omega(1,\hat{\mathbf{n}})^+;b)]\right\rangle\!\!\right\rangle. \tag{9.8}$$

As the LO radiation kernel $\mathcal{R}^{(0)}$ is given by a five-point amplitude, the waveform as a function of frequency ω, is simply the five-point amplitude up to the additional factor of ω.

The explicit form of Eq. (9.4) for electromagnetic scattering is given in Eq. (5.46) of Ref. [122] and reproduced as Eq. (C1). We evaluate it in Appendix C to obtain

$$\begin{aligned}
\mathcal{R}^{(0)}(\bar{k};b) &= \frac{Q_1^2Q_2}{m_1u_1\cdot\bar{k}}[u_2\cdot\bar{k}u_1\cdot\varepsilon - u_1\cdot\bar{k}u_2\cdot\varepsilon]I_3 - \frac{Q_1^2Q_2\gamma}{m_1u_1\cdot\bar{k}(\gamma^2-1)}[u_1\cdot\bar{k}(u_1-\gamma u_2)\cdot\varepsilon-(u_1-\gamma u_2)\cdot\bar{k}u_1\cdot\varepsilon]I_3\\
&\quad+\frac{Q_1^2Q_2\gamma e^{ib\cdot\bar{k}}}{m_1u_1\cdot\bar{k}}[u_1\cdot\bar{k}\,\tilde{b}\cdot\varepsilon-\tilde{b}\cdot\bar{k}u_1\cdot\varepsilon]\frac{i}{2\pi(\gamma^2-1)}K_1\left(\sqrt{-b^2}u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\right)+(1\leftrightarrow 2\text{ modulo phases}),\\
&=\frac{Q_1^2Q_2e^{ib\cdot\bar{k}}}{m_1u_1\cdot\bar{k}}[u_2\cdot\bar{k}u_1\cdot\varepsilon-u_1\cdot\bar{k}u_2\cdot\varepsilon]\frac{1}{2\pi\sqrt{\gamma^2-1}}K_0\left(\sqrt{-b^2}u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\right)\\
&\quad+\frac{Q_1^2Q_2\gamma e^{ib\cdot\bar{k}}}{m_1u_1\cdot\bar{k}}[u_1\cdot\bar{k}\,\tilde{b}\cdot\varepsilon-\tilde{b}\cdot\bar{k}u_1\cdot\varepsilon]\frac{i}{2\pi(\gamma^2-1)}K_1\left(\sqrt{-b^2}u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\right)+(1\leftrightarrow 2\text{ modulo phases}).
\end{aligned} \tag{9.9}$$

In this expression,

$$\tilde{b}^\mu = b^\mu/\sqrt{-b^2}. \tag{9.10}$$

A side calculation shows that (with ζ a null reference momentum),

$$\begin{aligned}
u_2\cdot\bar{k}u_1\cdot\varepsilon - u_1\cdot\bar{k}u_2\cdot\varepsilon &= \frac{1}{\sqrt{2}\langle\zeta\bar{k}\rangle}[\langle\bar{k}|u_2|\bar{k}]\langle\zeta|u_1|\bar{k}]-\langle\bar{k}|u_1|\bar{k}]\langle\zeta|u_2|\bar{k}]],\\
&=\frac{1}{\sqrt{2}}[\bar{k}|u_2u_1|\bar{k}]
\end{aligned} \tag{9.11}$$

for positive-helicity emission, and

$$\frac{1}{\sqrt{2}}\langle\bar{k}|u_2u_1|\bar{k}\rangle \tag{9.12}$$

for negative-helicity emission.

Then,

$$\begin{aligned}\mathcal{R}^{(0)}(\bar{k}^+;b) =& \frac{Q_1^2Q_2e^{ib\cdot\bar{k}}}{2\sqrt{2}\pi m_1u_1\cdot\bar{k}\sqrt{\gamma^2-1}}\left\{[\bar{k}|u_2u_1|\bar{k}]K_0\left(\sqrt{-b^2}u_1\cdot\bar{k}\Big/\sqrt{\gamma^2-1}\right)+\frac{i[\bar{k}|bu_1|\bar{k}]}{\sqrt{\gamma^2-1}\sqrt{-b^2}}K_1\left(\sqrt{-b^2}u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\right)\right\}\\ &+\frac{Q_1Q_2^2}{2\sqrt{2}\pi m_2u_2\cdot\bar{k}\sqrt{\gamma^2-1}}\left\{[\bar{k}|u_1u_2|\bar{k}]K_0\left(\sqrt{-b^2}u_2\cdot\bar{k}/\sqrt{\gamma^2-1}\right)+\frac{i[\bar{k}|bu_2|\bar{k}]}{\sqrt{\gamma^2-1}\sqrt{-b^2}}K_1\left(\sqrt{-b^2}u_2\cdot\bar{k}/\sqrt{\gamma^2-1}\right)\right\}.\end{aligned} \tag{9.13}$$

There is a similar result for the other photon helicity.

Using the integrals,

$$\begin{aligned}\int_0^\infty d\omega\,\omega e^{-i\omega(t+a_0)}K_0(\omega a_1) &= \frac{1}{a_1^2+(a_0+t)^2}-\frac{(t+a_0)}{[a_1^2+(a_0+t)^2]^{3/2}}\operatorname{arcsinh}\left(\frac{1}{a_1}(t+a_0)\right)-\frac{i\pi}{2}\frac{(t+a_0)}{[a_1^2+(a_0+t)^2]^{3/2}},\\ \int_0^\infty d\omega\,\omega e^{-i\omega(t+a_0)}K_1(\omega a_1) &= \frac{\pi a_1}{2[a_1^2+(a_0+t)^2]^{3/2}}-i\frac{(a_0+t)}{a_1[a_1^2+(a_0+t)^2]}-i\frac{a_1}{[a_1^2+(a_0+t)^2]^{3/2}}\operatorname{arcsinh}\left(\frac{1}{a_1}(t+a_0)\right),\end{aligned} \tag{9.14}$$

and defining

$$\begin{aligned}u_{i,\hat{\mathbf{n}}} &\equiv u_i\cdot\bar{k}/\omega = u_i\cdot(1,\hat{\mathbf{n}}),\\ \rho_1(t) &\equiv -b^2u_{1,\hat{\mathbf{n}}}^2+(\gamma^2-1)(t+\mathbf{b}\cdot\hat{\mathbf{n}})^2,\\ \rho_2(t) &\equiv -b^2u_{2,\hat{\mathbf{n}}}^2+(\gamma^2-1)t^2,\end{aligned} \tag{9.15}$$

along with

$$\begin{aligned}\Xi_{ia}^{\zeta}(t,\hat{\mathbf{n}};\mathbf{v}) &= \frac{\sqrt{\gamma^2-1}}{\rho_1(t)}-\zeta\frac{(\gamma^2-1)(t+\mathbf{v}\cdot\hat{\mathbf{n}})}{\rho_1^{3/2}(t)}\operatorname{arcsinh}\left(\frac{\sqrt{\gamma^2-1}}{\sqrt{-b^2}u_{1,\hat{\mathbf{n}}}}(t+\mathbf{v}\cdot\hat{\mathbf{n}})\right)-\frac{i\pi}{2}\frac{(\gamma^2-1)(t+\mathbf{v}\cdot\hat{\mathbf{n}})}{\rho_1^{3/2}(t)},\\ \Xi_{ib}(t,\hat{\mathbf{n}};\mathbf{v}) &= \frac{\pi u_{1,\hat{\mathbf{n}}}}{\rho_1^{3/2}(t)}+i\frac{\sqrt{\gamma^2-1}(t+\mathbf{v}\cdot\hat{\mathbf{n}})}{b^2u_{1,\hat{\mathbf{n}}}\rho_1(t)}-i\frac{u_{1,\hat{\mathbf{n}}}}{\rho_1^{3/2}(t)}\operatorname{arcsinh}\left(\frac{\sqrt{\gamma^2-1}}{\sqrt{-b^2}u_{1,\hat{\mathbf{n}}}}(t+\mathbf{v}\cdot\hat{\mathbf{n}})\right),\end{aligned} \tag{9.16}$$

we can write

$$\begin{aligned}\Phi_2^0(t,\hat{\mathbf{n}}) =& -\frac{ig^3Q_1^2Q_2}{(4\pi)^3\sqrt{2}m_1u_{1,\hat{\mathbf{n}}}}[\langle\hat{n}|u_2u_1|\hat{n}\rangle\Xi_{1a}^{+}(t,\hat{\mathbf{n}};\mathbf{b})-[\hat{n}|u_2u_1|\hat{n}]\Xi_{1a}^{-}(t,\hat{\mathbf{n}};\mathbf{b})+i(\langle\hat{n}|bu_1|\hat{n}\rangle-[\hat{n}|bu_1|\hat{n}])\Xi_{1b}(t,\hat{\mathbf{n}};\mathbf{b})]\\ &-\frac{ig^3Q_1Q_2^2}{(4\pi)^3\sqrt{2}m_2u_{2,\hat{\mathbf{n}}}}[\langle\hat{n}|u_1u_2|\hat{n}\rangle\Xi_{2a}^{+}(t,\hat{\mathbf{n}};\mathbf{0})-[\hat{n}|u_1u_2|\hat{n}]\Xi_{2a}^{-}(t,\hat{\mathbf{n}};\mathbf{0})+i(\langle\hat{n}|bu_2|\hat{n}\rangle-[\hat{n}|bu_2|\hat{n}])\Xi_{2b}(t,\hat{\mathbf{n}};\mathbf{0})].\end{aligned} \tag{9.17}$$

Here, $|\hat{n}\rangle$ and $|\hat{n}]$ are spinors built out of the null vector $(1,\hat{\mathbf{n}})$.

X. CONNECTION TO RADIATED MOMENTUM

In Sec. VI, we presented the general form for the waveform observable. We worked out the leading-order form in two-particle scattering in Sec. VIII, and computed the explicit form for electromagnetic scattering in the previous section. The appearance of the radiation kernel suggests a connection to the radiated momentum previously computed in Ref. [122]. Let us elucidate that connection in this section.

In Eq. (3.33) of Ref. [122], we find an expression for time-averaged radiated momentum,

$$R^\mu \equiv \langle k^\mu \rangle = {}_{\rm in}\langle\psi|S^\dagger \mathbb{K}^\mu S|\psi\rangle_{\rm in} = {}_{\rm in}\langle\psi|T^\dagger \mathbb{K}^\mu T|\psi\rangle_{\rm in}. \quad (10.1)$$

This quantity is also integrated over the entire celestial sphere; we need a more differential observable. Furthermore, this expression is related to the energy emitted, rather than the amplitude of the emitted wave.

We can use Mellin transforms to extract a more restricted observable, passing through the spectral waveform to relate the emitted power to the amplitude. Write the expectation of the observable $\langle (k^t)^{z-1}\rangle$,

$$R(z) \equiv \langle (k^t)^{z-1}\rangle = {}_{\rm in}\langle\psi|T^\dagger (\mathbb{K}^t)^{z-1} T|\psi\rangle_{\rm in}. \quad (10.2)$$

The inverse Mellin transform is related to the unpolarized energy density function,

$$f_\epsilon(E) = -iE\int_{c-i\infty}^{c+i\infty} dz\, E^{-z} R(z), \quad (10.3)$$

where the integral is taken along a line parallel to the imaginary axis, with $c \in (0,1)$ (or a deformation of that contour that does not cross any poles or branch points).[7] The total energy is given by the integral

$$E_{\rm tot} = \int_0^\infty dE\, f_\epsilon(E). \quad (10.4)$$

Using the form in Eq. (3.38) of Ref. [122], we can write

$$R(z) = \sum_X \int d\Phi(k) d\Phi(r_1) d\Phi(r_2) (k_X^t)^{z-1} \sum_\eta |\hat{\mathcal{R}}(k^\eta, r_X)|^2, \quad (10.5)$$

for the expression in the quantum theory. In this equation, $\hat{\mathcal{R}}$ represents the quantum radiation kernel, given by the integral over wave functions inside the absolute square in Eq. (3.38). The quantum radiation kernel is expressed directly in terms of a scattering amplitude.

[7] With our conventions, the expected power of $(2\pi)^{-1}$ is in the forward rather than the inverse Mellin transform.

In the classical limit, the density function is more naturally a function of frequency rather than of energy

$$f_{\epsilon,\rm cl}(\omega) = -i\omega \int_{c-i\infty}^{c+i\infty} dz\, \omega^{-z} R_{\rm cl}(z), \quad (10.6)$$

so that $R_{\rm cl}(z) = \hbar^{-z-1} R(z)$. We can use Eqs. (4.40)–(4.41) of Ref. [122] to write

$$R_{\rm cl}(z) = \sum_X \hbar^{-z-1} \left\langle\!\!\left\langle \int d\Phi(k) (k_X^t)^{z-1} \sum_\eta |\mathcal{R}(k^\eta, r_X)|^2 \right\rangle\!\!\right\rangle. \quad (10.7)$$

The radiation kernel here is expressed in terms of the appropriate limit of a quantum scattering amplitude.

We next need to restrict the measured radiation from the entire celestial sphere to a narrow cone in a given direction. We take the limit of the cone, and measure only the radiation in a given direction from the scattering event. We implicitly assume that the measurement distance is much larger than the impact parameter so that there is a unique and well-defined direction. It is not clear exactly what a formal expression for the operator would be, but what we want is

$$\mathbb{K}^\mu \delta^{(2)}(\hat{\mathbb{K}} - \hat{\mathbf{n}}) \quad (10.8)$$

for radiation in the $\hat{\mathbf{n}}$ direction. This operator is to be understood as inserting

$$\sum_{i\in \text{messengers}} k_i^\mu \delta^{(2)}(\hat{\mathbf{k}}_i - \hat{\mathbf{n}}), \quad (10.9)$$

into a sum over states or equivalently the phase-space integral. Focusing on the energy component, this can be understood as a light ray operator [105,159–162] given by

$$\mathbb{E}(\hat{\mathbf{n}}) = \int_{-\infty}^{+\infty} du \lim_{r\to\infty} r^2 \mathbb{T}_{uu}(u, r, \hat{\mathbf{n}}), \quad (10.10)$$

where u denotes the light cone time $u = t - r$ and $\mathbb{T}_{uu}(u, r, \hat{\mathbf{n}})$ is the (light cone) time-time component of the stress-energy tensor (in gravity, this will be replaced by the Bondi news squared operator [105]). By applying the saddle point approximation for the fields in the energy momentum tensor, the plane wave expansion will localize to the point on the sphere in the direction of propagation. Schematically we will have (see Refs. [163,164] for further details)

$$e^{ix\cdot k/\hbar} = e^{i\omega u + i\omega r(1-\hat{\mathbf{n}}\cdot\hat{\mathbf{k}})} \overset{r\to\infty}{\sim} \frac{1}{i\omega r} e^{i\omega u} \delta^{(2)}(\hat{\mathbf{n}} - \hat{\mathbf{k}}), \quad (10.11)$$

where $\omega = \bar{k}^t$. Then one finds

$$\mathbb{E}(\hat{\mathbf{n}}) = \sum_\eta \int d\Phi(k)\, k^t \delta^{(2)}(\hat{\mathbf{n}} - \hat{\mathbf{k}})[a^\dagger_{(\eta)}(k) a_{(\eta)}(k)], \quad (10.12)$$

where the action on on-shell particle states is equivalent to the time component of Eq. (10.9). The analogous Mellin kernel for $(\mathbb{K}^t)^{z-1}$ is presumably

$$(\mathbb{K}^t)^{z-1}\delta^{(2)}(\hat{\mathbb{K}} - \hat{\mathbf{n}}), \quad (10.13)$$

which is to be understood as inserting

$$\sum_{i\in \substack{\text{distinct}\\ \text{messengers}}} \Bigg(\sum_{\substack{j\|i\\ j\in\text{messengers}}} k_j^t\Bigg)^{z-1} \delta^{(2)}(\hat{\mathbf{k}}_i - \hat{\mathbf{n}}) \quad (10.14)$$

into a sum over states or the phase-space integral. The sum over distinct messengers is a sum over messengers which are not collinear; the sum over the collinear messengers is taken in the inner sum. The inner sum includes i itself.

This form is motivated by a subtlety about overlapping directions: if $\hat{\mathbf{k}}_j = \hat{\mathbf{k}}_l$ with the remaining directions distinct we want, then

$$\sum_{\substack{i\in\text{messengers}\\ i\neq j,l}} (k_i^t)^{z-1}\delta^{(2)}(\hat{\mathbf{k}}_i - \hat{\mathbf{n}}) + (k_j^t + k_l^t)^{z-1}\delta^{(2)}(\hat{\mathbf{k}}_j - \hat{\mathbf{n}}), \quad (10.15)$$

which is what Eq. (10.14) is designed to give. That is, we want collinear messengers to give a result indistinguishable from a lone messenger. This would be essential if we faced collinear divergences, absent in massive electrodynamics and in gravity. At leading order this subtlety is irrelevant.

The analog to Eq. (10.5) is

$$R(z,\hat{\mathbf{n}}) = \sum_{i\in \substack{\text{distinct}\\ \text{messengers}}} \sum_X \int d\Phi(k_i) d\Phi(r_1) d\Phi(r_2) \Bigg(\sum_{\substack{j\|i\\ j\in\text{messengers}}} k_j^t\Bigg)^{z-1} \delta^{(2)}(\hat{\mathbf{k}}_i - \hat{\mathbf{n}}) \sum_\eta |\hat{\mathcal{R}}(k_i^\eta, r_X)|^2, \quad (10.16)$$

and to Eq. (10.7),

$$R_{\mathrm{cl}}(z,\hat{\mathbf{n}}) = \sum_{i\in \substack{\text{distinct}\\ \text{messengers}}} \hbar^{-z-1} \Bigg\langle\!\!\Bigg\langle \int d\Phi(k_i) \Bigg(\sum_{\substack{j\|i\\ j\in\text{messengers}}} k_j^t\Bigg)^{z-1} \delta^{(2)}(\hat{\mathbf{k}}_i - \hat{\mathbf{n}}) \sum_\eta |\mathcal{R}(k_i^\eta, r_X)|^2 \Bigg\rangle\!\!\Bigg\rangle. \quad (10.17)$$

At LO, Eq. (10.17) simplifies to just

$$R^{(0)}_{\mathrm{cl}}(z,\hat{\mathbf{n}}) = g^6 \Bigg\langle\!\!\Bigg\langle \int d\Phi(\bar{k}) (\bar{k}^t)^{z-1} \delta^{(2)}(\hat{\mathbf{k}} - \hat{\mathbf{n}}) \sum_\eta |\mathcal{R}^{(0)}(\bar{k}^\eta; b)|^2 \Bigg\rangle\!\!\Bigg\rangle. \quad (10.18)$$

The corresponding result for the spectral density in the $\hat{\mathbf{n}}$ direction is

$$f_{\epsilon,\mathrm{cl}}(\omega,\hat{\mathbf{n}}) = g^6\omega \Bigg\langle\!\!\Bigg\langle \int d\Phi(\bar{k}) \frac{\hat{\delta}(\ln \bar{k}^t - \ln\omega)}{\bar{k}^t} \delta^{(2)}(\hat{\mathbf{k}} - \hat{\mathbf{n}}) \sum_\eta |\mathcal{R}^{(0)}(\bar{k}^\eta; b)|^2 \Bigg\rangle\!\!\Bigg\rangle. \quad (10.19)$$

Writing out

$$\begin{aligned} d\Phi(\bar{k}) &= \frac{d^3\bar{\mathbf{k}}}{2(2\pi)^3|\bar{\mathbf{k}}|} \\ &= \frac{|\bar{\mathbf{k}}| d|\bar{\mathbf{k}}| d\Omega_{\bar{\mathbf{k}}}}{2(2\pi)^3}, \end{aligned} \quad (10.20)$$

we can perform the integrals in Eq. (10.19) to obtain

$$f_{\epsilon,\mathrm{cl}}(\omega,\hat{\mathbf{n}}) = \frac{g^6\omega^2}{8\pi^2} \sum_\eta \langle\!\langle |\mathcal{R}^{(0)}(\omega(1,\hat{\mathbf{n}})^\eta; b)|^2 \rangle\!\rangle. \quad (10.21)$$

We can now compare this with the amplitude of each component of the waveform, expanded at the leading order order in the coupling: for $|f_{\mu\nu}M^{*\mu}N^\nu|$ and $|f_{\mu\nu}M^\mu N^\nu|$ we have, respectively,

$$|f_{\mu\nu}(\omega(1,\hat{\mathbf{n}}))M^{*\mu}N^{\nu}| = \frac{\omega}{16\pi}g^3|\langle\!\langle \mathcal{R}^{(0)}(\omega(1,\hat{\mathbf{n}})^-;b)\rangle\!\rangle|,$$
$$|f_{\mu\nu}(\omega(1,\hat{\mathbf{n}}))M^{\mu}N^{\nu}| = \frac{\omega}{16\pi}g^3|\langle\!\langle \mathcal{R}^{(0)}(\omega(1,\hat{\mathbf{n}})^+;b)\rangle\!\rangle|. \tag{10.22}$$

At LO, we can also write

$$\langle\!\langle |\mathcal{R}^{(0)}(\omega(1,\hat{\mathbf{n}})^{\eta};b)|^2\rangle\!\rangle = |\langle\!\langle \mathcal{R}^{(0)}(\omega(1,\hat{\mathbf{n}})^{\eta};b)\rangle\!\rangle|^2, \tag{10.23}$$

and therefore we can express the spectral density of emission from Eq. (10.21) in terms of the amplitudes of the two helicity components of the waveform,

$$f_{\epsilon,\mathrm{cl}}(\omega,\hat{\mathbf{n}}) = 32[|f_{\mu\nu}(\omega(1,\hat{\mathbf{n}}))M^{*\mu}N^{\nu}|^2 + |f_{\mu\nu}(\omega(1,\hat{\mathbf{n}}))M^{\mu}N^{\nu}|^2]. \tag{10.24}$$

This relation is the avatar of the relation between the energy of the wave and the squared amplitude of the wave, the only difference being that here we are measuring the momentum emitted in a given direction at a large distance r from the source. The emitted radiation observable provides information about the magnitude of the observed messenger wave but not about its phase. The direct derivation in previous sections adds that information.

A recently proposed generalization of a standard event shape is sensitive to amplitude phases [165]. It would be interesting to explore a possible connection to the waveform.

XI. CONCLUSIONS

In this paper, we have developed an observables-based formalism for computing classical waves from quantum scattering amplitudes. We have shown how to incorporate both outgoing and incoming narrowly sampled waves, via the "local" observables needed for the former, and scattering of waves needed for the latter.

Waveforms measured at gravitational wave observatories are "local" measurements, in the sense that the passing gravitational wave train is sampled only at the (small) spatial location of the observatory relative to the (very large) spatial extent of the gravitational wave. In this paper, our first major focus was on developing a quantum-field theoretic formalism to describe this kind of classical, local measurement. This is in contrast to previous work [59,122] on classical observables in quantum field theory, which discussed "global" observables, such as the total amount of energy-momentum radiated in a scattering event. Our formalism is very general, though in our explicit discussions we focused on the case of electromagnetic radiation, which has the pedagogical benefit of being slightly easier to work with. We look forward to applications of our formalism in gravity.

Scattering amplitudes are remarkably simple objects which can be computed efficiently. For this reason, it seems very promising that waveforms can be computed so directly in terms of amplitudes. In particular, it is clear from our work that there is no obstacle to using the double copy to compute gravitational waveforms (sourced by a scattering event) to any order of perturbation theory. It may be worth emphasizing that we do not need the Bern–Carrasco–Johansson (BCJ) formulation [11] of the double copy at loop level, which remains conjectural, to perform such a computation. The gravitational waveform at higher orders could be computed using the unitarity method, with only tree-level gravitational amplitudes required as inputs. For those amplitudes, the BCJ relations are proven. The insensitivity of the classical waveform to delta-function contributions localized on the worldlines of the particles offers another, potentially significant simplification: only a subset of all possible quantum factorization channels needs to be computed. The possibility of computing leading-order gravitational radiation using amplitudes and the double copy was previously discussed by Luna *et al.* [22], building on a leading-order worldline treatment by Goldberger and Ridgway [12]. The formalism presented here makes this computation possible to any order. Shen [13] has already computed the next-to-leading order waveform; it will be interesting to compare the efficiency of our methods, using the conventional double copy of amplitudes, with Shen's ingenious worldline implementation of the double copy.

One important theme in the calculation and exploration of scattering amplitudes is the search for the simplest forms in which to cast them. An early realization came through the focus on helicity amplitudes rather than covariant forms (in terms of polarization vectors and momenta). The former contain all physical information, and are simpler. This is especially true when they are expressed in terms of spinorial variables. The translation comes through a spinor-helicity formalism; historically, that of Xu, Zhang, and Chang [166] played an important role.

Remarkably, the same phenomenon occurs in classical field theory. NP scalars [123] are classical analogs of helicity amplitudes; indeed, as we have seen, the NP scalar Φ_2 is an integral over a helicity amplitude. The NP scalars can be defined by contracting tensorial quantities, such as the electromagnetic field strength $F^{\mu\nu}$, with a basis of four null vectors. This basis is a direct analog of the momentum of a particle, along with its two possible polarization vectors, and a gauge choice. Alternatively, the NP scalars can be constructed directly by passing from the tensorial field strength to its spinorial equivalent. In this formulation, a natural basis of spinors occurs classically in an exact analog of the spinor-helicity method in scattering amplitudes. It seems likely that further study will reveal more close connections between sophisticated approaches to classical physics and the methods of scattering amplitudes.

As a concrete application of our formalism, we computed a simple waveform: the electromagnetic radiation emitted as two charges scatter. We extracted the asymptotic spectral functions, as well as the relevant asymptotic Newman-Penrose scalar. At leading order, these quantities are closely related to five-point amplitudes. In the Fourier domain, they are built out of modified Bessel functions. At higher orders, the connection to five-point amplitudes will persist. We expect that an interesting class of functions, generalizing Bessel functions, will appear. In the time domain, the functions were simpler; we suspect that this may be an accident of low orders.

Our second major focus in this paper has been developing a quantum field-theoretic description of massless classical waves readily amenable to calculations using scattering amplitudes. Coherent states are key tools in extracting classical behavior from quantum field theory [134], so it is no surprise that we found them to be very helpful. Indeed, they mesh very naturally with amplitudes, and especially with the transition operator T whose matrix elements are the amplitudes. The reason is that the T matrix can be written out in terms of amplitudes and of creation and annihilation operators. These operators, in turn, act very simply on coherent states.

As an application of massless waves, we studied the scattering of a massless electromagnetic wave off a classical charge. We showed that the resulting outgoing wave is determined, at leading order, by the classical limit of the Compton four-point amplitude. We expect this final state to also be coherent. In Appendix B we provide evidence in favor of the coherence of this radiation.

Throughout our paper, the focus has been on scattering events. These are very naturally described using amplitudes. Scattering events in general relativity are interesting in themselves given the possibility that the tightly bound compact binaries observed by the LIGO and Virgo collaborations are created after a scattering event with a third object [3]. Of course, a major goal for the future will be to understand how gravitational waveforms from classically bound objects can also be computed using amplitudes. This will need a new understanding, perhaps building on the work [84,89] of Kälin and Porto in the context of conservative classical dynamics. Yet even without such a direct connection, it seems clear that our work can be used in the context of bound state physics by developing an effective action to enable the transfer of know-how from unbound to bound cases. The reader may also be interested in forthcoming work by Bautista, Guevara, Kavanagh, and Vines [143] on related subjects.

The future for gravitational wave physics is data-rich and high precision. We will need every good idea we can find to calculate waveform templates at the necessary precision. By now it is clear that amplitudes and the double copy will be a useful tool. The double copy, at least in its BCJ form, was a theoretical discovery which was a by-product of the drive for precision theory for LHC physics. New theoretical discoveries may well await us as we develop our understanding of gravitational amplitudes in the drive for precision gravitational-wave physics.

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APPENDIX A: BEAM SPREADING

Let us obtain a more refined picture of the time dependence of the classical wave in Eq. (3.38). Expand the square root in the exponent in that expression, keeping the next-to-leading term in the expansion,

$$\sqrt{\omega^2 + (\bar{k}^x)^2 + (\bar{k}^y)^2} = \omega + \frac{(\bar{k}^x)^2 + (\bar{k}^y)^2}{2\omega} + \cdots. \tag{A1}$$

Substituting this into Eq. (3.38), we obtain

$$A^\mu_{\rm cl}(x) = \sqrt{2} A_\odot \mathrm{Re}\, \varepsilon^\mu_\odot(\bar{k}_\odot) e^{-i\omega(t-z)} \mathcal{I}(\omega, \ell_\perp), \tag{A2}$$

where we have introduced the following scalar integral (recall that $\sigma_\perp = \ell_\perp^{-1}$):

$$\mathcal{I}(\omega, \ell_\perp) = \int d^2\bar{k}_\perp\, \delta_{\sigma_\perp}(\bar{k}^x) \delta_{\sigma_\perp}(\bar{k}^y) e^{i\bar{k}^x x} e^{i\bar{k}^y y} e^{-it\bar{k}_x^2/(2\omega)} e^{-it\bar{k}_y^2/(2\omega)}. \tag{A3}$$

Integrating over the angular variable, we find

$$\mathcal{I}(\omega, \ell_\perp) = 2\ell_\perp^2 \int_0^\infty dk\, k J_0\left(\sqrt{x^2+y^2}k\right) e^{-k^2[\ell_\perp^2 + it/(2\omega)]}. \tag{A4}$$

Performing the integral, we obtain

$$\mathcal{I}(\omega, l_\perp) = \frac{e^{-\frac{(x^2+y^2)}{4l_\perp^2}\left[1+i\frac{t}{2\omega l_\perp^2}\right]^{-1}}}{1+\frac{it}{2\omega l_\perp^2}}. \tag{A5}$$

Yet higher-order contributions may be computed by noticing that the electromagnetic field can be expressed—without expanding the square root in Eq. (A1)—as

$$A^{\mu}_{\rm cl}(x) = \sqrt{2}A_{\odot}{\rm Re}\,\varepsilon^{\mu}_{\odot}(\bar{k}_{\odot})e^{i\omega z - it\hat{H}(\omega)}\left[e^{-\frac{(x^2+y^2)}{4l_{\perp}^2}}\right], \tag{A6}$$

where we have introduced the operator $\hat{H}(\omega) = \sqrt{\omega^2 - \nabla^2_{(x,y)}}$. In this reformulation, the problem is now equivalent to computing the time evolution—for a relativistic Hamiltonian with effective mass ω—of a Gaussian wave packet. Restricting the time evolution to the nonrelativistic limit, we obtain the well-known result for the spread of a Gaussian wave packet in two dimensions, in agreement with Eq. (A5). In a similar way, we can easily generalize the computation by adding contributions from the expansion of the polarization vectors in the integrand as in Eq. (3.48).

APPENDIX B: FACTORIZATION AND UNITARITY IN THE CLASSICAL LIMIT

Our framework allows the computation of classical phenomena such as the electromagnetic field generated by the scattering of an incoming beam of light with a massive particle. In this Appendix, we address the question of whether the final state is coherent, in the context of a perturbative calculation. For coherence to hold, we must show that the mean value of the electromagnetic field operator on the final state factorizes. The final state is given by the evolution of the initial state,

$$|\psi\rangle_{\rm out} = \int d\Phi(p)\phi(p)e^{ib\cdot p/\hbar}S|p\alpha^{+}\rangle_{\rm in}. \tag{B1}$$

We say that the final state is coherent if the following correlation function vanishes in the classical limit,

$$\Delta = {}_{\rm out}\langle\psi|\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\alpha\beta}(y)|\psi\rangle_{\rm out} - {}_{\rm out}\langle\psi|\mathbb{F}^{\mu\nu}(x)|\psi\rangle_{\rm out}\,{}_{\rm out}\langle\psi|\mathbb{F}^{\alpha\beta}(y)|\psi\rangle_{\rm out}, \tag{B2}$$

where the electromagnetic field operator is given by Eq. (3.5). Let us prove that the previous correlation function vanishes at the first nontrivial order in the coupling g. The second term in Eq. (B2) is already known to this order as it matches the value of the electromagnetic field in Thomson scattering times its free counterpart. What is left is to compute is the first term. We can safely disregard contributions quadratic in the transfer matrix, leaving us to compute the classical limit of

$${}_{\rm out}\langle\psi|\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\alpha\beta}(y)|\psi\rangle_{\rm out} = F^{\mu\nu,(0)}(x)F^{\alpha\beta,(0)}(y) + i\,{}_{\rm in}\langle\psi|[\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\alpha\beta}(y), T]|\psi\rangle_{\rm in}, \tag{B3}$$

where $F^{(0)}_{\mu\nu}(x)$ denotes the free field. Expanding the electromagnetic field operator in terms of annihilation and creation operators,

$$\begin{aligned}\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\alpha\beta}(y) = -\frac{4}{\hbar^3}\sum_{\eta_1,\eta_2}\int & d\Phi(k_1)d\Phi(k_2)[a_{(\eta_1)}(k_1)a_{(\eta_2)}(k_2)k_1^{[\mu}\varepsilon^{(\eta_1)\nu]*}k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]*}e^{-i(k_1\cdot x+k_2\cdot y)/\hbar} \\ & + a^{\dagger}_{(\eta_1)}(k_1)a^{\dagger}_{(\eta_2)}(k_2)k_1^{[\mu}\varepsilon^{(\eta_1)\nu]}k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]}e^{i(k_1\cdot x+k_2\cdot y)/\hbar} - a^{\dagger}_{(\eta_2)}(k_2)a_{(\eta_1)}(k_1)k_1^{[\mu}\varepsilon^{(\eta_1)\nu]*}k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]}e^{-i(k_1\cdot x-k_2\cdot y)/\hbar} \\ & - a^{\dagger}_{(\eta_1)}(k_1)a_{(\eta_2)}(k_2)k_1^{[\mu}\varepsilon^{(\eta_1)\nu]}k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]*}e^{i(k_1\cdot x-k_2\cdot y)/\hbar} - \delta_{\Phi}(k_1-k_2)k_1^{[\mu}\varepsilon^{(\eta_1)\nu]*}k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]}e^{i(k_1\cdot x-k_2\cdot y)/\hbar}].\end{aligned} \tag{B4}$$

At leading order in the coupling, the T matrix reads

$$T = \sum_{\eta,\eta'}\int d\Phi(\tilde{k}',\tilde{k},\tilde{p}',\tilde{p})\langle\tilde{k}'^{\eta'}\tilde{p}'|T|\tilde{k}^{\eta}\tilde{p}\rangle a^{\dagger}_{(\eta')}(\tilde{k}')a^{\dagger}(\tilde{p}')a_{(\eta)}(\tilde{k})a(\tilde{p}) + \cdots. \tag{B5}$$

We can now evaluate the correlation function. The first two terms inside the bracket in Eq. (B4) can contribute only at higher order in the coupling, and can be safely neglected in the evaluation of the correlation function. As for the last term in Eq. (B4), we can see it is similar to (3.22), providing a quantum contribution at leading order in the coupling which will disappear in the classical limit. We are left with the following:

$$\begin{aligned}
[a^{\dagger}_{(\eta_1)}(k_1)a_{(\eta_2)}(k_2),T] &= \sum_{\eta}\int d\Phi(\tilde{p},\tilde{p}',\tilde{k})\langle k_2^{\eta_2}\tilde{p}'|T|\tilde{k}^{\eta}\tilde{p}\rangle a^{\dagger}_{(\eta_1)}(k_1)a^{\dagger}(\tilde{p}')a_{(\eta)}(\tilde{k})a(\tilde{p})\\
&\quad -\sum_{\eta'}\int d\Phi(\tilde{p},\tilde{p}',\tilde{k}')\langle \tilde{k}'^{\eta'}\tilde{p}'|T|k_1^{\eta_1}\tilde{p}\rangle a^{\dagger}_{(\eta')}(\tilde{k}')a^{\dagger}(\tilde{p}')a_{(\eta_2)}(k_2)a(\tilde{p}),\\
[a^{\dagger}_{(\eta_2)}(k_2)a_{(\eta_1)}(k_1),T] &= \sum_{\eta}\int d\Phi(\tilde{p},\tilde{p}',\tilde{k})\langle k_1^{\eta_1}\tilde{p}'|T|\tilde{k}^{\eta}\tilde{p}\rangle a^{\dagger}_{(\eta_2)}(k_2)a^{\dagger}(\tilde{p}')a_{(\eta)}(\tilde{k})a(\tilde{p})\\
&\quad -\sum_{\eta'}\int d\Phi(\tilde{p},\tilde{p}',\tilde{k}')\langle \tilde{k}'^{\eta'}\tilde{p}'|T|k_2^{\eta_2}\tilde{p}\rangle a^{\dagger}_{(\eta')}(\tilde{k}')a^{\dagger}(\tilde{p}')a_{(\eta_1)}(k_1)a(\tilde{p}).
\end{aligned} \tag{B6}$$

These results imply that

$$\begin{aligned}
{}_{\mathrm{out}}\langle\psi|\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\alpha\beta}(y)|\psi\rangle_{\mathrm{out}} &= F^{\mu\nu,(0)}(x)F^{\alpha\beta,(0)}(y)+\frac{8}{\hbar^3}\mathrm{Re}\sum_{\eta,\eta_1,\eta_2}\int d\Phi(k_1,k_2,\tilde{k},p,p')\phi(p)\phi^*(p')\\
&\quad\times[i\langle k_1^{\eta_1}p'|T|\tilde{k}^{\eta}p\rangle\langle\alpha^+|a^{\dagger}_{(\eta_2)}(k_2)a_{(\eta)}(\tilde{k})|\alpha^+\rangle k_1^{[\mu}\varepsilon^{(\eta_1)\nu]*}k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]}e^{-i(k_1\cdot x-k_2\cdot y)/\hbar}]\\
&\quad+\frac{8}{\hbar^3}\mathrm{Re}\sum_{\eta,\eta_1,\eta_2}\int d\Phi(k_1,k_2,\tilde{k},p,p')\phi(p)\phi^*(p')\\
&\quad\times[i\langle k_2^{\eta_2}p'|T|\tilde{k}^{\eta}p\rangle\langle\alpha^+|a^{\dagger}_{(\eta_1)}(k_1)a_{(\eta)}(\tilde{k})|\alpha^+\rangle k_1^{[\mu}\varepsilon^{(\eta_1)\nu]}k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]*}e^{i(k_1\cdot x-k_2\cdot y)/\hbar}].
\end{aligned} \tag{B7}$$

After some simple algebra, we find

$$\begin{aligned}
{}_{\mathrm{out}}\langle\psi|\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\alpha\beta}(y)|\psi\rangle_{\mathrm{out}} &= F^{\mu\nu,(0)}(x)F^{\alpha\beta,(0)}(y)+\frac{8}{\hbar^3}\mathrm{Re}\sum_{\eta_1,\eta_2}\int d\Phi(k_1,\tilde{k},p,p')\phi(p)\phi^*(p')\\
&\quad\times\left[i\langle k_1^{\eta_1}p'|T|\tilde{k}^{\eta}p\rangle\alpha(\tilde{k})k_1^{[\mu}\varepsilon^{(\eta_1)\nu]*}e^{-ik_1\cdot x/\hbar}\int d\Phi(k_2)\alpha^*(k_2)k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]}e^{ik_2\cdot y/\hbar}\right]\\
&\quad+\frac{8}{\hbar^3}\mathrm{Re}\sum_{\eta_1,\eta_2}\int d\Phi(k_1,k_2,\tilde{k},p,p')\phi(p)\phi^*(p')\\
&\quad\times\left[i\langle k_2^{\eta_2}p'|T|\tilde{k}^{\eta}p\rangle\alpha(\tilde{k})k_2^{[\alpha}\varepsilon^{(\eta_2)\beta]*}e^{-ik_2\cdot y/\hbar}\int d\Phi(k_1)\alpha^*(k_1)k_1^{[\mu}\varepsilon^{(\eta_1)\nu]}e^{ik_1\cdot x/\hbar}\right];
\end{aligned} \tag{B8}$$

reorganizing the terms we then obtain, as expected,

$$\begin{aligned}
{}_{\mathrm{out}}\langle\psi|\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\alpha\beta}(y)|\psi\rangle_{\mathrm{out}} &= F^{\mu\nu,(0)}(x)F^{\alpha\beta,(0)}(y)\\
&\quad+F^{\alpha\beta,(0)}(y)\frac{4}{\hbar^{3/2}}\mathrm{Re}\sum_{\eta_1}\int d\Phi(k_1,\tilde{k},p,p')\phi(p)\phi^*(p')[i\langle k_1^{\eta_1}p'|T|\tilde{k}^{\eta}p\rangle\alpha(\tilde{k})k_1^{[\mu}\varepsilon^{(\eta_1)\nu]*}e^{-ik_1\cdot x/\hbar}]\\
&\quad+F^{\mu\nu,(0)}(x)\frac{4}{\hbar^{3/2}}\mathrm{Re}\sum_{\eta_2}\int d\Phi(k_2,\tilde{k},p,p')\phi(p)\phi^*(p')[i\langle k_2^{\eta_2}p'|T|\tilde{k}^{\eta}p\rangle\alpha(\tilde{k})k_2^{[\mu}\varepsilon^{(\eta_2)\nu]*}e^{-ik_2\cdot y/\hbar}].
\end{aligned} \tag{B9}$$

From this result we conclude that

$$\Delta|_{g^2}=0. \tag{B10}$$

This demonstrates that the semiclassical state generated in Thomson scattering is a coherent state to this nontrivial order in the coupling.

APPENDIX C: INTEGRALS

We require explicit expressions for the integrals appearing in the leading-order radiation kernel, Eq. (5.46) of Ref. [122]. The integral is

$$\mathcal{R}^{(0)}(\bar{k};b) = 4\int \hat{d}^4\bar{w}_1\hat{d}^4\bar{w}_2\hat{\delta}(2p_1\cdot\bar{w}_1)\hat{\delta}(2p_2\cdot\bar{w}_2)\hat{\delta}^{(4)}(\bar{k}-\bar{w}_1-\bar{w}_2)e^{i\bar{w}_1\cdot b} \times\left\{\frac{Q_1^2Q_2}{\bar{w}_2^2}\left[-p_2\cdot\varepsilon+\frac{(p_1\cdot p_2)(\bar{w}_2\cdot\varepsilon)}{p_1\cdot\bar{k}}+\frac{(p_2\cdot\bar{k})(p_1\cdot\varepsilon)}{p_1\cdot\bar{k}}-\frac{(\bar{k}\cdot\bar{w}_2)(p_1\cdot p_2)(p_1\cdot\varepsilon)}{(p_1\cdot\bar{k})^2}\right]+(1\leftrightarrow 2)\right\}. \tag{C1}$$

We replace p_i^μ by $m_iu_i^\mu$, and introduce a fourth basis vector,

$$v_\mu = 4\epsilon_{\mu\nu\lambda\rho}u_1^\nu u_2^\lambda b^\rho. \tag{C2}$$

Its square is given by

$$v^2 = -2G(u_1,u_2,b), \tag{C3}$$

where G is the Gram determinant

$$G(\{p_i\}) = \det(2p_i\cdot p_j). \tag{C4}$$

The only nontrivial Lorentz invariants that can be built out of the u_i^μ, b^μ, and v^μ are

$$\gamma = u_1\cdot u_2, \tag{C5}$$

and b^2, as $u_i^2=1$.

We note that

$$v^2 = 16b^2(\gamma^2-1). \tag{C6}$$

It is convenient to introduce two rescaled four-vectors,

$$\tilde{b}^\mu = b^\mu/\sqrt{-b^2},$$
$$\tilde{v}^\mu = v^\mu/\sqrt{-v^2} = v^\mu/\left(4\sqrt{-b^2(\gamma^2-1)}\right). \tag{C7}$$

Let us also introduce as well new coordinates $z_{1,2,b,v}^{[i]}$,

$$\bar{w}_i^\mu = z_1^{[i]}u_1^\mu + z_2^{[i]}u_2^\mu + z_b^{[i]}\tilde{b}^\mu + z_v^{[i]}\tilde{v}^\mu. \tag{C8}$$

The Jacobian from the change of variables in each $\bar{w}_i$ is

$$|\epsilon_{\mu\nu\lambda\rho}\tilde{v}^\mu u_1^\nu u_2^\lambda\tilde{b}^\rho| = -\frac{v^2}{4\sqrt{-v^2}\sqrt{-b^2}} = \sqrt{\gamma^2-1}. \tag{C9}$$

We also have the following expression for each square,

$$\bar{w}_i^2 = (z_1^{[i]})^2 + 2\gamma z_1^{[i]}z_2^{[i]} + (z_2^{[i]})^2 - (z_b^{[i]})^2 - (z_v^{[i]})^2. \tag{C10}$$

There are four elementary integrals we need to evaluate,

$$\begin{aligned} I_1 &= \int \hat{d}^4\bar{w}_1\hat{d}^4\bar{w}_2\hat{\delta}(u_1\cdot\bar{w}_1)\hat{\delta}(u_2\cdot\bar{w}_2)\hat{\delta}^{(4)}(\bar{k}-\bar{w}_1-\bar{w}_2)\frac{e^{i\bar{w}_1\cdot b}}{\bar{w}_1^2},\\ I_2^\mu &= \int \hat{d}^4\bar{w}_1\hat{d}^4\bar{w}_2\hat{\delta}(u_1\cdot\bar{w}_1)\hat{\delta}(u_2\cdot\bar{w}_2)\hat{\delta}^{(4)}(\bar{k}-\bar{w}_1-\bar{w}_2)\frac{e^{i\bar{w}_1\cdot b}\bar{w}_1^\mu}{\bar{w}_1^2},\\ I_3 &= \int \hat{d}^4\bar{w}_1\hat{d}^4\bar{w}_2\hat{\delta}(u_1\cdot\bar{w}_1)\hat{\delta}(u_2\cdot\bar{w}_2)\hat{\delta}^{(4)}(\bar{k}-\bar{w}_1-\bar{w}_2)\frac{e^{i\bar{w}_1\cdot b}}{\bar{w}_2^2},\\ I_4^\mu &= \int \hat{d}^4\bar{w}_1\hat{d}^4\bar{w}_2\hat{\delta}(u_1\cdot\bar{w}_1)\hat{\delta}(u_2\cdot\bar{w}_2)\hat{\delta}^{(4)}(\bar{k}-\bar{w}_1-\bar{w}_2)\frac{e^{i\bar{w}_1\cdot b}\bar{w}_2^\mu}{\bar{w}_2^2}. \end{aligned} \tag{C11}$$

Start evaluating I_1 by using the fourfold delta function to evaluate the $\bar{w}_2$ integral,

$$I_1 = \int \hat{d}^4\bar{w}_1\hat{\delta}(u_1\cdot\bar{w}_1)\hat{\delta}(u_2\cdot\bar{w}_1-u_2\cdot\bar{k})\frac{e^{i\bar{w}_1\cdot b}}{\bar{w}_1^2}, \tag{C12}$$

and then make the change of variables (C8),

$$\frac{\sqrt{\gamma^2-1}}{(2\pi)^2}\int dz_1^{[1]}dz_2^{[1]}dz_b^{[1]}dz_v^{[1]}\delta(z_1^{[1]}+\gamma z_2^{[1]})\delta(\gamma z_1^{[1]}+z_2^{[1]}-u_2\cdot\bar{k})\frac{e^{-iz_b^{[1]}\sqrt{-b^2}}}{(z_1^{[1]})^2+2\gamma z_1^{[1]}z_2^{[1]}+(z_2^{[1]})^2-(z_b^{[1]})^2-(z_v^{[1]})^2}. \tag{C13}$$

Use the delta functions to perform the $z_{1,2}^{[1]}$ integrals,

$$\frac{1}{(2\pi)^2\sqrt{\gamma^2-1}}\int dz_b^{[1]}dz_v^{[1]}\frac{e^{-iz_b^{[1]}\sqrt{-b^2}}}{-(u_2\cdot\bar{k})^2/(\gamma^2-1)-(z_b^{[1]})^2-(z_v^{[1]})^2}. \tag{C14}$$

Perform the $z_v^{[1]}$ integral to obtain

$$-\frac{1}{4\pi\sqrt{\gamma^2-1}}\int dz_b^{[1]}\frac{e^{-iz_b^{[1]}\sqrt{-b^2}}}{\sqrt{(z_b^{[1]})^2+(u_2\cdot\bar{k})^2/(\gamma^2-1)}}. \tag{C15}$$

This can be evaluated as a Fourier transform,

$$I_1=-\frac{1}{2\pi\sqrt{\gamma^2-1}}K_0\left(\sqrt{-b^2}u_2\cdot\bar{k}/\sqrt{\gamma^2-1}\right), \tag{C16}$$

where K_0 is a modified Bessel function of the second kind.

The first two steps are the same for I_2^μ,

$$\begin{aligned}I_2^\mu&=\int\hat{d}^4\bar{w}_1\hat{\delta}(u_1\cdot\bar{w}_1)\hat{\delta}(u_2\cdot\bar{w}_1-u_2\cdot\bar{k})\frac{e^{i\bar{w}_1\cdot b}\bar{w}_1^\mu}{\bar{w}_1^2}\\&=\frac{\sqrt{\gamma^2-1}}{(2\pi)^2}\int dz_1^{[1]}dz_2^{[1]}dz_b^{[1]}dz_v^{[1]}\delta(z_1^{[1]}+\gamma z_2^{[1]})\delta(\gamma z_1^{[1]}+z_2^{[1]}-u_2\cdot\bar{k})\frac{e^{-iz_b^{[1]}\sqrt{-b^2}}(z_1^{[1]}u_1^\mu+z_2^{[1]}u_2^\mu+z_b^{[1]}\tilde{b}^\mu+z_v^{[1]}\tilde{v}^\mu)}{(z_1^{[1]})^2+2\gamma z_1^{[1]}z_2^{[1]}+(z_2^{[1]})^2-(z_b^{[1]})^2-(z_v^{[1]})^2}.\end{aligned} \tag{C17}$$

The $\tilde{v}^\mu$ term will vanish because of the antisymmetry in $z_v^{[1]}$; the $u_{1,2}^\mu$ terms will yield a result proportional to I_1,

$$\begin{aligned}I_{2a}^\mu&=\frac{u_2\cdot\bar{k}}{\gamma^2-1}(\gamma u_1^\mu-u_2^\mu)I_1\\&=-\frac{u_2\cdot\bar{k}}{2\pi(\gamma^2-1)^{3/2}}(\gamma u_1^\mu-u_2^\mu)K_0\left(\sqrt{-b^2}u_2\cdot\bar{k}/\sqrt{\gamma^2-1}\right).\end{aligned} \tag{C18}$$

The remaining ($\tilde{b}^\mu$) term is

$$\begin{aligned}I_{2b}^\mu&=-\frac{\tilde{b}^\mu}{4\pi\sqrt{\gamma^2-1}}\int dz_b^{[1]}\frac{e^{-iz_b^{[1]}\sqrt{-b^2}}z_b^{[1]}}{\sqrt{(z_b^{[1]})^2+(u_2\cdot\bar{k})^2/(\gamma^2-1)}}\\&=\frac{iu_2\cdot\bar{k}\tilde{b}^\mu}{2\pi(\gamma^2-1)}K_1\left(\sqrt{-b^2}u_2\cdot\bar{k}/\sqrt{\gamma^2-1}\right),\end{aligned} \tag{C19}$$

where we have dropped a delta-function contribution. The total is

$$I_2^\mu=I_{2a}^\mu+I_{2b}^\mu. \tag{C20}$$

In I_3, start by using the fourfold delta function to integrate out $\bar{w}_1$,

$$I_3=e^{ib\cdot\bar{k}}\int\hat{d}^4\bar{w}_2\hat{\delta}(u_1\cdot\bar{w}_2-u_1\cdot\bar{k})\hat{\delta}(u_2\cdot\bar{w}_2)\frac{e^{i\bar{w}_2\cdot b}}{\bar{w}_2^2}. \tag{C21}$$

This is proportional to I_1, with the exchange $u_1 \leftrightarrow u_2$,

$$I_3 = -\frac{e^{ib\cdot\bar{k}}}{2\pi\sqrt{\gamma^2-1}} K_0\left(\sqrt{-b^2}u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\right). \tag{C22}$$

Similarly for I_4^μ,

$$I_4^\mu = I_{4a}^\mu + I_{4b}^\mu, \tag{C23}$$

with

$$\begin{aligned} I_{4a}^\mu &= -\frac{u_1\cdot\bar{k}}{\gamma^2-1}(u_1^\mu - \gamma u_2^\mu) I_3 \\ &= \frac{u_1\cdot\bar{k}e^{ib\cdot\bar{k}}}{2\pi(\gamma^2-1)^{3/2}}(u_1^\mu-\gamma u_2^\mu)K_0\left(\sqrt{-b^2}u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\right), \\ I_{4b}^\mu &= \frac{iu_1\cdot\bar{k}e^{ib\cdot\bar{k}}\tilde{b}^\mu}{2\pi(\gamma^2-1)} K_1\left(\sqrt{-b^2}u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\right). \end{aligned} \tag{C24}$$

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