Protecting local and global symmetries in simulating $(1+1)$D non-Abelian gauge theories

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Efficient quantum simulation protocols for any quantum theories demand efficient protection protocols for its underlying symmetries. This task is nontrivial for gauge theories as it involves local symmetry/invariance. For non-Abelian gauge theories, protecting all the symmetries generated by a set of mutually noncommuting generators, is particularly difficult. In this paper, a global symmetry-protection protocol is proposed. Using the novel loop-string-hadron formalism of non-Abelian lattice gauge theory, we numerically demonstrate that all of the local symmetries get protected even for large time by this global symmetry protection scheme. With suitable protection strength, the dynamics of a $(1+1)$-dimensional SU(2) lattice gauge theory remains confined in the physical Hilbert space of the theory even in presence of explicit local symmetry violating terms in the Hamiltonian that may occur in both analog and digital simulation schemes as an error. The whole scheme holds for SU(3) gauge theory as well.

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I. INTRODUCTION

Gauge theories form the backbone of most of modern physics, ranging from the standard model of particle physics to condensed matter systems [1,2]. Typical calculations/computations for these theories are done via the path-integral approach that suits both analytical and numerical studies. Lattice regularized versions of gauge theories [3] provide an extremely useful platform to perform nonperturbative calculations using Monte-Carlo simulation toward understanding the strong interaction described by quantum chromodynamics (QCD) [4]. Despite its usefulness, the numerical lattice-QCD program faces a roadblock due to sign problem toward certain aspects of computation, such as computing dynamics or exploring the full phase diagram of QCD [5]. Of late, an interdisciplinary community has ramped up efforts to combine aspects of quantum technology with that of lattice gauge theories to tackle some of these outstanding problems [6–14]. There have been significant efforts to find suitable reformulations of lattice gauge theories within the Hamiltonian formulations to make them amenable to quantum simulations in noisy intermediate scale quantum (NISQ) era [15] devices and beyond [16–37]. Concurrently, there have been efforts for both digital and analog quantum simulations of LGT's on currently available technologies as well as proposals for the same attainable in near future [11,22,38–68].

The natural framework to study dynamics of gauge theories without any sign problem is a canonical/Hamiltonian framework. The Hamiltonian description of lattice gauge theories, put forth by Kogut and Susskind [69], provide such a framework. However, the exponentially growing dimension of the Hilbert space with lattice size, is not suitable for classical computation, and that is where quantum simulation/computation is expected to be useful. However, one additional complication that accompanies the gauge theory Hamiltonian is the local constraints that generate the gauge transformation. Preserving the gauge invariance of the simulated theory with state of the art quantum hardware is a major challenge in quantum simulating lattice gauge theories unless the gauge invariance arise naturally for a particular simulation protocol as in [18,48,62]. This article presents a protocol to protect the gauge invariance for two dimensional gauge field theories with a continuous non-Abelian gauge group such as SU(2) such that quantum simulation of the same is possible without imposing any additional symmetry that mimics gauge invariance.

Protecting all of the symmetries of the theory should lead to an efficient quantum simulation for the same [70]. The physical or gauge invariant Hilbert space for a gauge theory is defined to satisfy a set of local constraints, known as the Gauss law constraints. The Hamiltonian being gauge invariant, commutes with these constraints, keeping the dynamics confined in the physical Hilbert space. Construction of the gauge invariant Hilbert space of a

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gauge theory is a nontrivial task and becomes particularly involved/expensive for non-Abelian gauge groups. In Hamiltonian simulation on a classical computer, this results in an exponential rise in computational complexities [28]. Alternatively, there have been proposals to obtain alternate frameworks that utilize the Gauss law constraints in elimination of fermionic matter fields [35,37,44]. In terms of analog quantum simulation, imposing the Gauss law constraint is an additional burden in the simulation protocol [18,34,40,71,72] as the local constraints are not necessarily manifested in the simulating quantum mechanical Hamiltonian. Besides gauge invariance, a theory can have several global symmetries respected by the Hamiltonian, and the dynamics remain confined in each superselection sectors [56]. Restricting the dynamics in the physical Hilbert space as well as in a particular superselection sector is essential, yet a challenging task in the NISQ era quantum simulations due to the erroneous quantum hardware [22,46,56,59,65,73–78].

This has sparked a sharp interest in the past years to find out symmetry protection protocols in a quantum simulation [70,79–81] that is crucial for experimental demonstration of simulating a gauge theory, achieved over the past half a decade [54,61–64,67,68,82] and many more to be achieved in future. Recently, there have been proposals of studying breaking of gauge-invariance in Abelian gauge theories [81]. Halimeh *et al.* studied $\mathbb{Z}_2$ and U(1) gauge theory, whose theoretical and numerical modelling was based on an experimental realization of the same theory [82]. Furthermore, a non-Abelian study of gauge breaking was also proposed [80] that involves a single body protection protocol. In this work, we demonstrate that the complete symmetry protection in the dynamics of $1+1$ dimensional non-Abelian gauge theories can indeed be achieved using a comparatively less involved, namely a global protection protocol.

The original Kogut-Susskind Hamiltonian for SU(2) gauge theory has been recently mapped to a novel loop-string-hadron (LSH) framework [21] without losing any generality. The corresponding Hilbert space is spanned by a set of locally defined LSH basis vectors characterized by manifestly SU(2) invariant local quantum numbers. The Hamiltonian contains both diagonal and ladder operators acting locally on the LSH states. The LSH basis, being gauge-invariant as well as one-sparse, significantly reduces the computational cost in Hamiltonian simulation [28] and has been found to act as a suitable framework for quantum simulation using both analog [60] and digital schemes [59] in the recent past. Albeit being local and SU(2) invariant, the notion of nonlocality of the physical observables of a gauge theory, such as loops or strings, is still contained in the LSH framework in terms of a set of "on-link" constraints that is referred as Abelian Gauss law (AGL) constraints in the literature. The physical Hilbert space of the SU(2) gauge theory is thus constructed by local (on-site) LSH states weaved by local (on-link) AGL constraits.

The Hamiltonian simulation, based on the LSH framework, hence requires imposing local AGL constraints in order to confine dynamics within the physical Hilbert space [59,60]. Following the work of Halimeh *et al.* [81], we propose a AGL protection term for the LSH Hamiltonian, and explore its implications. Interestingly, the results of this paper demonstrates that, for two dimensional non-Abelian gauge theories, protecting a single Abelian global symmetry results in the complete protection of all of the local symmetries generated by AGL.

The organization of the paper is as follows: The consecutive sections describe the Hamiltonian, Hilbert space and the symmetries of the theory. A symmetry violating term in the Hamiltonian is introduced next and it is discussed how one can protect the symmetry. The protection scheme is validated by numerical evidences of protecting all the symmetries of the theory within this protocol. This work contains explicit results for SU(2), that can be generalized to SU(3).

II. HAMILTONIAN FRAMEWORKS FOR GAUGE THEORIES

The core ingredients necessary to study dynamics of a theory is a Hamiltonian and the Hilbert space. Lattice regularized version of gauge theories [3] provide a convenient path integral framework defined on a discretized space-time lattice that allows numerical simulation in order to go beyond the analytically solvable regime of the theory. However, an equivalent Hamiltonian simulation is more natural with the aim of computing real time dynamics of the theory without any sign problem. The Hamiltonian framework developed by Kogut and Susskind [69] provides a framework for such study.

A. Gauge symmetries and physical Hilbert space

The Hamiltonian is obtained from the Lagrangian by a temporal gauge fixing ($A_0(x) = 0, \forall x$) that results in temporal direction to be continuum while the spatial directions remain discretized. An additional structure that is associated with the Hamiltonian due to this particular gauge fixing is a constraint, namely the Gauss law constraints. These constraints are defined at every spatial point, for each color degrees of freedom for a gauge theory. In this work, we consider a SU(2) gauge group, that comes with three color degrees of freedom and hence there exists three Gauss law constraints $G^a(x)$ for $a = 1, 2, 3$ and for all site x. These Gauss law constraints are actually generators of the gauge transformation at each spatial point. By construction the Hamiltonian must commute with the constraints, i.e.,

$$[H, G^a(x)] = 0, \quad \forall \ a, x. \tag{1}$$

The above relation stands truly nontrivial for a non-Abelian group such as SU(2) where all of the generators are mutually noncommuting. The physical Hilbert space of the theory

consists of states, those are annihilated by the Gauss law constraints, i.e.,

$$G^a(x)|\Psi\rangle = 0 \Rightarrow |\Psi\rangle \in \mathcal{H}_{\text{phys}}, \quad \forall \ a, x. \tag{2}$$

The above two relations automatically suggest that if one prepares the initial state to lie in $\mathcal{H}_{\text{phys}}$, the dynamics of the theory will remain confined in $\mathcal{H}_{\text{phys}}$, that is a constrained sector of the whole gauge theory Hilbert space.

B. Hamiltonian dynamics: The challenges in simulation

The details of the Hamiltonian is discussed in Appendix A. Here we only highlight the important features of the same. The canonical conjugate variables in Kogut-Susskind Hamiltonian formulation are the three components of SU(2) colored electric fields $E^a_{L/R}(x)$ defined at both the ends L/R of a lattice and a holonomy or link operator $U_{\alpha\beta}(x)$ responsible for the parallel transport of gauge fields on a link connecting site x and $x + \hat{i}$ in spatial direction i. The link operators are group elements, that is written in the fundamental representation, as a 2×2 matrix for SU(2). The canonical commutation relations of these variables are given in Appendix A.

The principle of gauge invariance implies each term of the Hamiltonian to be color singlet. The electric field operators, being in the adjoint representation are combined as

$$\sum_{a=1,2,3} E^a_L(x) E^a_L(x) = \sum_{a=1,2,3} E^a_R(x) E^a_R(x) = E^2(x) \tag{3}$$

that provides the local contribution to the electric term of the Hamiltonian. The link operator, being in the fundamental representation can be coupled to the fundamental matter fields present at each lattice site to construct the matter-gauge interaction terms in the K-S Hamiltonian. In higher dimension, there exists further scope of forming gauge singlet operators by considering path ordered product of holonomies around a closed path and taking trace of the same, that is known as the Wilson loop operators. The smallest Wilson loop operator, i.e., loop around a plaquette, is the smallest possible closed path contribution to the Hamiltonian. However, as we are confined to only one spatial dimension for this work, this particular term do not appear in the Hamiltonian.

It also turns out to be a consequence of the principle of local gauge invariance that, all of the gauge invariant operators such as Wilson loops/ strings discussed above, are nonlocal. The gauge invariant states, defined by the action of such operators on "vacuum" are also nonlocal. Hence, one major obstacle to study Hamiltonian dynamics of a gauge theory is the fact that the physical Hilbert space is spanned by nonlocal basis vectors that is also over-complete. This has been one of the major obstacle in performing dynamical calculations of gauge theory, apart from the fact that the Hilbert space grows exponentially with the system size. For both the reasons, classical computers fail to provide enough resource. With the advent of quantum technology, handling exponential growth of the Hilbert space dimension in a quantum simulation seems feasible.

In order to quantum simulate only the physical dynamics, imposing the gauge symmetries is an essential task in terms of state preparation as well as error mitigation. Primarily, simulating physical dynamics involves working with nonlocal basis vectors comprising of the Wilson loops and strings as described earlier, and is indeed nontrivial. It is precisely the reason behind the loop approach to gauge theory [83] being not very successful. In addition to this, preparing gauge invariant state and maintaining gauge invariance (that involves preserving gauge symmetry at each spatial point of the system at all time steps) at each stage of the simulation is another difficult task especially in the NISQ era. In this work, we propose an effective solution to both of these issues.

C. A suitable framework: Loop-string-hadron framework

As mentioned earlier, the gauge invariant operators and states of a gauge theory are nonlocal. A series of developments in reformulating Hamiltonian lattice gauge theories in terms of prepotential operators over the past two decades [84–91] has led to an efficient and novel framework for calculating Hamiltonian dynamics in terms of the minimal and physical degrees of freedom that are local. Skipping all the microdetails of the construction, a part of which has been summarized in Appendix B, we make an effort to highlight the essential features of the loop-string-hadron (LSH) framework [21].

(1) The LSH states are defined locally at each lattice site and are manifestly gauge invariant. A local state, at a particular site is nothing but an on-site snapshot of the loops and strings passing through that site and are part of a global configuration of gauge invariant states of the lattice.
(2) A generic local snapshot may contain flow of electric flux from any direction to any direction of the lattice. The total flux flowing along any such direction is an integer multiple of $j = 1/2$ unit of flux for SU(2) gauge theory. Hence, a local loop can be characterized by a set of integers $(= 2j)$ for each pair of directions on the lattice. For one spatial dimension, there will be only one such direction and hence only one loop quantum number $n_l \in (0, \mathbb{Z}_+)$.
(3) Presence of fermions at each site in a gauge invariant way can occur in the following general cases:
    (a) A string, coming from any direction can start or end at a site and the string end can couple to the fermion to form a singlet that satisfy Gauss law. For non-Abelian gauge theory this involves forming an intertwiner involving $j = 1/2$ unit of flux

and a two component fermion. For one spatial dimension, there can only be an incoming string end or an outgoing string end located at each site.

(b) Two fermions located at each site can form a gauge singlet object that is a hadron at each site. In the LSH framework, the simultaneous presence of an incoming and an outgoing string at a site is equivalent to having a hadron [21].

(4) The collection of local snapshots of loops, strings and hadrons taken at each site would correspond to a valid global gauge invariant configuration of Wilson loops and strings if and only if there is a continuity of bosonic flux across neighboring sites. Hence, the total bosonic flux coming out of a site in any direction should equal to the total incoming bosonic flux at the next lattice site along that direction. Satisfying this conditions leads to a set of Abelian constraints between the LSH states across the links that is called the Abelian Gauss law constraints. This is discussed in detail in the later part of the work.

The goal of this work is to find a symmetry protection protocol in simulating the dynamics of gauge theories. As discussed above, doing the same in the original framework amounts to constructing the loop space and calculating the dynamics within the space of nonlocal loops. This is a nontrivial task for non-Abelian gauge theories. The LSH framework discussed above is expected to solve an array of difficulties in the same. We illustrate that for $d = 1$ case in the rest of the paper.

D. Loop-string-hadron dynamics

As mentioned earlier, the local LSH state captures the on-site snapshot of Wilson loops-strings and hadrons. For one spatial direction, the flux can only flow in one direction, the quanta of which is denoted by a positive semidefinite integer n_l. There can be a possible incoming and outgoing string end present on that site that correspond to (0,1) values for each of the two quantum numbers n_i, n_o. The LSH Hilbert space is thus characterized by the local basis states

$$|n_l, n_i, n_0\rangle_{(x)} \quad \forall\, x \tag{4}$$

where $0 \leq n_l \leq \infty; n_i, n_o \in \{0, 1\}$, designating the loop quantum number n_l to be bosonic and string quantum numbers n_i, n_o to be fermionic. Presence of a hadron is denoted by nonzero values of both the string quantum numbers in the LSH basis [21]. It is important to note that, the LSH basis is exactly equivalent to the particular linear combination of the angular momentum states at each site that satisfy the Gauss law constraints [28].

The global LSH Hilbert space, however is not a direct product of the local Hilbert spaces at each site. In order to identify the global LSH Hilbert space to contain physical states such as Wilson loops and strings, a constrained projection is considered. For SU(2) gauge theory, one single Abelian constraint per link serves the purpose and in one spatial dimension, that is given by:

$$n_l + n_o(1 - n_i)|_x = n_l + n_i(1 - n_o)|_{x+1} \tag{5}$$

where, n_l, n_i, n_o on both sides of the above equation are the LSH quantum numbers at site x and $x + 1$ respectively.

The development of the LSH framework has led to the original KS Hamiltonian written in the LSH basis. As described earlier, the Hamiltonian of the theory in $1 + 1$ dimension involve the terms:

$$H = H_{\text{electric}} + H_{\text{mass}} + H_{\text{interaction}} \tag{6}$$

that correspond to the contributions from total color-electric flux flowing in the lattice, mass of staggered fermions and matter-gauge interactions respectively. The details of the LSH Hamiltonian can be found in [21] and also in Appendix B.

The original KS Hamiltonian is local, i.e., each of the terms of the Hamiltonian has a contribution from each lattice site, and the full Hamiltonian is sum of all the on-site Hamiltonian terms. This fits into the LSH framework perfectly. A set of LSH operators are defined at each site that include: (i) LSH occupation number operators, and (ii) LSH ladder operators that acting on a LSH state, changes the LSH quantum numbers by ± 1 unit. The electric and mass terms are diagonal in the LSH basis, and is equivalent to a combination of LSH occupation number operators as given in (B26) and (B27). The matter gauge interaction term for 1-d spatial lattice is responsible for interesting dynamics of the theory. This particular term, being gauge invariant, acting on the strong coupling vacuum (a gauge invariant state) build up the gauge invariant Hilbert space. This term within the LSH framework [21] consists of local SU(2) invariant creation/annihilation operators corresponding to "string-ends" located at nearest neighbor sites.

$$H_{\text{int}} = x_0 \sum_x \hat{\eta}(x) \left[\sum_{\sigma=\pm} \mathcal{S}^{+,\sigma}_{\text{out}}(x) \mathcal{S}^{\sigma,-}_{\text{in}}(x+1) \right] \hat{\eta}(x+1) + \text{H.c.}, \tag{7}$$

where, x_0 is a dimensionless coupling and $\hat{\eta}(x)$ are diagonal operators in the LSH basis and is function of the LSH occupation number operators N_l, N_i, N_o defined in the Appendix B.

The string-end operators $\mathcal{S}^{\pm,\sigma}_{\text{out}}(x)$ and $\mathcal{S}^{\sigma,\pm}_{\text{in}}(x)$ are manifestly SU(2) singlet operators with both bosonic and fermionic field content that denote start/end of a string (out/in). These string-end operators create/annihilate ($\pm$) a fermionic field at both its ends and changes the electric flux on both ends of the link connecting neighboring sites in order to preserve local SU(2) invariance. The string end operators again factorize in terms of the LSH ladder operators as shown in (B19)–(B22). Acting on LSH states,

the interaction Hamiltonian changes the LSH quantum numbers at two neighboring sites as described in detail in the appendix. However, such interaction must preserve continuation of the bosonic flux lines across neighboring sites aka the AGL given in (5). In (7), this is guaranteed by the same σ index in both the string end terms located at sites x and $x+1$. The index $\sigma = \pm$ denotes creation/annihilation of one unit of electric flux along the link connecting sites x and $x+1$. As a consequence of (7) satisfying the AGL, the functions $\hat{\eta}(x)$ and $\hat{\eta}(x+1)$, being functions of the diagonal or occupation number LSH operators, yield identical eigenvalues while acting on a AGL satisfying state, and does not cause any violation of AGL.

III. THE SYMMETRIES

The primary symmetry that a gauge theory must satisfy is the gauge symmetry, imposed by a set of local constraints. For SU(2) gauge theory, the LSH framework preserves the local SU(2) symmetry by its manifestly SU(2) invariant operators and Hilbert space construction. However, as discussed earlier, the LSH Hamiltonian in $1+1$-d contains the following additional symmetries listed below:

(1) Local symmetries imposed by AGL: The LSH dynamics must satisfy the local "on-link" constraint:

$$n_{\text{out}}(x) = n_{\text{in}}(x+1) \tag{8}$$

where, $n_{\text{out/in}}(x)$ denotes the eigenvalues of the total bosonic occupation numbers (defined as $\mathcal{N}_L$ and $\mathcal{N}_R$ for a link connecting x and $x+1$ in Appendix B) at outgoing/ incoming links at a site x. In terms of the local LSH quantum numbers $\{n_l(x), n_i(x), n_o(x)\}$, these are obtained as:

$$n_{\text{out}}(x) = n_l(x) + n_o(x)(1 - n_i(x)); \tag{9}$$

$$n_{\text{in}}(x) = n_l(x) + n_i(x)(1 - n_o(x)). \tag{10}$$

(2) Global symmetries: In addition to the SU(2) gauge symmetries, any gauge theory also admits a set of global symmetries governed by a global SU(2) group. Being a non-Abelian group, the generators of SU(2) do not commute. The complete set of commuting observable (CSCO) for SU(2) contains two operators, namely the total angular momentum and the z-component of angular momentum in the angular momentum representation for the same. The LSH Hamiltonian, discussed before also admits a set of global SU(2) symmetries [28] and hence a global LSH state is characterized by two global quantum numbers corresponding to the CSCO of global SU(2). In the LSH framework, these two global quantum numbers are given by:

(a) Total fermionic occupation number:

$$Q = \sum_{x=0}^{N-1} [n_i(x) + n_o(x)] \tag{11}$$

For a N-site lattice, the value of Q can be any integer between $[0, 2N]$.

(b) The imbalance between incoming and outgoing strings: relates to the boundary fluxes

$$q = \sum_{x=0}^{N-1} [n_0(x) - n_i(x)] \tag{12}$$

For a particular Q value, q can take any value from $-Q$ to $+Q$ and defines different disconnected sectors of the larger gauge-invariant LSH Hilbert space.

The LSH Hamiltonian obeys both the Q, q symmetries and in turn results in a block diagonalized structure. The dynamics of the theory remain confined within each block, enabling computational benefit [28].

(3) Charge conjugation symmetry: The particle antiparticle symmetry of the theory identifies (Q, q) sector of the Hamiltonian to the $(Q, -q)$ sector.

A. An observation: Relating local and global symmetries

The electric and mass term of the LSH Hamiltonian are local as well as diagonal in the LSH basis, implying that they preserve all the local and global symmetries of the Hamiltonian discussed before. As previously mentioned, the AGL invariance of the LSH Hamiltonian is manifested by the same σ indices for both the ends of the string operator in (7). Appendix B contains the details of expressing the interaction Hamiltonian in terms of normalized ladder operators corresponding to the LSH quantum numbers n_l, n_i, n_o at each site. Interestingly the fermionic content of the particular combination of string end operators at two neighboring sites x, $x+1$ is obtained as:

$$\mathcal{S}^{++}_{\text{out}}(x)\mathcal{S}^{+-}_{\text{in}}(x+1) \approx \chi^{\dagger}_o(x)\chi_o(x+1) \tag{13}$$

$$\mathcal{S}^{+-}_{\text{out}}(x)\mathcal{S}^{--}_{\text{in}}(x+1) \approx \chi^{\dagger}_i(x)\chi_i(x+1) \tag{14}$$

where, the action of the normalized ladder operators on the local LSH quantum numbers are realized as[1]:

[1]In (13), (14), the left- and right-hand sides are related by $\approx$, as the full equality includes suitable ladder operators in the loop quantum number n_l as well as normalization factors as given in (B19)–(B22). The approximated form, illustrating the fermionic content is sufficient for understanding the symmetry protection protocol developed in this work. However, the numerical calculations use the exact expression.

$$\chi_i^\dagger|n_l,n_i,n_o\rangle = (1-n_i)|n_l,n_i+1,n_o\rangle \tag{15}$$

$$\chi_i|n_l,n_i,n_o\rangle = n_i|n_l,n_i-1,n_o\rangle \tag{16}$$

$$\chi_o^\dagger|n_l,n_i,n_o\rangle = (1-n_o)|n_l,n_i,n_o+1\rangle \tag{17}$$

$$\chi_o|n_l,n_i,n_o\rangle = n_o|n_l,n_i,n_o-1\rangle \tag{18}$$

From the above set of equations, one can readily suggest that the pairwise presence of the outgoing string operator at site x and the incoming string operator at site $x+1$, with the same σ index, i.e., $\sigma = +$ for (13) and $\sigma = -$ for (14) in effect preserves the total number of n_o excitation and n_i excitation respectively on a pair of neighboring sites. The entire Hamiltonian, with interaction terms present for each and every neighboring site, that satisfy the AGL, in turn preserves the global quantum numbers $\sum_x n_i(x)$ and $\sum_x n_o(x)$ for the lattice.

IV. VIOLATING THE SYMMETRIES

The symmetries of the SU(2) gauge theory, as discussed before includes the local as well as the global SU(2) symmetries as described above. The LSH framework manifestly solves the local SU(2) Gauss law constraints and is left with a local Abelian symmetry designated as the AGL mentioned before. Even though working within the LSH framework guarantees the complete protection of the non-Abelian gauge symmetries, the remnant AGLs are still prone to violation from several possible sources of error that is present for both analog and digital simulation schemes. Hence, those errors would definitely take the dynamics away from the physical Hilbert space. As an example, the analog simulation scheme of simulating LSH dynamics (as in [60]), where the two fermionic LSH degrees of freedom are mapped to up/down spins of the neutral atoms respectively may experience spin-flip error, which would result in violation of AGL in the dynamics as well as would couple multiple superselection sectors of the theory. For a digital simulation scheme, bit-flip errors would also cause a similar AGL violation. Such a consideration for a digital simulation has previously led to the construction of Gauss law oracles [59,73] to check for such error. We model this particular error by adding an extra term in the LSH Hamiltonian that would always violate the AGL and drag the dynamics away from the physical Hilbert space.

A term that would not satisfy AGL can be the same interaction Hamiltonian (7), but not necessarily with the same σ index as given below:

$$\hat{H'}_{\text{int}} = x_0 \sum_x \hat{\eta}(x) \left[\sum_{\sigma,\sigma'=\pm} \mathcal{S}_{\text{out}}^{+,\sigma}(x)\mathcal{S}_{\text{in}}^{\sigma',-}(x+1) \right] \hat{\eta}(x+1) + \text{H.c.} \tag{19}$$

Note that, violation of AGL would imply the eigenvalues of the diagonal operators $\hat{\eta}(x)$ and $\hat{\eta}(x+1)$ can be different, unlike the case in (7). With this particular interaction Hamiltonian, acting on the strong coupling vacuum, a local SU(2) invariant Hilbert space would be created. However, the global Hilbert space would rather be a tensor product space of individual on-site Hilbert spaces and the dynamics would be spanned all over that space. However, as mentioned before, the physical Hilbert space is only a constrained surface of the tensor-product space that satisfy AGL on each and every link. Note that, (19) contains the desired AGL preserving interaction (7) as well as terms that violate AGL with equal weight, but can be tuned separately to model individual cases as per the simulation schemes. In the next subsection we will illustrate a scheme for protecting the local symmetries of the theory generated by the AGL. In other words, we propose a scheme to simulate the constrained gauge theory dynamics on a tensor product Hilbert space without imposing any local constraint.

V. PROTECTING THE SYMMETRY

As elaborated in Sec. III A, the realization of AGLs in the LSH frameowrk in effect leads to the conservation of global quantities $\sum n_i(x)$ and $\sum n_o(x)$, that are the linear combination of global conserved quantities mentioned in (11) and (12). The aim of this work is to exploit this. We propose a symmetry protection protocol that protects the AGL in the dynamics. The global symmetry sectors of the theory results in a block diagonal structure for the Hamiltonian matrix written in a basis, such as the LSH basis and the dynamics in each sector remain confined in the same as long as the global symmetry is protected. The symmetry protection protocol presented in this work, projects the dynamics in one of the global symmetry sectors of the theory and in turn the AGL is also protected in the dynamics happening in that sector.

The interaction Hamiltonian given in (19) contains (7) that preserves AGL, but also contain contributions (for $\sigma \neq \sigma'$) that violates the same. As argued in Sec. III A, the AGL violation can also be related to global symmetry violation and it would result in mixing of the superselection sectors of the theory. This can indeed be realized if one expresses the AGL violating contributions in (19) in terms of the normalized ladder operators given in (15)–(18) as:

$$\mathcal{S}_{\text{out}}^{++}(x)\mathcal{S}_{\text{in}}^{--}(x+1) \approx \chi_o^\dagger(x)\chi_i(x+1) \tag{20}$$

$$\mathcal{S}_{\text{out}}^{+-}(x)\mathcal{S}_{\text{in}}^{+-}(x+1) \approx \chi_i^\dagger(x)\chi_o(x+1). \tag{21}$$

From (20) and (21), it is evident that the quantities $\sum_x n_i(x)$ and $\sum_x n_o(x)$ are not being conserved in presence of the interaction Hamiltonian (19).

It is also evident that, among the two globally conserved quantities defined in (11) and (12), these AGL violating

terms still preserve the global quantity Q likewise (7) as per (13), (14), (20), (21) and (15)–(18).

Hence, we relate that the violation of all of the local constraints is practically equivalent to violation of the global q symmetry given in (12).

In this regard, we propose that the dynamics of an erroneous Hamiltonian given by:

$$H' = H_{\text{electric}} + H_{\text{mass}} + H'_{\text{interaction}}$$

can be made to remain confined in the physical subspace consisting of Wilson loops, strings and hadrons by adding the following global protection term that protects the q symmetry to the above Hamiltonian:

$$H^q_{\text{protect}} = \Lambda\left[q - \sum_{x=0}^{N-1}(N_i(x) - N_o(x))\right], \tag{22}$$

where, q can be any integer in $[-Q, Q]$. In general, the erroneous Hamiltonian that is more realistic does not have any block diagonal structure in terms of q. However, with this particular protection scheme, the dynamics is made to occur in each of those superselection sectors as determined by the protection term. The bonus of projecting the dynamics to any of the particular Q, q sector is the automatic validation of the AGL constraints throughout the lattice as demonstrated by the numerical results in the next section.

Note that, the whole protection scheme is based on the principle of using Lagrange multipliers to find the low energy sector of the theory. The dynamics at larger is expected to get more contribution from the lower energy sprctrum. We demonstrate this to be the case through numerical calculation and establish effective dynamical protection of symmetries under the scheme described above in the next section.

VI. NUMERICAL DEMONSTRATION OF THE SYMMETRY PROTECTION PROTOCOL

Numerical benchmarking of the symmetry protection scheme described above is carried out using exact diagonalization on a 4-site lattice. The time evolution of a physical state under the erroneous Hamiltonian H' along with the protection term, i.e.,

$$\tilde{H} = H' + H^q_{\text{protect}} \tag{23}$$

is studied for different ratios of the dimensionless couplings x_0/Λ, where, x_0 is the dimensionless coupling coefficient with the interaction Hamiltonian in (19) and Λ is the dimensionless protection parameter in (22).

The following two observable are considered:

(1) Global observable:

$$q(t) = \langle\Psi(t)|\sum_{x=0}^{N-1}[N_i(x) - N_o(x)]|\Psi(t)\rangle$$

(2) Local observable, that measures AGL for each link connecting sites x and $x+1$:

$$AGL_{x,x+1}(t) = \langle\psi(t)|[\mathcal{N}_L(x) - \mathcal{N}_R(x+1)]|\psi(t)\rangle \tag{24}$$

where,

$$\begin{aligned}&\mathcal{N}_L(x)|n_l, n_i, n_o\rangle_x \\ &= [n_l + n_o(1-n_i)]_x|n_l, n_i, n_o\rangle_x \\ &\mathcal{N}_R(x+1)|n_l, n_i, n_o\rangle_{x+1} \\ &= [n_l + n_i(1-n_o)]_{x+1}|n_l, n_i, n_o\rangle_{x+1}\end{aligned}$$

by exact diagonalization technique for a 4-site lattice, and the time evolution of these two observable is studied using QuSpin [92,93] for sufficiently large time T at small intervals of $0.0001T$. The results presented are for open boundary condition, with zero incoming flux at the 0^{th} lattice site.

The numerical result is summarized below:

(i) Figure 1 demonstrates that $\langle q(t)\rangle$, time average of the global quantum number q converges to the value $\langle q(t)\rangle = q$ by using protection Hamiltonian H^q_{protect} with increasing protection strength, (i.e., decreasing x_0/Λ) for $q = 0, 2, 4$.

(ii) Figure 2 demonstrates that $\langle \text{AGL}_{x,x+1}(t)\rangle$, time average of the AGL quantum number for each link $(x, x+1)$ converges to the value 0 by using protection Hamiltonian H^q_{protect} with increasing protection strength, (i.e., decreasing x_0/Λ) for all the values of $q = 0, 2, 4$.

The erroneous Hamiltonian H' considered here does not possess any q-symmetry nor is it block-diagonalized for different q sector. The protection term in the Hamiltonian, determines the q value of the low energy sectors of the theory. Note that, in practice, the dynamics of all the states in the direct product Hilbert space is being projected to each superselection sectors defined by a common q-value. The numerical demonstration of this is presented in Fig. 1, where the time averaged value of the global observable is shown to approach the exact value q with increasing protection strength. Figure 2 does not require any additional numerical computation. The same calculation demonstrating q-protection yields simultaneous demonstration of observing AGL conservation in the simulation. For this purpose, the time averaged value of the AGL is measured on each link and for each of the q-protection. Figure 2,

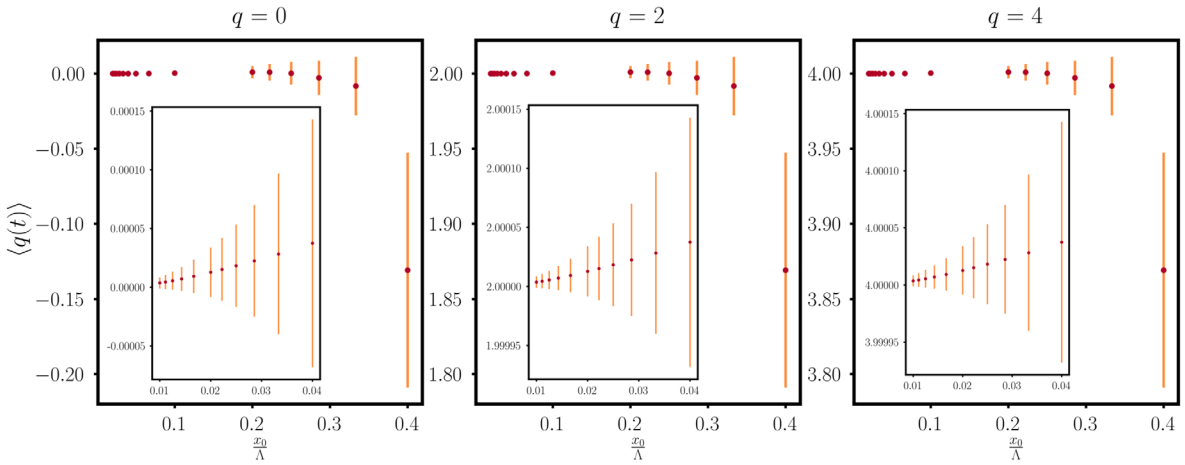

FIG. 1. Time average (for long time) of the global quantum number $q = \sum_x (n_o(x) - n_i(x))$ is plotted against the ratio of x_0, the dimensionless couplings that come with the symmetry violating interaction, and Λ the coupling with the symmetry protection term. The error bar denotes the deviation of the observed value from the mean time averaged value throughout the dynamics. With stronger protection strength, i.e., $\frac{x_0}{\Lambda} \to 0$ the global symmetry is better protected as manifested in the zoomed in region shown in the inset. The plots illustrate that the dynamics is successfully projected within each global symmetry sector $q = 0, 2, 4$ with each protection term H^q_{protect}.

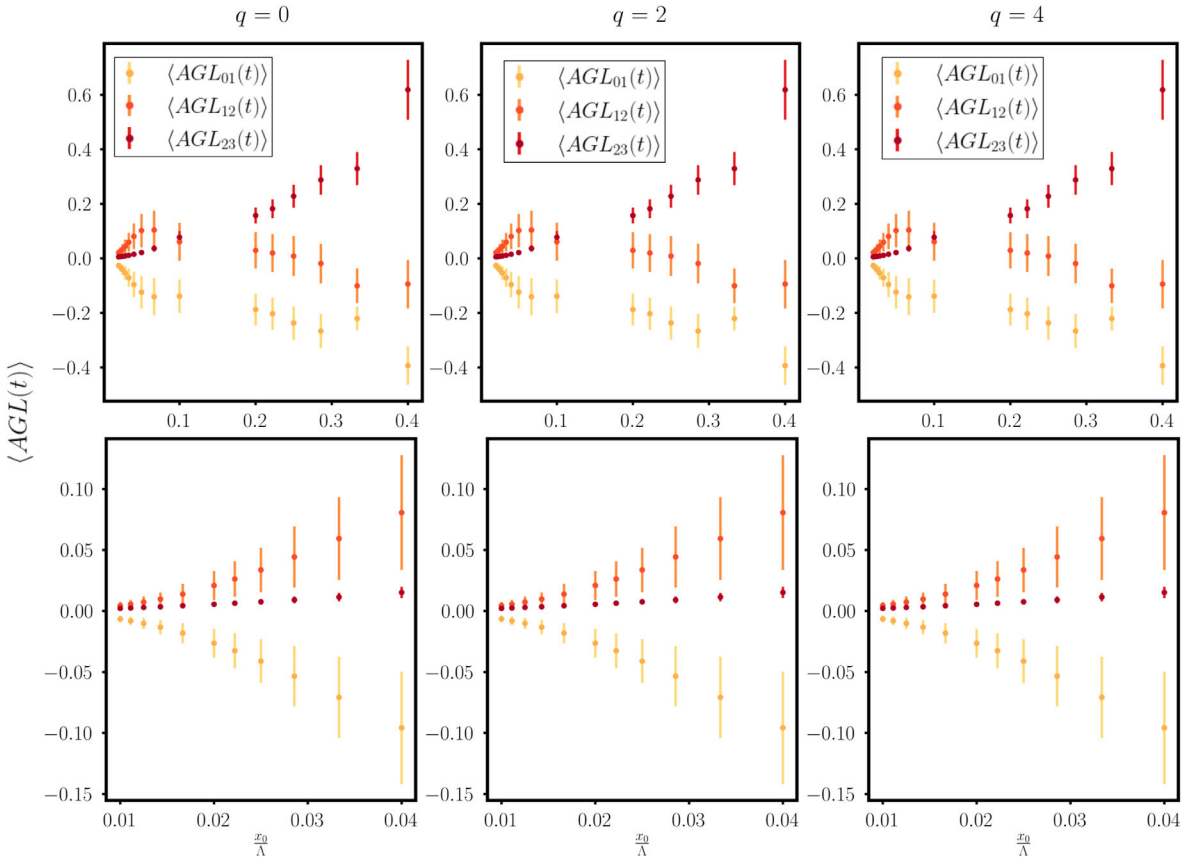

FIG. 2. The time averaged value of the AGL observables $\text{AGL}_{x,x+1}$ on each link of the 4 site lattice, while time evolved with the Hamiltonian that contains local AGL violating interaction terms with strength x_0 and global symmetry protection term with strength Λ, is plotted against x_0/Λ. For any q-sector of the protected global symmetry, the Abelian Gauss laws on all of the links are perfectly protected with increasing protection strength. The error bars denote the fluctuation of the observables from its mean value during the evolution. The second row of plots demonstrate systematically better protection (including systematic decrease in fluctuations as well) of all of the Abelian Gauss laws as x_0/Λ is decreased up to $1/100$ for a 4-site lattice.

demonstrates that, irrespective of the chosen q-sector, the AGL is thoroughly protected by the global q-protection scheme with sufficiently large protection strength. Hence, a simulation of the Hamiltonian without any local constraint yields the dynamics of the physical observables with the proposed protection scheme.

VII. CONCLUSION AND OUTLOOK

This article presents a novel idea of simulating the physical dynamics of a non-Abelian gauge theory in two dimensions described by a local Hamiltonian without imposing any local constraints. This was only made possible within the LSH framework, where the complete Abelianization of SU(2) gauge symmetries have been performed without introducing any nonlocal interaction. For $1+1$ dimensional case, the whole set of local Abelian symmetries has been made captured by a single global symmetry that is manifested in the LSH Hamiltonian construction as well. The whole study of protecting the symmetry is based on the idea of imposing a constraint in Hamiltonian dynamics with a large value of Lagrange multiplier. The current study on small lattice can be easily generalized to use state-of-the-art tensor network techniques to probe for larger system [94]. Tensor network ansatz can approximate ground states for 1D lattice theories [95]. Several efforts have been put toward that direction in context of Abelian and non-Abelian gauge theories that includes but does not exhaust [23,96–100]. Encoding global symmetries within the tensor network formulation is possible, and since gauge breaking in the LSH formulation is a global symmetry violation, it is possible to use the tensor network formulation to study these dynamics. It might even be possible to explore larger lattice sizes so that one can study the scaling effect of the protection strength.

This particular study removes a vast set of the difficulties (regarding imposing all the constraints) in quantum simulating non-Abelian gauge theories in lower dimension within the scope of NISQ era devices and also for tensor network calculations as the framework becomes free of local symmetries and yet the interactions remain local. A similar global protection scheme for a SU(3) gauge theory [87,91] is also possible to construct, provided there exists a LSH framework for SU(3) as well. Another important aspect of this study is to provide clear insight into the entanglement structure for a $1+1$-dimensional non-Abelian gauge theory. Works are in progress in these directions and will be reported shortly.

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APPENDIX A: KOGUT-SUSSKIND FORMULATION

The Hamiltonian formulation of lattice gauge theory was first put forth by Kogut and Susskind [69]. Here, the temporal direction is taken to be continuous while the spatial directions are discretized. The Hamiltonian for SU(2) LGT in $1+1$ D consists of the following terms:

$$H_{KS} = H_I + H_E + H_M \tag{A1}$$

Here, the H_I denotes the interaction term between the fermionic matter and the gauge degrees of freedom. The fermionic content is split into staggered sites, with matter and antimatter fields occupying even and odd sites respectively. The interaction part is of the following form

$$H_I = x_0 \sum_{x=0}^{N-2} [\Psi^\dagger(x) U(x) \Psi(x+1) + \text{H.c.}] \tag{A2}$$

where x_0 is a dimensionless parameter related to the lattice spacing the coupling constant g. The fermionic field is in the fundamental representation of SU(2) and consists of two components $\Psi = \binom{\Psi^1}{\Psi^2}$. The gauge link operator U is a 2×2 unitary matrix operator starting from site x and ending at site $x+1$. A temporal gauge is chosen such that the link operator along the time-direction is set to unity. The second term in the KG Hamiltonian H_E corresponds to the energy stored in the color electric fields

$$H_E = \sum_{x=0}^{N-1} E^2(x) \tag{A3}$$

Furthermore, $E^2 = (E^1)^2 + (E^2)^2 + (E^3)^2 \equiv E_L^2 = E_R^2$. E_L and E_R are the left and right color electric field operators that satisfy the SU(2) Lie algebra at each site:

$$\begin{aligned} [E_L^a, E_L^b] &= -i\epsilon^{abc} E_L^c \\ [E_R^a, E_R^b] &= i\epsilon^{abc} E_R^c \\ [E_L^a, E_R^b] &= 0 \end{aligned} \tag{A4}$$

where ϵ^{abc} is the Levi-Cevita tensor. The gauge fields and the electric fields satisfy the canonical commutation relations on each link:

$$\begin{aligned} [E_L^a, U] &= T^a U \\ [E_R^a, U] &= U T^a \end{aligned} \tag{A5}$$

where $T^a = \frac{1}{2}\tau^a$ and τ^a is the ath Pauli matrix.

The final term in the KG Hamiltonian is the staggered mass term:

$$H_M = \mu \sum_{x=0}^{N-1} [(-1)^x \Psi^\dagger(x) \cdot \Psi(x)] \tag{A6}$$

with μ being the dimensionless mass parameter.

The fermion charge density operator defined at each site is as follows:

$$\rho^a(x) \equiv \Psi^\dagger(x) T^a \Psi(x) \tag{A7}$$

which satisfies the SU(2) Lie algebra. Furthermore, the following commutator is also satisfied at each sites:

$$[\rho^a, \Psi] = -T^a \Psi \tag{A8}$$

The charge operator also commutes with the color electric field and the gauge link operator. Using all these commutation relations, it can be shown that the following operator commutes with the Hamiltonian:

$$G^a(x) = -E_L^a(x) + E_R^a(x-1) + \rho^a(x) \tag{A9}$$

The Hilbert space of this theory is then separated into different sectors which corresponds to the eigenvalues of the Gauss law operator G^a, and the physical sector corresponds to the zero eigenvalue of G^a.

APPENDIX B: LOOP-STRING-HADRON FRAMEWORK

The gauge invariant states of the Kogut-Susskind theory are the nonlocal Wilson loops and string. The idea behind the LSH formulation is to form a local description of the same. In order to do this, the color electric fields E and the gauge link operator U are described in terms of bilinears of harmonic oscillator doublets, dubbed as Schwinger bosons or prepotentials [89]. The electric field operators then become

$$\begin{aligned} E_L^a &\equiv a^\dagger(L) T^a a(L) \\ E_R^a &\equiv a^\dagger(R) T^a a(R) \end{aligned} \tag{B1}$$

Here, the harmonic oscillator doublets obey the following commutation relation at each site:

$$[a_\alpha(l), a_\beta^\dagger(l')] = \delta_{\alpha\beta}\delta_{ll'} \tag{B2}$$

where, $\alpha, \beta = 1, 2$ and $l, l' = L, R$. One can also define the number operators for the harmonic oscillator doublets

$$\mathcal{N}_L = a^\dagger(L).a(L) \tag{B3}$$

$$\mathcal{N}_R = a^\dagger(R).a(R) \tag{B4}$$

with eigenvalues $n_{L/R}$. From the condition that the left and right Casimirs of theory being equal, i.e., $E_L^2 = E_R^2$, we find an Abelian Gauss law relating the number operators at the end of each link.

$$n_L(x) \approx n_R(x+1) \tag{B5}$$

where, $\approx$ denotes this to be true only for number operators $\mathcal{N}_{L/R}$ acting on a "physical state." The link operator U, is defined on a link connecting site x denoted by L and $x+1$ denoted by R. In terms of the prepotentials, the same is reconstructed with the prepotentials at the L/R-end of the link and is given by:

$$U \equiv U_L U_R \tag{B6}$$

$$U_L \equiv \frac{1}{\sqrt{\mathcal{N}_L + 1}} \begin{bmatrix} a_2^\dagger(L) & a_1(L) \\ -a_1^\dagger(L) & a_2(L) \end{bmatrix} \tag{B7}$$

$$U_R \equiv \begin{bmatrix} a_1^\dagger(R) & a_2^\dagger(R) \\ -a_2(R) & a_1(R) \end{bmatrix} \frac{1}{\sqrt{\mathcal{N}_R + 1}} \tag{B8}$$

The above constructions also satisfy the Abelian Gauss law (B5). Incorporating this particular form of U in the interaction Hamiltonian given in (A2) one obtains local SU(2) invariant building blocks that are glued by the AGL along neighboring sites. One can actually write down the entire set of gauge-invariant SU(2) operators [21]. Using these one can directly transform the Kogut-Susskind Hamiltonian into the loop-string-hadron Hamiltonian, where the entire Hamiltonian is expressed in terms of local SU(2)-singlet operators and commute with the AGLs on each and every link of the lattice.

The electric part of the Hamiltonian in the LSH formalism is as follows:

$$\begin{aligned} H_E^{\mathrm{KG}} \to H_E^{\mathrm{LSH}} \equiv \sum_x \Bigg[&\frac{1}{2}\mathcal{N}_R(x)\left(\frac{1}{2}\mathcal{N}_R(x) + 1\right) \\ &+ \frac{1}{2}\mathcal{N}_L(x)\left(\frac{1}{2}\mathcal{N}_L(x) + 1\right)\Bigg] \end{aligned} \tag{B9}$$

Where $\mathcal{N}_{L/R}$ are the total Schwinger boson occupation number defined in (B3), (B4).

The staggered mass term is written in terms of the quark number operator, $\mathcal{N}_\Psi = \Psi^\dagger \cdot \Psi$

$$H_M^{\mathrm{KG}} \to H_M^{\mathrm{LSH}} \equiv \mu \sum_x (-1)^x \mathcal{N}_\Psi(x) \tag{B10}$$

And finally, using the definition of the link operator in terms of the left and right harmonic oscillator doublets, we can see that

$$\Psi^\dagger(x) U \Psi(x) \to \Psi^\dagger(x) U_L U_R \Psi(x). \tag{B11}$$

Using Eq. (B6), we arrive at

$$\Psi^\dagger(x)U_L(x) = \frac{1}{\sqrt{\mathcal{N}_L(x)+1}}\begin{pmatrix}\mathcal{S}^{++}_{\text{out}}(x) & \mathcal{S}^{+-}_{\text{out}}(x)\end{pmatrix}, \tag{B12}$$

$$U_R(x)\Psi(x) = \begin{pmatrix}\mathcal{S}^{+-}_{\text{in}}(x)\\ \mathcal{S}^{--}_{\text{in}}(x)\end{pmatrix}\frac{1}{\sqrt{\mathcal{N}_R(x)+1}}. \tag{B13}$$

Finally, the interaction term can be summarized succinctly as:

$$H_I^{\text{KG}} \to H_I^{\text{LSH}} \tag{B14}$$

$$\begin{aligned}&\equiv x_0\sum_x \frac{1}{\sqrt{\mathcal{N}_L(x)+1}}[\mathcal{S}^{++}_{\text{out}}(x)\mathcal{S}^{+-}_{\text{in}}(x+1)\\ &\quad+\mathcal{S}^{+-}_{\text{out}}(x)\mathcal{S}^{--}_{\text{in}}(x+1)]\times\frac{1}{\sqrt{\mathcal{N}_R(x+1)+1}}\\ &\quad+\sum_x\frac{1}{\sqrt{\mathcal{N}_R(x+1)+1}}[\mathcal{S}^{-+}_{\text{in}}(x+1)\mathcal{S}^{--}_{\text{out}}(x)\\ &\quad+\mathcal{S}^{++}_{\text{in}}(x+1)\mathcal{S}^{-+}_{\text{out}}(x)]\times\frac{1}{\sqrt{\mathcal{N}_L(x)+1}}\end{aligned} \tag{B15}$$

Here, $S^{\sigma,\sigma'}_{\text{in/out}}$ are the string operators. They consist of pairs of fermionic/bosonic creation and annihilation operators. Up until this point, the basis of the LSH hamiltonian has not be described clearly. The natural choice would be to use the occupation number basis, i.e., considering $\mathcal{N}_R$, $\mathcal{N}_L$, $\mathcal{N}_\Psi$ as the complete set of commuting operators. However, we define a loop-string-hadron basis [21] that is more useful. The LSH basis is characterized by quantum numbers n_l, n_i, n_0, which are defined as the eigenvalues of the LSH occupation numbers $\mathcal{N}_l$, $\mathcal{N}_i$, $\mathcal{N}_o$. n_l, n_i, n_0 denotes the occupation number for each individual mode of loop-string "quanta" present at each lattice site. The details of this basis construction can be found in [21]. The relations between the LSH quantum numbers to bosonic-fermionic occupation numbers are as follows:

$$n_\Psi = n_i + n_o \tag{B16}$$

$$n_L = n_l + n_o(1-n_i) \tag{B17}$$

$$n_R = n_l + n_i(1-n_o) \tag{B18}$$

at any particular site x (index omitted in the above equation).

An entire dictionary of normalized ladder operators has been constructed, which when acting on the LSH basis, gives back normalized states. This second layer of operator redefinition ensures that the states generated after sequential action of these operators remain normalized at all times, with the appropriate cofactors hidden in the operator definitions. The explicit form of the interaction Hamiltonian in terms of the fundamental bosonic/fermionic operators and the newly defined normalized operators is as follows:

$$\begin{aligned}&\mathcal{S}^{++}_{\text{out}}(x)\mathcal{S}^{+-}_{\text{in}}(x+1)\\ &=[\chi_o^\dagger]_x[\chi_o]_{x+1}\\ &\quad\times[(1-\mathcal{N}_i)+\Lambda^+\mathcal{N}_i]_x[\mathcal{N}_i+\Lambda^+(1-\mathcal{N}_i)]_{x+1}\\ &\quad\times[\sqrt{\mathcal{N}_l-\mathcal{N}_i+2}]_x[\sqrt{\mathcal{N}_l-(1-\mathcal{N}_i)+2}]_{x+1},\end{aligned} \tag{B19}$$

$$\begin{aligned}&\mathcal{S}^{--}_{\text{out}}(x)\mathcal{S}^{-+}_{\text{in}}(x+1)\\ &=[\chi_o]_x[\chi_0^\dagger]_{x+1}\\ &\quad\times[(1-\mathcal{N}_i)+\Lambda^-\mathcal{N}_i]_x[\mathcal{N}_i+\Lambda^-(1-\mathcal{N}_i]_{x+1}\\ &\quad\times[\sqrt{\mathcal{N}_l+2(1-\mathcal{N}_i)}]_x[\sqrt{\mathcal{N}_l+2\mathcal{N}_i}]_{x+1}\end{aligned} \tag{B20}$$

$$\begin{aligned}&\mathcal{S}^{+-}_{\text{out}}(x)\mathcal{S}^{--}_{\text{in}}(x+1)\\ &=[\chi_i^\dagger]_x[\chi_i]_{x+1}\\ &\quad\times[\mathcal{N}_o+\Lambda^-(1-\mathcal{N}_o)]_x[(1-\mathcal{N}_o)+\Lambda^-\mathcal{N}_o]_{x+1}\\ &\quad\times[\sqrt{\mathcal{N}_l+2\mathcal{N}_o}]_x[\sqrt{\mathcal{N}_l+2(1-\mathcal{N}_o)}]_{x+1},\end{aligned} \tag{B21}$$

$$\begin{aligned}&\mathcal{S}^{-+}_{\text{out}}(x)\mathcal{S}^{++}_{\text{in}}(x+1)\\ &=[\chi_i]_x[\chi_i^\dagger]_{x+1}\\ &\quad\times[\mathcal{N}_o+\Lambda^+(1-\mathcal{N}_o]_x[(1-\mathcal{N}_o)+\Lambda^+\mathcal{N}_o]_{x+1}\\ &\quad\times[\sqrt{\mathcal{N}_l+\mathcal{N}_o+1}]_x[\sqrt{\mathcal{N}_l+(1-\mathcal{N}_o)+1}]_{x+1}\end{aligned} \tag{B22}$$

Here, we have only shown the different two body string operator terms. The full interaction term would include these operators sandwiched between the $\mathcal{N}_{L/R}$ operators. The $\chi_{i,o}$ are the SU(2) invariant fermionic modes, which act as ladder operators on the LSH basis states. The (i,o) denote incoming and outgoing modes

$$\{\chi_{q'},\chi_q\} = \{\chi^\dagger_{q'},\chi^\dagger_q\} = 0 \quad (\text{q} = \text{i},\text{o}) \tag{B23}$$

$$\{\chi_{q'},\chi^\dagger_q\} = \delta_{q'q} \tag{B24}$$

and $\Lambda^\pm$ are the loop quantum number (n_l) ladder operators, defined as:

$$\Lambda^\pm \equiv \mathcal{L}^{\pm\pm}\frac{1}{\sqrt{(\mathcal{N}_l+\frac{1}{2}\pm\frac{1}{2})(\mathcal{N}_l+\frac{3}{2}\pm\frac{1}{2}+(\mathcal{N}_i\oplus\mathcal{N}_o))}}. \tag{B25}$$

Briefly, the above defined $\chi_{i,o}$ act as ladder operators for the n_i, n_o quantum numbers and $\Lambda^\pm$ act as ladder operator for the n_l quantum number. One can also substitute the definitions given in (B16) into the definitions of the electric (B9) and mass part (B10) of the LSH Hamiltonian to construct their corresponding form in terms of the $\mathcal{N}_l$, $\mathcal{N}_i$, $\mathcal{N}_o$ operators as follows:

$$H_E = \frac{g_0^2}{4}\sum_x \Bigg\{ \left[\frac{1}{2}(\mathcal{N}_l + \mathcal{N}_o(1-\mathcal{N}_i))\right]_x \times \left[\frac{1}{2}(\mathcal{N}_l + \mathcal{N}_o(1-\mathcal{N}_i)) + 1\right]_x + \left[\frac{1}{2}(\mathcal{N}_l + \mathcal{N}_i(1-\mathcal{N}_o))\right]_x \times \left[\frac{1}{2}(\mathcal{N}_l + \mathcal{N}_i(1-\mathcal{N}_o)) + 1\right]_x \Bigg\}. \tag{B26}$$

The mass term is given by:

$$H_M = m_0 \sum_x (-)^x (\mathcal{N}_i(x) + \mathcal{N}_o(x)). \tag{B27}$$

This completes the construction of the LSH Hamiltonian in an explicitly gauge-invariant form.

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