

One-loop corrections to dihadron production in DIS at small x

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We calculate the one-loop corrections to dihadron production in deep inelastic scattering (DIS) at small x using the color glass condensate formalism. We show that all UV and soft singularities cancel while the collinear divergences are absorbed into quark and antiquark-hadron fragmentation functions. Rapidity divergences lead to Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner evolution of dipoles and quadrupoles describing multiple scatterings of the quark antiquark dipole on the target proton/nucleus. The resulting cross section is finite and can be used for phenomenological studies of dihadron angular correlations at small x in a future Electron-Ion Collider.

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I. INTRODUCTION

Since the experimental observation of the fast rise of parton (especially gluon) distribution functions at HERA [1], the phenomenon of gluon saturation [2,3] at small x has been an active field of study. The color glass condensate formalism [4–7] is an effective theory of QCD at small x which provides a robust platform that can be used to investigate saturation dynamics. Nevertheless and despite the intense ongoing theoretical and experimental efforts, clear and unambiguous evidence for gluon saturation remains elusive. There is hope that the proposed Electron-Ion Collider (EIC) [8] will be able to unambiguously discover gluon saturation and to establish the kinematic region in which it is applicable. Perhaps the most promising process in which to discover gluon saturation is two-particle angular correlations which have been extensively studied using leading-order (LO) expressions obtained in the color glass condensate formalism [9–52]. There are already strong hints for the presence of gluon saturation dynamics in the observed disappearance of away side hadrons in inclusive dihadron production in deuteron (proton)-gold collisions at RHIC

[53–55]. Nevertheless the EIC will measure dihadrons with much more accuracy and with much better control over the range of target x contributing to the process. Therefore it is essential to go beyond a LO calculation in order to achieve a reasonable quantitative accuracy. Toward this end we calculate the one-loop corrections to inclusive dihadron production in DIS here. As expected there are several divergences that appear when going beyond a tree level calculation. We show that all such divergences either cancel or can be absorbed into evolution (scale dependence) of physical quantities. Our conclusion on factorization/cancellation of all singularities are in agreement with recent studies of one-loop corrections to dijet production in DIS [45,56], nevertheless our treatment of collinear divergences and their absorption into evolution of hadron fragmentation functions is new. Our final results are completely finite and can be used to investigate gluon saturation dynamics in inclusive dihadron production and angular correlations in DIS at small x .

In the small x limit scattering of a virtual photon on a target hadron or nucleus can be understood as a two-step process; first the virtual photon splits into a quark antiquark pair (a dipole), which then scatters from the target hadron or nucleus. The total virtual photon-target scattering probability is expressed as a convolution of the probability for a photon to split into a quark at transverse position $\mathbf{x}_1$ and an antiquark at position $\mathbf{x}_2$, with the probability for this dipole to scatter from the target. The leading-order double inclusive production cross section can be written as

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$$\frac{d\sigma^{r^*A \rightarrow q\bar{q}X}}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{e^2 Q^2 (z_1 z_2)^2 N_c}{(2\pi)^7} \delta(1 - z_1 - z_2) \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ \times e^{i\mathbf{p} \cdot (\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q} \cdot (\mathbf{x}'_2 - \mathbf{x}_2)} \left[4z_1 z_2 K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) + (z_1^2 + z_2^2) \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2'}}{|\mathbf{x}_{12}||\mathbf{x}_{1'2'}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2'}|Q_1) \right], \quad (1)$$

where the first and second terms above correspond to the contribution of the longitudinal and transverse polarizations of the virtual photon. Here l^μ is the momentum of the virtual photon with $l^2 = -Q^2$. We set the transverse momentum of the photon to zero without any loss of generality. Also p^μ (q^μ) is the momentum of the outgoing quark (antiquark) and z_1 (z_2) is its longitudinal momentum fraction relative to the photon. $\mathbf{x}_1$ ($\mathbf{x}_2$) is the transverse coordinate of the quark (antiquark), and primed coordinates are used in the conjugate amplitude. Note that we write the differential variables on the left side using quark and antiquark rapidities y_1 and y_2 which are related to their momentum fractions z_1 and z_2 via $dy_i = dz_i/z_i$. The electric charge of the quark is written as e , and in principle one needs to sum over quark flavors. We also use the following shorthand notations:

$$Q_i = Q\sqrt{z_i(1-z_i)}, \quad \mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j, \\ d^8\mathbf{x} = d^2\mathbf{x}_1 d^2\mathbf{x}_2 d^2\mathbf{x}_{1'} d^2\mathbf{x}_{2'}. \quad (2)$$

All the dynamics of the strong interactions and gluon saturation are contained in the dipoles S_{ij} and quadrupoles S_{ijkl} normalized correlation functions of two and four Wilson lines

$$S_{ij} = \frac{1}{N_c} \text{tr} \langle V_i V_j^\dagger \rangle, \quad S_{ijkl} = \frac{1}{N_c} \text{tr} \langle V_i V_j^\dagger V_k V_l^\dagger \rangle, \quad (3)$$

where the index i refers to the transverse coordinate $\mathbf{x}_i$ and the following notation is used for Wilson lines:

$$V_i = \hat{P} \exp \left(ig \int dx^+ A^-(x^+, \mathbf{x}_i) \right). \quad (4)$$

The Wilson lines efficiently resum the multiple scatterings of the quark and antiquark from the target hadron or nucleus. The angle brackets in Eq. (3) signify *color averaging*. It is important to keep in mind that as this is a classical result the cross section has no nontrivial x (or rapidity/energy) dependence. It is also easy to check that if one integrates over the phase space of the quark and antiquark, one recovers the standard expressions for the virtual photon-target total cross section at small x .

II. ONE-LOOP CORRECTIONS

One-loop corrections to the LO cross section above involve radiation of a gluon from either the quark or the antiquark. The radiated gluon can either be real (radiated in the amplitude and absorbed in the complex conjugate amplitude) or virtual (radiated and absorbed in either the amplitude or the complex conjugate amplitude). The real contributions were already computed in [57,58], here we will repeat the calculation and reproduce the previous results. The virtual corrections have also been computed, but for a different process, namely inclusive dijets [45,56] rather than dihadrons. Furthermore we use spinor helicity methods to evaluate the Dirac algebra which leads to a tremendous simplification of the calculations. The real corrections are shown in Fig. 1 and are given by¹ for quark initiated radiation,

$$i\mathcal{M}_1^a = -ieg \int \frac{d^3k_1}{(2\pi)^3} d^4\mathbf{x} \frac{16(l^+)^2 N_1 t^a V_1 V_2^\dagger}{k_1^2 (k_1 - l)^2} e^{i\mathbf{k}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2)} e^{-i(\mathbf{p} + \mathbf{k}) \cdot \mathbf{x}_1} e^{-i\mathbf{q} \cdot \mathbf{x}_2}, \quad k_1^+ = p^+ + k^+, \\ i\mathcal{M}_3^a = -eg \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} d^6\mathbf{x} \frac{16(l^+)^2 N_3 (2k^+) V_1 t^b V_2^\dagger U_3^{ab} e^{i\mathbf{k}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2)} e^{i\mathbf{k}_2 \cdot (\mathbf{x}_3 - \mathbf{x}_1)} e^{-i(\mathbf{p} \cdot \mathbf{x}_1 + \mathbf{q} \cdot \mathbf{x}_2 + \mathbf{k} \cdot \mathbf{x}_3)}}{k_1^2 k_2^2 (k_1 - l)^2 (k_1 - k_2)^2}, \\ k_1^+ = p^+ + k^+, \quad k_2^+ = k^+. \quad (5)$$

¹We define $i\mathcal{M}_i$ via $i\mathcal{A}_i = 2\pi\delta(l^+ - p^+ - q^+)i\mathcal{M}_i$.

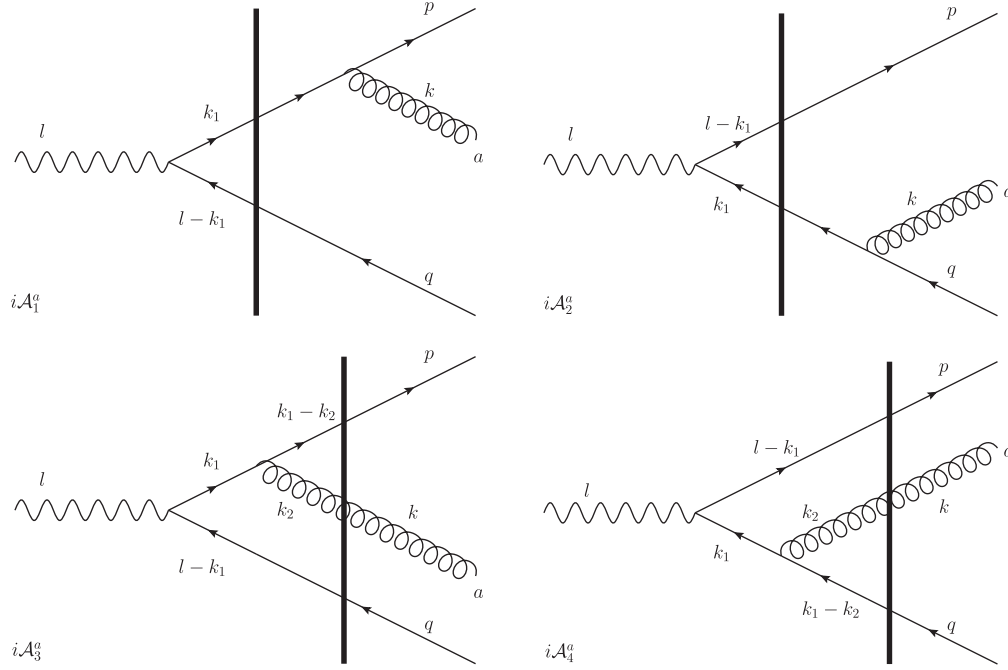


FIG. 1. The real corrections $i\mathcal{A}_1^a, \dots, i\mathcal{A}_4^a$. The arrows on Fermion lines indicate Fermion number flow, all momenta flow to the right. The thick solid line indicates interaction with the target.

Here we use $\frac{d^3k}{(2\pi)^3} = \frac{dk^-}{(2\pi)} \frac{d^2\mathbf{k}}{(2\pi)^2}$ and the Dirac numerators N_i are defined as

$$N_1 \equiv \frac{1}{16(l^+)^2} \frac{\bar{u}(p)\not{\epsilon}(k)^*(\not{p} + \not{k})\not{k}_1\not{\epsilon}(l)(\not{k}_1 - \not{l})\not{v}(q)}{(p+k)^2},$$

$$N_3 \equiv \frac{1}{16(l^+)^2} \bar{u}(p)\not{\epsilon}(k_1 - \not{k}_2)\gamma^\mu\not{k}_1\not{\epsilon}(l)(\not{k}_1 - \not{l})\not{v}(q)d_{\mu\nu}(k_2)\epsilon^\nu(k)^*. \quad (6)$$

We define the gauge vector n^μ in $(+, -, \perp)$ notation as $(0, 1, \mathbf{0})$. The antiquark diagrams $i\mathcal{M}_2$ and $i\mathcal{M}_4$ are related to the quark diagrams via swapping quark and antiquark momenta (i.e.: $\mathbf{p} \leftrightarrow \mathbf{q}$ and $z_1 \leftrightarrow z_2$). Note that when a gluon line multiply scatters from the target one gets a Wilson line in the adjoint representation U_j^{ab} ,

$$U_j^{ab} = \hat{P} \exp \left(ig \int dx^+ A_c^-(x^+, \mathbf{x}_j) T^c \right)_{ab}, \quad \text{where } T_{ab}^c = if^{abc}. \quad (7)$$

Virtual diagrams are shown in Fig. 2, and the amplitudes can be written as follows:

$$i\mathcal{M}_5 = ieg^2 \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^4k_3}{(2\pi)^4} d^6\mathbf{x} \frac{N_5 e^{i\mathbf{k}_1 \cdot (\mathbf{x}_3 - \mathbf{x}_2)} e^{i\mathbf{k}_2 \cdot (\mathbf{x}_1 - \mathbf{x}_3)} e^{i\mathbf{k}_3 \cdot (\mathbf{x}_3 - \mathbf{x}_1)} e^{-i\mathbf{p} \cdot \mathbf{x}_3} e^{-i\mathbf{q} \cdot \mathbf{x}_2}}{k_1^2 k_2^2 k_3^2 (k_1 - l)^2 (k_2 - k_1)^2 (k_3 - p)^2} 2(k_3^+ - p^+) \\ \times t^a [\theta(k_3^+) V_1 - \theta(-k_3^+) V_1^\dagger] t^b V_2^\dagger [\theta(k_3^+ - p^+) U_3^{ba} - \theta(p^+ - k_3^+) U_3^{\dagger ba}], \quad k_2^+ = k_3^+, \quad k_1^+ = p^+. \quad (8)$$

$$i\mathcal{M}_7 = ieg^2 \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^4k_3}{(2\pi)^4} d^6\mathbf{x} \frac{N_7 e^{i\mathbf{k}_3 \cdot (\mathbf{x}_3 - \mathbf{x}_1)} e^{i\mathbf{k}_2 \cdot (\mathbf{x}_2 - \mathbf{x}_3)} e^{i\mathbf{k}_1 \cdot (\mathbf{x}_3 - \mathbf{x}_1)} e^{-i\mathbf{p} \cdot \mathbf{x}_3} e^{-i\mathbf{q} \cdot \mathbf{x}_2}}{k_1^2 k_2^2 k_3^2 (l - k_1)^2 (k_1 - k_2)^2 (k_3 - p)^2} 2(k_3^+ - p^+) \\ \times t^a [\theta(k_3^+) V_1 - \theta(-k_3^+) V_1^\dagger] t^b V_2^\dagger [\theta(k_3^+ - p^+) U_3^{ba} - \theta(p^+ - k_3^+) U_3^{\dagger ba}], \quad k_2^+ = q^+, \quad k_1^+ = l^+ - k_3^+. \quad (9)$$

$$i\mathcal{M}_9 = \frac{-eg^2}{p^2} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^4k_2}{(2\pi)^4} d^4\mathbf{x} \frac{N_9 t^a t^a V_1 V_2^\dagger e^{i\mathbf{k}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2)} e^{-i(\mathbf{p} \cdot \mathbf{x}_1 + \mathbf{q} \cdot \mathbf{x}_2)}}{k_1^2 k_2^2 (k_1 - l)^2 (k_2 - p)^2}, \quad k_1^+ = p^+. \quad (10)$$

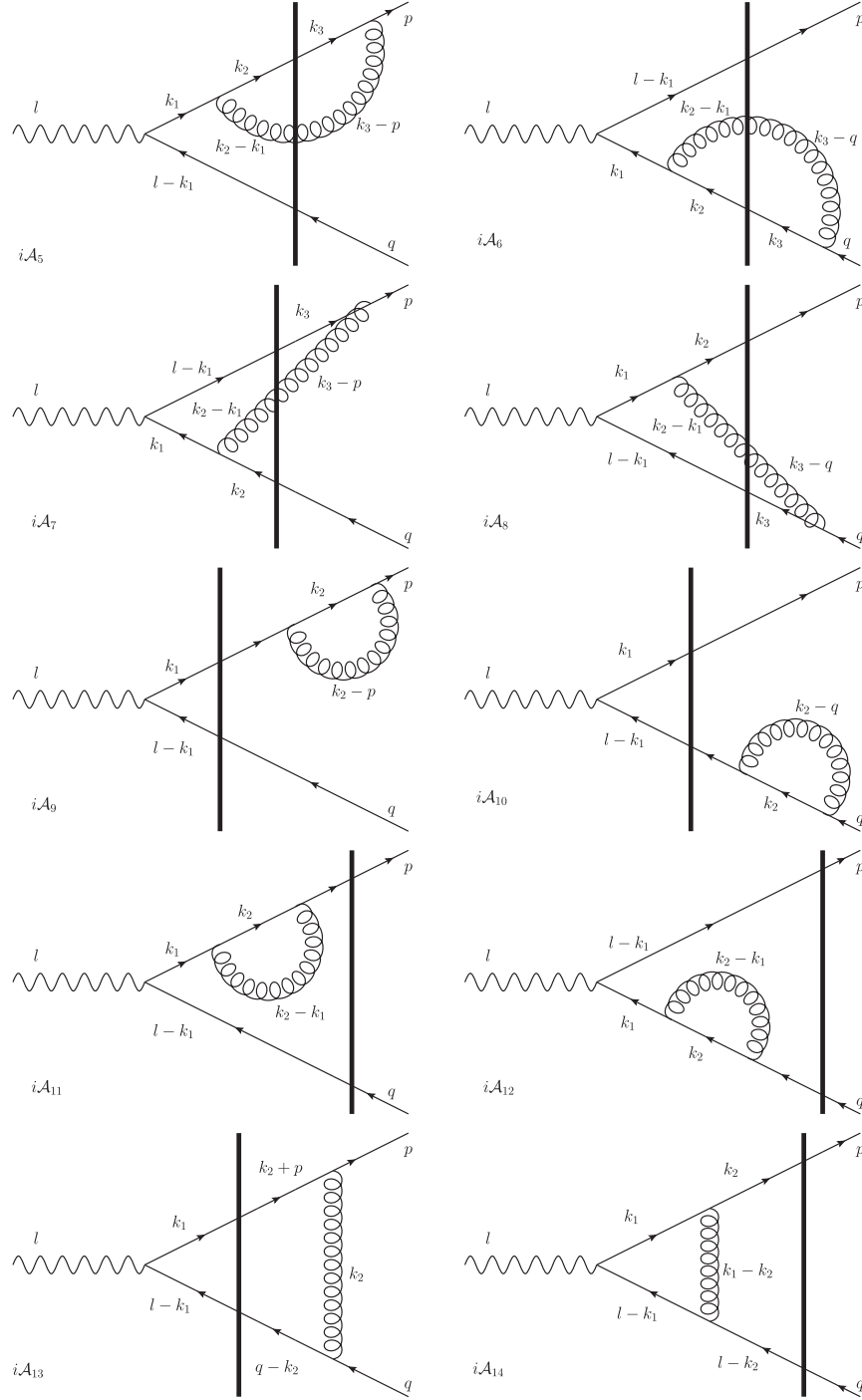


FIG. 2. The ten virtual NLO diagrams $i\mathcal{A}_5, \dots, i\mathcal{A}_{14}$. The arrows on fermion lines indicate fermion number flow, all momenta flow to the right, *except* for gluon momenta. The thick solid line indicates interaction with the target.

$$i\mathcal{M}_{11} = -eg^2 \int \frac{d^3k_1}{(2\pi)^3} \frac{d^4k_2}{(2\pi)^4} d^4\mathbf{x} \frac{N_{11}V_1 t^a V_2^\dagger e^{i\mathbf{k}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2)} e^{-i(\mathbf{p} \cdot \mathbf{x}_1 + \mathbf{q} \cdot \mathbf{x}_2)}}{k_1^4 k_2^2 (k_1 - l)^2 (k_1 - k_2)^2}, \quad k_1^+ = p^+. \quad (11)$$

$$i\mathcal{M}_{13} = eg^2 \int \frac{d^3k_1}{(2\pi)^3} \frac{d^4k_2}{(2\pi)^4} d^4\mathbf{x} \frac{N_{13} e^{i\mathbf{k}_1 \cdot \mathbf{x}_{12}} e^{i\mathbf{k}_2 \cdot \mathbf{x}_{21}} e^{-i(\mathbf{p} \cdot \mathbf{x}_1 + \mathbf{q} \cdot \mathbf{x}_2)}}{k_1^2 k_2^2 (k_1 - l)^2 (k_2 + p)^2 (k_2 - q)^2} \\ \times t^a [\theta(k_2^+ + p^+) V_1 - \theta(-k_2^+ - p^+) V_1^\dagger] [\theta(k_2^+ - q^+) V_2 - \theta(q^+ - k_2^+) V_2^\dagger] t^a, \quad k_1^+ = k_2^+ + p^+. \quad (12)$$

$$i\mathcal{M}_{14} = -eg^2 \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^3 k_2}{(2\pi)^3} d^4 \mathbf{x} \frac{N_{14} V_1 t^a t^a V_2^\dagger e^{i\mathbf{k}_2 \cdot (\mathbf{x}_1 - \mathbf{x}_2)} e^{-i(\mathbf{p} \cdot \mathbf{x}_1 + \mathbf{q} \cdot \mathbf{x}_2)}}{k_1^2 k_2^2 (k_1 - l)^2 (k_2 - l)^2 (k_1 - k_2)^2}, \quad k_2^+ = p^+. \quad (13)$$

with the virtual Dirac numerators defined as

$$N_5 \equiv \bar{u}(p) \gamma^\mu \not{k}_3 \not{l} \not{k}_2 \gamma^\nu \not{k}_1 \not{l} (\not{k}_1 - \not{l}) \not{v}(q) d_{\mu\sigma} (k_2 - k_1) d_\mu^\sigma (k_3 - p), \quad (14)$$

$$N_7 \equiv \bar{u}(p) \gamma^\mu \not{k}_3 \not{l} (\not{l} - \not{k}_1) \not{l} \not{k}_1 \gamma^\nu \not{k}_2 \not{v}(q) d_{\nu\sigma} (k_2 - k_1) d_\mu^\sigma (k_3 - p), \quad (15)$$

$$N_9 \equiv \bar{u}(p) \gamma^\mu (\not{p} - \not{k}_2) \gamma^\nu \not{p} \not{k}_1 \not{l} (\not{k}_1 - \not{l}) \not{v}(q) d_{\mu\nu} (k_2), \quad (16)$$

$$N_{11} \equiv \bar{u}(p) \not{k}_1 \gamma^\mu \not{k}_2 \gamma^\nu \not{k}_1 \not{l} (\not{k}_1 - \not{l}) \not{v}(q) d_{\mu\nu} (k_1 - k_2), \quad (17)$$

$$N_{13} \equiv \bar{u}(p) \gamma^\mu (\not{k}_2 + \not{p}) \not{k}_1 \not{l} (\not{k}_1 - \not{l}) \not{k}_2 - \not{q} \gamma^\nu \not{v}(q) d_{\mu\nu} (k_2), \quad (18)$$

$$N_{14} \equiv \bar{u}(p) \not{k}_2 \gamma^\mu \not{k}_1 \not{l} (\not{k}_1 - \not{l}) \gamma^\nu (\not{k}_2 - \not{l}) \not{v}(q) d_{\mu\nu} (k_1 - k_2). \quad (19)$$

Here again the antiquark diagrams ($i\mathcal{M}_6, i\mathcal{M}_8, i\mathcal{M}_{10}, i\mathcal{M}_{12}$) are related to the corresponding quark diagrams via swapping quark and antiquark momenta. The tensor $d_{\mu\nu}$ is defined as follows:

$$d_{\mu\nu}(p) = -g_{\mu\nu} + \frac{p_\mu n_\nu + p_\nu n_\mu}{n \cdot p}. \quad (20)$$

In this study, we focus on the contribution from longitudinally polarized photons and we compute the numerators using the spinor helicity formalism. The necessary real numerators are in Table I and the virtual numerators are in Eqs. (21)–(26).

We also need the Dirac numerators for the virtual diagrams; they are

$$N_5^{L;+} = \frac{2^5 (l^+)^2 Q (z_1 z_2)^{3/2}}{(z_1 - z_3)^2} \left\{ z_1^2 \left[\left(\mathbf{k}_3 - \frac{z_3}{z_1} \mathbf{p} \right) \cdot \boldsymbol{\epsilon} \right] \left[\left(\mathbf{k}_2 - \frac{z_3}{z_1} \mathbf{k}_1 \right) \cdot \boldsymbol{\epsilon}^* \right] + z_3^2 \left[\left(\mathbf{k}_3 - \frac{z_3}{z_1} \mathbf{p} \right) \cdot \boldsymbol{\epsilon}^* \right] \left[\left(\mathbf{k}_2 - \frac{z_3}{z_1} \mathbf{k}_1 \right) \cdot \boldsymbol{\epsilon} \right] \right\}, \quad (21)$$

$$N_7^{L;+} = \frac{2^5 (l^+)^2 Q z_3 \sqrt{z_1 z_2}}{(z_1 - z_3)^2} \left\{ z_1 z_2 \left[\left(\mathbf{k}_3 - \frac{z_3}{z_1} \mathbf{p} \right) \cdot \boldsymbol{\epsilon} \right] [(z_2 \mathbf{k}_1 - (1 - z_3) \mathbf{k}_2) \cdot \boldsymbol{\epsilon}^*] \right. \\ \left. + z_3 (1 - z_3) \left[\left(\mathbf{k}_3 - \frac{z_3}{z_1} \mathbf{p} \right) \cdot \boldsymbol{\epsilon}^* \right] [(z_2 \mathbf{k}_1 - (1 - z_3) \mathbf{k}_2) \cdot \boldsymbol{\epsilon}] \right\}, \quad (22)$$

$$N_9^{L;+} = 2^4 Q (l^+)^2 (z_1 z_2)^{3/2} \left[k_2^2 + \frac{(2z_1 - z)}{z} (k_2 - p)^2 - \frac{[z_1^2 + (z_1 - z)^2]}{z_1 z} p^2 \right], \quad (23)$$

$$N_{11}^{L;+} = 2^4 (l^+)^2 Q (z_1 z_2)^{3/2} \left[\frac{[(k_2^+)^2 + (p^+)^2]}{p^+ (k_2^+ - p^+)} k_1^2 + \frac{(p^+ + k_2^+)}{(p^+ - k_2^+)} k_2^2 + (k_1 - k_2)^2 \right], \quad (24)$$

$$N_{13}^{L;+} = 2^5 Q (l^+)^2 (z_1 + z_3) (z_2 - z_3) \sqrt{z_1 z_2} \left[z_1 z_3 \left(\frac{\mathbf{p} \cdot \boldsymbol{\epsilon}}{z_1} - \frac{\mathbf{q} \cdot \boldsymbol{\epsilon}}{z_2} \right) \left(\frac{\mathbf{p} \cdot \boldsymbol{\epsilon}^*}{z_1} - \frac{\mathbf{k}_2 \cdot \boldsymbol{\epsilon}^*}{z_3} \right) + z_3^2 \left(\frac{\mathbf{k}_2 \cdot \boldsymbol{\epsilon}}{z_3} - \frac{\mathbf{q} \cdot \boldsymbol{\epsilon}}{z_2} \right) \left(\frac{\mathbf{p} \cdot \boldsymbol{\epsilon}^*}{z_1} - \frac{\mathbf{k}_2 \cdot \boldsymbol{\epsilon}^*}{z_3} \right) \right. \\ \left. - z_2 z_3 \left(\frac{\mathbf{k}_2 \cdot \boldsymbol{\epsilon}}{z_3} - \frac{\mathbf{q} \cdot \boldsymbol{\epsilon}}{z_2} \right) \left(\frac{\mathbf{p} \cdot \boldsymbol{\epsilon}^*}{z_1} - \frac{\mathbf{q} \cdot \boldsymbol{\epsilon}^*}{z_2} \right) - p \cdot q - \frac{(z_1 + z_3)}{2z_3} (k_2 - q)^2 + \frac{(z_2 - z_3)}{2z_3} (k_2 + p)^2 \right], \quad (25)$$

$$N_{14}^{L;+} = \frac{2^5 Q (l^+)^2 (z_1 z_2)^{3/2} z_3}{(z_3 - z_1)} \left[2l^+ (1 - z_3) (k_1^- - k_2^-) + \frac{(z_3 - z_2)}{z_3} (\mathbf{k}_1 \cdot \boldsymbol{\epsilon}) (\mathbf{k}_1 \cdot \boldsymbol{\epsilon}^*) + \frac{(z_1 - z_3)}{z_2} (\mathbf{k}_2 \cdot \boldsymbol{\epsilon}) (\mathbf{k}_1 \cdot \boldsymbol{\epsilon}^*) \right. \\ \left. + \frac{(1 - z_3)(z_1 - z_3)}{z_1 z_3} (\mathbf{k}_1 \cdot \boldsymbol{\epsilon}) (\mathbf{k}_2 \cdot \boldsymbol{\epsilon}^*) + \frac{(z_2 - z_3)(1 - z_3)}{z_1 z_2} (\mathbf{k}_2 \cdot \boldsymbol{\epsilon}) (\mathbf{k}_2 \cdot \boldsymbol{\epsilon}^*) \right]. \quad (26)$$

TABLE I. The minimal set of real numerators N_1 to N_4 in momentum fraction notation. Complex conjugation results in the numerator with all helicities flipped (while leaving longitudinal helicities unchanged), and any numerator where $\lambda_q = \lambda_{\bar{q}}$ is zero. We have also defined z_3 to be the gluon's momentum fraction relative to the photon ($z_3 = k^+/l^+$).

Numerator	$\lambda_\gamma; \lambda_q, \lambda_g$	$N_i^{\lambda_\gamma; \lambda_q, \lambda_g}$
N_1	$L; +, +$	$-Q(z_1 z_2)^{3/2} (1 - z_2) \frac{[(z_1 \mathbf{k} - z_3 \mathbf{p}) \cdot \boldsymbol{\epsilon}]}{(z_1 \mathbf{k} - z_3 \mathbf{p})^2}$
	$L; +, -$	$-Q(z_2)^{3/2} \sqrt{z_1} (1 - z_2)^2 \frac{[(z_1 \mathbf{k} - z_3 \mathbf{p}) \cdot \boldsymbol{\epsilon}]}{(z_1 \mathbf{k} - z_3 \mathbf{p})^2}$
N_2	$L; +, +$	$Q(z_1)^{3/2} \sqrt{z_2} (1 - z_1)^2 \frac{[(z_2 \mathbf{k} - z_3 \mathbf{q}) \cdot \boldsymbol{\epsilon}]}{(z_2 \mathbf{k} - z_3 \mathbf{q})^2}$
	$L; +, -$	$Q(z_1 z_2)^{3/2} (1 - z_1) \frac{[(z_2 \mathbf{k} - z_3 \mathbf{q}) \cdot \boldsymbol{\epsilon}^*]}{(z_2 \mathbf{k} - z_3 \mathbf{q})^2}$
N_3	$L; +, +$	$Q(z_1 z_2)^{3/2} (1 - z_2) \left(\frac{\mathbf{k}_2 \cdot \boldsymbol{\epsilon}}{z_3} - \frac{\mathbf{k}_1 \cdot \boldsymbol{\epsilon}}{1 - z_2} \right)$
	$L; +, -$	$Q(z_2)^{3/2} \sqrt{z_1} (1 - z_2)^2 \left(\frac{\mathbf{k}_2 \cdot \boldsymbol{\epsilon}^*}{z_3} - \frac{\mathbf{k}_1 \cdot \boldsymbol{\epsilon}^*}{1 - z_2} \right)$
N_4	$L; +, +$	$-Q(z_1)^{3/2} \sqrt{z_2} (1 - z_1)^2 \left(\frac{\mathbf{k}_2 \cdot \boldsymbol{\epsilon}}{z_3} - \frac{\mathbf{k}_1 \cdot \boldsymbol{\epsilon}}{1 - z_1} \right)$
	$L; +, -$	$-Q(z_1 z_2)^{3/2} (1 - z_1) \left(\frac{\mathbf{k}_2 \cdot \boldsymbol{\epsilon}^*}{z_3} - \frac{\mathbf{k}_1 \cdot \boldsymbol{\epsilon}^*}{1 - z_1} \right)$

In all these virtual numerators, z_3 is the momentum fraction for the remaining internal + momentum [see Eqs. (8)–(13)]. We have also defined the transverse vector $\boldsymbol{\epsilon}$ as

$$\boldsymbol{\epsilon} = \frac{1}{\sqrt{2}}(1, i). \quad (27)$$

III. RESULTS

To calculate the $\mathcal{O}(\alpha_s)$ corrections to the production cross section we need to multiply the helicity amplitudes with the

corresponding conjugate amplitudes (real amplitudes need to be multiplied with each other, virtual amplitudes need to be multiplied by the leading-order amplitude $i\mathcal{M}$), include the phase space factors $d\Phi^n$, and divide by the incident flux $\mathcal{F}$ which we will take to be $2l^+$. We also relabel and shift (where necessary) on the remaining z_3 integral in each expression so that the gluon's momentum fraction is always labeled z . We will write the real corrections as $\sigma_{i \times j}$ for $i, j = 1, \dots, 4$ and the virtual corrections as σ_i for $i = 5, \dots, 14$:

$$d\sigma_{\text{NLO}}^L = \sum_{i,j=1}^4 d\sigma_{i \times j}^L + 2\text{Re} \sum_{i=5}^{14} d\sigma_i^L, \quad (28)$$

$$d\sigma_{i \times j}^L = \frac{1}{\mathcal{F}} \int_{z, \mathbf{k}} [(i\mathcal{M}_i^a)(i\mathcal{M}_j^a)^*] d\Phi^{(3)},$$

$$d\sigma_i^L = \frac{1}{\mathcal{F}} [(i\mathcal{M}_i)(i\mathcal{M})^*] d\Phi^{(2)}, \quad (29)$$

$$d\Phi^{(3)} = 2l^+ \frac{d^2 \mathbf{p} d^2 \mathbf{q} d^2 \mathbf{k} dy_1 dy_2 dz}{(2\pi)^8 (4l^+)^2 z} \delta(1 - z_1 - z_2 - z),$$

$$d\Phi^{(2)} = 2l^+ \frac{d^2 \mathbf{p} d^2 \mathbf{q} dy_1 dy_2}{2(2\pi)^5 (2l^+)^2} \delta(1 - z_1 - z_2). \quad (30)$$

Here the L signifies that we are including contributions only from longitudinally polarized photons, and imply that we have summed over all outgoing polarizations. We note that all real corrections come with a $\delta(1 - z_1 - z_2 - z)$ whereas all virtual corrections come with a $\delta(1 - z_1 - z_2)$. We omit these delta functions here for brevity and restore them at the end. In many cases, it is easiest to write the results in coordinate space with the radiation kernel $\Delta_{ij}^{(3)}$ defined as follows:

$$\Delta_{ij}^{(3)} = \frac{\mathbf{x}_{3i} \cdot \mathbf{x}_{3j}}{\mathbf{x}_{3i}^2 \mathbf{x}_{3j}^2}. \quad (31)$$

The real corrections are

$$\frac{d\sigma_{1 \times 1}^L}{d^2 \mathbf{p} d^2 \mathbf{q} dy_1 dy_2} = \frac{2e^2 g^2 Q^2 N_c^2 z_2^3 (1 - z_2)^2 (z_1^2 + (1 - z_2)^2)}{(2\pi)^{10} z_1} \int \frac{dz}{z} \int d^{10} x K_0(|\mathbf{x}_{12}| Q_2) K_0(|\mathbf{x}_{1'2'}| Q_2) \Delta_{11'}^{(3)}$$

$$\times [S_{122'1'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p} \cdot (\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q} \cdot (\mathbf{x}'_2 - \mathbf{x}_2)} e^{i\frac{z}{z_1} \mathbf{p} \cdot (\mathbf{x}'_1 - \mathbf{x}_1)}, \quad (32)$$

$$\frac{d\sigma_{2 \times 2}^L}{d^2 \mathbf{p} d^2 \mathbf{q} dy_1 dy_2} = \frac{2e^2 g^2 Q^2 N_c^2 z_1^3 (1 - z_1)^2 (z_2^2 + (1 - z_1)^2)}{(2\pi)^{10} z_2} \int \frac{dz}{z} \int d^{10} \mathbf{x} K_0(|\mathbf{x}_{12}| Q_1) K_0(|\mathbf{x}_{1'2'}| Q_1) \Delta_{22'}^{(3)}$$

$$\times [S_{122'1'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{q} \cdot (\mathbf{x}'_2 - \mathbf{x}_2)} e^{i\mathbf{p} \cdot (\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\frac{z}{z_2} \mathbf{q} \cdot (\mathbf{x}'_2 - \mathbf{x}_2)}, \quad (33)$$

$$\frac{d\sigma_{1 \times 2}^L}{d^2 \mathbf{p} d^2 \mathbf{q} dy_1 dy_2} = \frac{-2e^2 g^2 Q^2 N_c^2 z_1 z_2 (1 - z_1)(1 - z_2)(z_1(1 - z_1) + z_2(1 - z_2))}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10} \mathbf{x} K_0(|\mathbf{x}_{12}| Q_2)$$

$$\times K_0(|\mathbf{x}_{1'2'}| Q_1) \Delta_{12'}^{(3)} [S_{12} S_{1'2'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p} \cdot (\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q} \cdot (\mathbf{x}'_2 - \mathbf{x}_2)} e^{i\frac{z}{z_1} \mathbf{p} \cdot (\mathbf{x}_3 - \mathbf{x}_1)} e^{i\frac{z}{z_2} \mathbf{q} \cdot (\mathbf{x}'_2 - \mathbf{x}_3)}, \quad (34)$$

$$\frac{d\sigma_{3\times 3}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{2e^2g^2Q^2N_c^2z_1z_2^3(z_1^2 + (1-z_2)^2)}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(QX) K_0(QX') \Delta_{11'}^{(3)} \times [S_{11'}S_{22'} - S_{13}S_{23} - S_{1'3}S_{2'3} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)}, \quad (35)$$

$$\frac{d\sigma_{4\times 4}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{2e^2g^2Q^2N_c^2z_1^3z_2(z_2^2 + (1-z_1)^2)}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(QX) K_0(QX') \Delta_{22'}^{(3)} \times [S_{11'}S_{22'} - S_{13}S_{23} - S_{1'3}S_{2'3} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)}, \quad (36)$$

$$\frac{d\sigma_{3\times 4}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2z_1^2z_2^2(z_1(1-z_1) + z_2(1-z_2))}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(QX) K_0(QX') \Delta_{12'}^{(3)} \times [S_{11'}S_{22'} - S_{13}S_{23} - S_{1'3}S_{2'3} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)}, \quad (37)$$

$$\frac{d\sigma_{1\times 3}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2z_2^3(1-z_2)(z_1^2 + (1-z_2)^2)}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_2) K_0(QX') \Delta_{11'}^{(3)} \times [S_{122'3}S_{1'3} - S_{1'3}S_{2'3} - S_{12} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} e^{i\frac{z}{z_1}\mathbf{p}\cdot(\mathbf{x}_3-\mathbf{x}_1)}, \quad (38)$$

$$\frac{d\sigma_{1\times 4}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{2e^2g^2Q^2N_c^2z_1z_2^2(1-z_2)(z_1(1-z_1) + z_2(1-z_2))}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_2) K_0(QX') \Delta_{12'}^{(3)} \times [S_{122'3}S_{1'3} - S_{1'3}S_{2'3} - S_{12} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} e^{i\frac{z}{z_1}\mathbf{p}\cdot(\mathbf{x}_3-\mathbf{x}_1)}, \quad (39)$$

$$\frac{d\sigma_{2\times 3}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{2e^2g^2Q^2N_c^2z_1^2z_2(1-z_1)(z_1(1-z_1) + z_2(1-z_2))}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(QX') \Delta_{21'}^{(3)} \times [S_{1231'}S_{2'3} - S_{1'3}S_{2'3} - S_{12} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} e^{i\frac{z}{z_2}\mathbf{q}\cdot(\mathbf{x}_3-\mathbf{x}_2)}, \quad (40)$$

$$\frac{d\sigma_{2\times 4}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2z_1^3(1-z_1)(z_2^2 + (1-z_1)^2)}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(QX') \Delta_{22'}^{(3)} \times [S_{1231'}S_{2'3} - S_{1'3}S_{2'3} - S_{12} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} e^{i\frac{z}{z_1}\mathbf{q}\cdot(\mathbf{x}_3-\mathbf{x}_2)}. \quad (41)$$

This concludes the real corrections.

Next we have the virtual corrections:

$$\frac{d\sigma_5^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{2e^2g^2Q^2N_c^2z_2^3z_1}{(2\pi)^{10}} \int_0^{z_1} \frac{dz}{z} d^{10}\mathbf{x} [S_{322'1'}S_{13} - S_{13}S_{23} - S_{1'2'} + 1] (z_1^2 + (z_1-z)^2) \times \frac{K_0(QX_5)K_0(|\mathbf{x}_{1'2'}|Q_1)}{\mathbf{x}_{31}^2} e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} e^{-i\frac{z}{z_1}\mathbf{p}\cdot(\mathbf{x}_3-\mathbf{x}_1)}, \quad (42)$$

$$\frac{d\sigma_6^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{2e^2g^2Q^2N_c^2z_1^3z_2}{(2\pi)^{10}} \int_0^{z_2} \frac{dz}{z} d^{10}\mathbf{x} [S_{132'1'}S_{23} - S_{13}S_{23} - S_{1'2'} + 1] (z_2^2 + (z_2-z)^2) \times \frac{K_0(QX_6)K_0(|\mathbf{x}_{1'2'}|Q_1)}{\mathbf{x}_{32}^2} e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} e^{-i\frac{z}{z_2}\mathbf{q}\cdot(\mathbf{x}_3-\mathbf{x}_2)}, \quad (43)$$

$$\frac{d\sigma_7^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2z_2^2z_1}{(2\pi)^{10}} \int_0^{z_1} \frac{dz(z_1-z)}{z} d^{10}\mathbf{x} [S_{322'1'}S_{13} - S_{13}S_{23} - S_{1'2'} + 1] (z_1z_2 + (z_1-z)(z_2+z)) \times K_0(QX_5)K_0(|\mathbf{x}_{1'2'}|Q_1)\Delta_{12}^{(3)} e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} e^{-i\frac{z}{z_1}\mathbf{p}\cdot(\mathbf{x}_3-\mathbf{x}_1)}, \quad (44)$$

$$\frac{d\sigma_8^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2z_1^2z_2}{(2\pi)^{10}} \int_0^{z_2} \frac{dz(z_2-z)}{z} d^{10}\mathbf{x}[S_{132'1'}S_{23} - S_{13}S_{23} - S_{1'2'} + 1](z_1z_2 + (z_2-z)(z_1+z)) \\ \times K_0(QX_6)K_0(|\mathbf{x}_{1'2'}|Q_1)\Delta_{12}^{(3)}e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)}e^{-i\frac{z}{z_2}\mathbf{q}\cdot(\mathbf{x}_3-\mathbf{x}_2)}, \quad (45)$$

$$\frac{d\sigma_9^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^8} \int d^8\mathbf{x}[S_{122'1'} - S_{12} - S_{1'2'} + 1]K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \\ \times \int_0^{z_1} \frac{dz}{z} \frac{[z_1^2 + (z_1-z)^2]}{z_1^2} \int \frac{d^2\mathbf{k}_2}{(2\pi)^2} \frac{1}{(\mathbf{k}_2 - \frac{z}{z_1}\mathbf{p})^2}, \quad (46)$$

$$\frac{d\sigma_{10}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^8} \int d^8\mathbf{x}[S_{122'1'} - S_{12} - S_{1'2'} + 1]K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \\ \times \int_0^{z_2} \frac{dz}{z} \frac{[z_2^2 + (z_2-z)^2]}{z_2^2} \int \frac{d^2\mathbf{k}_2}{(2\pi)^2} \frac{1}{(\mathbf{k}_2 - \frac{z}{z_2}\mathbf{q})^2}, \quad (47)$$

$$\frac{d\sigma_{11}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^7} \int d^8\mathbf{x}[S_{122'1'} - S_{12} - S_{1'2'} + 1]K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \\ \times \int_0^{z_1} \frac{dz}{z} \frac{[z_1^2 + (z_1-z)^2]}{z_1^2} \int \frac{d^2\mathbf{k}_2}{(2\pi)^2} \int \frac{d^2\mathbf{k}_1}{(2\pi)^2} \frac{e^{i\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{[\mathbf{k}_1^2 + Q_1^2] \left[(\mathbf{k}_2 - \frac{z}{z_1}\mathbf{k}_1)^2 + \frac{z(z_1-z)}{z_1^2z_2} \mathbf{k}_1^2 + \frac{z}{z_1}(z_1-z)Q^2 \right]}, \quad (48)$$

$$\frac{d\sigma_{12}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^7} \int d^8\mathbf{x}[S_{122'1'} - S_{12} - S_{1'2'} + 1]K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \\ \times \int_0^{z_2} \frac{dz}{z} \frac{[z_2^2 + (z_2-z)^2]}{z_2^2} \int \frac{d^2\mathbf{k}_2}{(2\pi)^2} \int \frac{d^2\mathbf{k}_1}{(2\pi)^2} \frac{e^{i\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{[\mathbf{k}_1^2 + Q_1^2] \left[(\mathbf{k}_2 - \frac{z}{z_2}\mathbf{k}_1)^2 + \frac{z(z_2-z)}{z_2^2z_1} \mathbf{k}_1^2 + \frac{z}{z_2}(z_2-z)Q^2 \right]}, \quad (49)$$

$$\frac{d\sigma_{13(1)}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-e^2g^2Q^2N_c^2(z_1z_2)^2}{(2\pi)^8} \int_0^{z_2} \frac{dz}{z} (z_1+z)(z_2-z) \int d^8\mathbf{x}K_0(|\mathbf{x}_{12}|Q\sqrt{(z_1+z)(z_2-z)})K_0(|\mathbf{x}_{1'2'}|Q_1) \\ \times e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)}[S_{12}S_{1'2'} - S_{12} - S_{1'2'} + 1] \\ \times \int \frac{d^2\mathbf{k}}{(2\pi)^2} \left[\frac{-\frac{z(z_2-z)[z_1z_2+(z_1+z)(z_2-z)]}{z_1^2z_2^2} (z_2\mathbf{p} - z_1\mathbf{q})^2}{\left[(\mathbf{k} + (z_2-z)\mathbf{p} - (z_1+z)\mathbf{q})^2 - \frac{(z_1+z)(z_2-z)}{z_1z_2} (z_2\mathbf{p} - z_1\mathbf{q})^2 - i\epsilon \right] (\mathbf{k} - \frac{z}{z_2}\mathbf{q})^2} \right. \\ \left. + \left[\frac{\frac{z[z_1(z_1+z)+z_2(z_2-z)]}{z_1z_2}}{\left[(\mathbf{k} + (z_2-z)\mathbf{p} - (z_1+z)\mathbf{q})^2 - \frac{(z_1+z)(z_2-z)}{z_1z_2} (z_2\mathbf{p} - z_1\mathbf{q})^2 - i\epsilon \right]} - \frac{\frac{[z_1z_2+(z_1+z)(z_2-z)]}{z_1z_2}}{(\mathbf{k} - \frac{z}{z_2}\mathbf{q})^2} \right] e^{i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)}, \quad (50)$$

$$\frac{d\sigma_{13(2)}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-e^2g^2Q^2N_c^2(z_1z_2)^2}{(2\pi)^8} \int_0^{z_1} \frac{dz}{z} (z_2+z)(z_1-z) \int d^8\mathbf{x}K_0(|\mathbf{x}_{12}|Q\sqrt{(z_2+z)(z_1-z)})K_0(|\mathbf{x}_{1'2'}|Q_1) \\ \times e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)}[S_{12}S_{1'2'} - S_{12} - S_{1'2'} + 1] \\ \times \int \frac{d^2\mathbf{k}}{(2\pi)^2} \left[\frac{-\frac{z(z_1-z)[z_1z_2+(z_2+z)(z_1-z)]}{z_2^2z_1^2} (z_2\mathbf{p} - z_1\mathbf{q})^2}{\left[(\mathbf{k} + (z_1-z)\mathbf{q} - (z_2+z)\mathbf{p})^2 - \frac{(z_2+z)(z_1-z)}{z_1z_2} (z_2\mathbf{p} - z_1\mathbf{q})^2 + i\epsilon \right] (\mathbf{k} - \frac{z}{z_1}\mathbf{p})^2} \right. \\ \left. + \left[\frac{\frac{z[z_2(z_2+z)+z_1(z_1-z)]}{z_1z_2}}{\left[(\mathbf{k} + (z_1-z)\mathbf{q} - (z_2+z)\mathbf{p})^2 - \frac{(z_2+z)(z_1-z)}{z_1z_2} (z_2\mathbf{p} - z_1\mathbf{q})^2 + i\epsilon \right]} - \frac{\frac{[z_1z_2+(z_2+z)(z_1-z)]}{z_1z_2}}{(\mathbf{k} - \frac{z}{z_1}\mathbf{p})^2} \right] e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}. \quad (51)$$

Here we have kept the $i\epsilon$ in the long denominators, since it proves useful for evaluating the $\mathbf{k}$ integral:

$$\begin{aligned} \frac{d\sigma_{14(1)}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{e^2g^2Q^2N_c^2(z_1z_2)^2}{(2\pi)^7} \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] K_0(|\mathbf{x}_{1'2'}|Q_1) e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \\ &\times \int_0^{z_1} \frac{dz}{z} \frac{d^2\mathbf{k}_1}{(2\pi)^2} \frac{d^2\mathbf{k}_2}{(2\pi)^2} e^{i\mathbf{k}_2\cdot(\mathbf{x}_1-\mathbf{x}_2)} \left[\frac{(z_1z_2 + (z_1-z)(z_2+z))}{[\mathbf{k}_2^2 + Q_1^2] \left[(\mathbf{k}_1 - \frac{z_1-z}{z_1}\mathbf{k}_2)^2 + \frac{z(z_1-z)}{z_2z_1^2}\mathbf{k}_2^2 + \frac{z}{z_1}(z_1-z)Q^2 \right]} \right. \\ &+ \frac{\frac{(z_1-z)(z_2+z)}{z_1z_2} (z_1z_2 + (z_1-z)(z_2+z))}{[\mathbf{k}_1^2 + (z_1-z)(z_2+z)Q^2] \left[(\mathbf{k}_1 - \frac{z_1-z}{z_1}\mathbf{k}_2)^2 + \frac{z(z_1-z)}{z_2z_1^2}\mathbf{k}_2^2 + \frac{z}{z_1}(z_1-z)Q^2 \right]} \\ &- \frac{\frac{z(z_1-z)}{z_1} (z_1z_2 + (z_1-z)(z_2+z))Q^2}{[\mathbf{k}_1^2 + (z_1-z)(z_2+z)Q^2][\mathbf{k}_2^2 + Q_1^2] \left[(\mathbf{k}_1 - \frac{z_1-z}{z_1}\mathbf{k}_2)^2 + \frac{z(z_1-z)}{z_2z_1^2}\mathbf{k}_2^2 + \frac{z}{z_1}(z_1-z)Q^2 \right]} \\ &\left. - \frac{z(z_1(z_1-z) + z_2(z_2+z))}{[\mathbf{k}_1^2 + (z_1-z)(z_2+z)Q^2][\mathbf{k}_2^2 + Q_1^2]} \right], \end{aligned} \quad (52)$$

$$\begin{aligned} \frac{d\sigma_{14(2)}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{e^2g^2Q^2N_c^2(z_1z_2)^2}{(2\pi)^7} \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] K_0(|\mathbf{x}_{1'2'}|Q_1) e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \\ &\times \int_0^{z_2} \frac{dz}{z} \frac{d^2\mathbf{k}_1}{(2\pi)^2} \frac{d^2\mathbf{k}_2}{(2\pi)^2} e^{i\mathbf{k}_2\cdot(\mathbf{x}_1-\mathbf{x}_2)} \left[\frac{(z_1z_2 + (z_1+z)(z_2-z))}{[\mathbf{k}_2^2 + Q_1^2] \left[(\mathbf{k}_1 - \frac{z_2-z}{z_2}\mathbf{k}_2)^2 + \frac{z(z_2-z)}{z_1z_2^2}\mathbf{k}_2^2 + \frac{z}{z_2}(z_2-z)Q^2 \right]} \right. \\ &+ \frac{\frac{(z_1+z)(z_2-z)}{z_1z_2} (z_1z_2 + (z_1+z)(z_2-z))}{[\mathbf{k}_1^2 + (z_1+z)(z_2-z)Q^2] \left[(\mathbf{k}_1 - \frac{z_2-z}{z_2}\mathbf{k}_2)^2 + \frac{z(z_2-z)}{z_1z_2^2}\mathbf{k}_2^2 + \frac{z}{z_2}(z_2-z)Q^2 \right]} \\ &- \frac{\frac{z(z_2-z)}{z_2} (z_1z_2 + (z_1+z)(z_2-z))Q^2}{[\mathbf{k}_1^2 + (z_1+z)(z_2-z)Q^2][\mathbf{k}_2^2 + Q_1^2] \left[(\mathbf{k}_1 - \frac{z_2-z}{z_2}\mathbf{k}_2)^2 + \frac{z(z_2-z)}{z_1z_2^2}\mathbf{k}_2^2 + \frac{z}{z_2}(z_2-z)Q^2 \right]} \\ &\left. - \frac{z(z_2(z_2-z) + z_1(z_1+z))}{[\mathbf{k}_1^2 + (z_1+z)(z_2-z)Q^2][\mathbf{k}_2^2 + Q_1^2]} \right]. \end{aligned} \quad (53)$$

These expressions constitute the full result for the one-loop corrections to the inclusive quark antiquark production cross section. We have written these results all in terms of the dipole and quadrupole functions defined in Eq. (3) in the large N_c limit and ignored all the subleading N_c terms. We have also used the following notation for the coordinate dependence of some of the Bessel functions:

$$\begin{aligned} X &= \sqrt{z_1z_2\mathbf{x}_{12}^2 + z_1z\mathbf{x}_{13}^2 + z_2z\mathbf{x}_{23}^2}, \\ X_5 &= \sqrt{z_2(z_1-z)\mathbf{x}_{12}^2 + z(z-z)\mathbf{x}_{13}^2 + z_2z\mathbf{x}_{23}^2}, \\ X_6 &= \sqrt{z_1(z_2-z)\mathbf{x}_{12}^2 + z_1z\mathbf{x}_{13}^2 + z(z_2-z)\mathbf{x}_{23}^2}. \end{aligned} \quad (54)$$

Note that when $z \rightarrow 0$ these all become $|\mathbf{x}_{12}|\sqrt{z_1z_2}$. The primed version X' that appears in some real corrections is the same as X above but with $\mathbf{x}_1, \mathbf{x}_2 \rightarrow \mathbf{x}'_1, \mathbf{x}'_2$.

IV. DIVERGENCES

The expressions for the one-loop corrections to quark antiquark inclusive production cross section given above are formal, i.e. they contain divergences that need to be regulated and/or eliminated before one obtains a meaningful result. There are four categories of divergences:

- (i) Ultraviolet (UV) divergences when loop momentum $\mathbf{k} \rightarrow \infty$ or equivalently in coordinate space, when the transverse coordinate of the radiated gluon approaches the transverse coordinate $\mathbf{x}_i$ of either

quark or antiquark when integrated, i.e. $\mathbf{x}_3 \rightarrow \mathbf{x}_i$ such that $|\mathbf{x}_3 - \mathbf{x}_i| \rightarrow 0$. We will show that the real corrections are UV finite and that all UV divergences in the virtual corrections cancel against each other so that the virtual corrections are also UV finite by themselves.

- (ii) Soft divergences when $k^\mu \rightarrow 0$, which in this context corresponds to *both* transverse momentum in the loop $\mathbf{k}$ and the radiated gluon momentum fraction z go to zero simultaneously, $\mathbf{k}, z \rightarrow 0$. Both the real and virtual corrections contain soft divergences; however, all soft divergences cancel between real and virtual corrections.
- (iii) Collinear divergences when the radiated gluon momentum becomes parallel to either quark or antiquark momentum, i.e. when the angle θ between the radiated gluon 3-momentum and its parent quark (antiquark) 3-momentum goes to zero at *finite* $\mathbf{k}$ and z (note that finite in this context means that $\mathbf{k}$ is neither zero nor infinite). These appear, for example, as $\frac{1}{(p+k)^2} \sim \frac{1}{(z_1 \mathbf{k} - z \mathbf{p})^2} \sim \frac{1}{1 - \cos \theta}$ for radiation from a quark and similarly for radiation from an antiquark ($z_1, \mathbf{p} \rightarrow z_2, \mathbf{q}$). These collinear divergences are absorbed into quark-hadron (antiquark-hadron) fragmentation functions which makes the fragmentation functions scale dependent, i.e. they evolve with the renormalization scale μ^2 .
- (iv) Rapidity divergences when the momentum fraction z of the gluon goes to zero while the transverse momentum $\mathbf{k}$ of the gluon remains finite. These rapidity divergences are absorbed into (x) renormalization of the dipoles and quadrupoles making them energy/rapidity dependent and evolving according to the Balitsky-Kovchegov (BK) equation and Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner (JIMWLK) evolution equations [59–66].

Here we give more details of the divergences and their cancellation/absorption into renormalized quantities. The easiest way to see cancellations of divergences is via power counting which is what we will use in this section. We will also provide results using dimensional regularization in the $\overline{\text{MS}}$ scheme which is the more common method. Using dimensional regularization raises issues when using spinor helicity methods as one changes the number of spacetime dimensions from 4 (where spinor helicity methods are formulated) to $4 - 2\epsilon$. There are various ways of dealing with this issue; here we will use the four-dimensional helicity (FDH) scheme which treats both internal and external states in four dimensions. As expected, the finite terms after regulating divergences will depend on the regularization scheme.

A. UV divergences

For the real corrections [Eqs. (32)–(41)] all momentum integrations have been performed and results are given in

terms of a two-dimensional coordinate space integration $d^2\mathbf{x}_3$ and an integration over momentum fraction z [relative to the LO result in Eq. (1)]. In this context UV divergences manifest themselves as poles when $\mathbf{x}_3 \rightarrow \mathbf{x}_1$ or $\mathbf{x}_3 \rightarrow \mathbf{x}_2$, i.e. when the transverse coordinate of the radiated gluon $\mathbf{x}_3$ approaches that of the quark $\mathbf{x}_1$ or the antiquark $\mathbf{x}_2$. We focus on the quark case and consider the limit $\mathbf{x}_3 \rightarrow \mathbf{x}_1$. The case of radiation from the antiquark is identical.

We note that the only possible UV singularity in our expressions is in the gluon radiation kernel $\Delta_{ij}^{(3)}$ which appears in all real corrections. Writing it as

$$\Delta_{ij}^{(3)} = \frac{\mathbf{x}_{3i} \cdot \mathbf{x}_{3j}}{\mathbf{x}_{3i}^2 \mathbf{x}_{3j}^2} = \frac{1}{2} \left[\frac{1}{\mathbf{x}_{3i}^2} + \frac{1}{\mathbf{x}_{3j}^2} - \frac{\mathbf{x}_{ij}^2}{\mathbf{x}_{3i}^2 \mathbf{x}_{3j}^2} \right], \quad (55)$$

we see that the integral of this kernel over $\mathbf{x}_3$ is finite as $\mathbf{x}_3 \rightarrow \mathbf{x}_i$ due to cancellations between the first and third terms. Similarly, as $\mathbf{x}_3 \rightarrow \mathbf{x}_j$ there is a cancellation between the second and third terms. The other factors appearing in the real corrections are all nonsingular in the UV limit. We therefore conclude that there are no UV divergences in the real corrections and that *all real corrections are UV finite*.

For the virtual corrections, several of the diagrams are UV divergent. However when combined they all cancel each other in the UV limit, specifically

$$\begin{aligned} [d\sigma_5 + d\sigma_{11}]_{\text{UV}} &= 0, \\ [d\sigma_6 + d\sigma_{12}]_{\text{UV}} &= 0, \\ [d\sigma_9 + d\sigma_{10} + d\sigma_{14(1)} + d\sigma_{14(2)}]_{\text{UV}} &= 0. \end{aligned} \quad (56)$$

The remaining virtual diagrams (7,8,13) are all UV finite. Therefore, *all UV divergences are canceled* when adding up all the relevant pieces of the differential cross section. The UV finiteness of the production cross section holds for any value of z . We provide more details in Appendix A.

B. Soft divergences

Soft divergences appear when all components of a loop momentum go to zero, i.e. $k^\mu \rightarrow 0$ which in this context means $\mathbf{k}$ and z *both* go to zero (in coordinate space this becomes $\mathbf{x}_3 \rightarrow \infty$ and $z \rightarrow 0$). Soft divergences in the real corrections generically appear either as

$$\int \frac{d^2\mathbf{k}}{\mathbf{k}^2} e^{i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_j)} \quad \text{or} \quad \int \frac{d^2\mathbf{k}}{\mathbf{k}^2} \quad (57)$$

in the virtual corrections (after possibly shifting the transverse momentum). In coordinate space soft divergences are contained in

$$\int d^2\mathbf{x}_3 \Delta_{ij}^{(3)} = \frac{1}{2} \int d^2\mathbf{x}_3 \left[\frac{1}{(\mathbf{x}_3 - \mathbf{x}_i)^2} + \frac{1}{(\mathbf{x}_3 - \mathbf{x}_j)^2} - \frac{(\mathbf{x}_i - \mathbf{x}_j)^2}{(\mathbf{x}_3 - \mathbf{x}_i)^2(\mathbf{x}_3 - \mathbf{x}_j)^2} \right] \quad (58)$$

from which it is clear that the first two terms in $\Delta_{ij}^{(3)}$ are divergent when $\mathbf{x}_3 \rightarrow \infty$. Adding all the singular parts of the production cross section we get the following cancellations between the various terms:

$$\begin{aligned} [d\sigma_{1\times 1} + 2d\sigma_9]_{\text{soft}} &= 0, \\ [d\sigma_{2\times 2} + 2d\sigma_{10}]_{\text{soft}} &= 0, \\ [d\sigma_{1\times 2} + d\sigma_{13(1)} + d\sigma_{13(2)}]_{\text{soft}} &= 0, \\ [d\sigma_{3\times 3} + d\sigma_{4\times 4} + 2d\sigma_{3\times 4}]_{\text{soft}} &= 0, \\ [d\sigma_{1\times 3} + d\sigma_{1\times 4}]_{\text{soft}} &= 0, \\ [d\sigma_{2\times 3} + d\sigma_{2\times 4}]_{\text{soft}} &= 0, \\ [d\sigma_5 + d\sigma_7]_{\text{soft}} &= 0, \\ [d\sigma_6 + d\sigma_8]_{\text{soft}} &= 0, \\ [d\sigma_{11} + d\sigma_{14(1)}]_{\text{soft}} &= 0, \\ [d\sigma_{12} + d\sigma_{14(2)}]_{\text{soft}} &= 0. \end{aligned} \quad (59)$$

The necessary relative factors of 2 in these expressions can be seen from Eq. (28). These cancellations are easiest to see when the cross section is written in coordinate space in the soft z limit (Appendix B). It is worth noting that there are singular terms $\sim \frac{1}{z} \log(\frac{1}{z})$ which can be seen in dimensional regularization of the remaining integrals which

cancel between $d\sigma_{11,\text{soft}}$ and $d\sigma_{14(1),\text{soft}}$ and similarly between $d\sigma_{12,\text{soft}}$ and $d\sigma_{14(2),\text{soft}}$ (this is shown in Appendix C). Thus there are no soft singularities (both $\mathbf{k}, z \rightarrow 0$) left after adding all the real and virtual terms.

C. Rapidity divergences

From Eqs. (32)–(53) it is clear that the production cross section contains terms which are singular in the $\int \frac{dz}{z}$ integral as $z \rightarrow 0$. This happens at finite $\mathbf{k}$ (or $\mathbf{x}_3$) so that rapidity divergences are different from the soft divergences that have been shown to cancel among various terms. Following [45] we introduce a longitudinal momentum fraction factorization scale z_f and divide the z integration into two regions: $z > z_f$ and $z < z_f$ (here we write the upper limit as 1, in our expressions the upper limit is always either z_1 or z_2),

$$\int_0^1 \frac{dz}{z} f(z) = \left\{ \int_0^{z_f} \frac{dz}{z} + \int_{z_f}^1 \frac{dz}{z} \right\} f(z). \quad (60)$$

In the region where $z > z_f$ there are formally no divergences left anymore (there are still collinear divergences in this region but they will be absorbed into fragmentation functions—this is the topic of the next section) so the cross section is completely finite. On the other hand in the second region where $z < z_f$ we encounter the familiar rapidity divergence $y = \log(1/z)$ which we expect will be absorbed into JIMWLK evolution of the quadrupoles and dipoles. We now focus on the $z \rightarrow 0$ limit of our expressions while $\mathbf{k}$ (or equivalently $\mathbf{x}_3$) is kept finite (see Appendix B),

$$\begin{aligned} \frac{d\sigma_{\text{NLO}}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_f} \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} \delta(1 - z_1 - z_2) \\ &\times \left\{ [S_{122'1'} - S_{12} - S_{1'2'} + 1] \left(\frac{3}{\mathbf{x}_{31}^2} + \frac{3}{\mathbf{x}_{32}^2} + \frac{1}{\mathbf{x}_{31'}^2} + \frac{1}{\mathbf{x}_{32'}^2} - \frac{8}{\mathbf{x}_3^2} - \tilde{\Delta}_{11'} - \tilde{\Delta}_{22'} - 2\tilde{\Delta}_{12} \right) \right. \\ &+ [S_{12}S_{1'2'} - S_{12} - S_{1'2'} + 1] \left(\frac{2}{\mathbf{x}_{32}^2} - \frac{2}{\mathbf{x}_{32'}^2} + 2\tilde{\Delta}_{12'} - 2\tilde{\Delta}_{12} \right) \\ &+ [S_{11'}S_{22'} - S_{13}S_{23} - S_{1'3}S_{2'3} + 1] \left(-\frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{31'}^2} + \frac{1}{\mathbf{x}_{32}^2} - \frac{1}{\mathbf{x}_{32'}^2} - \tilde{\Delta}_{11'} - \tilde{\Delta}_{22'} + 2\tilde{\Delta}_{12'} \right) \\ &+ [S_{122'3}S_{1'3} - S_{1'3}S_{2'3} - S_{12} + 1] \left(\frac{2}{\mathbf{x}_{32'}^2} - \frac{2}{\mathbf{x}_{31'}^2} + 2\tilde{\Delta}_{11'} - 2\tilde{\Delta}_{12'} \right) \\ &+ [S_{1231'}S_{2'3} - S_{1'3}S_{2'3} - S_{12} + 1] \left(\frac{2}{\mathbf{x}_{31'}^2} - \frac{2}{\mathbf{x}_{32'}^2} + 2\tilde{\Delta}_{22'} - 2\tilde{\Delta}_{21'} \right) \\ &+ [S_{322'1'}S_{13} - S_{13}S_{23} - S_{1'2'} + 1] \left(\frac{2}{\mathbf{x}_{31}^2} - \frac{2}{\mathbf{x}_{32}^2} + 2\tilde{\Delta}_{12} \right) \\ &\left. + [S_{132'1'}S_{23} - S_{13}S_{23} - S_{1'2'} + 1] \left(\frac{2}{\mathbf{x}_{32}^2} - \frac{2}{\mathbf{x}_{31}^2} + 2\tilde{\Delta}_{12} \right) \right\}. \end{aligned} \quad (61)$$

Here we have defined

$$\tilde{\Delta}_{ij} = \frac{\mathbf{x}_{ij}^2}{\mathbf{x}_{3i}^2 \mathbf{x}_{3j}^2}. \quad (62)$$

Inside the curly brackets, the first line comes from adding $\sigma_{1 \times 1} + \sigma_{2 \times 2} + 2\sigma_9 + 2\sigma_{10} + 2\sigma_{11} + 2\sigma_{12} + 2\sigma_{14(1)} + 2\sigma_{14(2)}$. The second line is $2\sigma_{1 \times 2} + 2\sigma_{13(1)} + 2\sigma_{13(2)}$. The third line

is $\sigma_{3 \times 3} + \sigma_{4 \times 4} + 2\sigma_{3 \times 4}$. Fourth line is $2\sigma_{1 \times 3} + 2\sigma_{1 \times 4}$. Fifth line is $2\sigma_{2 \times 3} + 2\sigma_{2 \times 4}$. Sixth line is $2\sigma_5 + 2\sigma_7$, and the last line is $2\sigma_6 + 2\sigma_8$. Notice that cross terms have been doubled since we need to add their complex conjugate. To simplify the expression we use a combination of shifting $\mathbf{x}_3$ where possible and using symmetry between primed and unprimed coordinates. We find that everything can be simplified down to the following expression:

$$\begin{aligned} \frac{d\sigma_{\text{NLO}}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \delta(1-z_1-z_2) \int_0^{z_f} \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \\ &\times e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} \left\{ (\tilde{\Delta}_{12} + \tilde{\Delta}_{22'} - \tilde{\Delta}_{12'}) S_{132'1'} S_{23} + (\tilde{\Delta}_{1'2'} + \tilde{\Delta}_{22'} - \tilde{\Delta}_{21'}) S_{1'321} S_{2'3} \right. \\ &+ (\tilde{\Delta}_{12} + \tilde{\Delta}_{11'} - \tilde{\Delta}_{21'}) S_{322'1'} S_{13} + (\tilde{\Delta}_{1'2'} + \tilde{\Delta}_{11'} - \tilde{\Delta}_{12'}) S_{32'21} S_{1'3} \\ &- (\tilde{\Delta}_{11'} + \tilde{\Delta}_{22'} + \tilde{\Delta}_{12} + \tilde{\Delta}_{1'2'}) S_{122'1'} - (\tilde{\Delta}_{12} + \tilde{\Delta}_{1'2'} - \tilde{\Delta}_{12'} - \tilde{\Delta}_{21'}) S_{12} S_{1'2'} \\ &\left. - (\tilde{\Delta}_{11'} + \tilde{\Delta}_{22'} - \tilde{\Delta}_{12'} - \tilde{\Delta}_{21'}) S_{11'} S_{22'} - 2\tilde{\Delta}_{12} (S_{13} S_{23} - S_{12}) - 2\tilde{\Delta}_{1'2'} (S_{1'3} S_{2'3} - S_{1'2'}) \right\}. \quad (63) \end{aligned}$$

Here the first seven terms inside the curly bracket give the JIMWLK evolution of the quadrupole $S_{122'1'}$ with the factorization scale z_f [see Eq. (5) in [67]]. The last two terms correspond to the rapidity evolution of the dipoles S_{12} and $S_{1'2'}$ as given by the BK equation [59,60]. This demonstrates that the rapidity divergences can be absorbed into rapidity evolution of the LO cross section. The contribution of the z integration above the factorization scale z_f is finite and constitutes part of the next-to-leading-order (NLO) corrections to the leading-order cross section.

D. Collinear divergences

1. Leading order dihadron production

We now focus on the collinear divergences present in some of the one-loop corrections in Eqs. (32)–(53) and show that they can be absorbed into evolution of quark-hadron and antiquark-hadron fragmentation functions. As this is completely new we will provide the full details of the calculation. We start by writing the inclusive dihadron production cross section in terms of the partonic cross section as

$$\frac{d\sigma^{\gamma^*A \rightarrow h_1 h_2 X}}{d^2\mathbf{p}_h d^2\mathbf{q}_h dy_1 dy_2} = \int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{1}{z_{h_1}^2 z_{h_2}^2} \frac{d\sigma^{\gamma^*A \rightarrow q\bar{q}X}}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} D_{h_1/q}(z_{h_1}) D_{h_2/\bar{q}}(z_{h_2}), \quad (64)$$

where $\mathbf{p}, \mathbf{q}$ are the quark and antiquark transverse momenta while $\mathbf{p}_h, \mathbf{q}_h$ are the transverse momenta of the two produced hadrons. As partons and hadrons are assumed to be massless, their rapidities are the same so that $y_1 \equiv y_p = y_h$ (and similarly for the antiquark). Hadronization is assumed to be describable in terms of a nonperturbative quark (or antiquark-) hadron fragmentation function $D_{h/q}(z_{h_1})$ which is then perturbatively renormalized (becomes μ^2 dependent) after absorption of the collinear divergences. The momentum fractions z_{h_1}, z_{h_2} are defined via $p_{h_1}^\mu = z_{h_1} p^\mu$ and $q_{h_2}^\mu = z_{h_2} q^\mu$ so that $0 < z_{h_i} < 1$. Using our leading-order result from Eq. (1) (only the longitudinal part) we can write this as

$$\begin{aligned} \frac{d\sigma_{LO}^{\gamma^*A \rightarrow h_1 h_2 X}}{d^2\mathbf{p}_h d^2\mathbf{q}_h dy_1 dy_2} &= \int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{4e^2Q^2(z_1z_2)^3 N_c}{(2\pi)^7 (z_{h_1} z_{h_2})^2} \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ &\times K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) e^{i\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2 - \mathbf{x}_2)} D_{h_1/q}(z_{h_1}) D_{h_2/\bar{q}}(z_{h_2}) \delta(1-z_1-z_2), \quad (65) \end{aligned}$$

where z_1 and z_2 are related to hadronic momentum fractions by

$$z_1 = \frac{p_h^+}{z_{h_1} l^+}, \quad z_2 = \frac{q_h^+}{z_{h_2} l^+}. \quad (66)$$

Anticipating factorization of the one-loop corrections and to make our expression more compact we define the common factor $H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_1)$ as

$$H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_1) \equiv [S_{122'1'} - S_{12} - S_{1'2'} + 1] K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) e^{i\mathbf{p} \cdot (\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q} \cdot (\mathbf{x}'_2 - \mathbf{x}_2)}, \quad (67)$$

with $Q_1^2 \equiv z_1(1 - z_1)Q^2$ so that the common factor implicitly depends on Q^2 . The leading-order cross section can be then expressed in the following compact form:

$$\frac{d\sigma_{LO}^{\gamma^* A \rightarrow h_1 h_2 X}}{d^2\mathbf{p}_h d^2\mathbf{q}_h dy_1 dy_2} = \int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{4e^2 Q^2 (z_1 z_2)^3 N_c}{(2\pi)^7 (z_{h_1} z_{h_2})^2} \int d^8\mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) \delta(1 - z_1 - z_2) D_{h_1/q}(z_{h_1}) D_{h_2/\bar{q}}(z_{h_2}). \quad (68)$$

2. The virtual correction

First we will focus on the fragmentation of the quark into a hadron h_1 . Treatment of antiquark fragmentation will be identical. Schematically, the fragmentation function receives corrections from the diagrams in Fig. 3.

In Fig. 3 the first diagram after the equals sign corresponds to the leading-order hadronic cross section computed in the previous subsection. The two $\mathcal{O}(\alpha_s)$ corrections correspond to our virtual correction σ_9 and the real correction $\sigma_{1 \times 1}$. So, let us now compute the hadronic cross section contribution from the virtual correction σ_9 [combining Eq. (46) with Eq. (64)],

$$\begin{aligned} \frac{d\sigma_9^{\gamma^* A \rightarrow h_1 h_2 X}}{d^2\mathbf{p}_h d^2\mathbf{q}_h dy_1 dy_2} &= - \int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{4e^2 Q^2 (z_1 z_2)^3 N_c}{(2\pi)^7 (z_{h_1} z_{h_2})^2} \int d^8\mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_1/q}^0(z_{h_1}) D_{h_2/\bar{q}}^0(z_{h_2}) \\ &\quad \times \alpha_s C_F \int_0^{z_1} \frac{dz}{z} \left[\frac{z_1^2 + (z_1 - z)^2}{z_1^2} \right] \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{(\mathbf{k} - \frac{z}{z_1}\mathbf{p})^2} \delta(1 - z_1 - z_2). \end{aligned} \quad (69)$$

Here we relaxed the large N_c approximation taking one factor of $N_c \rightarrow 2C_F$, and have also used $g^2 \rightarrow 4\pi\alpha_s$. Let us define a new variable for the z integration,

$$\xi = \frac{z_1 - z}{z_1}. \quad (70)$$

This defines ξ to be the fraction of longitudinal momentum that the quark carries after radiating the gluon. In terms of this, we get

$$\begin{aligned} \frac{d\sigma_9^{\gamma^* A \rightarrow h_1 h_2 X}}{d^2\mathbf{p}_h d^2\mathbf{q}_h dy_1 dy_2} &= - \int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{4e^2 Q^2 (z_1 z_2)^3 N_c}{(2\pi)^7 (z_{h_1} z_{h_2})^2} \int d^8\mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_2/\bar{q}}^0(z_{h_2}) D_{h_1/q}^0(z_{h_1}) \\ &\quad \times \alpha_s C_F \int_0^1 d\xi \frac{(1 + \xi^2)}{(1 - \xi)} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{(\mathbf{k} - (1 - \xi)\mathbf{p})^2} \delta(1 - z_1 - z_2). \end{aligned} \quad (71)$$

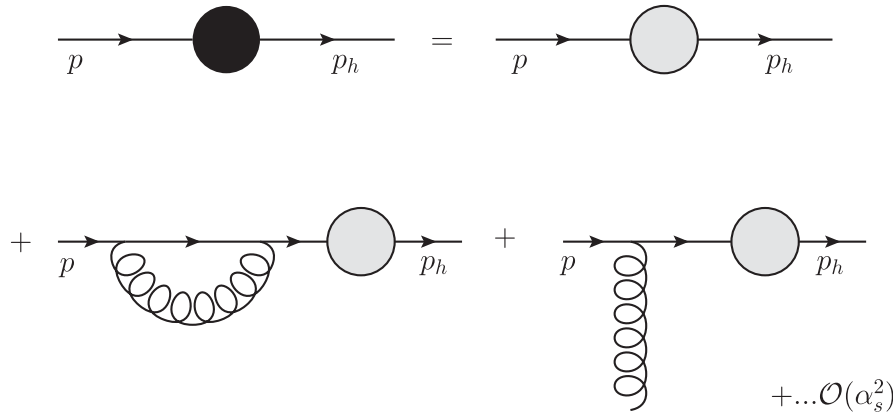


FIG. 3. The full fragmentation function $D_{h_1/q}(z_{h_1}, \mu^2)$ (solid black circle) can be written as the bare fragmentation function (faint gray circle) plus the $\mathcal{O}(\alpha_s)$ corrections shown. These need to be added at the cross section level, and we will keep terms only up to $\mathcal{O}(\alpha_s)$.

This can now be added to the leading-order hadronic cross section. Note that we actually need to add $2\sigma_9$ because it appears twice in the formula since it is a cross term [Eq. (28)].

3. The real correction

Looking at the relevant diagrams in Fig. 3, the last one we need for the $\mathcal{O}(\alpha_s)$ corrections to the quark fragmentation function is the real correction $\sigma_{1\times 1}$,

$$\begin{aligned} \frac{d\sigma_{1\times 1}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{2e^2g^2Q^2N_c^2z_2^3(1-z_2)^2(z_1^2+(1-z_2)^2)}{(2\pi)^8z_1} \int d^8x K_0(|\mathbf{x}_{12}|Q_2) K_0(|\mathbf{x}_{1'2'}|Q_2) \\ &\times [S_{122'1'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \int \frac{dz}{z} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}}{(\mathbf{k}-\frac{z}{z_1}\mathbf{p})^2} \delta(1-z_1-z_2-z). \end{aligned} \quad (72)$$

We have rewritten the integral over the radiation kernel [in Eq. (32)] in momentum space using

$$e^{i\frac{z}{z_1}\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} \int d^2\mathbf{x}_3 \Delta_{11'}^{(3)} = \int d^2\mathbf{k} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}}{(\mathbf{k}-\frac{z}{z_1}\mathbf{p})^2}. \quad (73)$$

Looking at Eq. (64) we see that this correction contributes the following expression to the hadronic cross section:

$$\begin{aligned} \frac{d\sigma_{1\times 1}^{\gamma^*A\rightarrow h_1h_2X}}{d^2\mathbf{p}_h d^2\mathbf{q}_h dy_1 dy_2} &= \int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{2e^2g^2Q^2N_c^2z_2^3(1-z_2)^2(z_1^2+(1-z_2)^2)}{(2\pi)^8z_1(z_{h_1}z_{h_2})^2} \int d^8x K_0(|\mathbf{x}_{12}|Q_2) \\ &\times K_0(|\mathbf{x}_{1'2'}|Q_2) [S_{122'1'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \\ &\times \int \frac{dz}{z} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}}{(\mathbf{k}-\frac{z}{z_1}\mathbf{p})^2} \delta(1-z_1-z_2-z) D_{h_1/q}^0(z_{h_1}) D_{h_2/\bar{q}}^0(z_{h_2}). \end{aligned} \quad (74)$$

Now we will write this using $H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2)$ defined in Eq. (67) and again relax the large N_c approximation taking one factor of N_c to $2C_F$, and also write $g^2 = 4\pi\alpha_s$,

$$\begin{aligned} \frac{d\sigma_{1\times 1}^{\gamma^*A\rightarrow h_1h_2X}}{d^2\mathbf{p}_h d^2\mathbf{q}_h dy_1 dy_2} &= \int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{8e^2Q^2N_c(z_1z_2)^3}{(2\pi)^7(z_{h_1}z_{h_2})^2} \int d^8x H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_1/q}^0(z_{h_1}) D_{h_2/\bar{q}}^0(z_{h_2}) \\ &\times \alpha_s C_F \int \frac{dz}{z} \frac{(z_1+z)^2(z_1^2+(z_1+z)^2)}{z_1^4} \delta(1-z_1-z_2-z) \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}}{(\mathbf{k}-\frac{z}{z_1}\mathbf{p})^2}. \end{aligned} \quad (75)$$

We factored out $(z_1z_2)^3$ to match the leading-order expression and used the delta function to write $1-z_2$ as z_1+z . We then perform a change of variables in the z integral by defining

$$\xi = \frac{z_1}{z_1+z}. \quad (76)$$

This ξ_1 is then the fraction of the outgoing quark's momentum relative to the parent quark that emitted the gluon (i.e. the same definition as we used for the virtual correction),

$$\begin{aligned} \frac{d\sigma_{1\times 1}^{\gamma^*A\rightarrow h_1h_2X}}{d^2\mathbf{p}_h d^2\mathbf{q}_h dy_1 dy_2} &= \int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{8e^2Q^2N_c(z_1z_2)^3}{(2\pi)^7(z_{h_1}z_{h_2})^2} \int d^8x H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_1/q}^0(z_{h_1}) D_{h_2/\bar{q}}^0(z_{h_2}) \\ &\times \alpha_s C_F \int \frac{d\xi}{\xi^5} \frac{(1+\xi^2)}{(1-\xi)} \delta(1-z_2-z_1/\xi) D_{h_1/q}^0(z_{h_1}) \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}}{(\mathbf{k}-\frac{(1-\xi)}{\xi}\mathbf{p})^2}. \end{aligned} \quad (77)$$

Now, z_1 is no longer an external variable; as remarked earlier it can be written in terms of z_{h_1} using Eq. (66). So one can then write z_1 in terms of z_{h_1} ,

$$\begin{aligned} \frac{d\sigma_{1\times 1}^{\gamma^* A \rightarrow h_1 h_2 X}}{d^2\mathbf{p}_h d^2\mathbf{q}_h dy_1 dy_2} &= \int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{8e^2 Q^2 N_c (p_h^+)^3 z_2^3}{(2\pi)^7 (l^+)^3 z_{h_1}^5 z_{h_2}^2} \int d^8\mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_2/\bar{q}}^0(z_{h_2}) \\ &\times \alpha_s C_F \int \frac{d\xi}{\xi^5} \frac{(1+\xi^2)}{(1-\xi)} \delta\left(1 - z_2 - \frac{p_h^+}{z_{h_1} l^+ \xi}\right) D_{h_1/q}^0(z_{h_1}) \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)}}{\left(\mathbf{k} - \frac{(1-\xi)}{\xi}\mathbf{p}\right)^2}. \end{aligned} \quad (78)$$

Now that the z_{h_1} dependence is all explicit, we make a substitution in the z_{h_1} integral, defining the new variable

$$z'_{h_1} = z_{h_1} \xi, \quad dz'_{h_1} = \xi dz_{h_1}. \quad (79)$$

In order to complete this substitution, we need to take $z_{h_1} \rightarrow \frac{z'_{h_1}}{\xi}$ everywhere. This also means that the bounds on the z'_{h_1} integral go from 0 to ξ . We will write this using a step function $\theta(\xi - z'_{h_1})$ so we can keep the explicit bounds from 0 to 1,

$$\begin{aligned} \frac{d\sigma_{1\times 1}^{\gamma^* A \rightarrow h_1 h_2 X}}{d^2\mathbf{p}_h d^2\mathbf{q}_h dy_1 dy_2} &= \int_0^1 dz'_{h_1} \int_0^1 dz_{h_2} \frac{8e^2 Q^2 N_c (p_h^+)^3 z_2^3}{(2\pi)^7 (l^+)^3 z_{h_1}'^5 z_{h_2}^2} \int d^8\mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_2/\bar{q}}^0(z_{h_2}) \\ &\times \alpha_s C_F \int \frac{d\xi}{\xi} \frac{(1+\xi^2)}{(1-\xi)} D_{h_1/q}^0\left(\frac{z'_{h_1}}{\xi}\right) \theta(\xi - z'_{h_1}) \delta\left(1 - z_2 - \frac{p_h^+}{z'_{h_1} l^+}\right) \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)}}{\left(\mathbf{k} - \frac{(1-\xi)}{\xi}\mathbf{p}\right)^2}. \end{aligned} \quad (80)$$

We can now write things back in terms of z_1 to get rid of p_h^+ and l^+ to match the leading-order and virtual correction. We can also use the step function to set the bounds on the ξ integral. Finally, we remove the primes from z'_{h_1} since it is an integration variable,

$$\begin{aligned} \frac{d\sigma_{1\times 1}^{\gamma^* A \rightarrow h_1 h_2 X}}{d^2\mathbf{p}_h d^2\mathbf{q}_h dy_1 dy_2} &= \int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{8e^2 Q^2 N_c (z_1 z_2)^3}{(2\pi)^7 (z_{h_1} z_{h_2})^2} \int d^8\mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_2/\bar{q}}^0(z_{h_2}) \\ &\times \delta(1 - z_1 - z_2) \alpha_s C_F \int_{z_{h_1}}^1 \frac{d\xi}{\xi} \frac{(1+\xi^2)}{(1-\xi)} D_{h_1/q}^0\left(\frac{z_{h_1}}{\xi}\right) \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)}}{\left(\mathbf{k} - \frac{(1-\xi)}{\xi}\mathbf{p}\right)^2}. \end{aligned} \quad (81)$$

4. Evolution of the fragmentation function

Now we would like to add up the three terms in Fig. 3. These terms are Eqs. (68), (71), and (81). Note that we also need to double Eq. (71) since it is a cross term. Adding up these three gives the following:

$$\begin{aligned} &\int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{4e^2 Q^2 (z_1 z_2)^3 N_c}{(2\pi)^7 (z_{h_1} z_{h_2})^2} \int d^8\mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_2/\bar{q}}^0(z_{h_2}) \delta(1 - z_1 - z_2) \\ &\times \left[D_{h_1/q}^0(z_{h_1}) + 2\alpha_s C_F \left\{ \int_{z_{h_1}}^1 \frac{d\xi}{\xi} \frac{(1+\xi^2)}{(1-\xi)} D_{h_1/q}^0\left(\frac{z_{h_1}}{\xi}\right) \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)}}{(\mathbf{k} - (1-\xi)\mathbf{p})^2} \right. \right. \\ &\left. \left. - D_{h_1/q}^0(z_{h_1}) \int_0^1 \frac{d\xi}{\xi} \frac{(1+\xi^2)}{(1-\xi)} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{(\mathbf{k} - (1-\xi)\mathbf{p})^2} \right\} \right]. \end{aligned} \quad (82)$$

Now we turn our attention to the two transverse momentum integrals. The one with the exponential comes from the real correction $\sigma_{1\times 1}$, and contains both a soft divergence, $\mathbf{k} \rightarrow 0$ and $z \rightarrow 0$ (equivalently $\xi \rightarrow 1$) at the same time, and a collinear divergence as $\mathbf{k} \rightarrow (1-\xi)\mathbf{p}$. The integral without an exponential (from the virtual correction σ_9) contains a soft divergence, a collinear divergence, and a UV divergence as $|\mathbf{k}| \rightarrow \infty$. We showed in Sec. IV B that the soft divergences cancel between these two terms. In Sec. IV A we also showed that the UV divergence in the virtual correction is canceled with other terms in the cross section. Therefore, the only divergence left to consider here is the collinear divergence in both integrals.

To see how regulating these divergences leads to renormalization of the quark-hadron fragmentation function it is perhaps easiest to demonstrate this using an explicit cutoff where the lower limit on $|\mathbf{k}|$ is Λ and the upper limit is μ :

$$\int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{4e^2 Q^2 (z_1 z_2)^3 N_c}{(2\pi)^7 (z_{h_1} z_{h_2})^2} \int d^8 \mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_2/\bar{q}}^0(z_{h_2}) \delta(1 - z_1 - z_2) \\ \times \left[D_{h_1/q}^0(z_{h_1}) + \frac{\alpha_s C_F}{2\pi} \left\{ \int_{z_{h_1}}^1 \frac{d\xi}{\xi} \frac{(1 + \xi^2)}{(1 - \xi)} D_{h_1/q}^0\left(\frac{z_{h_1}}{\xi}\right) - \int_0^1 d\xi \frac{(1 + \xi^2)}{(1 - \xi)} D_{h_1/q}^0(z_{h_1}) \right\} \log\left(\frac{\mu^2}{\Lambda^2}\right) \right]. \quad (83)$$

We would like to combine the terms inside the curly brackets; to do so we will define the quark-quark splitting function $P_{qq}(\xi)$,

$$P_{qq}(\xi) \equiv C_F \left[\frac{(1 + \xi^2)}{(1 - \xi)_+} + \frac{3}{2} \delta(1 - \xi) \right], \quad (84)$$

where the $+$ distribution is defined inside an integral via

$$\int_z^1 d\xi \frac{f(\xi)}{(1 - \xi)_+} \equiv \int_z^1 \frac{d\xi}{(1 - \xi)} (f(\xi) - f(1)) + f(1) \log(1 - z). \quad (85)$$

This allows us to write Eq. (83) as

$$\int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{4e^2 Q^2 (z_1 z_2)^3 N_c}{(2\pi)^7 (z_{h_1} z_{h_2})^2} \int d^8 \mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_2/\bar{q}}^0(z_{h_2}) \delta(1 - z_1 - z_2) \\ \times \left[D_{h_1/q}^0(z_{h_1}) + \frac{\alpha_s}{2\pi} \left\{ \int_{z_{h_1}}^1 \frac{d\xi}{\xi} P_{qq}(\xi) D_{h_1/q}^0\left(\frac{z_{h_1}}{\xi}\right) \right\} \log\left(\frac{\mu^2}{\Lambda^2}\right) \right]. \quad (86)$$

Here we have used the following result:

$$\int_{z_{h_1}}^1 \frac{d\xi}{\xi} P_{qq}(\xi) D_{h_1/q}^0\left(\frac{z_{h_1}}{\xi}\right) = \int_{z_{h_1}}^1 \frac{d\xi}{\xi} \frac{(1 + \xi^2)}{(1 - \xi)} D_{h_1/q}^0\left(\frac{z_{h_1}}{\xi}\right) - \int_0^1 d\xi \frac{(1 + \xi^2)}{(1 - \xi)} D_{h_1/q}^0(z_{h_1}). \quad (87)$$

To prove this, one first expands the P_{qq} function using Eq. (84), then further expands the term with the plus distribution according to Eq. (85). One of the terms is then exactly the first term on the right side of Eq. (87). In the remaining term with a ξ integral one separates into two terms using $\int_{z_{h_1}}^1 = \int_0^1 - \int_0^{z_{h_1}}$. One can then use the fact that

$$\frac{2}{1 - \xi} = \frac{(1 + \xi^2)}{(1 - \xi)} + 1 + \xi. \quad (88)$$

The first term here paired with the $\int_0^1$ integrals then gives exactly the second term on the right side of Eq. (87). In all the other terms, the integral over ξ can be performed and these terms all cancel in the full expression.

Next, we can also combine with the leading-order term (first term in the square brackets) by including a ξ integral that evaluates to 1:

$$\int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{4e^2 Q^2 (z_1 z_2)^3 N_c}{(2\pi)^7 (z_{h_1} z_{h_2})^2} \int d^8 \mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_2/\bar{q}}^0(z_{h_2}) \delta(1 - z_1 - z_2) \\ \times \int_{z_{h_1}}^1 \frac{d\xi}{\xi} D_{h_1/q}^0\left(\frac{z_{h_1}}{\xi}\right) \left[\delta(1 - \xi) + \frac{\alpha_s}{2\pi} P_{qq}(\xi) \log\left(\frac{\mu^2}{\Lambda^2}\right) \right]. \quad (89)$$

Then, we define the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolved quark-hadron fragmentation function $D_{h_1/q}(z_{h_1}, \mu^2)$ as

$$D_{h_1/q}(z_{h_1}, \mu^2) = \int_{z_{h_1}}^1 \frac{d\xi}{\xi} D_{h_1/q}^0\left(\frac{z_{h_1}}{\xi}\right) \left[\delta(1 - \xi) + \frac{\alpha_s}{2\pi} P_{qq}(\xi) \log\left(\frac{\mu^2}{\Lambda^2}\right) \right], \quad (90)$$

in terms of which our expression becomes

$$\frac{d\sigma_{LO+9+1\times 1}^{\gamma^* A \rightarrow h_1 h_2 X}}{d^2\mathbf{p}_h d^2\mathbf{q}_h dy_1 dy_2} = \int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{4e^2 Q^2 (z_1 z_2)^3 N_c}{(2\pi)^7 (z_{h_1} z_{h_2})^2} \int d^8\mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_2/\bar{q}}^0(z_{h_2}) D_{h_1/q}(z_{h_1}, \mu^2) \delta(1 - z_1 - z_2). \quad (91)$$

It must be noted that here we used an explicit cutoff μ to regulate these collinear divergences. However the most common way to handle these divergences is via dimensional regularization (dim reg). This is done in the next subsection.

5. Evolution of the fragmentation function in dimensional regularization

As most parametrizations of the fragmentation functions are defined in the $\overline{\text{MS}}$ scheme, it is important to relate our result with that given in the $\overline{\text{MS}}$ scheme and using dimensional regularization as this would affect the finite pieces. The real integral can be regulated without complication via dimensional regularization,

$$\begin{aligned} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)}}{\left(\mathbf{k} - \frac{(1-\xi)}{\xi}\mathbf{p}\right)^2} &\rightarrow \mu^{2-d} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)}}{\left(\mathbf{k} - \frac{(1-\xi)}{\xi}\mathbf{p}\right)^2} \\ &= \mu^{2-d} e^{i\frac{(1-\xi)}{\xi}\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)}}{\mathbf{k}^2} = \frac{\mu^{2-d} e^{i\frac{(1-\xi)}{\xi}\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)}}{(2\pi)^d} \int dk k^{d-3} \int d\Omega_d e^{ik|\mathbf{x}'_1 - \mathbf{x}_1| \cos\theta}. \end{aligned} \quad (92)$$

The limit $d \rightarrow 2$ is safe in the angular integral,

$$\begin{aligned} \frac{\mu^{2-d} e^{i\frac{(1-\xi)}{\xi}\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)}}{(2\pi)^{d-1}} \int dk k^{d-3} J_0(k|\mathbf{x}'_1 - \mathbf{x}_1|) &= \frac{\mu^{2-d} e^{i\frac{(1-\xi)}{\xi}\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)}}{(2\pi)^{d-1}} 2^{d-3} |\mathbf{x}'_1 - \mathbf{x}_1|^{2-d} \frac{\Gamma[d/2 - 1]}{\Gamma[2 - d/2]}, \quad 2 < d < 7/2, \\ &= \frac{e^{i\frac{(1-\xi)}{\xi}\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)}}{2\pi} \left[\frac{1}{\epsilon} - \log(\pi e^{\gamma_E} \mu |\mathbf{x}'_1 - \mathbf{x}_1|) \right] + \mathcal{O}(\epsilon), \quad \epsilon = d - 2 > 0. \end{aligned} \quad (93)$$

This completes our result for regulating the real integral via dimensional regularization. Next we look at the virtual integral which, after a shift, becomes

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2}. \quad (94)$$

This divergent integral is zero in dimensional regularization [68,69]. This can be understood as the result of the collinear (here looks soft due to the shift) divergence that exactly cancels the ultraviolet divergence in dimensional regularization. One can see this by separating the integration into two terms with a cutoff and performing each separately [45,70,71]. Now, it is important to note that in our case we have already used the UV divergence to cancel other UV divergences in the virtual corrections (see Sec. IV A). Thus we need to be careful with this integral. Following [45] let us separate into two regions,

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2} \rightarrow \mu^{2-d} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{\mathbf{k}^2} = \frac{\mu^{2-d} 2\pi^{d/2}}{(2\pi)^d \Gamma[d/2]} \left[\int_0^\Lambda dk k^{d-3} + \int_\Lambda^\infty dk k^{d-3} \right]. \quad (95)$$

The first term contains the infrared (collinear) divergence, and the second term contains the UV divergence. These integrals can both be done, which yields

$$\frac{\mu^{2-d} 2\pi^{d/2}}{(2\pi)^d \Gamma[d/2]} \left[\frac{\Lambda^{-\epsilon_{\text{UV}}}}{\epsilon_{\text{UV}}} - \frac{\Lambda^{-\epsilon_{\text{IR}}}}{\epsilon_{\text{IR}}} \right]. \quad (96)$$

Here $\epsilon = 2 - d$ with $\epsilon_{\text{UV}} > 0$ and $\epsilon_{\text{IR}} < 0$. If one formally sets $\epsilon_{\text{UV}} = \epsilon_{\text{IR}}$, then one finds that the expression is zero. For our purposes, let us write both expressions separately and expand each for small ϵ :

$$\left[\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2} \right]_{\text{IR}} = \frac{1}{2\pi} \left[-\frac{1}{\epsilon_{\text{IR}}} - \frac{1}{2} \log\left(\frac{4\pi e^{-\gamma_E} \mu^2}{\Lambda^2}\right) \right] + \mathcal{O}(\epsilon). \quad (97)$$

$$\left[\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2} \right]_{\text{UV}} = \frac{1}{2\pi} \left[\frac{1}{\epsilon_{\text{UV}}} + \frac{1}{2} \log\left(\frac{4\pi e^{-\gamma_E} \mu^2}{\Lambda^2}\right) \right] + \mathcal{O}(\epsilon). \quad (98)$$

Notice that the finite terms cancel if one adds both of these together to form the full integral [which gives

Eq. (5.45) in [45]]. We would like for the finite term to look similar to the finite term in the result for the real integral [Eq. (93)], so let us add a finite term to the IR piece and also subtract it from the UV piece:

$$\left[\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2} \right]_{\text{IR}} = \frac{1}{2\pi} \left[-\frac{1}{\epsilon_{\text{IR}}} - \frac{1}{2} \log \left(\frac{4\pi e^{-\gamma_E} \mu^2}{\Lambda^2} \right) + \log \left(\frac{2e^{-\frac{3}{2}\gamma_E}}{\pi^{1/2} \Lambda |\mathbf{x}'_1 - \mathbf{x}_1|} \right) \right] + \mathcal{O}(\epsilon). \quad (99)$$

$$\left[\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2} \right]_{\text{UV}} = \frac{1}{2\pi} \left[\frac{1}{\epsilon_{\text{UV}}} + \frac{1}{2} \log \left(\frac{4\pi e^{-\gamma_E} \mu^2}{\Lambda^2} \right) - \log \left(\frac{2e^{-\frac{3}{2}\gamma_E}}{\pi^{1/2} \Lambda |\mathbf{x}'_1 - \mathbf{x}_1|} \right) \right] + \mathcal{O}(\epsilon). \quad (100)$$

We have added and subtracted the same quantity from the full expression. Combining the logarithms in both terms gives

$$\left[\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2} \right]_{\text{IR}} = \frac{1}{2\pi} \left[-\frac{1}{\epsilon_{\text{IR}}} - \log(e^{\gamma_E} \pi \mu |\mathbf{x}'_1 - \mathbf{x}_1|) \right] + \mathcal{O}(\epsilon). \quad (101)$$

$$\left[\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2} \right]_{\text{UV}} = \frac{1}{2\pi} \left[\frac{1}{\epsilon_{\text{UV}}} + \log(e^{\gamma_E} \pi \mu |\mathbf{x}'_1 - \mathbf{x}_1|) \right] + \mathcal{O}(\epsilon). \quad (102)$$

Then, since ϵ_{IR} is negative, we can label it as $-\epsilon$ with $\epsilon > 0$ to match the real case. This addition and subtraction of a finite piece can also be thought of as choosing a particular value for the cutoff Λ . So, Eq. (101) will contribute to the fragmentation function, while Eq. (102) will contribute to the rest of the NLO corrections. The divergence in Eq. (101) is collinear and will be absorbed into the bare quark hadron fragmentation function to make it scale dependent. The divergence in Eq. (102) is canceled with the other UV divergences in virtual diagrams as before. Finally we note that the real integral [Eq. (93)] has an overall exponential which we set to 1. This can be motivated by noticing that it does not affect the nature of the collinear singularity and the splitting function favors $\xi \rightarrow 1$ (soft radiation).

Using the results of our transverse integrals in the expression for the hadronic cross section we get

$$\int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{4e^2 Q^2 (z_1 z_2)^3 N_c}{(2\pi)^7 (z_{h_1} z_{h_2})^2} \int d^8\mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_2/\bar{q}}^0(z_{h_2}) \delta(1 - z_1 - z_2) \times \left[D_{h_1/q}^0(z_{h_1}) + \frac{\alpha_s C_F}{\pi} \left\{ \int_{z_{h_1}}^1 \frac{d\xi}{\xi} \frac{(1 + \xi^2)}{(1 - \xi)} D_{h_1/q}^0\left(\frac{z_{h_1}}{\xi}\right) - D_{h_1/q}^0(z_{h_1}) \int_0^1 d\xi \frac{(1 + \xi^2)}{(1 - \xi)} \right\} \left(\frac{1}{\epsilon} - \log(\pi e^{\gamma_E} \mu |\mathbf{x}'_1 - \mathbf{x}_1|) \right) \right]. \quad (103)$$

Next we again introduce the quark-quark splitting function as in the previous subsection (Eq. (84)). In terms of this, the full expression becomes

$$\int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{4e^2 Q^2 (z_1 z_2)^3 N_c}{(2\pi)^7 (z_{h_1} z_{h_2})^2} \int d^8\mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_2/\bar{q}}^0(z_{h_2}) \delta(1 - z_1 - z_2) \times \int_{z_{h_1}}^1 \frac{d\xi}{\xi} D_{h_1/q}^0\left(\frac{z_{h_1}}{\xi}\right) \left[\delta(1 - \xi) + \frac{\alpha_s}{\pi} P_{qq}(\xi) \left(\frac{1}{\epsilon} - \log(\pi e^{\gamma_E} \mu |\mathbf{x}'_1 - \mathbf{x}_1|) \right) \right]. \quad (104)$$

Then, we define the DGLAP evolved quark-hadron fragmentation function $D_{h_1/q}(z_{h_1}, \mu^2)$ in dimensional regularization as

$$D_{h_1/q}(z_{h_1}, \mu^2) = \int_{z_{h_1}}^1 \frac{d\xi}{\xi} D_{h_1/q}^0\left(\frac{z_{h_1}}{\xi}\right) \left[\delta(1 - \xi) + \frac{\alpha_s}{\pi} P_{qq}(\xi) \left(\frac{1}{\epsilon} - \log(\pi e^{\gamma_E} \mu |\mathbf{x}'_1 - \mathbf{x}_1|) \right) \right], \quad (105)$$

in terms of which our expression becomes

$$\frac{d\sigma_{\text{LO}+9+1 \times 1}^{\gamma^* A \rightarrow h_1 h_2 X}}{d^2\mathbf{p}_h d^2\mathbf{q}_h dy_1 dy_2} = \int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{4e^2 Q^2 (z_1 z_2)^3 N_c}{(2\pi)^7 (z_{h_1} z_{h_2})^2} \int d^8\mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_2) D_{h_2/\bar{q}}^0(z_{h_2}) D_{h_1/q}(z_{h_1}, \mu^2) \delta(1 - z_1 - z_2), \quad (106)$$

where we recall that

$$z_1 = \frac{p_h^+}{z_{h_1} l^+}, \quad z_2 = \frac{q_h^+}{z_{h_2} l^+}. \quad (107)$$

Finally, if one repeats the exact same procedure for the antiquark line (i.e.: adding up $LO + 10 + 2 \times 2$), one gets

$$\begin{aligned} & \frac{d\sigma_{LO+10+2 \times 2}^{\gamma^* A \rightarrow h_1 h_2 X}}{d^2 \mathbf{p}_h d^2 \mathbf{q}_h dy_1 dy_2} \\ &= \int_0^1 dz_{h_1} \int_0^1 dz_{h_2} \frac{4e^2 Q^2 (z_1 z_2)^3 N_c}{(2\pi)^7 (z_{h_1} z_{h_2})^2} \int d^8 \mathbf{x} H(\mathbf{p}, \mathbf{q}, \mathbf{x}, z_1) \\ & \times D_{h_1/q}^0(z_{h_1}) D_{h_2/\bar{q}}(z_{h_2}, \mu^2) \delta(1 - z_1 - z_2). \end{aligned} \quad (108)$$

This shows that absorption of collinear divergences in $\sigma_9, \sigma_{1 \times 1}, \sigma_{10}$, and $\sigma_{2 \times 2}$ into bare quark and antiquark-hadron fragmentation functions lead to evolution, i.e. scale dependence of these fragmentation functions.

Therefore our final result for the regulated dihadron production cross section can be written as a sum of several terms [Eq. (109)]. The first term contains the z integration region below z_f where the leading-order cross section is evolved with the BK/JIMWLK evolution equations. The second term includes the integration region $z > z_f$ where the leading-order cross section multiplies the DGLAP evolved fragmentation functions for both quark and antiquark. Finally the last term constitutes all the remaining contributions to the NLO cross section which is finite,

$$\begin{aligned} d\sigma^{\gamma^* A \rightarrow h_1 h_2 X} &= d\sigma_{LO} \otimes \text{JIMWLK} \\ &+ d\sigma_{LO} \otimes D_{h_1/q}(z_{h_1}, \mu^2) D_{h_2/\bar{q}}(z_{h_2}, \mu^2) \\ &+ d\sigma_{NLO}^{\text{finite}}. \end{aligned} \quad (109)$$

Here we imply the presence of the bare fragmentation functions in the first and last terms.

In summary, we have calculated the one-loop corrections to inclusive quark antiquark production in DIS at small x

for longitudinal photons. We have shown that all divergences that appear at the one-loop level are either canceled or absorbed into JIMWLK evolution of dipoles and quadrupoles, and into DGLAP evolution of parton-hadron fragmentation functions. These results are well suited for further phenomenological studies of angular correlations of the dihadrons produced in DIS at small x [72]. A particularly interesting limit is the so-called back-to-back limit where a suppression of the away side peak is observed experimentally. Here one will be sensitive to Sudakov radiation which will be significant [18,56,73] and must be included in a full phenomenological study. One can also use our results here and integrate over the phase space of one of the outgoing particles, thus obtaining the single inclusive hadron production cross section in DIS at small x at next-to-leading order. This is under investigation and will be reported elsewhere [74].

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APPENDIX A: CANCELLATION OF UV DIVERGENCES

UV divergences appear in the corrections $d\sigma_5, d\sigma_6, d\sigma_9, d\sigma_{10}, d\sigma_{11}, d\sigma_{12}, d\sigma_{14(1)}$, and $d\sigma_{14(2)}$. In this appendix, we will show that these divergences all cancel between these corrections. First, we note that the UV divergence in $d\sigma_5$ appears in the limit $\mathbf{x}_3 \rightarrow \mathbf{x}_1$. If we take this limit in our result [Eq. (42)] everywhere except where it causes the divergence, we get the following:

$$\begin{aligned} \frac{d\sigma_{5,UV}^L}{d^2 \mathbf{p} d^2 \mathbf{q} dy_1 dy_2} &= \frac{2e^2 g^2 Q^2 N_c^2 (z_1 z_2)^3}{(2\pi)^{10}} \int_0^{z_1} \frac{dz}{z} \left[\frac{z_1^2 + (z_1 - z)^2}{z_1^2} \right] \int d^8 \mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ &\times K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) e^{i\mathbf{p} \cdot (\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q} \cdot (\mathbf{x}'_2 - \mathbf{x}_2)} \int_{\mathbf{x}_3 \rightarrow \mathbf{x}_1} \frac{d^2 \mathbf{x}_3}{\mathbf{x}_{31}^2}. \end{aligned} \quad (A1)$$

Here we have used the notation $\int_{\mathbf{x}_3 \rightarrow \mathbf{x}_1}$ to signify that the limit has been taken inside the $\mathbf{x}_3$ integral. The divergence as $\mathbf{x}_3 \rightarrow \mathbf{x}_1$ can equivalently be written as a divergent momentum integral as the transverse momentum goes to infinity: $\int_{\mathbf{k} \rightarrow \infty} \frac{d^2 \mathbf{k}}{\mathbf{k}^2}$. Making this substitution in $d\sigma_5$, and similarly for the UV divergence as $\mathbf{x}_3 \rightarrow \mathbf{x}_2$ in $d\sigma_6$, and writing the UV divergent parts of all the other UV divergent diagrams, we have

$$\begin{aligned} \frac{d\sigma_{5,UV}^L}{d^2 \mathbf{p} d^2 \mathbf{q} dy_1 dy_2} &= \frac{2e^2 g^2 Q^2 N_c^2 (z_1 z_2)^3}{(2\pi)^{10}} \int_0^{z_1} \frac{dz}{z} \left[\frac{z_1^2 + (z_1 - z)^2}{z_1^2} \right] \int d^8 \mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ &\times K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) e^{i\mathbf{p} \cdot (\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q} \cdot (\mathbf{x}'_2 - \mathbf{x}_2)} \int_{|\mathbf{k}| \rightarrow \infty} \frac{d^2 \mathbf{k}}{\mathbf{k}^2}, \end{aligned} \quad (A2)$$

$$\frac{d\sigma_{6,\text{UV}}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_2} \frac{dz}{z} \left[\frac{z_2^2 + (z_2 - z)^2}{z_2^2} \right] \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ \times K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \int_{|\mathbf{k}|\rightarrow\infty} \frac{d^2\mathbf{k}}{\mathbf{k}^2}, \quad (\text{A3})$$

$$\frac{d\sigma_{9,\text{UV}}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_1} \frac{dz}{z} \left[\frac{z_1^2 + (z_1 - z)^2}{z_1^2} \right] \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ \times K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \int_{|\mathbf{k}|\rightarrow\infty} \frac{d^2\mathbf{k}}{\mathbf{k}^2}, \quad (\text{A4})$$

$$\frac{d\sigma_{10,\text{UV}}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_2} \frac{dz}{z} \left[\frac{z_2^2 + (z_2 - z)^2}{z_2^2} \right] \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ \times K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \int_{|\mathbf{k}|\rightarrow\infty} \frac{d^2\mathbf{k}}{\mathbf{k}^2}, \quad (\text{A5})$$

$$\frac{d\sigma_{11,\text{UV}}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_1} \frac{dz}{z} \left[\frac{z_1^2 + (z_1 - z)^2}{z_1^2} \right] \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ \times K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \int_{|\mathbf{k}|\rightarrow\infty} \frac{d^2\mathbf{k}}{\mathbf{k}^2}, \quad (\text{A6})$$

$$\frac{d\sigma_{12,\text{UV}}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_2} \frac{dz}{z} \left[\frac{z_2^2 + (z_2 - z)^2}{z_2^2} \right] \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ \times K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \int_{|\mathbf{k}|\rightarrow\infty} \frac{d^2\mathbf{k}}{\mathbf{k}^2}, \quad (\text{A7})$$

$$\frac{d\sigma_{14(1),\text{UV}}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_1} \frac{dz}{z} \left[1 + \frac{(z_1 - z)(z_2 + z)}{z_1z_2} - z \frac{z_1(z_1 - z) + z_2(z_2 + z)}{z_1z_2} \right] \\ \times \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \int_{|\mathbf{k}|\rightarrow\infty} \frac{d^2\mathbf{k}}{\mathbf{k}^2}, \quad (\text{A8})$$

$$\frac{d\sigma_{14(2),\text{UV}}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_2} \frac{dz}{z} \left[1 + \frac{(z_1 + z)(z_2 - z)}{z_1z_2} - z \frac{z_1(z_1 + z) + z_2(z_2 - z)}{z_1z_2} \right] \\ \times \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \int_{|\mathbf{k}|\rightarrow\infty} \frac{d^2\mathbf{k}}{\mathbf{k}^2}. \quad (\text{A9})$$

One can immediately verify from the above expressions that the UV divergent part of σ_5 is canceled by the UV divergent part of σ_{11} . Similarly $\sigma_{6,\text{UV}}$ cancels with $\sigma_{12,\text{UV}}$. The remaining terms can be written (after some simplification of the z factors) as follows:

$$\frac{d\sigma_{9+14(1),\text{UV}}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_1} dz \left[\frac{2(z_1 - z)}{z_1} - \frac{z}{z_1^2} \right] \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ \times K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \int_{|\mathbf{k}|\rightarrow\infty} \frac{d^2\mathbf{k}}{\mathbf{k}^2}, \quad (\text{A10})$$

$$\begin{aligned} \frac{d\sigma_{10+14(2),UV}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_2} dz \left[\frac{2(z_2 - z)}{z_2} - \frac{z}{z_2^2} \right] \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ &\times K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \int_{|\mathbf{k}|\rightarrow\infty} \frac{d^2\mathbf{k}}{\mathbf{k}^2}. \end{aligned} \quad (\text{A11})$$

The integrals over z in these two expressions can now be done, and we find the following:

$$\begin{aligned} \frac{d\sigma_{9+14(1),UV}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \left[z_1 - \frac{1}{2} \right] \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ &\times K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \int_{|\mathbf{k}|\rightarrow\infty} \frac{d^2\mathbf{k}}{\mathbf{k}^2}, \end{aligned} \quad (\text{A12})$$

$$\begin{aligned} \frac{d\sigma_{10+14(2),UV}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \left[z_2 - \frac{1}{2} \right] \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ &\times K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)}e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \int_{|\mathbf{k}|\rightarrow\infty} \frac{d^2\mathbf{k}}{\mathbf{k}^2}. \end{aligned} \quad (\text{A13})$$

Now, if we add up σ_9 , σ_{10} , $\sigma_{14(1)}$, and $\sigma_{14(2)}$ we see that the result is proportional to $z_1 + z_2 - 1$ which is zero by virtue of the delta function (not shown here). So, what we have found is precisely Eq. (56). Therefore, *all UV divergences are canceled* when adding up all the relevant pieces of the differential cross section.

APPENDIX B: COORDINATE SPACE RESULTS AT SOFT z

Here we have taken the limit $z \rightarrow 0$ in all our expressions everywhere except where it causes a divergence. The result for each expression has also been written in coordinate space where we have recalled the definition of the radiation kernel [Eq. (57)]. The expressions in this appendix are useful both for showing that soft divergences are canceled and that rapidity divergences can be combined to show evolution of the dipoles and quadrupoles according to the BK and JIMWLK equations,

$$\begin{aligned} \frac{d\sigma_{1\times 1}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{31'}^2} - \frac{\mathbf{x}_{11'}^2}{\mathbf{x}_{31}^2\mathbf{x}_{31'}^2} \right] \\ &\times [S_{122'1'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1 - z_1 - z_2), \end{aligned} \quad (\text{B1})$$

$$\begin{aligned} \frac{d\sigma_{2\times 2}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{1}{\mathbf{x}_{32}^2} + \frac{1}{\mathbf{x}_{32'}^2} - \frac{\mathbf{x}_{22'}^2}{\mathbf{x}_{32}^2\mathbf{x}_{32'}^2} \right] \\ &\times [S_{122'1'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} \delta(1 - z_1 - z_2), \end{aligned} \quad (\text{B2})$$

$$\begin{aligned} \frac{d\sigma_{1\times 2}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{-2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{32'}^2} - \frac{\mathbf{x}_{12'}^2}{\mathbf{x}_{31}^2\mathbf{x}_{32'}^2} \right] \\ &\times [S_{12}S_{1'2'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1 - z_1 - z_2), \end{aligned} \quad (\text{B3})$$

$$\begin{aligned} \frac{d\sigma_{3\times 3}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{31'}^2} - \frac{\mathbf{x}_{11'}^2}{\mathbf{x}_{31}^2\mathbf{x}_{31'}^2} \right] \\ &\times [S_{11'}S_{22'} - S_{13}S_{23} - S_{1'3}S_{2'3} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1 - z_1 - z_2), \end{aligned} \quad (\text{B4})$$

$$\begin{aligned} \frac{d\sigma_{4\times 4}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{1}{\mathbf{x}_{32}^2} + \frac{1}{\mathbf{x}_{32'}^2} - \frac{\mathbf{x}_{22'}^2}{\mathbf{x}_{32}^2\mathbf{x}_{32'}^2} \right] \\ &\times [S_{11'}S_{22'} - S_{13}S_{23} - S_{1'3}S_{2'3} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1 - z_1 - z_2), \end{aligned} \quad (\text{B5})$$

$$\frac{d\sigma_{3\times 4}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{32'}^2} - \frac{\mathbf{x}_{12'}^2}{\mathbf{x}_{31}^2\mathbf{x}_{32'}^2} \right] \\ \times [S_{11'}S_{22'} - S_{13}S_{23} - S_{1'3}S_{2'3} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1-z_1-z_2), \quad (\text{B6})$$

$$\frac{d\sigma_{1\times 3}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_2) K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{31'}^2} - \frac{\mathbf{x}_{11'}^2}{\mathbf{x}_{31}^2\mathbf{x}_{31'}^2} \right] \\ \times [S_{122'3}S_{1'3} - S_{1'3}S_{2'3} - S_{12} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1-z_1-z_2), \quad (\text{B7})$$

$$\frac{d\sigma_{1\times 4}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_2) K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{32'}^2} - \frac{\mathbf{x}_{12'}^2}{\mathbf{x}_{31}^2\mathbf{x}_{32'}^2} \right] \\ \times [S_{122'3}S_{1'3} - S_{1'3}S_{2'3} - S_{12} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1-z_1-z_2), \quad (\text{B8})$$

$$\frac{d\sigma_{2\times 3}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{1}{\mathbf{x}_{32}^2} + \frac{1}{\mathbf{x}_{31'}^2} - \frac{\mathbf{x}_{21'}^2}{\mathbf{x}_{32}^2\mathbf{x}_{31'}^2} \right] \\ \times [S_{1231'}S_{2'3} - S_{1'3}S_{2'3} - S_{12} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1-z_1-z_2), \quad (\text{B9})$$

$$\frac{d\sigma_{2\times 4}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{1}{\mathbf{x}_{32}^2} + \frac{1}{\mathbf{x}_{32'}^2} - \frac{\mathbf{x}_{22'}^2}{\mathbf{x}_{32}^2\mathbf{x}_{32'}^2} \right] \\ \times [S_{1231'}S_{2'3} - S_{1'3}S_{2'3} - S_{12} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1-z_1-z_2), \quad (\text{B10})$$

$$\frac{d\sigma_5^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{4e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_1} \frac{dz}{z} d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \frac{1}{\mathbf{x}_{31}^2} \\ \times [S_{322'1'}S_{13} - S_{13}S_{23} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1-z_1-z_2), \quad (\text{B11})$$

$$\frac{d\sigma_6^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{4e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_2} \frac{dz}{z} d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \frac{1}{\mathbf{x}_{32}^2} \\ \times [S_{132'1'}S_{23} - S_{13}S_{23} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1-z_1-z_2), \quad (\text{B12})$$

$$\frac{d\sigma_7^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_1} \frac{dz}{z} d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{32}^2} - \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{31}^2\mathbf{x}_{32}^2} \right] \\ \times [S_{322'1'}S_{13} - S_{13}S_{23} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1-z_1-z_2), \quad (\text{B13})$$

$$\frac{d\sigma_8^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_2} \frac{dz}{z} d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{32}^2} - \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{31}^2\mathbf{x}_{32}^2} \right] \\ \times [S_{132'1'}S_{23} - S_{13}S_{23} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1-z_1-z_2), \quad (\text{B14})$$

$$\frac{d\sigma_9^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_1} \frac{dz}{z} d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \frac{1}{\mathbf{x}_3^2} \\ \times [S_{122'1'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1-z_1-z_2), \quad (\text{B15})$$

$$\frac{d\sigma_{10}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_2} \frac{dz}{z} d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \frac{1}{\mathbf{x}_3^2} \\ \times [S_{122'1'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \delta(1-z_1-z_2), \quad (\text{B16})$$

$$\frac{d\sigma_{11}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-4e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_1} \frac{dz}{z} d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \frac{1}{\mathbf{x}_3^2} \\ \times [S_{122'1'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2 - \mathbf{x}_2)} \delta(1 - z_1 - z_2), \quad (\text{B17})$$

$$\frac{d\sigma_{12}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-4e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_2} \frac{dz}{z} d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \frac{1}{\mathbf{x}_3^2} \\ \times [S_{122'1'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2 - \mathbf{x}_2)} \delta(1 - z_1 - z_2), \quad (\text{B18})$$

$$\frac{d\sigma_{13(1)}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_2} \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{32}^2} - \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{31}^2\mathbf{x}_{32}^2} \right] \\ \times [S_{12}S_{1'2'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2 - \mathbf{x}_2)} \delta(1 - z_1 - z_2), \quad (\text{B19})$$

$$\frac{d\sigma_{13(2)}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_1} \frac{dz}{z} \int d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{32}^2} - \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{31}^2\mathbf{x}_{32}^2} \right] \\ \times [S_{12}S_{1'2'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2 - \mathbf{x}_2)} \delta(1 - z_1 - z_2), \quad (\text{B20})$$

$$\frac{d\sigma_{14(1)}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_1} = \frac{e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_1} \frac{dz}{z} d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{2}{\mathbf{x}_3^2} + \frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{32}^2} - \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{31}^2\mathbf{x}_{32}^2} \right] \\ \times [S_{122'1'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2 - \mathbf{x}_2)} \delta(1 - z_1 - z_2), \quad (\text{B21})$$

$$\frac{d\sigma_{14(2)}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_1} = \frac{e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^{10}} \int_0^{z_2} \frac{dz}{z} d^{10}\mathbf{x} K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \left[\frac{2}{\mathbf{x}_3^2} + \frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{32}^2} - \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{31}^2\mathbf{x}_{32}^2} \right] \\ \times [S_{122'1'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2 - \mathbf{x}_2)} \delta(1 - z_1 - z_2). \quad (\text{B22})$$

APPENDIX C: DIMENSIONAL REGULARIZATION

The results in Sec. III include divergent integrals over the transverse momentum $\mathbf{k}$ or in coordinate space as divergent integrals over $\mathbf{x}_3$. Ideally, one would like to regulate these divergences using dimensional regularization. In many of the results, we have not found a way to evaluate the regularized integral, but in several it is possible. Here we show the ones that can be done:

$$\frac{d\sigma_{1\times 1}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{2e^2g^2Q^2N_c^2z_2^3(1-z_2)^2(z_1^2 + (1-z_2)^2)}{(2\pi)^9z_1} \int d^8x K_0(|\mathbf{x}_{12}|Q_2) K_0(|\mathbf{x}_{1'2'}|Q_2) [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ \times e^{i\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2 - \mathbf{x}_2)} \int \frac{dz}{z} e^{i\frac{z}{z_1}\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)} \left[\frac{1}{\epsilon} - \log(e^{\gamma_E}\pi\mu|\mathbf{x}'_1 - \mathbf{x}_1|) \right], \quad (\text{C1})$$

$$\frac{d\sigma_{2\times 2}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{2e^2g^2Q^2N_c^2z_2^3(1-z_1)^2(z_2^2 + (1-z_1)^2)}{(2\pi)^9z_2} \int d^8x K_0(|\mathbf{x}_{12}|Q_2) K_0(|\mathbf{x}_{1'2'}|Q_2) [S_{122'1'} - S_{12} - S_{1'2'} + 1] \\ \times e^{i\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2 - \mathbf{x}_2)} \int \frac{dz}{z} e^{i\frac{z}{z_2}\mathbf{q}\cdot(\mathbf{x}'_2 - \mathbf{x}_2)} \left[\frac{1}{\epsilon} - \log(e^{\gamma_E}\pi\mu|\mathbf{x}'_2 - \mathbf{x}_2|) \right], \quad (\text{C2})$$

$$\frac{d\sigma_{1\times 2}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-e^2g^2Q^2N_c^2z_1z_2(1-z_1)(1-z_2)(z_1(1-z_1) + z_2(1-z_2))}{(2\pi)^9} \int \frac{dz}{z} \int d^8\mathbf{x} K_0(|\mathbf{x}_{12}|Q_2) K_0(|\mathbf{x}_{1'2'}|Q_1) \\ \times [S_{12}S_{1'2'} - S_{12} - S_{1'2'} + 1] e^{i\mathbf{p}\cdot(\mathbf{x}'_1 - \mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2 - \mathbf{x}_2)} \left[(e^{i\frac{z}{z_1}\mathbf{p}\cdot\mathbf{x}_{2'1}} + e^{i\frac{z}{z_2}\mathbf{q}\cdot\mathbf{x}_{2'1}}) \left(\frac{1}{\epsilon} - \log(e^{\gamma_E}\pi\mu|\mathbf{x}_1 - \mathbf{x}'_2|) \right) \right. \\ \left. - \left(\frac{z}{z_1z_2} \right)^2 (z_2\mathbf{p} - z_1\mathbf{q})^2 \int_0^1 dx e^{iz(\frac{z}{z_1}\mathbf{p} + \frac{(1-x)}{z_2}\mathbf{q})\cdot\mathbf{x}_{2'1}} K_0\left(|\mathbf{x}_{12'}| \frac{z}{z_1z_2} |z_2\mathbf{p} - z_1\mathbf{q}| \sqrt{x(1-x)} \right) \right], \quad (\text{C3})$$

$$\frac{d\sigma_{11}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^9} \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \\ \times e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \int_0^{z_1} \frac{dz}{z} \frac{[z_1^2 + (z_1 - z)^2]}{z_1^2} \left(\frac{1}{\epsilon} + \frac{1}{2} \log \left(\frac{2\pi z_1^2 z_2 |\mathbf{x}_{12}| \mu^2}{z(z_1 - z)Q_1} \right) \right), \quad (C4)$$

$$\frac{d\sigma_{12}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-2e^2g^2Q^2N_c^2(z_1z_2)^3}{(2\pi)^9} \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) \\ \times e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \int_0^{z_2} \frac{dz}{z} \frac{[z_2^2 + (z_2 - z)^2]}{z_2^2} \left(\frac{1}{\epsilon} + \frac{1}{2} \log \left(\frac{2\pi z_2^2 z_1 |\mathbf{x}_{12}| \mu^2}{z(z_2 - z)Q_1} \right) \right), \quad (C5)$$

$$\frac{d\sigma_{14(1)}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_1} = \frac{e^2g^2Q^2N_c^2(z_1z_2)^2}{(2\pi)^9} \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \\ \times \int_0^{z_1} \frac{dz}{z} \left[[z_1z_2 + (z_1 - z)(z_2 + z)] \left(\frac{1}{\epsilon} - \frac{\gamma_E}{2} + \frac{1}{2} \log \left[\frac{4\pi\mu^2 z_1^2 (z_2 + z)}{z^2 (z_1 - z) Q^2} \right] \right) \right. \\ \left. - z[z_1(z_1 - z) + z_2(z_2 + z)] \left(\frac{1}{\epsilon} - \frac{\gamma_E}{2} + \frac{1}{2} \log \left[\frac{4\pi\mu^2}{(z_1 - z)(z_2 + z) Q^2} \right] \right) \right], \quad (C6)$$

$$\frac{d\sigma_{14(2)}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_1} = \frac{e^2g^2Q^2N_c^2(z_1z_2)^2}{(2\pi)^9} \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{1'2'} + 1] K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) e^{i\mathbf{p}\cdot(\mathbf{x}'_1-\mathbf{x}_1)} e^{i\mathbf{q}\cdot(\mathbf{x}'_2-\mathbf{x}_2)} \\ \times \int_0^{z_2} \frac{dz}{z} \left[[z_1z_2 + (z_2 - z)(z_1 + z)] \left(\frac{1}{\epsilon} - \frac{\gamma_E}{2} + \frac{1}{2} \log \left[\frac{4\pi\mu^2 z_2^2 (z_1 + z)}{z^2 (z_2 - z) Q^2} \right] \right) \right. \\ \left. - z[z_1(z_1 + z) + z_2(z_2 - z)] \left(\frac{1}{\epsilon} - \frac{\gamma_E}{2} + \frac{1}{2} \log \left[\frac{4\pi\mu^2}{(z_2 - z)(z_1 + z) Q^2} \right] \right) \right]. \quad (C7)$$

Note: these last two expressions are only valid for $z \neq 0$. We have defined $\epsilon = 2 - d$ where d is the number of dimensions for the transverse integral. Another type of divergence becomes apparent when looking at the results in this way; there is a $\frac{\log z}{z}$ divergence as $z \rightarrow 0$ in the virtual corrections σ_{11} , σ_{12} , $\sigma_{14(1)}$, and $\sigma_{14(2)}$. However, we find that taking the limit $z \rightarrow 0$ inside the z integrals shows that the $\frac{\log z}{z}$ cancels between σ_{11} and $\sigma_{14(1)}$, and similarly between σ_{12} and $\sigma_{14(2)}$:

$$\frac{d\sigma_{13(1)}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{-e^2g^2Q^2N_c^2(z_1z_2)^2}{(2\pi)^8} \int_0^{z_2} \frac{dz}{z} (z_1 + z)(z_2 - z) \int d^8\mathbf{x} K_0(|\mathbf{x}_{12}|Q\sqrt{(z_1 + z)(z_2 - z)}) K_0(|\mathbf{x}_{1'2'}|Q_1) \\ \times e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} [S_{12}S_{1'2'} - S_{12} - S_{1'2'} + 1] \\ \times \left[- \int_0^1 dx \frac{\frac{z(z_2 - z)[z_1z_2 + (z_1 + z)(z_2 - z)](z_2\mathbf{p} - z_1\mathbf{q})^2}{z_1^2 z_2^2} e^{i\left[\frac{xz_1(z_2 - z) + z}{z_2}\mathbf{q} - x(z_2 - z)\mathbf{p}\right]\cdot\mathbf{x}_{21}} i|\mathbf{x}_{12}|K_1\left(-i|\mathbf{x}_{12}|\sqrt{\frac{x(z_2 - z)(z_2\mathbf{p} - z_1\mathbf{q})^2(z + xz_1(z_2 - z))}{z_1^2 z_2^2}}\right)}{4\pi\sqrt{\frac{x(z_2 - z)(z_2\mathbf{p} - z_1\mathbf{q})^2(z + xz_1(z_2 - z))}{z_1^2 z_2^2}}} \right. \\ \left. + \frac{z[z_1(z_1 + z) + z_2(z_2 - z)]}{4z_1z_2} e^{i[(z_1 + z)\mathbf{q} - (z_2 - z)\mathbf{p}]\cdot\mathbf{x}_{21}} iH_0^{(1)}\left(\frac{(z_1 + z)(z_2 - z)(z_2\mathbf{p} - z_1\mathbf{q})^2|\mathbf{x}_{12}|}{z_1z_2}\right) \right. \\ \left. - \frac{[z_1z_2 + (z_1 + z)(z_2 - z)]}{z_1z_2} \frac{e^{i\frac{z}{z_2}\mathbf{q}\cdot\mathbf{x}_{21}}}{2\pi} \left[\frac{1}{\epsilon} - \log(e^{\gamma_E}\pi\mu|\mathbf{x}_{12}|) \right] \right] \delta(1 - z_1 - z_2). \quad (C8)$$

Here $H_0^{(1)}(x)$ is the Hankel function of the first kind:

$$\begin{aligned}
& \frac{d\sigma_{13(2)}^L}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} \\
&= \frac{-e^2g^2Q^2N_c^2(z_1z_2)^2}{(2\pi)^8} \int_0^{z_1} \frac{dz}{z} (z_2+z)(z_1-z) \int d^8\mathbf{x} K_0(|\mathbf{x}_{12}|Q\sqrt{(z_2+z)(z_1-z)}) K_0(|\mathbf{x}_{1'2'}|Q_1) \\
&\quad \times e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} [S_{12}S_{1'2'} - S_{12} - S_{1'2'} + 1] \\
&\quad \times \left[- \int_0^1 dx \frac{\frac{z(z_1-z)[z_1z_2+(z_2+z)(z_1-z)](z_2\mathbf{p}-z_1\mathbf{q})^2}{z_2^2z_1^3} e^{i[\frac{xz_2(z_1-z)+z}{z_1}\mathbf{p}-x(z_1-z)\mathbf{q}]\cdot\mathbf{x}_{12}} i|\mathbf{x}_{12}| K_1\left(-i|\mathbf{x}_{12}|\sqrt{\frac{x(z_1-z)(z_2\mathbf{p}-z_1\mathbf{q})^2(z+xz_2(z_1-z))}{z_2z_1^3}}\right)}{4\pi\sqrt{\frac{x(z_1-z)(z_2\mathbf{p}-z_1\mathbf{q})^2(z+xz_2(z_1-z))}{z_2z_1^3}}} \right. \\
&\quad + \frac{z[z_2(z_2+z)+z_1(z_1-z)]}{4z_1z_2} e^{i[(z_2+z)\mathbf{p}-(z_1-z)\mathbf{q}]\cdot\mathbf{x}_{12}} (-i)H_0^{(2)}\left(\frac{(z_2+z)(z_1-z)(z_2\mathbf{p}-z_1\mathbf{q})^2|\mathbf{x}_{12}|}{z_1z_2}\right) \\
&\quad \left. - \frac{[z_1z_2+(z_2+z)(z_1-z)]}{z_1z_2} \frac{e^{\frac{iz_1}{\epsilon}\mathbf{p}\cdot\mathbf{x}_{12}}}{2\pi} \left[\frac{1}{\epsilon} - \log(e^{\gamma_E}\pi\mu|\mathbf{x}_{12}|)\right] \right] \delta(1-z_1-z_2). \tag{C9}
\end{aligned}$$

Here the sign change in the $i\epsilon$ going from $\sigma_{13(1)}$ to $\sigma_{13(2)}$ results in taking $iH_0^{(1)} \rightarrow -iH_0^{(2)}$ in the middle term.

APPENDIX D: USEFUL RELATIONS

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 + A^2} = \frac{1}{2\pi} K_0(|A||\mathbf{x}|), \quad A \in \mathbb{R}. \tag{D1}$$

$$\int \frac{d^2\mathbf{k}_1}{(2\pi)^2} \frac{d^2\mathbf{k}_2}{(2\pi)^2} \frac{(Ak_1+Bk_2)^i e^{i\mathbf{k}_1\cdot\mathbf{x}} e^{i\mathbf{k}_2\cdot\mathbf{y}}}{[\mathbf{k}_1^2+CQ^2]\left[Q^2+\frac{\mathbf{k}^2}{D}+\frac{\mathbf{k}_2^2}{E}+\frac{(\mathbf{k}_1-\mathbf{k}_2)^2}{F}\right]} = \frac{-iAF}{(2\pi)^2} \frac{y^i}{\mathbf{y}^2} K_0\left(Q\sqrt{C\left(\mathbf{x}-\frac{A}{B}\mathbf{y}\right)^2-\frac{AF}{B}\mathbf{y}^2}\right). \tag{D2}$$

Note: This result is valid only if C, D, E, F are all positive, $\frac{C}{F}\left(\frac{D+F}{D}-\frac{E}{E+F}\right)=1$, and $A+\frac{BE}{E+F}=0$. We have used these relations to eliminate D and E in the result. One can confirm that these two relations hold for all the cases we will need.

$$\int \frac{d^2\mathbf{k}_3}{(2\pi)^2} \frac{(\mathbf{k}_3-\mathbf{k})^i}{(\mathbf{k}_3-\mathbf{k})^2} e^{i\mathbf{k}_3\cdot\mathbf{x}} = \frac{i}{2\pi} \frac{x^i}{\mathbf{x}^2} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \text{equivalently} \quad \frac{\mathbf{k}^i}{\mathbf{k}^2} = \frac{i}{2\pi} \int d^2\mathbf{x} \frac{\mathbf{x}^i}{\mathbf{x}^2} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{D3}$$

$$\int \frac{d^2\mathbf{k}_1}{(2\pi)^2} \frac{1}{[\mathbf{k}_1^2+A][(\mathbf{k}_1+\mathbf{k}_2)^2+B]} = \frac{1}{4\pi} \frac{\log\left(\frac{A+B+\mathbf{k}_2^2+\sqrt{(A-B)^2+2(A+B)\mathbf{k}_2^2+\mathbf{k}_2^4}}{A+B+\mathbf{k}_2^2-\sqrt{(A-B)^2+2(A+B)\mathbf{k}_2^2+\mathbf{k}_2^4}}\right)}{\sqrt{(A-B)^2+2(A+B)\mathbf{k}_2^2+\mathbf{k}_2^4}} \quad \text{for } A, B > 0. \tag{D4}$$

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot\mathbf{x}} \log(A(\mathbf{k}^2+B^2))}{\mathbf{k}^2+B^2} = \frac{-K_0(|\mathbf{x}||B|)}{2\pi} \left(\gamma_E + \log\left(\frac{|\mathbf{x}|}{2A|B|}\right)\right), \quad A > 0, B \in \mathbb{R}. \tag{D5}$$

$$\begin{aligned}
\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2+A^2} &\rightarrow \mu^{2-d} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{\mathbf{k}^2+A^2} = \frac{\mu^{2-d}2\pi^{d/2}}{(2\pi)^d\Gamma[d/2]} \int_0^\infty dk \frac{k^{d-1}}{k^2+A^2} = \frac{\mu^{2-d}\pi^{1-\frac{d}{2}}}{2^d\Gamma[d/2]} \frac{A^{d-2}}{\sin(\pi d/2)} \\
&= \frac{1}{2\pi} \left[\frac{1}{\epsilon} + \frac{1}{2} \left(\log\left[\frac{4\pi\mu^2}{A^2}\right] - \gamma_E\right)\right] + \mathcal{O}(\epsilon), \quad \epsilon = 2-d, \quad 0 < d < 2. \tag{D6}
\end{aligned}$$

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{(\mathbf{k}-\mathbf{p})^2} \rightarrow \mu^{2-d} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{1}{\mathbf{k}^2} = \frac{\mu^{2-d} 2\pi^{d/2}}{(2\pi)^d \Gamma[d/2]} \int_0^\infty dk k^{d-3} = \frac{\mu^{2-d} 2\pi^{d/2}}{(2\pi)^d \Gamma[d/2]} \left[\frac{k^{d-2}}{d-2} \right]_0^\infty. \quad (\text{D7})$$

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}}{\mathbf{k}^2} = \int \frac{d^2\mathbf{x}_3}{(2\pi)^2} \Delta_{ij}^{(3)} = \int \frac{d^2\mathbf{x}_3}{(2\pi)^2} \frac{(\mathbf{x}_3-\mathbf{x}_i) \cdot (\mathbf{x}_3-\mathbf{x}_j)}{(\mathbf{x}_3-\mathbf{x}_i)^2 (\mathbf{x}_3-\mathbf{x}_j)^2} \rightarrow \frac{1}{2\pi} \left[\frac{1}{\epsilon} - \log(e^{\gamma_E} \pi \mu |\mathbf{x}_i - \mathbf{x}_j|) \right] + \mathcal{O}(\epsilon). \quad (\text{D8})$$

$$z_1 = \frac{p_h^+}{z_{h_1} l^+}, \quad z_2 = \frac{q_h^+}{z_{h_2} l^+}. \quad (\text{D9})$$

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