Covariant tetraquark equations in quantum field theory

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We derive exact yet practical covariant equations of quantum field theory describing a tetraquark in terms of a mix of four-quark states $2q2\bar{q}$, and two-quark states $q\bar{q}$. A feature of our approach is that it avoids the overcounting problems that usually plague quantum field theory formulations of few-body covariant equations (the only exception being the two-body Bethe-Salpeter equation). This is achieved by describing the coupling of $2q2\bar{q}$ to $q\bar{q}$ states through the use of model operators that contract a four-quark $q\bar{q}$-irreducible Green function down to a two-quark $q\bar{q}$ Bethe-Salpeter kernel. Although the model chosen in the current work describes the four-quark dynamics in terms of meson-meson and diquark-antidiquark states, the derived equations have a form that is exact, as all corrections due to the use of a particular model are taken into account through the use of a well-defined special four-point amplitude Δ entering the equations. The equations are in agreement with those obtained previously by consideration of disconnected interactions; however, despite being more general, they have been derived here in a much simpler and more transparent way.

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I. INTRODUCTION

In quantum field theory (QFT) the number of particles is not conserved. This fact necessitates a careful consideration of the theoretical description, as well as the precise definition of an exotic particle. In particular, this applies to the case of a tetraquark, an exotic bound state of two quarks and two antiquarks ($2q2\bar{q}$) whose existence has recently been evidenced [1–3]. That the $2q2\bar{q}$ system couples to $q\bar{q}$ states makes the tetraquark a more complicated object than often assumed. This is made clear in Fig. 1, which expresses the $2q2\bar{q}$ Green function $G^{(4)}$ in terms of its $q\bar{q}$-irreducible[1] part $G^{(4)}_{ir}$ and its $q\bar{q}$-reducible part $M^{(4-2)}_{ir}G^{(2)}M^{(2-4)}_{ir}$. Not only is the last, $q\bar{q}$-reducible term of Fig. 1 necessary for a complete description of a tetraquark, but its presence also demonstrates that any pole in the two-body $q\bar{q}$ Green function $G^{(2)}$ will automatically appear in $G^{(4)}$, thus making a pole in $G^{(4)}$ (the signature of a $2q2\bar{q}$ bound state), an inadequate criterion for a tetraquark.

In this paper we are concerned with the formulation of covariant equations describing the $2q2\bar{q}$ bound state while taking into account the coupling to $q\bar{q}$ channels as illustrated in Fig. 1. We shall refer to these equations as "tetraquark equations" even though our formulation does not depend on any specific definition of a tetraquark; nevertheless, we point out that the context of our derivation provides an ideal setting for considering such a precise definition, a task which we will return to in a separate work.

Our approach is motivated by recent efforts to describe tetraquarks using covariant few-body equations where the underlying dynamics is dominated by meson-meson (MM) and diquark-antidiquark ($D\bar{D}$) components [4–6]. The initial such formulation [4] was based on an analysis of only the $q\bar{q}$-irreducible part of the $2q2\bar{q}$ Green function, $G^{(4)}_{ir}$, although this fact was not emphasized at the time. The present paper therefore addresses the more recent formulations of covariant few-body equations describing the four-body $2q2\bar{q}$ system where coupling to two-body $q\bar{q}$ states is included [5,6]. The ultimate goal of such equations is to describe tetraquarks in terms of identical poles in the full $2q2\bar{q}$ and $q\bar{q}$ Green functions, $G^{(4)}$ and $G^{(2)}$, respectively. Unfortunately there is currently no consensus on the form such equations take [6].

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[1]In this work we use the commonly used definition of "irreducibility," namely, a Feynman diagram with any number of external quark legs is n-particle irreducible if it cannot be divided into two parts separating initial states from final states by cutting n quark lines where at least one of the cut quark lines is internal. In particular, we apply this definition to skeleton Feynman diagrams as we assume all propagators and vertices in such diagrams are fully dressed.

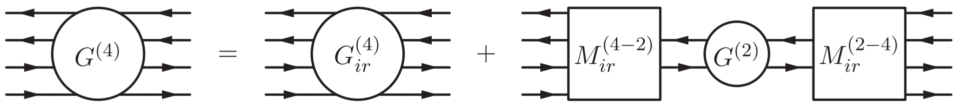

FIG. 1. Field theoretic structure of the $2q2\bar{q}$ Green function $G^{(4)}$, where $G^{(4)}_{ir}$ is the $q\bar{q}$-irreducible part of $G^{(4)}$, $G^{(2)}$ is the $q\bar{q}$ Green function, with $M^{(4-2)}_{ir}$ and $M^{(2-4)}_{ir}$ being $q\bar{q}$-irreducible $2q2\bar{q} \leftarrow q\bar{q}$ and $q\bar{q} \leftarrow 2q2\bar{q}$ transition amplitudes, respectively.

A serious issue facing all relativistically covariant derivations of equations that couple four-body to two-body states, is the appearance of overcounted terms. This type of overcounting first came to light in the analogous problem of formulating covariant few-body equations for the pion-two-nucleon (πNN) system where coupling to two-nucleon (NN) states is included [7]; there, it was found that, in order to attain overcounting-free equations where all possible two-body interactions are retained, special subtraction terms needed to be introduced and certain three-body forces needed to be retained. Although covariant few-body equations for the coupled $q\bar{q} - 2q2\bar{q}$ system can be derived in an analogous manner to that for the $NN - \pi NN$ system, as in Ref. [7], current interest is in formulating a less detailed approach that applies specifically to the case where the $q\bar{q} - 2q2\bar{q}$ system is dominated by MM and $D\bar{D}$ components. It is in this context that we would like to revisit the formulation of the coupled $q\bar{q} - 2q2\bar{q}$ system. In the process, we aim to help resolve the differences between the aforementioned tetraquark equations by presenting a derivation that does not involve the introduction of any explicit disconnected contributions to two-body ($q\bar{q}$) interactions, viewed as potentially problematic in Ref. [6], but which were pivotal to our previous derivation [5].

The key idea of our new approach is to first construct a model for the Green function $G^{(4)}_{ir}$ describing the $2q2\bar{q}$ system without coupling to $q\bar{q}$ states, and then express the $q\bar{q}$ kernel $K^{(2)}$ (the driving kernel for $G^{(2)}$) in terms of $G^{(4)}_{ir}$ as

$$K^{(2)} = \Delta + A^{(2-4)} G^{(4)}_{ir} A^{(4-2)}, \tag{1}$$

where operators $A^{(4-2)}$ ($A^{(2-4)}$) describe the transitions $2q2\bar{q} \leftarrow q\bar{q}$ ($q\bar{q} \leftarrow 2q2\bar{q}$) in the chosen model, and Δ is defined as the $q\bar{q}$ four-point function consisting of all contributions not accounted for by the last term of Eq. (1). A feature of this approach is that the usual problems of overcounting (encountered in formulations of covariant few-body equations) are avoided, and the final tetraquark equations have a coupled form that is exact no matter what model is chosen for $G^{(4)}_{ir}$ and $A^{(4-2)}$ ($A^{(2-4)}$). Within the MM-$D\bar{D}$ model, these equations take the form given by Eq. (46).

Despite the very different approach presented here, the tetraquark equations resulting from the new derivation are in agreement with those of Ref. [5], and moreover, are obtained in a simpler and much more transparent way.

II. TETRAQUARK POLES AND WAVE FUNCTIONS

In the context of QFT, to formulate a few-body approach for a system of particles where some of them can be absorbed by others (e.g., π by N in the πNN system or a $q\bar{q}$ pair that is annihilated in the $2q2\bar{q}$ system) one starts with the general structure of the full few-body Green function, which in the case of the $2q2\bar{q}$ system is manifested by the decomposition

$$G^{(4)} = G^{(4)}_{ir} + G^{(4-2)}_{ir} G^{(2)-1}_0 G^{(2)} G^{(2)-1}_0 G^{(2-4)}_{ir}, \tag{2}$$

where $G^{(2)}_0$ is the disconnected part of the two-body $q\bar{q}$ Green function $G^{(2)}$ corresponding to the independent propagation of q and $\bar{q}$ in the s channel, and $G^{(2-4)}_{ir}$ ($G^{(4-2)}_{ir}$) is the sum of all $q\bar{q}$-irreducible diagrams corresponding to the transition $q\bar{q} \leftarrow 2q2\bar{q}$ ($2q2\bar{q} \leftarrow q\bar{q}$). Equation (2) is illustrated in Fig. 1 where $G^{(4-2)}_{ir} G^{(2)-1}_0 \equiv M^{(4-2)}_{ir}$ and $G^{(2)-1}_0 G^{(2-4)}_{ir} \equiv M^{(2-4)}_{ir}$. The main problem is then to express $G^{(2-4)}_{ir}$ and $G^{(4-2)}_{ir}$ in terms of $G^{(4)}_{ir}$, while $G^{(4)}_{ir}$ can be expressed in terms of four-body scattering equations that are valid in the absence of $q\bar{q}$ annihilation.

The contribution of $G^{(4)}_{ir}$ to the $q\bar{q}$ Green function $G^{(2)}$ takes place through the Bethe-Salpeter kernel $K^{(2)}$, which, by definition, is $q\bar{q}$ irreducible and is related to $G^{(2)}$ through the Dyson equation[2]

$$G^{(2)} = G^{(2)}_0 + G^{(2)}_0 K^{(2)} G^{(2)}. \tag{3}$$

In particular, $G^{(4)}_{ir}$ contributes to $K^{(2)}$ via a $2q2\bar{q}$-reducible term of the form

$$K^{(2)}_{4q-\mathrm{red}} = A^{(2-4)} G^{(4)}_{ir} A^{(4-2)}, \tag{4}$$

where $A^{(4-2)}$ ($A^{(2-4)}$) is some amplitude (of course $q\bar{q}$ irreducible) corresponding to the transition $2q2\bar{q} \leftarrow q\bar{q}$ ($q\bar{q} \leftarrow 2q2\bar{q}$). Note that the full $2q2\bar{q}$-reducible part of $K^{(2)}$ consists of $K^{(2)}_{4q-\mathrm{red}}$, as defined by Eq. (4), together with extra $2q2\bar{q}$-reducible terms. More specifically, with $G^{(4)}_{ir}$ being the sum of all four-body diagrams that are $q\bar{q}$

[2]It is worth noting that our definition of $K^{(2)}$ is different from the one used in Ref. [6].

irreducible, the expression $A^{(2-4)}G_{ir}^{(4)}A^{(4-2)}$ either contains all $4q$-reducible diagrams some of which are overcounted, or only a part of them with no overcounting. In the first case, $4q$-reducible terms need to be subtracted from $A^{(2-4)}G_{ir}^{(4)}A^{(4-2)}$ in order to obtain the full $2q2\bar{q}$-reducible part of $K^{(2)}$, whereas in the second case, the missing $4q$-reducible terms should be added. The case of overcounting is rooted in the ambiguity of some $4q$ cuts, in contrast to $2q$ cuts where there is no such ambiguity. The full analysis of these terms and the function $A^{(2-4)}$ ($A^{(4-2)}$) would be similar to the one used for the πNN system in the covariant approach of Ref. [7], but this will be investigated elsewhere.

Ultimately, we shall be interested in the case where $G^{(4)}$ and $G^{(2)}$ display simultaneous poles corresponding to a tetraquark of mass M, so that as $P^2 \to M^2$ where P is the total momentum of each system,

$$G^{(4)} \to i\frac{\Psi\bar{\Psi}}{P^2 - M^2}, \qquad G^{(2)} \to i\frac{G_0^{(2)}\Gamma^*\bar{\Gamma}^* G_0^{(2)}}{P^2 - M^2}. \quad (5)$$

In Eq. (5), Ψ is the tetraquark four-body $(2q2\bar{q})$ bound state wave function, while Γ^* is the form factor for the disintegration of a tetraquark into a $q\bar{q}$ pair. We note that the definition of Ψ and Γ^* via the pole parts of $G^{(4)}$ and $G^{(2)}$ in Eq. (5), together with Eq. (2) relating $G^{(4)}$ and $G^{(2)}$, leads to the relation between Ψ and Γ^*,

$$\Psi = G^{(4-2)}\Gamma^*. \quad (6)$$

As is evident from Eq. (3) and the second of the relations in Eq. (5), a tetraquark state will also satisfy the two-body (not four-body) equation,

$$\Gamma^* = K^{(2)}G_0^{(2)}\Gamma^*. \quad (7)$$

It is Eq. (7) that will be used in this paper to formulate the tetraquark equations. This will be achieved by first constructing $G_{ir}^{(4)}$ using the four-body equations of Khvedelidze and Kvinikhidze [8], then making approximations where a pole ansatz is used for all quark pair scattering amplitudes and where single-scattering terms are neglected in the four-body t matrix expression of Eq. (16), and then using Eq. (4) to generate the essential part of the $q\bar{q}$ kernel.

III. TETRAQUARK FEW-BODY EQUATIONS

The approach used here to derive covariant equations for the coupled $q\bar{q} - 2q2\bar{q}$ system is different from that employed by us in Ref. [5]. Instead of incorporating coupling to $q\bar{q}$ states right from the outset, as embodied in the full four-body Green function $G^{(4)}$, here we first consider a formulation of four-body tetraquark equations for the case where there is no coupling to $q\bar{q}$ states; that is, we first consider a formulation based on $G_{ir}^{(4)}$, the $q\bar{q}$-irreducible part of $G^{(4)}$. Coupling to $q\bar{q}$ states is then achieved by generating the $q\bar{q}$ kernel $K^{(2)}$ through a simple contraction of four-body to two-body states as in Eq. (4).

One can express $G_{ir}^{(4)}$ in terms of the $q\bar{q}$-irreducible four-body interaction kernel $K_{ir}^{(4)}$ through the Dyson equation

$$G_{ir}^{(4)} = G_0^{(4)} + G_0^{(4)}K_{ir}^{(4)}G_{ir}^{(4)}, \quad (8)$$

where $G_0^{(4)}$ is the fully disconnected part of $G^{(4)}$. For simplicity, we start out by treating the quarks as distinguishable particles; however, the full antisymmetry of quark states will be taken into account shortly. The kernel $K_{ir}^{(4)}$ can be formally expressed as

$$K_{ir}^{(4)} = K_{2F}^{(4)} + K_{3F}^{(4)}, \quad (9)$$

where $K_{2F}^{(4)}$ consists of only pair-wise interactions, and $K_{3F}^{(4)}$ consists of all other contributions, necessarily involving three- and four-body forces. Assigning labels 1,2 to the quarks and 3,4 to the antiquarks, one can write $K_{2F}^{(4)}$ as a sum of three terms whose structure is illustrated in Fig. 2, and correspondingly expressed as

$$K_{2F}^{(4)} = \sum_{aa'} K_{aa'}^{(4)} = \sum_{\alpha} K_{\alpha}^{(4)}, \quad (10)$$

where the index $a \in \{12, 13, 14, 23, 24, 34\}$ enumerates six possible pairs of particles, the double index $aa' \in \{(13,24), (14,23), (12,34)\}$ enumerates three possible two pairs of particles, and the Greek index α is used as an abbreviation for aa' such that $\alpha = 1$ denotes $aa' = (13,24)$, $\alpha = 2$ denotes $aa' = (14,23)$, and $\alpha = 3$ denotes $aa' = (12,34)$. Thus $K_{aa'}^{(4)}$ describes the part of the four-body kernel where all interactions are switched off except those within the pairs a and a'. As is well known

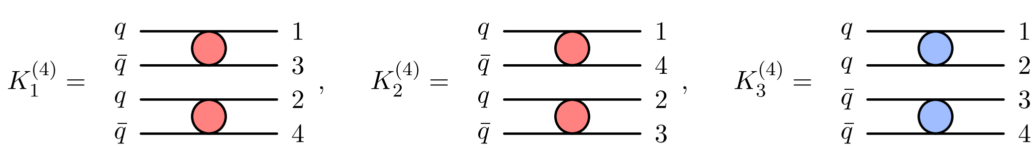

FIG. 2. Structure of the terms $K_\alpha^{(4)}$ ($\alpha = 1, 2, 3$) making up the four-body kernel $K_{2F}^{(4)}$ where only two-body forces are included. The three terms are summed as in Eq. (10).

[4,5,8], $K^{(4)}_{aa'}$ can be expressed in terms of the two-body kernels $K^{(2)}_a$ and $K^{(2)}_{a'}$ as

$$K^{(4)}_{aa'} = K^{(2)}_a G^{0\,-1}_{a'} + K^{(2)}_{a'} G^{0-1}_a - K^{(2)}_a K^{(2)}_{a'}, \qquad (11)$$

where $K^{(2)}_a K^{(2)}_{a'}$ is a subtraction term needed to avoid overcounting (present even when coupling to two-body channels is neglected), and where G^0_a ($G^0_{a'}$) is the two-body disconnected Green function for particle pair a (a'). It is also useful to introduce the corresponding four-body $q\bar{q}$-irreducible t matrix $T^{(4)}_{ir}$ defined by equation

$$G^{(4)}_{ir} = G^{(4)}_0 + G^{(4)}_0 T^{(4)}_{ir} G^{(4)}_0. \qquad (12)$$

One can similarly express $T^{(4)}_{ir}$ as a sum of three terms [8]

$$T^{(4)}_{ir} = \sum_{aa'} \mathcal{T}^{(4)}_{aa'} = \sum_\alpha \mathcal{T}^{(4)}_\alpha \qquad (13)$$

with components $\mathcal{T}^{(4)}_\alpha$ satisfying Faddeev-like equations

$$\mathcal{T}^{(4)}_\alpha = T^{(4)}_\alpha + \sum_\beta T^{(4)}_\alpha \bar{\delta}_{\alpha\beta} G^{(4)}_0 \mathcal{T}^{(4)}_\beta, \qquad (14)$$

where $\bar{\delta}_{\alpha\beta} = 1 - \delta_{\alpha\beta}$ and where the Greek subscripts run over the three possible "two pairs" of particles as in Eq. (10). In Eq. (14), $T^{(4)}_\alpha$ is the t matrix corresponding to kernel $K^{(4)}_\alpha$, that is

$$T^{(4)}_\alpha = K^{(4)}_\alpha + K^{(4)}_\alpha G^{(4)}_0 T^{(4)}_\alpha, \qquad (15)$$

with $T^{(4)}_\alpha$ being expressed in terms of two-body t matrices $T^{(2)}_a$ and $T^{(2)}_{a'}$ as

$$T^{(4)}_\alpha = T^{(4)}_{aa'} = T^{(2)}_a G^{0\,-1}_{a'} + T^{(2)}_{a'} G^{0-1}_a + T^{(2)}_a T^{(2)}_{a'}. \qquad (16)$$

A. Tetraquark equations with no coupling to $q\bar{q}$ states

To compare with the existing approaches [4–6], our aim is to describe the tetraquark using two-body equations that couple identical meson-meson (MM), and diquark-antidiquark ($D\bar{D}$) channels. To this end we consider $G^{(4)}_{ir}$ in the approximation

$$T^{(4)}_{aa'} = T^{(2)}_a T^{(2)}_{a'}, \qquad (17a)$$

where the two-body t matrices $T^{(2)}_a$ and $T^{(2)}_{a'}$ are expressed in the bound state pole approximation

$$T^{(2)}_a = i\Gamma_a D_a \bar{\Gamma}_a, \qquad (17b)$$

where $D_a(P_a) = 1/(P_a^2 - m_a^2)$ is the propagator for the bound particle (diquark, antidiquark, or meson) in the two-body channel a. Showing explicit dependence on momentum variables, $T^{(2)}_a$, for $a = 12$, can be expressed as $T^{(2)}_{12}(p_1'p_2', p_1p_2) = i\Gamma(p_1'p_2')D_{12}(P)\bar{\Gamma}(p_1p_2)$, where $P = p_1 + p_2$ is the total off-mass-shell momentum of the bound particle.

In the current context, the signature for a tetraquark is the existence of a pole in $G^{(4)}_{ir}$. In turn, this means the existence of a four-body tetraquark wave function $\Psi_{ir} \equiv G^{(4)}_0 \psi$ for the case where all coupling to $q\bar{q}$ states is switched off. We therefore begin by considering the corresponding bound state form factor ψ for the case of two indistinguishable quarks and two indistinguishable antiquarks, and relate it to the corresponding form factor ψ^d for distinguishable quarks as

$$\psi = \frac{1}{4}(1 - \mathcal{P}_{12})(1 - \mathcal{P}_{34})\psi^d, \qquad (18)$$

where $\mathcal{P}_{ij}$ is the operator exchanging the quantum numbers of particles i and j. The Faddeev-like equations for ψ^d are [4,8]

$$\psi^d = \sum_\alpha \psi^d_\alpha, \qquad (19a)$$

$$\psi^d_\alpha = \sum_\beta T^{(4)}_\alpha \bar{\delta}_{\alpha\beta} G^{(4)}_0 \psi^d_\beta, \qquad (19b)$$

where $\bar{\delta}_{\alpha\beta} = 1 - \delta_{\alpha\beta}$ and the Greek subscripts run over the three possible "two pairs" of particles as in Eq. (10). Using the approximations of Eq. (17), one can write

$$T^{(4)}_\alpha = T^{(2)}_a T^{(2)}_{a'} = i^2 \Gamma_a \Gamma_{a'} D_a D_{a'} \bar{\Gamma}_a \bar{\Gamma}_{a'} \equiv -\Gamma_\alpha D_\alpha \bar{\Gamma}_\alpha, \qquad (20)$$

where $\Gamma_\alpha \equiv \Gamma_a \Gamma_{a'}$, $\bar{\Gamma}_\alpha \equiv \bar{\Gamma}_a \bar{\Gamma}_{a'}$, and $D_\alpha \equiv D_a D_{a'}$. Further, defining the vertex functions ϕ^d_α by the relation

$$\psi^d_\alpha = \Gamma_\alpha D_\alpha \phi^d_\alpha, \qquad (21)$$

it follows from Eq. (19b) that

$$\phi^d_\alpha = \sum_\beta V_{\alpha\beta} D_\beta \phi^d_\beta, \qquad (22)$$

where

$$V_{\alpha\beta} = -\bar{\delta}_{\alpha\beta} \bar{\Gamma}_\alpha G^{(4)}_0 \Gamma_\beta. \qquad (23)$$

Noting that $\Gamma_{12} = -\Gamma_{21}$ and $\Gamma_{34} = -\Gamma_{43}$, it follows that $V_{12} = V_{21}$, $V_{23} = -V_{13}$, and $V_{32} = -V_{31}$.

We can now use Eq. (18) to derive MM and $D\bar{D}$ components of the tetraquark form factor ψ in the case of indistinguishable quarks. These are defined by the pole contributions to ψ at $p_{13}^2 = M_\pi^2$, $p_{24}^2 = M_\pi^2$, $p_{14}^2 = M_\pi^2$, $p_{23}^2 = M_\pi^2$, $p_{12}^2 = M_D^2$, and $p_{34}^2 = M_D^2$, where $p_{ij} = p_i + p_j$ is the total momentum of particles i and j, M_π is the mass of the meson and M_D is the mass of the diquark or antidiquark. To this end consider the use of Eq. (22) in Eq. (18):

$$\begin{aligned}\psi &= \frac{1}{4}(1-\mathcal{P}_{14})(1-\mathcal{P}_{34})[\Gamma_1 D_1 \phi_1^d(p_{13},p_{24}) + \Gamma_2 D_2 \phi_2^d(p_{14},p_{23}) + \Gamma_3 D_3 \phi_3^d(p_{12},p_{34})],\\ &= \frac{1}{4}\Gamma_{13}\Gamma_{24}D_1[\phi_1^d(p_{13},p_{24}) + \phi_1^d(p_{24},p_{13})] - \frac{1}{4}\Gamma_{14}\Gamma_{23}D_2[\phi_1^d(p_{23},p_{14}) + \phi_1^d(p_{14},p_{23})]\\ &\quad + \frac{1}{4}\Gamma_{14}\Gamma_{23}D_2[\phi_2^d(p_{14},p_{23}) + \phi_2^d(p_{23},p_{14})] - \frac{1}{4}\Gamma_{13}\Gamma_{24}D_1[\phi_2^d(p_{24},p_{13}) + \phi_2^d(p_{13},p_{24})] + \Gamma_{12}\Gamma_{34}D_3\phi_3^d(p_{12},p_{34}),\\ &= \frac{1}{2}\Gamma_{13}\Gamma_{24}D_1[\phi_1^S(p_{13},p_{24}) - \phi_2^S(p_{13},p_{24})] + \frac{1}{2}\Gamma_{14}\Gamma_{23}D_2[\phi_2^d(p_{14},p_{23}) - \phi_1^d(p_{14},p_{23})] + \Gamma_{12}\Gamma_{34}D_3\phi_3^d(p_{12},p_{34}),\\ &= \frac{1}{2}\Gamma_{13}\Gamma_{24}D_1\phi_M(p_{13},p_{24}) - \frac{1}{2}\Gamma_{14}\Gamma_{23}D_2\phi_M(p_{14},p_{23}) + \Gamma_{12}\Gamma_{34}D_3\phi_D(p_{12},p_{34}),\end{aligned} \tag{24}$$

where

$$\phi_1^S(p,q) = \frac{1}{2}[\phi_1^d(p,q) + \phi_1^d(q,p)], \tag{25a}$$

$$\phi_2^S(p,q) = \frac{1}{2}[\phi_2^d(p,q) + \phi_2^d(q,p)] \tag{25b}$$

are symmetric functions under the exchange of the meson quantum numbers, and

$$\phi_M(p,q) = \phi_1^S(p,q) - \phi_2^S(p,q), \tag{26a}$$

$$\phi_D(p,q) = \phi_3^d(p,q) \tag{26b}$$

define the MM and $D\bar{D}$ components of the tetraquark form factor ψ where quarks are identical.

To derive equations for the tetraquark vertex functions for identical quarks, we first write out Eq. (22) for distinguishable quarks using notation $V_{13} = V_{1D}$, $V_{23} = V_{2D}$, $V_{31} = V_{D1}$, $V_{32} = V_{D2}$, and $\phi_D = \phi_{12,34}^d$:

$$\phi_1^d = V_{12}D_2\phi_2^d + V_{1D}D_3\phi_D, \tag{27a}$$

$$\phi_2^d = V_{21}D_1\phi_1^d + V_{2D}D_3\phi_D = V_{12}D_1\phi_1^d - V_{1D}D_3\phi_D, \tag{27b}$$

$$\phi_D = V_{D1}D_1\phi_1^d + V_{D2}D_2\phi_2^d = V_{D1}(D_1\phi_1^d - D_2\phi_2^d). \tag{27c}$$

Then, subtracting the second line from the first, we obtain a set of two equations for $\phi_- = \phi_1^d - \phi_2^d$ and ϕ_D,

$$\begin{aligned}\phi_- &= -V_{12}D_2\phi_- + 2V_{1D}D_3\phi_D,\\ &= -2V_{12}\left(\frac{1}{2}MM\right)\phi_- + 2V_{1D}D\bar{D}\phi_D,\end{aligned} \tag{28a}$$

$$\phi_D = V_{D1}D_1\phi_- = 2V_{D1}\left(\frac{1}{2}MM\right)\phi_-, \tag{28b}$$

where we used $D_1 = D_2 \equiv MM$, $D_3 \equiv D\bar{D}$. Equation (28) can be written in matrix form as

$$\begin{pmatrix}\phi_-\\ \phi_D\end{pmatrix} = 2\begin{pmatrix}-V_{12} & V_{1D}\\ V_{D1} & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2}MM & 0\\ 0 & D\bar{D}\end{pmatrix}\begin{pmatrix}\phi_-\\ \phi_D\end{pmatrix}. \tag{29}$$

To finally derive the tetraquark equations in the case of indistinguishable quarks, note that according Eq. (25),

$$\begin{aligned}\phi_M &= \Phi_1^S - \Phi_2^S = \Phi_-^S = \frac{1}{2}[\phi_-(p,q) + \phi_-(q,p)],\\ &= \frac{1}{2}(1+\mathcal{P})\phi_-,\end{aligned} \tag{30}$$

where $\mathcal{P}$ is permutation operator of the meson state labels. Thus, symmetrizing Eq. (28) with respect to meson legs gives

$$\begin{aligned}\phi_M &= -2\frac{1}{2}(1+\mathcal{P})V_{12}\left(\frac{1}{2}MM\right)\phi_- + 2\frac{1}{2}(1+\mathcal{P})V_{1D}D\bar{D}\phi_D,\\ &= -2\frac{1}{2}(1+\mathcal{P})V_{12}\left(\frac{1}{2}MM\right)\phi_M + 2V_{1D}D\bar{D}\phi_D,\end{aligned} \tag{31a}$$

$$\phi_D = 2V_{D1}\left(\frac{1}{2}MM\right)\phi_- = 2V_{D1}\left(\frac{1}{2}MM\right)\phi_M, \tag{31b}$$

where we have used the following symmetry properties of V_{12} and V_{1D}:

$$(1+\mathcal{P})V_{12} = (1+\mathcal{P})V_{12}\frac{1}{2}(1+\mathcal{P}), \tag{32a}$$

$$\frac{1}{2}(1+\mathcal{P})V_{1D} = V_{1D}. \tag{32b}$$

Equation (31) can be written in matrix form as

$$\phi = VG_0^M\phi, \tag{33}$$

where

$$\phi = \begin{pmatrix} \phi_M \\ \phi_D \end{pmatrix}, \qquad G_0^M = \begin{pmatrix} \frac{1}{2}MM & 0 \\ 0 & D\bar{D} \end{pmatrix}, \tag{34}$$

and

$$V = \begin{pmatrix} -(1+\mathcal{P})V_{12} & 2V_{1D} \\ 2V_{D1} & 0 \end{pmatrix}, \tag{35}$$

thereby revealing V of Eq. (35) to be the interaction kernel for the coupled $MM - D\bar{D}$ system. The elements of V involve the potentials

$$V_{12} = -\bar{\Gamma}_1 G_0^{(4)}\Gamma_2 = -\bar{\Gamma}_{13}\bar{\Gamma}_{24}G_0^{(4)}\Gamma_{14}\Gamma_{23}, \tag{36a}$$

$$V_{1D} = -\bar{\Gamma}_1 G_0^{(4)}\Gamma_3 = -\bar{\Gamma}_{13}\bar{\Gamma}_{24}G_0^{(4)}\Gamma_{12}\Gamma_{34}, \tag{36b}$$

$$V_{D1} = -\bar{\Gamma}_3 G_0^{(4)}\Gamma_1 = -\bar{\Gamma}_{12}\bar{\Gamma}_{34}G_0^{(4)}\Gamma_{13}\Gamma_{24}, \tag{36c}$$

as illustrated in Fig. 3. With the kernel matrix V established, one can determine the t matrix T defined by

$$T = V + VG_0^M T, \tag{37}$$

and thereafter $G_0^M + G_0^M T G_0^M$, which is the matrix Green function in coupled MM-$D\bar{D}$ space corresponding to $G_{ir}^{(4)}$.

B. Tetraquark equations with coupling to $q\bar{q}$ states

In the previous section we derived the tetraquark equations without coupling to $q\bar{q}$ states, Eq. (33), and we established the model form for the Green function $G_{ir}^{(4)}$, which carries the signature pole for the $2q2\bar{q}$ bound state in the absence of coupling to $q\bar{q}$ states. We now use this form to establish tetraquark equations with coupling to $q\bar{q}$ states, where the $2q2\bar{q}$ bound state is now signaled by a pole in the full Green function $G^{(4)}$.[3]

As indicated by Eq. (4), the contribution of $G_{ir}^{(4)}$ to the 4q-reducible part of the two-body $q\bar{q}$ kernel $K^{(2)}$, occurs via the term $K^{(2)}_{4q-\mathrm{red}}$, which is constructed by sandwiching $G_{ir}^{(4)}$ between amplitudes $A^{(4-2)}$ and $A^{(2-4)}$ corresponding to the transitions $2q2\bar{q} \leftarrow q\bar{q}$ and $q\bar{q} \leftarrow 2q2\bar{q}$, respectively. These amplitudes contract 4q states to 2q states and, in general, should be chosen within a model or

[3]Although it is not necessary for there to be a bound state pole in $G_{ir}^{(4)}$ in order for there to be a bound state pole in $G^{(4)}$, it is worth noting that any bound state pole in $G_{ir}^{(4)}$ will not appear $G^{(4)}$. This is further discussed in the Appendix.

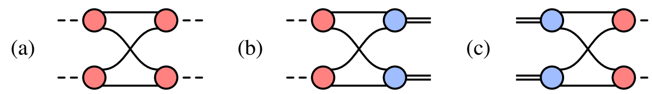

FIG. 3. The potentials making up the elements of the coupled channel $MM - D\bar{D}$ kernel matrix V of Eq. (35): (a) V_{12}, (b) V_{1D}, and (c) V_{D1}. Solid lines represent quarks or antiquarks, dashed lines represent mesons, and double lines represent diquarks and antidiquarks.

approximation. In the present case of coupled MM-$D\bar{D}$ channels, this contraction can be expressed as

$$K^{(2)}_{4q-\mathrm{red}} = \bar{N}(G_0^M + G_0^M T G_0^M)N, \tag{38}$$

where $\bar{N} = (\bar{N}_M, \bar{N}_D)$ is the two-component amplitude whose elements $\bar{N}_M$ and $\bar{N}_D$ describe transitions of two-meson and diquark-antidiquark states to the quark-antiquark state, $q\bar{q} \leftarrow M(p)M(k)$ and $q\bar{q} \leftarrow D(p)\bar{D}(k)$, respectively. Similarly, $N = (N_M, N_D)$ describes transitions $M(p)M(k) \leftarrow q\bar{q}$ and $D(p)\bar{D}(k) \leftarrow q\bar{q}$. Explicitly, these transition amplitudes are given by

$$\bar{N}_M = S_{23}(\Gamma_{13}^p\Gamma_{24}^k + \Gamma_{13}^k\Gamma_{24}^p), \quad \bar{N}_D = S_{23}\Gamma_{12}^p\Gamma_{34}^k, \tag{39a}$$

$$N_M = (\bar{\Gamma}_{13}^p\bar{\Gamma}_{24}^k + \bar{\Gamma}_{13}^k\bar{\Gamma}_{24}^p)S_{23}, \quad N_D = \bar{\Gamma}_{12}^p\bar{\Gamma}_{34}^k S_{23}, \tag{39b}$$

where S_{23} is the quark propagator connecting quark lines 2 and 3. Equations (39a) and (39b) are illustrated in Fig. 4.

Using the formal solution to Eq. (37),

$$T = (1 - VG_0^M)^{-1}G_0^{M-1} - G_0^{M-1}, \tag{40}$$

we can write the general expression for the two-body $q\bar{q}$ kernel as

$$\begin{aligned} K^{(2)} &= \Delta + K^{(2)}_{4q-\mathrm{red}}, \\ &= \Delta + \bar{N}G_0^M(1 - VG_0^M)^{-1}N, \end{aligned} \tag{41}$$

where Δ is defined to be the sum of all $q\bar{q}$-irreducible contributions allowed by QFT that are not accounted for by

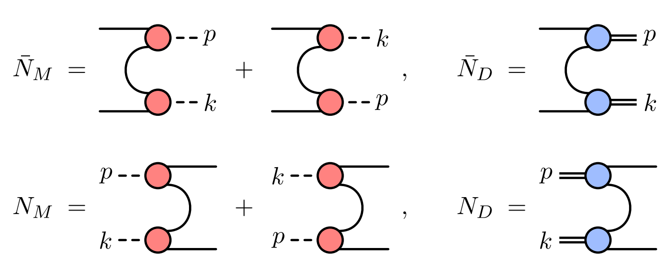

FIG. 4. Illustration of Eqs. (39a) and (39b). Lines have the same meaning as in Fig. 3.

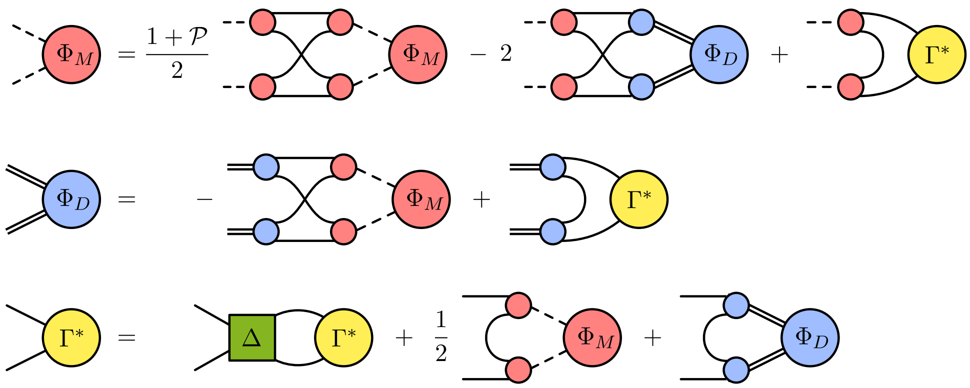

FIG. 5. Illustration of the tetraquark equations, (46a)–(46c), with coupling to $q\bar{q}$ states included. Tetraquark form factors Φ_M (displayed in red) couple to two mesons (dashed lines), tetraquark form factors Φ_D (displayed in blue) couple to diquark and antidiquark states (double lines), and the tetraquark form factors Γ^* (displayed in yellow) couple to $q\bar{q}$ states (solid lines). The amplitude Δ (displayed in green) represents all contributions to the $q\bar{q}$ kernel $K^{(2)}$ that are not included in the last term of Eq. (41).

the last term of Eq. (41). In particular, Δ includes correction terms that account for the difference between the approximations used in Eq. (17), and exact QFT, thus making Eq. (41) an exact expression for $K^{(2)}$. Just this clear definition of Δ allows one to improve the precision of the equations by taking into account effects in a systematic way. It is important to note that in this model, none of the $2q2\bar{q}$-reducible diagrams in the last term of Eq. (41) are overcounted, therefore Δ should not contain counterterms for eliminating overcounting.[4] As such, Δ can be used in future studies to take into account effects such as one-gluon exchange, one-meson exchange, etc.

Equation (33) constitutes the matrix form of the tetraquark equations without coupling to $q\bar{q}$ states. It expresses the column matrix ϕ of tetraquark form factors ϕ_M and ϕ_D, in terms of potentials contained in matrix V. To derive the corresponding tetraquark equations with coupling to $q\bar{q}$ states, we simply use the kernel $K^{(2)}$ of Eq. (41) in Eq. (7), the bound state equation for the tetraquark form factor Γ^*:

$$\begin{aligned}\Gamma^* &= K^{(2)} G_0^{(2)} \Gamma^*,\\ &= [\Delta + \bar{N} G_0^M (1 - V G_0^M)^{-1} N] G_0^{(2)} \Gamma^*,\\ &= \Delta G_0^{(2)} \Gamma^* + \bar{N} G_0^M \Phi,\end{aligned} \tag{42}$$

where

$$\Phi = V G_0^M \Phi + N G_0^{(2)} \Gamma^*. \tag{43}$$

Equation (43) is the matrix form of the sought-after tetraquark equations with coupling to $q\bar{q}$ states. It expresses the column matrix Φ of tetraquark form factors Φ_M and Φ_D in terms of both the potentials contained in matrix V, and the tetraquark form factor Γ^* describing the disintegration of a tetraquark into a $q\bar{q}$ pair. We can write Eq. (43) explicitly as

$$\begin{pmatrix}\Phi_M\\ \Phi_D\end{pmatrix} = \begin{pmatrix}(1+\mathcal{P})\bar{\Gamma}_M G_0^{(4)} \mathcal{P}_{34} \Gamma_M & -2\bar{\Gamma}_M G_0^{(4)} \Gamma_D\\ -2\bar{\Gamma}_D G_0^{(4)} \Gamma_M & 0\end{pmatrix} \times \begin{pmatrix}\frac{1}{2}MM & 0\\ 0 & D\bar{D}\end{pmatrix}\begin{pmatrix}\Phi_M\\ \Phi_D\end{pmatrix} + \begin{pmatrix}N_M\\ N_D\end{pmatrix} G_0^{(2)} \Gamma^*, \tag{44}$$

where $\Gamma_M = \Gamma_{13}\Gamma_{24}, \bar{\Gamma}_M = \bar{\Gamma}_{13}\bar{\Gamma}_{24}, \Gamma_D = \Gamma_{12}\Gamma_{34}, \bar{\Gamma}_D = \bar{\Gamma}_{12}\bar{\Gamma}_{34}$, and $\mathcal{P}_{ij}$ is the operator exchanging quarks i and j, therefore

$$\begin{aligned}\bar{\Gamma}_M G_0^{(4)} \mathcal{P}_{34} \Gamma_M &= \bar{\Gamma}_{13}\bar{\Gamma}_{24} G_0^{(4)} \Gamma_{14}\Gamma_{23},\\ \bar{\Gamma}_M G_0^{(4)} \Gamma_D &= \bar{\Gamma}_{13}\bar{\Gamma}_{24} G_0^{(4)} \Gamma_{12}\Gamma_{34}.\end{aligned} \tag{45}$$

Thus the tetraquark equations with coupling to $q\bar{q}$ included take the form of three coupled equations

$$\Phi_M = (1+\mathcal{P})\bar{\Gamma}_M G_0^{(4)} \mathcal{P}_{34} \Gamma_M \frac{MM}{2} \Phi_M - 2\bar{\Gamma}_M G_0^{(4)} \Gamma_D D\bar{D} \Phi_D + N_M G_0^{(2)} \Gamma^*, \tag{46a}$$

$$\Phi_D = -2\bar{\Gamma}_D G_0^{(4)} \Gamma_M \frac{MM}{2} \Phi_M + N_D G_0^{(2)} \Gamma^*, \tag{46b}$$

$$\Gamma^* = \Delta G_0^{(2)} \Gamma^* + \bar{N}_M \frac{MM}{2} \Phi_M + \bar{N}_D D\bar{D} \Phi_D, \tag{46c}$$

which are illustrated in Fig. 5. Since Δ is defined in a way that makes the expression used for $K^{(2)}$ exact, Eq. (46) represents an exact form of the tetraquark equations in QFT.

A nice feature of the present approach is the flexibility in the choice of the $4q \leftrightarrow 2q$ transition amplitudes $A^{(2-4)}$ ($A^{(4-2)}$) used to expose the contribution of the $4q$ Green

[4]In fact our choice of the last term of Eq. (41) in this note is motivated by physics arguments and the possibility of close comparison with existing studies.

function $G_{ir}^{(4)}$ in the $q\bar{q}$ kernel $K^{(2)}$—see Eq. (4) defining $K^{(2)}_{4q-\mathrm{red}}$. No matter what model is chosen to describe $G_{ir}^{(4)}$ (the present MM-$D\bar{D}$ model being just one such example), these transition amplitudes can be chosen accordingly, in this way turning the universal form for $K^{(2)}$ given by Eq. (1), into a model-specific one like that of Eq. (41) (of course each of the quantities $\bar{N}$, G_0^M, T, and N would then need to be redefined according to the chosen model). If there is any overcounting or undercounting due to the particular choice made for $A^{(2-4)}$ ($A^{(4-2)}$), as discussed in Sec. II, then compensating terms would be included in the term Δ of Eq. (1) defining $K^{(2)}$. With this understanding, Eq. (46) can be viewed as representative of exact tetraquark equations using any chosen model. It is noteworthy that such equations will contain the term $\Delta G_0^{(2)}\Gamma^*$, as in the last of the three tetraquark equations of Eq. (46), which is essential for including important mechanisms outside the considered model (for example one gluon exchange in the $MM - D\bar{D}$ model we have been considering). A feature of this term is the fact that Δ is clearly defined, thereby allowing for a precise control of what is included and what is neglected in any calculation.

Particularly noteworthy is how easy it is to deal with overcounted terms in the above approach compared with the handling of such terms in other formulations of covariant few-body equations (as in Refs. [5,7], for example): if the last term of Eq. (1) contains overcounted terms for any chosen model, then subtraction terms are simply included in Δ (being a four-point amplitude, Δ does not suffer from overcounting itself). However, in the present case of the MM-$D\bar{D}$ model, the transition amplitudes $A^{(4-2)}$ and $A^{(2-4)}$ are taken from our 2014 paper [5], and this choice does not lead to overcounting at all.

IV. CONCLUSIONS

We have derived a set of covariant coupled equations for the tetraquark, Eq. (46), using a model of recent interest where the two-body $q\bar{q}$, qq, and $\bar{q}\,\bar{q}$ interactions are dominated by the formation of a meson, a diquark, and an antidiquark, respectively. Despite the use of this model, the derived equations constitute an exact form of tetraquark equations in QFT since all differences between the model used and exact QFT are accounted for by correction terms included in the term Δ. Using our approach, similar equations easily follow for any choice of model of quark interactions. Equations (46) determine the form factors Φ_M, Φ_D, and Γ^* of the tetraquark, describing its disintegration into two identical mesons, a diquark-antidiquark pair, and a quark-antiquark pair. As such, they extend the purely four-body ($4q$) tetraquark equations of Ref. [4] to include coupling to two-body($2q$) $q\bar{q}$ states.

The motivation for the present work comes from the need to resolve the lack of agreement between two previous attempts to derive tetraquark equations with $4q$-$2q$ mixing. The first of these was our derivation of 2014 [5] using a careful but involved incorporation of disconnected $q\bar{q}$ interactions as a means of incorporating $q\bar{q}$ annihilation into a $4q$ description. The second of these was a recent derivation [6] where coupling to $2q$ channels was included phenomenologically, and where some doubt was expressed regarding the incorporation of disconnected $q\bar{q}$ interactions. Our present derivation of Eq. (46) has therefore been based on a method that avoids any explicit introduction of disconnected $q\bar{q}$ interactions, and which, in the absence of approximations for Δ, provides an exact field-theoretic description. It is therefore gratifying to note that in the absence of the term Δ, Eq. (46) coincides with the equations derived by us in Ref. [5]. Indeed, setting $\Delta = 0$ in Eq. (46c) and substituting into Eq. (44) gives Φ in the form presented in Ref. [5][5]:

$$\Phi = \left[\begin{pmatrix} (1+\mathcal{P})\bar{\Gamma}_M G_0^{(4)}\mathcal{P}_{34}\Gamma_M & -2\bar{\Gamma}_M G_0^{(4)}\Gamma_D \\ -2\bar{\Gamma}_D G_0^{(4)}\Gamma_M & 0 \end{pmatrix} + N G_0^{(2)}\bar{N}\right] \times \begin{pmatrix} \frac{1}{2}MM & 0 \\ 0 & D\bar{D} \end{pmatrix}\Phi. \tag{47}$$

Here $NG_0^{(2)}\bar{N}$ is the $q\bar{q}$ reducible part of the kernel which is denoted by $V_{q\bar{q}}$ in Ref. [5]. It accounts for the $q\bar{q}$ admixture through the $q\bar{q}$ propagator $G_0^{(2)}$. By contrast, the tetraquark equations of Ref. [6] are not consistent with the general form prescribed by Eq. (46).

Finally, it is worth noting that in comparison with our previous derivation [5], the approach taken in the present work allows for the derivation of the tetraquark equations in a much simpler and more clear way.

ACKNOWLEDGMENTS

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APPENDIX: POLE STRUCTURE OF $G^{(4)}$

In our approach, we first consider equations for the bound state of the $2q2\bar{q}$ system in the absence of coupling to $q\bar{q}$ channels, and in this case the $4q$ bound state is signaled by a pole at $P^2 = M_0^2$ in the $q\bar{q}$-irreducible Green function $G_{ir}^{(4)}$, where P is the total momentum and M_0 is a "bare" tetraquark mass. Then, after incorporating coupling to $q\bar{q}$ channels, the bound state is instead signaled by a pole at $P^2 = M^2$ in the full $2q2\bar{q}$ Green function $G^{(4)}$, where M is the "physical" tetraquark mass. In view of this, and the

[5]The expression in the square bracket in Eq. (47) may appear to come with an opposite sign in Ref. [5], but this is not the case as the definitions of $\bar{\Gamma}_M$, Γ_M, $\bar{\Gamma}_D$, Γ_D, $\bar{N}$, and N used in Ref. [5] differ from the ones used here.

direct connection between $G^{(4)}_{ir}$ and $G^{(4)}$ expressed by Fig. 1, it should be emphasized that $G^{(4)}$ will only exhibit just a single pole (at the physical mass M) even if $G^{(4)}_{ir}$ still has a pole at a different mass M_0. Although this may be self-evident, here we shall prove it mathematically. To do this, we write Eq. (2) as

$$G^{(4)} = G^{(4)}_{ir} + M^{(4-2)}_{ir} G^{(2)} M^{(2-4)}_{ir} \tag{A1}$$

and note that the pole in $G^{(4)}_{ir}$ originates from the use of a corresponding kernel $K^{(4)}_{ir}$ in the equation

$$G^{(4)}_{ir} = G^{(4)}_0 + G^{(4)}_0 K^{(4)}_{ir} G^{(4)}_{ir}, \tag{A2}$$

and similarly, the pole in $G^{(2)}$ originates from the use of a kernel $K^{(2)}$ in the equation

$$G^{(2)} = G^{(2)}_0 + G^{(2)}_0 K^{(2)} G^{(2)}. \tag{A3}$$

Then, by exposing the term $G^{(4)}_{ir}$ within $M^{4-2}_{ir}, M^{2-4}_{ir}$ and $K^{(2)}$ as follows:

$$M^{4-2}_{ir} \equiv G^{(4)}_{ir} A + A_s, \qquad M^{2-4}_{ir} \equiv \bar{A} G^{(4)}_{ir} + \bar{A}_s,$$
$$K^{(2)} \equiv \Delta_s + \bar{A} G^{(4)}_{ir} A, \tag{A4}$$

we show below that the above equations can be used in Eq. (A1) to write $G^{(4)}$ as

$$G^{(4)} = [1 + A_s G^{(2)}_\Delta \bar{A}] G^{(4)}_+ [1 + A G^{(2)}_\Delta \bar{A}_s] + A_s G^{(2)}_\Delta \bar{A}_s \tag{A5}$$

and similarly $G^{(2)}$ as

$$G^{(2)} = G^{(2)}_\Delta + G^{(2)}_\Delta \bar{A} G^{(4)}_+ A G^{(2)}_\Delta, \tag{A6}$$

where $G^{(2)}_\Delta$ is the Green function driven by kernel Δ_s, and $G^{(4)}_+$ is the solution to the four-body equation driven by the sum of the kernel $K^{(4)}_{ir}$ and $A G^{(2)}_\Delta \bar{A}$,

$$G^{(4)}_+ = G^{(4)}_0 + G^{(4)}_0 [K^{(4)}_{ir} + A G^{(2)}_\Delta \bar{A}] G^{(4)}_+. \tag{A7}$$

What Eqs. (A5) and (A6) show is that $G^{(4)}$ and $G^{(2)}$ each have just one pole, this being the shared pole due to the Green function $G^{(4)}_+$. Moreover, Eq. (A7) shows that this shared pole is shifted with respect to the pole of $G^{(4)}_{ir}$ generated in Eq. (A2). It is also clear that because $K^{(4)}_{ir}$ is combined in the sum $K^{(4)}_{ir} + A G^{(2)}_\Delta \bar{A}$, it does not generate a pole separately.

1. Proofs of (A5)–(A7)

By pulling out $G^{(4)}_{ir}$ from M^{4-2}_{ir} and M^{2-4}_{ir}, and exposing $G^{(4)}_{ir}$ in the kernel $K^{(2)}$ driving Eq. (A3), one obtains Eq. (A4) where the main (physically important) parts of the terms A_s, $\bar{A}_s$, and Δ_s are $4q$-irreducible, although they also contain some $4q$ reducible parts for fixing overcounting or undercounting suffered by $\bar{A} G^{(4)}_{ir} A$ (recall that subscript "ir" stands for "$2q$ irreducible").

With $K^{(2)}$ being a sum of two terms Δ_s and $\bar{A} G^{(4)}_{ir} A$, it will be useful to consider Eq. (A3) as a special case of the generic two-potential equation

$$G = G_0 + G_0 (K_1 + K_2) G, \tag{A8}$$

which may be expressed equivalently as

$$G = G_1 + G_1 K_2 G, \tag{A9}$$

where

$$G_1 = G_0 + G_0 K_1 G_1. \tag{A10}$$

Equation (A9) can also be written as

$$G = G_1 + G_1 T_2 G_1, \tag{A11}$$

where $K_2 G = T_2 G_1$ and $G K_2 = G_1 T_2$. One also has that

$$T_2 = K_2 + K_2 G_1 T_2, \tag{A12}$$

$$= K_2 + K_2 G K_2. \tag{A13}$$

Setting $K_1 = \Delta_s$, $K_2 = \bar{A} G^{(4)}_{ir} A$, $G = G^{(2)}$, $G_0 = G^{(2)}_0$, and $G_1 = G^{(2)}_\Delta$, Eq. (A3) can thus be expressed equivalently as

$$G^{(2)} = G^{(2)}_\Delta + G^{(2)}_\Delta \bar{A} G^{(4)}_{ir} A G^{(2)}, \tag{A14}$$

where

$$G^{(2)}_\Delta = G^{(2)}_0 + G^{(2)}_0 \Delta_s G^{(2)}_\Delta. \tag{A15}$$

Further defining

$$G^{(4)}_+ = G^{(4)}_{ir} + G^{(4)}_{ir} A G^{(2)} \bar{A} G^{(4)}_{ir}, \tag{A16}$$

and multiplying this equation from the left by $\bar{A}$ and from the right by A results in Eq. (A13) with

$$T_2 = \bar{A} G^{(4)}_+ A. \tag{A17}$$

Equation (A11) then becomes Eq. (A6). One then has that

$$\begin{aligned} G^{(4)} &= G_{ir}^{(4)} + M_{ir}^{4-2} G^{(2)} M_{ir}^{2-4}, \\ &= G_{ir}^{(4)} + G_{ir}^{(4)} A G^{(2)} \bar{A} G_{ir}^{(4)} + A_s G^{(2)} \bar{A} G_{ir}^{(4)} + G_{ir}^{(4)} A G^{(2)} \bar{A}_s + A_s G^{(2)} \bar{A}_s, \\ &= G_+^{(4)} + A_s G_\Delta^{(2)} \bar{A} G_+^{(4)} + G_+^{(4)} A G_\Delta^{(2)} \bar{A}_s + A_s [G_\Delta^{(2)} + G_\Delta^{(2)} \bar{A} G_+^{(4)} A G_\Delta^{(2)}] \bar{A}_s, \\ &= [1 + A_s G_\Delta^{(2)} \bar{A}] G_+^{(4)} [1 + A G_\Delta^{(2)} \bar{A}_s] + A_s G_\Delta^{(2)} \bar{A}_s, \end{aligned} \tag{A18}$$

where Eq. (A6) and the following relation has been used:

$$\begin{aligned} G_{ir}^{(4)} A G^{(2)} &= G_{ir}^{(4)} A [1 + G^{(2)} \bar{A} G_{ir}^{(4)} A] G_\Delta^{(2)}, \\ &= [G_{ir}^{(4)} + G_{ir}^{(4)} A G^{(2)} \bar{A} G_{ir}^{(4)}] A G_\Delta^{(2)} \\ &= G_+^{(4)} A G_\Delta^{(2)}. \end{aligned} \tag{A19}$$

This proves Eq. (A5). To prove Eq. (A7), reset the generic variables to $K_1 = K_{ir}^{(4)}$, $K_2 = A G_\Delta^{(2)} \bar{A}$, $G = G_+^{(4)}$, $G_0 = G_0^{(4)}$, and $G_1 = G_{ir}^{(4)}$. Then Eq. (A2) is just a statement of Eqs. (A10) and (A16), after the use of Eq. (A19), which is just a statement of Eq. (A9). Equation (A7) then follows immediately.

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