Toward the discovery of novel B_c states: Radiative and hadronic transitions

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The properties of the B_c-meson family ($c\bar{b}$) are still not well determined experimentally because the specific mechanisms of formation and decay remain poorly understood. Unlike heavy quarkonia, i.e., the hidden heavy quark-antiquark sectors of charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$), the B_c mesons cannot annihilate into gluons and they are, consequently, more stable. The excited B_c states, lying below the lowest strong-decay BD threshold, can only undergo radiative decays and hadronic transitions to the B_c ground state, which then decays weakly. As a result of this, a rich spectrum of narrow excited states below the BD threshold appear, whose total widths are 2 orders of magnitude smaller than those of the excited levels of charmonium and bottomonium. In a different article, we determined bottom-charmed meson masses using a nonrelativistic constituent quark model which has been applied to a wide range of hadron physical observables, and thus the model parameters are completely constrained. Herein, continuing to our study of the B_c sector, we calculate the relevant radiative decay widths and hadronic transition rates between $c\bar{b}$ states which are below the BD threshold. This shall provide the most promising signals for discovering excited B_c states that are below the lowest strong-decay BD threshold. Finally, our results are compared with other models to measure the reliability of the predictions and point out differences.

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I. INTRODUCTION

The feasibility of studying experimentally the family of $c\bar{b}$ mesons was demonstrated by the CDF Collaboration at the Tevatron collider in 1998 [1,2] with the observation of the $B_c(1^1S_0)$ bound state.[1] However, low production cross sections, large backgrounds, and relatively-easy misidentifications eluded the discovery of new bottom-charmed mesons until 2014, when the ATLAS Collaboration [3] observed a peak at $6842 \pm 4 \pm 5$ MeV/c^2, which was interpreted as either the $B_c^*(2^3S_1)$ excited state or an unresolved pair of peaks from the decays $B_c(2^1S_0) \to B_c(1^1S_0)\pi^+\pi^-$ and $B_c^*(2^3S_1) \to B_c^*(1^3S_1)\pi^+\pi^-$ followed by $B_c(1^3S_1) \to B_c(1^1S_0)\gamma$. It was not until 2019 when the CMS [4] and LHCb [5] Collaborations released signals consistent with the $B_c(2S)$ and $B_c^*(2S)$ states, observed in the $B_c(1S)\pi^+\pi^-$ invariant mass spectrum. More results on B_c mesons are expected to be reported in the near future.

On the theoretical side, the B_c-meson family provides another opportunity to test nonrelativistic quark models that have been successfully applied to charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) systems. This is because the B_c states share dynamical properties with both the $c\bar{c}$ and $b\bar{b}$ sectors, but they consist of two heavy quarks with different flavors that make the B_c states very stable, with narrow widths, since annihilation into gluons is forbidden. In fact, their results [6–11] can be contrasted with those from other theoretical frameworks such as relativistic quark models [12–19], QCD sum rules [20,21], continuum functional methods for QCD [22–24], effective field theories [25–28], and lattice QCD [29–31]. A collection of all of these results should provide a reliable template from which to compare

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[1]The spectroscopic notation $n^{2S+1}L_J$ is used, where $n = 1$ indicates the ground state and $n = 2, 3, \ldots$, the respective excited states with higher energies but equal J^P (following the notation of PDG), the total spin of the two valence quarks is denoted by S, while L is their relative angular momentum where $S, P, D, F\ldots$ implies, respectively, $L = 0, 1, 2, 3, \ldots$, and J is the total angular momentum of the system.

the future experimental findings. In fact, there is some agreement about which conventional B_c states must exist below the lowest strong-decay BD threshold. There should be two sets of S-wave states, the $1P$ multiplet and some or all of the $2P$ states, one multiplet of D-wave nature, and the lowest F-wave case should be located so close to threshold that its member states may be narrow due to angular momentum barrier suppression of the Okubo-Zweig-Iizuka (OZI) rule [32–34].

Complications with the B_c spectroscopy are expected to begin at the energy region in which strong-decay meson-meson thresholds could play an important role in the formation of B_c (-like) structures. This has been vigorously manifested in the heavy quarkonium spectrum with the discovery of many charmonium- and bottomonium-like XYZ states [35–37]. In Ref. [38], we studied the influence of two-meson thresholds on the B_c states finding, for instance, dynamically generated additional states in the $J^P = 1^+$ and 2^+ channels very close to the DB^* and D^*B^* thresholds, respectively. In that article, however, we did not perform any study related with decay properties and possible ways of finding low-lying states located either below or around the lowest strong-decay meson-meson thresholds.

The theoretical methods used to study the spectroscopy of bottom-charmed mesons can be extended to their decay properties. The excited B_c states lying below the BD threshold can only undergo radiative decays and hadronic transitions to the B_c ground state, which then decays weakly. Therefore, radiative and hadronic decay rates almost comprise the total decay width of the lowest excited B_c states, making them narrow with total widths 2 orders of magnitude smaller than those of the excited levels of charmonium and bottomonium, for which annihilation channels are significant. Moreover, such electromagnetic and hadronic processes are interesting by themselves because they allow experimental access to excited levels of heavy quarkonia which are below the lowest strong-decay meson-meson threshold and provide information about the internal structure and quantum numbers.

In this article we extend our previous investigation of the B_c spectrum [38] to potentially interesting radiative decays and hadronic transitions. Our theoretical framework is a nonrelativistic constituent quark model [39] in which quark-antiquark and meson-meson degrees of freedom can be incorporated at the same time (see Refs. [40,41] for reviews). The naive model, and its successive improvements, has been successfully applied to the charmonium and bottomonium sectors, studying their spectra [42–46], their electromagnetic, weak, and strong decays and reactions [47–51], their coupling with meson-meson thresholds [52–56], and, lately, phenomenological explorations of multiquark structures [57–61].

Electromagnetic transitions have been treated traditionally within the potential model approach. However, in the last decade, progress has been made using effective field theories (see [62,63] and references therein) and lattice-regularized QCD [64,65]. We shall use the formulae described in Ref. [35], but adapting it to our nonrelativistic constituent quark model approach. Although such expressions have been used since the early days of hadron spectroscopy, a brief description can be found below. Focusing now on the hadronic transitions, since the energy difference between the initial and final B_c states is expected to be small, the emitted gluons are rather soft. In Ref. [66], Gottfried pointed out that this gluon radiation can be treated in a multipole expansion, since the wavelengths of the emitted gluons are large compared to the size of the heavy mesons. The multipole expansion within QCD (QCDME) has been studied by many authors [66–71], but Yan was the first one to present a gauge-invariant formulation in Refs. [72,73] (see also the interesting advances made very recently in Refs. [74,75]). We shall follow the updated review [76] and references therein to calculate the hadronic transitions within our quark model formalism.

The manuscript is arranged as follows. After this introduction, the theoretical framework is presented in Sec. II; we explain first the quark model Hamiltonian and then the consistent formulation of radiative and hadronic decays. Section III is mostly devoted to the analysis and discussion of our theoretical results; we end this section by discussing some strategies for searching for excited B_c mesons and studying their spectroscopy. Finally, we summarize and give some prospects in Sec. IV.

II. THEORETICAL FRAMEWORK

In this section we are going to present, first, a detailed description of all of the different terms of the interacting potential. Later on, the standard formula that describes radiative transitions between low-lying B_c states is shown, which includes the dominant E1 and M1 multipole electromagnetic decay rates. And, finally, the latter subsection is dedicated to the hadronic transitions following the QCD multipole expansion method. It consists of a two-step process in which the gluons are first emitted from the heavy quarks and then recombine into light quarks. A multipole expansion of the color gauge field is employed to describe the emission process, whereas the intermediate color octet state is modeled by some sort of quark-antiquark-gluon hybrid wave function.

A. Constituent Quark Model

In the heavy quark sector, chiral symmetry is explicitly broken and, thus, the interaction between quarks due to Goldstone-boson exchanges does not take place. Therefore, one-gluon exchange and confinement are the only interactions remaining. The one-gluon exchange potential contains central, tensor, and spin-orbit contributions given by

$$
\begin{aligned}
V_{\mathrm{OGE}}^{\mathrm{C}}(\vec{r}_{ij}) &= \frac{1}{4}\alpha_s(\vec{\lambda}_i^c\cdot\vec{\lambda}_j^c)\left[\frac{1}{r_{ij}}-\frac{1}{6m_im_j}(\vec{\sigma}_i\cdot\vec{\sigma}_j)\frac{e^{-r_{ij}/r_0(\mu)}}{r_{ij}r_0^2(\mu)}\right],\\
V_{\mathrm{OGE}}^{\mathrm{T}}(\vec{r}_{ij}) &= -\frac{1}{16}\frac{\alpha_s}{m_im_j}(\vec{\lambda}_i^c\cdot\vec{\lambda}_j^c)\left[\frac{1}{r_{ij}^3}-\frac{e^{-r_{ij}/r_g(\mu)}}{r_{ij}}\left(\frac{1}{r_{ij}^2}+\frac{1}{3r_g^2(\mu)}+\frac{1}{r_{ij}r_g(\mu)}\right)\right]S_{ij},\\
V_{\mathrm{OGE}}^{\mathrm{SO}}(\vec{r}_{ij}) &= -\frac{1}{16}\frac{\alpha_s}{m_i^2m_j^2}(\vec{\lambda}_i^c\cdot\vec{\lambda}_j^c)\left[\frac{1}{r_{ij}^3}-\frac{e^{-r_{ij}/r_g(\mu)}}{r_{ij}^3}\left(1+\frac{r_{ij}}{r_g(\mu)}\right)\right]\\
&\quad\times[((m_i+m_j)^2+2m_im_j)(\vec{S}_+\cdot\vec{L})+(m_j^2-m_i^2)(\vec{S}_-\cdot\vec{L})], \qquad (1)
\end{aligned}
$$

with $\vec{\lambda}^c$ being the $SU(3)$ color matrices and α_s is the quark-gluon coupling constant. The regulators $r_0(\mu)=\hat{r}_0\frac{\mu_{nn}}{\mu_{ij}}$ and $r_g(\mu)=\hat{r}_g\frac{\mu_{nn}}{\mu_{ij}}$ depend on μ_{ij} which is the reduced mass of the interacting $q\bar{q}$ pair. The quark tensor operator is $S_{ij}=3(\vec{\sigma}_i\cdot\hat{r}_{ij})(\vec{\sigma}_j\cdot\hat{r}_{ij})-\vec{\sigma}_i\cdot\vec{\sigma}_j$, with σ_i denoting the Pauli matrices; and $\vec{S}_\pm=\vec{S}_i\pm\vec{S}_j$ with $S_i=\sigma_i/2$. The contact term of the central potential has been regularized as

$$\delta(\vec{r}_{ij})\sim\frac{1}{4\pi r_0^2}\frac{e^{-r_{ij}/r_0}}{r_{ij}}. \qquad (2)$$

The wide energy range needed to provide a consistent description of light, strange, and heavy mesons requires an effective scale-dependent strong coupling constant. We use the frozen coupling constant [77]

$$\alpha_s(\mu)=\frac{\alpha_0}{\ln(\frac{\mu^2+\mu_0^2}{\Lambda_0^2})}, \qquad (3)$$

in which μ is the reduced mass of the $q\bar{q}$ pair and α_0, μ_0, and Λ_0 are parameters of the model determined by a global fit to the meson spectra.

Lattice gauge Wilson loop computations convincingly demonstrate that the interquark interaction grows linearly with distance. These computations are made in the pure gauge theory where color sources and sinks are taken to be static and virtual quarks are omitted. In this so-called quenched approximation a funnel potential containing a linear plus a color-Coulomb term is well established [78]. Spin- and velocity-dependent corrections to this form have been obtained as well [79].

When sea quarks are incorporated (unquenched approximation) the long-distance behavior of the static potential may change dramatically. As a matter of fact, it has been shown in QCD at finite temperature [80], in SU(2) Yang-Mills theory [81], and in $n_f=2$ lattice QCD at zero temperature [82] that the potential saturates, i.e., it gets a constant value from a certain distance. Physically the saturation of the potential is related to screening: light $q\bar{q}$ pairs are created out of the vacuum between the heavy-quark source and heavy-antiquark sink, giving rise to a screening of their color charges.

The constituent quark model is similar to quenched lattice gauge theory in the sense that nonvalence quark effects are neglected; being more precise, virtual quark effects are absorbed by the model parameters since they are fitted to a body of data, including higher excited states. Incorporating virtual quark-antiquark loops such as meson creation effects in the quark model has been a longstanding goal of hadronic physics [83]. A rough approximation to the effects of unquenching is the use of a screened-linear confining potential in the constituent quark model.[2] This is what we have done herein; therefore, the different pieces of the confinement potential are

$$
\begin{aligned}
V_{\mathrm{CON}}^{\mathrm{C}}(\vec{r}_{ij}) &= [-a_c(1-e^{-\mu_c r_{ij}})+\Delta](\vec{\lambda}_i^c\cdot\vec{\lambda}_j^c),\\
V_{\mathrm{CON}}^{\mathrm{SO}}(\vec{r}_{ij}) &= -(\vec{\lambda}_i^c\cdot\vec{\lambda}_j^c)\frac{a_c\mu_c e^{-\mu_c r_{ij}}}{4m_i^2m_j^2r_{ij}}[((m_i^2+m_j^2)(1-2a_s)\\
&\quad+4m_im_j(1-a_s))(\vec{S}_+\cdot\vec{L})\\
&\quad+(m_j^2-m_i^2)(1-2a_s)(\vec{S}_-\cdot\vec{L})], \qquad (4)
\end{aligned}
$$

where a_s controls the mixture between the scalar and vector Lorentz structures of the confinement. At short distances this potential presents a linear behavior with an effective confinement strength $\sigma=-a_c\mu_c(\vec{\lambda}_i^c\cdot\vec{\lambda}_j^c)$, while it becomes constant at large distances. This type of potential shows a threshold defined by $V_{\mathrm{thr}}=\{-a_c+\Delta\}(\vec{\lambda}_i^c\cdot\vec{\lambda}_j^c)$.

Among the different methods to solve the Schrödinger equation in order to find the quark-antiquark bound states, we use the Gaussian expansion method [89] which provides enough accuracy and it simplifies the subsequent evaluation of the needed matrix elements.

This procedure provides the radial wave function solution of the Schrödinger equation as an expansion in terms of basis functions

[2]The interested reader is referred to [84,85] for a detailed discussion of the physics behind such a mechanism and Refs. [57,86–88] for applications.

TABLE I. Quark model parameters.

Quark masses	m_c (MeV)	1763
	m_b (MeV)	5110
OGE (one-gluon exchange)	α_0	2.118
	Λ_0 (fm^{-1})	0.113
	μ_0 (MeV)	36.976
	$\hat{r}_0$ (fm)	0.181
	$\hat{r}_g$ (fm)	0.259
CON (confinement)	a_c (MeV)	507.4
	μ_c (fm^{-1})	0.576
	Δ (MeV)	184.432
	a_s	0.81

$$R_\alpha(r) = \sum_{n=1}^{n_{\max}} c_n^\alpha \phi_{nl}^G(r), \tag{5}$$

where α refers to the channel quantum numbers. The coefficients, c_n^α, and the eigenvalue, E, are determined from the Rayleigh-Ritz variational principle

$$\sum_{n=1}^{n_{\max}} [(T_{n'n}^\alpha - EN_{n'n}^\alpha)c_n^\alpha + \sum_{\alpha'} V_{n'n}^{\alpha\alpha'} c_n^{\alpha'} = 0], \tag{6}$$

where $T_{n'n}^\alpha$, $N_{n'n}^\alpha$, and $V_{n'n}^{\alpha\alpha'}$ are the matrix elements of the kinetic energy, the normalization, and the potential, respectively. $T_{n'n}^\alpha$ and $N_{n'n}^\alpha$ are diagonal, whereas the mixing between different channels is given by $V_{n'n}^{\alpha\alpha'}$.

Following Ref. [89], we employ Gaussian trial functions with ranges in geometric progression. This facilitates the optimization of ranges employing a small number of free parameters. Moreover, the geometric progression is dense at short distances, so that it enables the description of the dynamics mediated by short range potentials. The fast damping of the Gaussian tail does not represent an issue, since we can choose the maximal range much longer than the hadronic size.

Finally, the model parameters can be found in Table I. They have been fixed following hadron phenomenology described in, for instance, the literature mentioned previously in the Introduction.

B. Radiative decays

The decay rate for E1 transitions between an initial state $n^{2S+1}L_J$ and a final state $n'^{2S'+1}L'_{J'}$ can be written as

$$\Gamma_{E1}(n^{2S+1}L_J \to n'^{2S'+1}L'_{J'}) = \frac{4\alpha e_Q^2 k^3}{3}(2J'+1)S_{fi}^E \delta_{SS'}|\mathcal{E}_{fi}|^2 \frac{E_f}{M_i}, \tag{7}$$

where $e_Q = (e_c m_b - e_b m_c)/(m_c + m_b)$, $k = (M_i^2 - M_f^2)/2M_i$ is the emitted photon momentum with M_i (M_f) the mass of the initial (final) state, E_f/M_i is a relativistic correction where E_f the energy of the final state. The statistical factor, S_{fi}^E, is given by

$$S_{fi}^E = \max(L, L') \begin{Bmatrix} J & 1 & J' \\ L' & S & L \end{Bmatrix}^2. \tag{8}$$

If the full momentum dependence is retained, the overlap integral, $\mathcal{E}_{fi}$, is

$$\mathcal{E}_{fi} = \frac{3}{k}\int_0^\infty R_{\alpha'}(r)\left[\frac{kr}{2}j_0\left(\frac{kr}{2}\right) - j_1\left(\frac{kr}{2}\right)\right] R_\alpha(r) r^2 dr, \tag{9}$$

where $j_i(x)$ are the spherical Bessel functions of the first kind and α (α') are the initial (final) meson quantum numbers.

The M1 radiative transitions can be evaluated with the following expression:

$$\Gamma_{M1}(n^{2S+1}L_J \to n'^{2S'+1}L'_{J'}) = \frac{4\alpha e_Q^2 k^3}{3m_c m_b}(2J'+1)S_{fi}^M |\mathcal{M}_{fi}|^2 \frac{E_f}{M_i}, \tag{10}$$

where we use the same notation as in the E1 transitions but now

$$S_{fi}^M = 6(2S+1)(2S'+1)\begin{Bmatrix} J & 1 & J' \\ S' & L & S \end{Bmatrix}^2 \begin{Bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & S' & S \end{Bmatrix}^2, \tag{11}$$

and

$$\mathcal{M}_{fi} = \int_0^\infty R_{\alpha'}(r) j_0\left(\frac{kr}{2}\right) R_\alpha(r) r^2 dr. \tag{12}$$

C. Hadronic decays

One can refer to a hadronic transition in the following general way:

$$\Phi_I \to \Phi_F + h, \tag{13}$$

where h denotes the light hadron(s) emerging from the emitted gluons; they are kinematically dominated by either single-particle ($\pi^0, \eta, \omega, \ldots$) or two-particle ($2\pi, 2K, \ldots$) states. The initial and final states of B_c mesons are named Φ_I and Φ_F, respectively.

The emitted gluons are rather soft because the energy difference between the initial and final charm-beauty states is small. Gottfried pointed out in Ref. [66] that the gluon radiation can be expanded in multipoles since the wavelengths of emitted gluons are larger than the size of B_c-meson states. After the expansion of the gluon field, the Hamiltonian of the system can be decomposed as

$$\mathcal{H}_{\rm QCD}^{\rm eff} = \mathcal{H}_{\rm QCD}^{(0)} + \mathcal{H}_{\rm QCD}^{(1)} + \mathcal{H}_{\rm QCD}^{(2)}, \tag{14}$$

with $\mathcal{H}_{\rm QCD}^{(0)}$ the sum of the kinetic and potential energies of the bottom-charmed meson, and $\mathcal{H}_{\rm QCD}^{(1)}$ and $\mathcal{H}_{\rm QCD}^{(2)}$ are defined by

$$\begin{aligned} \mathcal{H}_{\rm QCD}^{(1)} &= Q_a A_0^a(x,t), \\ \mathcal{H}_{\rm QCD}^{(2)} &= -d_a E^a(x,t) - m_a B^a(x,t), \end{aligned} \tag{15}$$

in which Q_a is the color charge and the color electric and magnetic dipole moments are represented by d_a and m_a, respectively. Since we are working with $c\bar{b}$ pairs that form a color singlet object, there is no contribution from the $\mathcal{H}_{\rm QCD}^{(1)}$ and only E_l and B_m transitions can take place.

A multipole expansion within QCD is now necessary in order to continue with the computation of the hadronic transitions between B_c states. A brief description of the derived formulas following the updated review [76] can be found below.

1. Spin-nonflip $\pi\pi$ and η transitions

The spin-nonflip $\pi\pi$ decay is dominated by the double electric-dipole term (E1-E1) in the QCD multipole expansion, and thus the transition amplitude can be written as follows:

$$\mathcal{M}_{E1E1} = i\frac{g_E^2}{6}\langle \Phi_F h|\vec{x}\cdot\vec{E}\frac{1}{E_I - H_{\rm QCD}^{(0)} - iD_0}\vec{x}\cdot\vec{E}|\Phi_I\rangle, \tag{16}$$

where $\vec{x}$ is the separation between the c-quark and $\bar{b}$-antiquark, and $(D_0)_{bc} \equiv \delta_{bc}\partial_0 - g_s f_{abc}A_0^a$.

Inserting a complete set of intermediate states, the transition amplitude, Eq. (16), becomes

$$\mathcal{M}_{E1E1} = i\frac{g_E^2}{6}\sum_{KL}\frac{\langle\Phi_F|x_k|KL\rangle\langle KL|x_l|\Phi_I\rangle}{E_I - E_{KL}}\langle\pi\pi|E_k^a E_l^a|0\rangle, \tag{17}$$

where E_{KL} is the energy eigenvalue of the intermediate state $|KL\rangle$ with the principal quantum number K and the orbital angular momentum L.

The intermediate states in the hadronic transition can be considered as hybrid mesons consisting of a color-octet $c\bar{b}$ pair plus gluon(s). They are very difficult to calculate in QCD from first principles when the quark-antiquark pair is open flavor; however, it is worth mentioning herein that there exist nonrelativistic effective field theories [90–92] and lattice-regularized QCD [93–95] computations of, at least, the first multiplet of quark-gluon hybrid mesons when the quark and antiquark are of the same heavy flavor. We shall take a reasonable model which has been already used for the study of similar hadronic transitions in the charmonium and bottomonium sectors [45,96,97]; moreover, such a model provides in Ref. [96] ground-state masses of hybrid charmonium and bottomoium mesons with quantum numbers $J^P = 1^{--}$. They are, respectively, 4.35 and 10.79 GeV, which can be compared with the latest lattice-QCD data of 4.41 [94] and 10.95 GeV [95].

One can see in Eq. (17) that the transition amplitude splits into two factors. The first one concerns the wave functions and energies of the initial and final quarkonium states as well as those of the intermediate hybrid mesons. All of these quantities can be calculated using suitable quark-gluon models. The second one describes the conversion of the emitted gluons into light hadrons. As the momenta involved are very low, this matrix element cannot be calculated using perturbative QCD and one needs to resort to a phenomenological approach based on soft-pion techniques [98]. In the center-of-mass frame, the two pion momenta q_1 and q_2 are the only independent variables describing this matrix element which, in the nonrelativistic limit, can be parametrized as [72,73,76,98]

$$\begin{aligned} &\frac{g_E^2}{6}\langle\pi_\alpha(q_1)\pi_\beta(q_2)|E_k^a E_l^a|0\rangle \\ &= \frac{\delta_{\alpha\beta}}{\sqrt{(2\omega_1)(2\omega_2)}} \\ &\quad\times\left[C_1\delta_{kl}q_1^\mu q_{2\mu} + C_2\left(q_{1k}q_{2l} + q_{1l}q_{2k} - \frac{2}{3}\delta_{kl}\vec{q}_1\cdot\vec{q}_2\right)\right], \end{aligned} \tag{18}$$

where C_1 and C_2 are two unknown constants, related to our ignorance about the mechanism of the conversion of the emitted gluons into light hadron(s). The C_1 term is isotropic, while the C_2 term has a $L = 2$ angular dependence. Thus, C_1 is involved in hadronic transitions where $\Delta l = l_f - l_i = 0$, while C_2 begins to participate when $\Delta l = 2$.

It is also important to mention here that the above parameters are considered theoretically as Wilson coefficients and thus they depend on the characteristic energy scale of the physical process. They have been fixed in our previous studies of hadronic transitions within the charmonium and bottomonium sectors [96,97] and, in order to gain predictive power, we use here the values corresponding to the bottomonium case.

Finally, the transition rate is given by

$$\Gamma(\Phi_I({}^{2s+1}l_{IJ_I}) \to \Phi_F({}^{2s+1}l_{FJ_F}) + \pi\pi) = \delta_{l_I l_F}\delta_{J_I J_F}\left(G|C_1|^2 - \frac{2}{3}H|C_2|^2\right)\left|\sum_L (2L+1)\begin{pmatrix} l_I & 1 & L \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L & 1 & l_I \\ 0 & 0 & 0 \end{pmatrix} f_{IF}^{L11}\right|^2$$
$$+ (2l_I+1)(2l_F+1)(2J_F+1)\sum_k (2k+1)(1+(-1)^k)\begin{Bmatrix} s & l_F & J_F \\ k & J_I & l_I \end{Bmatrix}^2 H|C_2|^2$$
$$\times \left|\sum_L (2L+1)\begin{pmatrix} l_F & 1 & L \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L & 1 & l_I \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} l_I & L & 1 \\ 1 & k & l_F \end{Bmatrix} f_{IF}^{L11}\right|^2, \quad (19)$$

with

$$f_{IF}^{LP_IP_F} = \sum_K \frac{1}{M_I - M_{KL}}\left[\int dr r^{2+P_F} R_F(r) R_{KL}(r)\right]\left[\int dr' r'^{2+P_I} R_{KL}(r') R_I(r')\right]. \quad (20)$$

$R_{KL}(r)$ is the radial wave function of the intermediate quark-gluon states, whereas $R_I(r)$ and $R_F(r)$ are the radial wave functions of the initial and final states, respectively. The mass of the decaying meson is M_I, whereas the ones corresponding to the hybrid states are M_{KL}. The quantities G and H are phase-space integrals

$$G = \frac{3}{4}\frac{M_F}{M_I}\pi^3 \int dM_{\pi\pi}^2 k \left(1 - \frac{4m_\pi^2}{M_{\pi\pi}^2}\right)^{1/2} (M_{\pi\pi}^2 - 2m_\pi^2)^2,$$
$$H = \frac{1}{20}\frac{M_F}{M_I}\pi^3 \int dM_{\pi\pi}^2 k \left(1 - \frac{4m_\pi^2}{M_{\pi\pi}^2}\right)^{1/2} \times \left[(M_{\pi\pi}^2 - 4m_\pi^2)^2 \left(1 + \frac{2}{3}\frac{k^2}{M_{\pi\pi}^2}\right) + \frac{8k^4}{15M_{\pi\pi}^4}(M_{\pi\pi}^4 + 2m_\pi^2 M_{\pi\pi}^2 + 6m_\pi^4)\right], \quad (21)$$

with the momentum k given by

$$k = \frac{\sqrt{[(M_I + M_F)^2 - M_{\pi\pi}^2][(M_I - M_F)^2 - M_{\pi\pi}^2]}}{2M_I}. \quad (22)$$

The leading multipoles of spin-nonflip η transitions between spin-triplet S-wave states are M1-M1 and E1-M2. Therefore, the matrix element is given schematically by

$$\mathcal{M}({}^3S_1 \to {}^3S_1 + \eta) = \mathcal{M}_{M1M1} + \mathcal{M}_{E1M2}. \quad (23)$$

After some algebra and assuming that $\mathcal{M}_{M1M1} = 0$ (see Ref. [73] for details), the decay rate can be written as

$$\Gamma(\Phi_I({}^3S_1) \to \Phi_F({}^3S_1) + \eta) = \frac{8\pi^2}{27}\frac{M_f C_3^2}{M_i m_Q m_{Q'}} |f_{IF}^{111}|^2 |\vec{q}|^3, \quad (24)$$

where $\vec{q}$ is the momentum of η, the function f_{IF}^{111} is defined in Eq. (20), and C_3 is a new parameter.

2. Spin-flip $\pi\pi$ and η transitions

The spin-flip $\pi\pi$ and η transitions between B_c mesons are induced by an E1-M1 multipole amplitude. Within the hadronization approach presented above, the description of this kind of decay implies the introduction of another phenomenological constant which should be fixed by experiment. Therefore, as one can deduce, the decay model for hadronic transitions begins to lose its predictive power.

In order to avoid this undesirable feature, the term which describes the conversion of the emitted gluons into light hadrons can be computed assuming a duality argument between the physical light hadron final state and the associated two-gluon final state [73]:

$$\Gamma(\Phi_I \to \Phi_F + \pi\pi) \sim \Gamma(\Phi_I \to \Phi_F gg),$$
$$\Gamma(\Phi_I \to \Phi_F + \eta) \sim \Gamma(\Phi_I \to \Phi_F (gg)_{0^-}), \quad (25)$$

where in the second line the two gluons are projected into a $J^P = 0^-$ state to simulate the η meson. The advantage of this approach is that we have now only two free parameters, g_E and g_M, in order to fix the spin-nonflip and spin-flip $\pi\pi$- and η-hadronic transitions. The values used herein are those reported in Ref. [97].

Explicit expressions within this new approach of the decay rates for the spin-nonflip $\pi\pi$ and η transitions can be found in Refs. [73,76]. The decay rates for the spin-flip $\pi\pi$ and η transitions are

$$\Gamma(\Phi_I(^3l_{IJ_I}) \to \Phi_F(^1l_{FJ_F}) + \pi\pi) = \frac{g_E^2 g_M^2}{36 m_Q m_{Q'}} \frac{(M_I - M_F)^7}{315\pi^3}(2l_F + 1)\begin{pmatrix} l_F & 1 & l_I \\ 0 & 0 & 0 \end{pmatrix}^2 |f_{IF}^{l_F 10} + f_{IF}^{l_I 01}|^2,$$
$$\Gamma(\Phi_I(^3S_{J_I}) \to \Phi_F(^1P_{J_F}) + \eta) = \frac{g_M^2}{g_E^2}\frac{E_F}{M_I}|\vec{q}|\frac{\pi}{1144 m_Q m_{Q'}}\left(\frac{4\pi}{\sqrt{6}} f_\pi m_\eta^2\right)^2 |f_{IF}^{110} + f_{IF}^{001}|^2. \tag{26}$$

The decay rate of the spin-flip η transition in Eq. (26) can be read from the decay rate of the isospin violating hadronic transition [76]

$$\Gamma(\Phi_I(^3S_{J_I}) \to \Phi_F(^1P_{J_F}) + \pi^0) = \frac{g_M^2}{g_E^2}\frac{E_F}{M_I}|\vec{q}| \times \frac{\pi}{1144 m_Q m_{Q'}}\left(\frac{4\pi}{\sqrt{2}}\frac{m_d - m_u}{m_d + m_u} f_\pi m_\pi^2\right)^2 \times |f_{IF}^{110} + f_{IF}^{001}|^2, \tag{27}$$

in which the factor $(m_d - m_u)/(m_d + m_u) \approx 0.35$ reflects the violation of isospin.

3. A model for hybrid mesons

One might expect to have bound states in which the gluon field itself is excited and carries J^{PC} quantum numbers. Quantum chromodynamics does not forbid this and, in fact, it should be expected from its general properties. The gluonic quantum numbers couple to those of the quark-antiquark pair, giving rise to the so-called exotic J^{PC} mesons, but also can produce hybrid mesons with natural quantum numbers. We are interested in the last ones because they are involved in the calculation of hadronic transitions within the QCDME approach.

An extension of the nonrelativistic constituent quark model described above to include hybrid states was presented in [96] (see also Refs. [45,97]). This extension is inspired on the Buchmuller-Tye quark-confining string model [99–101] in which the meson is composed of a quark and antiquark linked by an appropriate color electric flux line (the string).

The string can carry energy momentum only in the region between the quark and the antiquark. The string and the quark-antiquark pair can rotate as a unit and also vibrate. Ignoring its vibrational motion, the equation which describes the dynamics of the quark-antiquark pair linked by the string should be the usual Schrödinger equation with a confinement potential. Gluon excitation effects are described by the vibration of the string. These vibrational modes provide new states beyond the naive meson picture.

The coupled equations that describe the dynamics of the string and the quark sectors are very nonlinear so that there is no hope of solving them completely. Using the Bohr-Sommerfeld quantization, the vibrational potential energy can be estimated as a function of the interquark distance and then, via the Bohr-Oppenheimer method, these vibrational energies are inserted into the meson equation as an effective potential, $V_n(r)$.

Therefore, the potential for hybrid mesons derived from our nonrelativistic constituent quark model has the following expression:

$$V_{\mathrm{hyb}}(r) = V_{\mathrm{OGE}}^{\mathrm{C}}(r) + V_{\mathrm{CON}}^{\mathrm{C}}(r) + [V_n(r) - \sigma(r)r], \tag{28}$$

where $V_{\mathrm{OGE}}^{\mathrm{C}}(r) + V_{\mathrm{CON}}^{\mathrm{C}}(r)$ would be the naive quark-antiquark potential, $V_n(r)$ the vibrational one, and the definition of $\sigma(r)$ is

$$\sigma(r) = \frac{16}{3} a_c \left(\frac{1 - e^{-\mu_c r}}{r}\right). \tag{29}$$

We must subtract the term $\sigma(r)r$ because it appears twice, once in $V_{\mathrm{CON}}^{\mathrm{C}}(r)$ and the other one in $V_n(r)$. This potential does not include new model parameters and depends only on those coming from the original quark model. In this sense, the calculation of the hybrid states is parameter free. More explicitly, our different contributions are

$$V_{\mathrm{OGE}}^{\mathrm{C}}(r) = -\frac{4\alpha_s}{3r},$$
$$V_{\mathrm{CON}}^{\mathrm{C}}(r) = \frac{16}{3}[a_c(1 - e^{-\mu_c r}) - \Delta],$$
$$V_n(r) = \sigma(r)r\left\{1 + \frac{2n\pi}{\sigma(r)[(r - 2d)^2 + 4d^2]}\right\}^{1/2}, \tag{30}$$

where the vibrational potential energy can be estimated using the Bohr-Sommerfeld quantization and assuming the quark mass to be very heavy so that the ends of the string are fixed [100]. In order to relax the last assumption one can define a parameter d given by

$$d(m_Q, r, \sigma, n) = \frac{\sigma(r) r^2 \alpha_n}{4(m_Q + m_{Q'} + \sigma(r) r \alpha_n)}, \tag{31}$$

in which α_n relates to the shape of the vibrating string [100], and can take the values $1 \le \alpha_n^2 \le 2$.

An important feature of our hybrid model is that, just like the naive quark model, the hybrid potential has a threshold from which no more states can be found and so we have a finite number of hybrid states in the spectrum. Hybrid meson masses calculated in the B_c sector using our model are shown in Table II.

TABLE II. Hybrid meson masses, in MeV, calculated in the $c\bar{b}$ sector. The variations of the parameter α_n which range between $1 < \alpha_n < \sqrt{2}$ modifies the energy as much as 30 MeV; we have taken $\alpha_n = \sqrt{1.5}$.

K	$L = 0$	$L = 1$	$L = 2$
1	7328	7567	7733
2	7667	7828	7956
3	7910	8034	8136
4	8102	8199	8281
5	8255	8333	8399
6	8378	8441	8493
7	8477	8525	8566
8	8553	8588	…
Threshold = 8595 MeV			

III. RESULTS

Table III shows the predicted masses of the low-lying B_c states which are expected to be either below or around BD threshold (7144–7149 MeV) [102]. One can see that there are two S-wave multiplets with spin-parity 0^- and 1^-; another two P-wave multiplets with quantum numbers $J^P = 0^+$, 1^+, and 2^+; one D-wave multiplet with $J^P = 1^-$, 2^-, and 3^-; and one F-wave multiplet with $J^P = 2^+$ very close to the BD threshold. The proliferation of states in the spin-parity channels 1^+ and 2^- is due to the coupling of the $S = 0$ and $S = 1$ channels given by the antisymmetric spin-orbit term of the quark-antiquark potential.

We compare our results with the scarce experimental data collected by the PDG [102]. These experimental results only cover the lowest-lying states of the $J^P = 0^-$ sector. To compare other sectors we included recent lattice QCD studies, such as the quenched $2 + 1$ [29] and the $2 + 1 + 1$ flavors [31] calculations of the HPQCD Collaboration and the $2 + 1 + 1$ flavors analysis of Ref. [30]. An overall good agreement with the available lattice/experimental data for the B_c spectra below the lowest BD threshold is obtained. Finally, our predicted masses are also compared with those obtained by a significant sample of phenomenological models [7,12,15]. Within the expected theoretical accuracy, the different models are in remarkable agreement for the most part of the spectrum. The spin-dependent splittings are also in reasonable agreement; the only significant difference is the larger spread ($\approx$70 MeV) for the $1D$ multiplet center of gravity predictions. Potential models can therefore be used as a reliable guide in searching for the B_c excited states.

Above the aforementioned BD threshold, coupled-channels effects may appear. The influence of coupling bare $c\bar{b}$ states with open channels depends on the relative position of the $c\bar{b}$ mass and the open threshold. When the value of the threshold energy E is greater than the $q\bar{q}$ mass M, the effective potential is repulsive and it is unlikely that the coupling can generate a bound state rather than a dressing effect of the bare state. However, if $M > E$ the potential becomes negative, and an extra bound state with a large molecular probability may appear. All this is explained in

TABLE III. Predicted masses, in MeV, of the B_c states which are expected to be either below or around the BD threshold. All spin and orbital partial waves compatible with total spin and parity quantum numbers are considered in the coupled-channels Schrödinger equation and, thus, the fourth column indicates the dominant channel. We compare with available experimental data [102], recent lattice QCD studies [30,31], and some other model predictions [7,12,15].

State	J^P	n	$^{2S+1}L_J$	Theory	Experiment [102]	Reference [30]	Reference [31]	Reference [12]	Reference [15]	Reference [7]
B_c	0^-	1	$^1S_0$	6277	6274.47 ± 0.32	6276 ± 7	6278 ± 9	6271	6270	6264
		2	$^1S_0$	6868	6871.2 ± 1.0	⋯	6894 ± 21	6855	6835	6856
B_{c0}^*	0^+	1	$^3P_0$	6689	⋯	6712 ± 19	6707 ± 16	6706	6699	6700
		2	$^3P_0$	7109	⋯	⋯	⋯	7122	7091	7108
B_c^*	1^-	1	$^3S_1$	6328	⋯	6331 ± 7	6332 ± 9	6338	6332	6337
		2	$^3S_1$	6898	⋯	⋯	6922 ± 21	6887	6881	6899
		3	$^3D_1$	6999	⋯	⋯	⋯	7028	7072	7012
B_{c1}	1^+	1	$^3P_1$	6723	⋯	6736 ± 18	6742 ± 16	6741	6734	6730
		2	$^1P_1$	6731	⋯	⋯	⋯	6750	6749	6736
		3	$^3P_1$	7135	⋯	⋯	⋯	7145	7126	7135
		4	$^1P_1$	7142	⋯	⋯	⋯	7150	7145	7142
B_{c2}	2^-	1	$^1D_2$	7002	⋯	⋯	⋯	7036	7079	7009
		2	$^3D_2$	7011	⋯	⋯	⋯	7041	7077	7012
B_{c2}^*	2^+	1	$^3P_2$	6742	⋯	⋯	⋯	6768	6762	6747
		2	$^3P_2$	7151	⋯	⋯	⋯	7164	7156	7153
B_{c3}^*	3^-	1	$^3D_3$	7009	⋯	⋯	⋯	7045	7081	7005
BD threshold = 7144–7149 MeV [102]										

TABLE IV. The radiative E1 electromagnetic transitions for dominant S-wave states. We compare with some other model predictions [7,15,17].

Initial state	Final state	Γ_{The} (keV)	Reference [17]	Reference [15]	Reference [7]
$B_c^*(2S)$	$\gamma B_{c0}^*(1P)$	8.8	2.9	3.78	7.8
	$\gamma B_{c1}(1P)$	20	4.7	5.05	14.5
	$\gamma B_{c1}(2P)$	1.5×10^{-3}	0.7	0.63	0.0
	$\gamma B_{c2}^*(1P)$	29	5.7	5.18	17.7
$B_c(2S)$	$\gamma B_{c1}(1P)$	4.8×10^{-3}	1.3	1.02	0.0
	$\gamma B_{c1}(2P)$	35	6.1	3.72	5.2

Ref. [38] where an example of the influence of two-meson thresholds on the B_c states in the $J^P = 0^+$, 1^+, and 2^+ channels is shown.

Our predictions for the radiative E1 electromagnetic transitions for dominant S-wave states are shown in Table IV. Since the $B_c^{(*)}(2S)$ states have been already seen by the ATLAS [4] and LHCb [5] experiments at CERN, they can be the gateway for the exploration of first and second P-wave multiplets. In fact, the $B_c^*(2S)$ state has partial widths ranging from a few keV to tens of keV, and the $B_c(2S)$ has a decay rate of 35 keV. We compare our results with those from some other model predictions [7,15,17]; in general, ours are larger than those of Refs. [15,17] and of the same order of magnitude as the ones collected in Ref. [7]. The differences are associated with both quark model assumptions and solving, or not, a coupled-channels Schrödinger equation, because the work reported in Ref. [7] is closer to ours. As one can see, the decay rates are sensible to the mixing between different partial waves in a given wave function and such mixing is

TABLE V. The radiative E1 electromagnetic transitions for dominant P-wave states. We compare with some other model predictions [7,15,17].

Initial state	Final state	Γ_{The} (keV)	Reference [17]	Reference [15]	Reference [7]
$B_{c0}^*(1P)$	$\gamma B_c^*(1S)$	119	55	67.2	79.2
$B_{c0}^*(2P)$	$\gamma B_c^*(1S)$	28	1	⋯	21.9
	$\gamma B_c^*(2S)$	77	42	29.2	41.2
	$\gamma B_c^*(1D)$	17	4.2	0.036	6.9
$B_{c1}(1P)$	$\gamma B_c(1S)$	1.5×10^{-4}	13	18.4	0.0
	$\gamma B_c^*(1S)$	146	60	78.9	99.5
$B_{c1}(2P)$	$\gamma B_c(1S)$	173	80	132	56.4
	$\gamma B_c^*(1S)$	1.8×10^{-3}	11	13.6	0.1
$B_{c1}(3P)$	$\gamma B_c(1S)$	1.4	⋯	⋯	⋯
	$\gamma B_c(2S)$	2.4	⋯	⋯	⋯
	$\gamma B_c^*(1S)$	50	⋯	⋯	⋯
	$\gamma B_c^*(2S)$	88	⋯	⋯	⋯
	$\gamma B_c^*(1D)$	6.4	⋯	⋯	⋯
	$\gamma B_{c2}(1D)$	14	⋯	⋯	⋯
	$\gamma B_{c2}(2D)$	5.1	⋯	⋯	⋯
$B_{c1}(4P)$	$\gamma B_c(1S)$	79	⋯	⋯	⋯
	$\gamma B_c(2S)$	101	⋯	⋯	⋯
	$\gamma B_c^*(1S)$	1.5	⋯	⋯	⋯
	$\gamma B_c^*(2S)$	1.9	⋯	⋯	⋯
	$\gamma B_c^*(1D)$	0.14	⋯	⋯	⋯
	$\gamma B_{c2}(1D)$	10	⋯	⋯	⋯
	$\gamma B_{c2}(2D)$	15	⋯	⋯	⋯
$B_{c2}^*(1P)$	$\gamma B_c^*(1S)$	156	83	107	112.6
$B_{c2}^*(2P)$	$\gamma B_c^*(1S)$	67	14	⋯	25.8
	$\gamma B_c^*(2S)$	96	55	57.3	73.8
	$\gamma B_c^*(1D)$	0.27	0.1	0.035	0.2
	$\gamma B_{c2}(1D)$	2.4	0.7	0.113	⋯
	$\gamma B_{c2}(2D)$	1.9	0.6	0.269	3.2
	$\gamma B_{c3}^*(1D)$	24	6.8	1.59	17.8

TABLE VI. The radiative E1 electromagnetic transitions for dominant D-wave states. We compare with some other model predictions [7,15,17].

Initial state	Final state	$\Gamma_{\rm The}$ (keV)	Reference [17]	Reference [15]	Reference [7]
$B_c^*(1D)$	$\gamma B_{c0}^*(1P)$	98	55	128	88.6
	$\gamma B_{c1}(1P)$	64	28	73.8	49.3
	$\gamma B_{c1}(2P)$	1.3×10^{-3}	4.4	7.66	0.0
	$\gamma B_{c2}^*(1P)$	3.7	1.4	5.52	2.7
$B_{c2}(1D)$	$\gamma B_{c1}(1P)$	58	7	7.25	$\cdots$
	$\gamma B_{c1}(2P)$	71	63	116	92.5
	$\gamma B_{c2}^*(1P)$	18	8.8	12.8	$\cdots$
$B_{c2}(2D)$	$\gamma B_{c1}(1P)$	62	64	112	88.8
	$\gamma B_{c1}(2P)$	84	15	14.1	0.1
	$\gamma B_{c2}^*(1P)$	19	9.6	27.5	24.7
$B_{c3}(1D)$	$\gamma B_{c2}^*(1P)$	149	78	102	98.7

completely fixed in our computation through the tensor and the antisymmetric spin-orbit potentials, which are solved nonperturbatively through their exact treatment in the Schrödinger equation.

Table V compares our results on the radiative E1 electromagnetic transitions for dominant P-wave states with those of Refs. [7,15,17]. One can observe that the differences between models are less cumbersome, although they still exist; and, again, our results are in better agreement with those of Ref. [7]. Table V also shows that there are radiative E1 electromagnetic transitions from P-wave to S-wave states that have rates of the order of tens to hundreds keV. Some remarkable examples are the reactions in which the P-wave states decay to $B_c(nS)$ and $B_c^*(nS)$ mesons, and thus making these transitions the most feasible ones to be explored by experiments in the near future.

The radiative E1 electromagnetic transitions for dominant D-wave states are collected in Table VI. Again, we compare with some other model predictions [7,15,17]. Our results are mostly in accordance with those reported by Ref. [7] and are similar, with some discrepancies, with the results of the remaining references [15,17]. Table VI shows that D-wave states are also feasible to measure performing energy scans around their predicted masses when looking at their electromagnetic decay into P-wave states, whose masses are almost equally predicted in any theoretical framework mentioned in the Table III.

We collect in Table VII our predictions for the radiative M1 electromagnetic transitions and compare them with the results of Refs. [7,15,17]. Let us give some comments on these results: First, these decay rates are very small, ranging from hundreds to tenths of eV, and even smaller in some cases. Second, the largest rates are found for the radiative M1 electromagnetic transitions between S-wave states; although the $B_{c2}^*(2P)\to\gamma B_{c1}(1P)$ and $B_{c2}^*(2P)\to\gamma B_{c1}(2P)$ decays have sizeable widths. And third, the theoretical predictions are scarce but, when it is possible to

TABLE VII. The radiative M1 electromagnetic transitions. We compare with some other model predictions [7,15,17].

Initial state	Final state	$\Gamma_{\rm The}$ (eV)	Reference [17]	Reference [15]	Reference [7]
$B_c^*(1S)$	$\gamma B_c(1S)$	52	80	33	154.5
$B_c^*(2S)$	$\gamma B_c(1S)$	650	600	428	123.4
	$\gamma B_c(2S)$	10	10	17	28.9
$B_c(2S)$	$\gamma B_c^*(1S)$	250	300	488	93.3
$B_{c2}^*(1P)$	$\gamma B_{c1}(1P)$	0.65	$\cdots$	$\cdots$	$\cdots$
	$\gamma B_{c1}(2P)$	0.27	$\cdots$	$\cdots$	$\cdots$
$B_{c2}^*(2P)$	$\gamma B_{c1}(1P)$	40	$\cdots$	$\cdots$	$\cdots$
	$\gamma B_{c1}(2P)$	51	$\cdots$	$\cdots$	$\cdots$
	$\gamma B_{c1}(3P)$	0.24	$\cdots$	$\cdots$	$\cdots$
	$\gamma B_{c1}(4P)$	0.18	$\cdots$	$\cdots$	$\cdots$
$B_{c2}(1D)$	$\gamma B_c^*(1D)$	2.1×10^{-6}	$\cdots$	$\cdots$	$\cdots$
$B_{c2}(2D)$	$\gamma B_c^*(1D)$	0.52	$\cdots$	$\cdots$	$\cdots$
	$\gamma B_{c3}^*(1D)$	1.3×10^{-4}	$\cdots$	$\cdots$	$\cdots$
$B_{c3}^*(1D)$	$\gamma B_{c2}(1D)$	0.20	$\cdots$	$\cdots$	$\cdots$

TABLE VIII. Decay rates, in keV, of the spin-nonflip $\pi\pi$ hadronic transitions between spin triplets and between spin singlets. When possible, we compare with Ref. [17].

Initial state	Final state	Γ_{The} (keV)	Reference [17]
2^1S_0	$\pi\pi + 1^1S_0$	42	57
2^3S_1	$\pi\pi + 1^3S_1$	41	57
2^3P_0	$\pi\pi + 1^3P_0$	12	0.97
	$\pi\pi + 1^3P_1$	0	0
	$\pi\pi + 1^3P_2$	5.5×10^{-3}	5.5×10^{-2}
2^3P_1	$\pi\pi + 1^3P_0$	0	0
	$\pi\pi + 1^3P_1$	11	2.7
	$\pi\pi + 1^3P_2$	1.2×10^{-2}	3.7×10^{-2}
2^1P_1	$\pi\pi + 1^1P_1$	11	1.2
2^3P_2	$\pi\pi + 1^3P_0$	1.8×10^{-2}	1.1×10^{-2}
	$\pi\pi + 1^3P_1$	2.0×10^{-2}	2.1×10^{-2}
	$\pi\pi + 1^3P_2$	11	1.0
1^3D_1	$\pi\pi + 1^3S_1$	0.75	4.3
1^1D_2	$\pi\pi + 1^1S_0$	1.1	2.2
1^3D_2	$\pi\pi + 1^3S_1$	0.87	2.2
1^3D_3	$\pi\pi + 1^3S_1$	0.84	4.3

compare, our calculation is in reasonable agreement with those of Refs. [7,15,17].

Let us now turn our attention to some, but most relevant, hadronic transitions between B_c mesons. Table VIII shows our prediction for the decay rates of the spin-nonflip $\pi\pi$ hadronic transitions between spin triplets, and between spin singlets. We compare our results with those reported in Ref. [17]. In most cases we predict the same order of magnitude, but the diversity of the results makes it difficult to provide general statements. We can mention that the $2^1S_0 \to \pi\pi + 1^1S_0$ and $2^3S_1 \to \pi\pi + 1^3S_1$ decay rates reported in Ref. [17] have been fitted following some experimental guidance, whereas they are predictions in our case. In general, our values are larger for spin-nonflip $\pi\pi$ hadronic transitions between P-wave states, except in those cases in which the decay width is very small and we predict similar figures. The transitions between D-wave states and S-wave ones are small due to the only contribution of C_2 term in the formulas and our values are slightly smaller than those collected in Ref. [17].

From an experimental point of view, independently of the discrepancies between the two theoretical estimations, the $2^1S_0 \to \pi\pi + 1^1S_0$ and $2^3S_1 \to \pi\pi + 1^3S_1$ transitions have decay rates of about 50 keV and thus they are potentially observable in experiments. This is in fact the case; however, there is still a lack of statistics which avoids a quantitative study and even to discern if the initial state is either 2^1S_0 or 2^3S_1. We find that the $2^3P_J \to \pi\pi + 1^3P_J$ transitions have decay rates of the order of 10 keV, making them potentially detectable in experiments. Note that Ref. [17] predicts an order of magnitude smaller, but what is clear is that transitions $2^3P_J \to \pi\pi + 1^3P_J'$ are very small, with no hope of measuring. And, finally, it seems impossible to explore the D-wave states of the B_c system using as an experimental tool the spin-nonflip $\pi\pi$ hadronic transitions.

Table IX shows other relevant hadronic transitions between B_c states. Most of them are spin-flip $\pi\pi$ reactions because we are focusing our attention on the B_c mesons which lie below the lowest strong-decay BD threshold and thus there is not enough phase space to accommodate many light hadrons as part of the final state. As one can see, all decay rates are predicted to be very small with the largest ones being 5.5 and 2.7 keV for the $2^3P_0 \to \pi\pi + 1^1S_0$ and $2^3P_1 \to \pi\pi + 1^1S_0$ hadronic transition, respectively. It is worth mentioning herein that the isospin-violating transition $2^3S_1 \to \pi^0 + 1^1P_1$ has a decay of 0.48 keV, which is of the same order of magnitude as most of the widths collected in Table IX; this gives one an idea of the smallness of these decay rates. The theoretical computations of these decays are scarce and, if they exist, the way of computing the decay rates results is not very clear and thus we have decided to not collect them in Table IX.

Finally, it is worth mentioning here that our numerical values of hadronic transitions could be sensitive with respect to the spectrum of hybrid $c\bar{b}g$ mesons. If one focuses on Eq. (20), the contribution of each hybrid to the hadronic decay amplitude depends on the mass difference between the conventional hadron and the hybrid state, producing a smaller contribution when the difference in mass is greater. Since we have focused on computing the physical properties of B_c mesons that are below the lowest BD threshold, their masses are all well below the range of hybrid masses (see Table III versus Table II). Therefore, no singular effects are found in our perturbative calculation, all decay widths converge smoothly to their asymptotic value,

TABLE IX. Other relevant hadronic transitions between B_c states, most of them spin-flip $\pi\pi$ reactions. Decay rates are shown in keV.

Initial state	Final state	Γ_{The} (keV)
2^3S_1	$\eta + 1^3S_1$	0.20
	$\pi^0 + 1^1P_1$	0.48
1^3P_0	$\pi\pi + 1^1S_0$	0.58
2^3P_0	$\pi\pi + 1^1S_0$	5.5
	$\pi\pi + 2^1S_0$	7.4×10^{-2}
1^3P_1	$\pi\pi + 1^1S_0$	1.1
2^3P_1	$\pi\pi + 1^1S_0$	2.7
	$\pi\pi + 2^1S_0$	0.15
1^3P_2	$\pi\pi + 1^1S_0$	1.6
2^3P_2	$\pi\pi + 1^1S_0$	0.85
	$\pi\pi + 2^1S_0$	2.2×10^{-2}
1^3D_1	$\pi\pi + 1^1P_1$	0.13
1^3D_2	$\pi\pi + 1^1P_1$	0.17
1^3D_3	$\pi\pi + 1^1P_1$	0.16

and their sensitivity is quite limited as long as a reasonable hybrid spectrum is used.

IV. EPILOGUE

The properties of the B_c-meson family ($c\bar{b}$) are still not well determined experimentally because the specific mechanisms of formation and decay remain poorly understood. In this article, we have extended our previous investigation of the B_c spectrum to potentially interesting radiative decays and hadronic transitions between B_c states that lie below the lowest strong-decay BD threshold. It is expected that the decay rates of these kinds of reactions constitute the total decay width of such mesons and thus such processes can play an important role in the discovery and quantitative analysis of the B_c-meson family.

Our theoretical framework is a nonrelativistic constituent quark model in which quark-antiquark and meson-meson degrees of freedom can be incorporated at the same time. Below the BD threshold it is sufficient to work out the naive model which has been widely applied to the charmonium and bottomonium phenomenology, and one expects that it works reasonably well within the B_c sector. The formulas describing radiative E1 and M1 dominant multipole electromagnetic transitions have been used since the early days of heavy quarkonium spectroscopy; we have adapted it to the $c\bar{b}$ sector and our nonrelativistic constituent quark model approach. The calculation of the hadronic decay rates has been performed using the QCDME approach whose unknown constants parametrize the conversion of the emitted gluons into light hadron(s) and have been fitted in previous works. This formalism requires the computation of a hybrid meson spectrum. We have calculated the hybrid states using a natural, parameter-free extension of our quark model based on the quark confining string scheme.

Among the results we describe, the following are of particular interest:

(i) Below the lowest strong-decay BD threshold, there are two S-wave multiplets with spin-parity 0^- and 1^-; another two P-wave multiplets with quantum numbers $J^P = 0^+$, 1^+ and 2^+; and one D-wave multiplet with $J^P = 1^-$, 2^-, and 3^-. Moreover, compared with other theoretical approaches, the predicted spectra are very similar among each other.

(ii) The radiative E1 electromagnetic transitions between low-lying B_c states present decay rates which range from a few to hundreds of keV. Among the large variety of predictions, it is important to mention that all S-, P-, and D-wave states present some electromagnetic decay channels with large widths which would allow their observation, and even their quantitative analysis. Additionally, the radiative M1 electromagnetic transitions are characterized by very small decay rates, ranging from hundreds to tenths of eV; the largest rates are found for the $B^*_{c2}(2S) \to \gamma B_c(1S)$ and $B_c(2S) \to \gamma B^*_c(1S)$ reactions.

(iii) The predicted decay rates of the most relevant hadronic transitions indicate that the spin-nonflip $\pi\pi$ reactions are larger than those where a spin-flip exists. Furthermore, the spin-nonflip $\pi\pi$ hadronic transitions are around 50, 10, and 1 keV between S-, P-, and D-wave states, respectively, whereas most of the spin-flip $\pi\pi$ hadronic transitions are of the order of tenths of keV, similar to the case of isospin violating transition $1^3S_1 \to \pi^0 + 1^1P_1$.

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[1] F. Abe *et al.* (CDF Collaboration), Phys. Rev. Lett. **81**, 2432 (1998).
[2] F. Abe *et al.* (CDF Collaboration), Phys. Rev. D **58**, 112004 (1998).
[3] G. Aad *et al.* (ATLAS Collaboration), Phys. Rev. Lett. **113**, 212004 (2014).
[4] A. M. Sirunyan *et al.* (CMS Collaboration), Phys. Rev. Lett. **122**, 132001 (2019).
[5] R. Aaij *et al.* (LHCb Collaboration), Phys. Rev. Lett. **122**, 232001 (2019).
[6] W.-k. Kwong and J. L. Rosner, Phys. Rev. D **44**, 212 (1991).
[7] E. J. Eichten and C. Quigg, Phys. Rev. D **49**, 5845 (1994).
[8] E. J. Eichten and C. Quigg, Phys. Rev. D **99**, 054025 (2019).
[9] L. P. Fulcher, Phys. Rev. D **60**, 074006 (1999).
[10] Q. Li, M.-S. Liu, L.-S. Lu, Q.-F. Lü, L.-C. Gui, and X.-H. Zhong, Phys. Rev. D **99**, 096020 (2019).
[11] L. Tang, T.-y. Li, C.-h. Wang, C.-q. Pang, and X. Liu, arXiv:2204.14258.
[12] S. Godfrey and N. Isgur, Phys. Rev. D **32**, 189 (1985).
[13] J. Zeng, J. W. Van Orden, and W. Roberts, Phys. Rev. D **52**, 5229 (1995).

[14] S. N. Gupta and J. M. Johnson, Phys. Rev. D **53**, 312 (1996).
[15] D. Ebert, R. N. Faustov, and V. O. Galkin, Phys. Rev. D **67**, 014027 (2003).
[16] S. M. Ikhdair and R. Sever, Int. J. Mod. Phys. A **19**, 1771 (2004).
[17] S. Godfrey, Phys. Rev. D **70**, 054017 (2004).
[18] S. M. Ikhdair and R. Sever, Int. J. Mod. Phys. A **20**, 4035 (2005).
[19] S. M. Ikhdair and R. Sever, Int. J. Mod. Phys. A **20**, 6509 (2005).
[20] Z.-G. Wang, Eur. Phys. J. A **49**, 131 (2013).
[21] Z.-G. Wang, Eur. Phys. J. C **73**, 2559 (2013).
[22] L. Chang, M. Chen, X.-q. Li, Y.-x. Liu, and K. Raya, Few Body Syst. **62**, 4 (2021).
[23] M. Chen, L. Chang, and Y.-x. Liu, Phys. Rev. D **101**, 056002 (2020).
[24] P.-L. Yin, C. Chen, G. a. Krein, C. D. Roberts, J. Segovia, and S.-S. Xu, Phys. Rev. D **100**, 034008 (2019).
[25] N. Brambilla and A. Vairo, Phys. Rev. D **62**, 094019 (2000).
[26] A. A. Penin, A. Pineda, V. A. Smirnov, and M. Steinhauser, Phys. Lett. B **593**, 124 (2004); **677**, 343(E) (2009).
[27] C. Peset, A. Pineda, and J. Segovia, J. High Energy Phys. 09 (2018) 167.
[28] C. Peset, A. Pineda, and J. Segovia, Phys. Rev. D **98**, 094003 (2018).
[29] I. F. Allison, C. T. H. Davies, A. Gray, A. S. Kronfeld, P. B. Mackenzie, and J. N. Simone (HPQCD Fermilab Lattice, UKQCD Collaborations), Phys. Rev. Lett. **94**, 172001 (2005).
[30] N. Mathur, M. Padmanath, and S. Mondal, Phys. Rev. Lett. **121**, 202002 (2018).
[31] R. J. Dowdall, C. T. H. Davies, T. C. Hammant, and R. R. Horgan, Phys. Rev. D **86**, 094510 (2012).
[32] S. Okubo, Phys. Lett. **5**, 165 (1963).
[33] G. Zweig, Report No. CERN-TH-412, NP-8419, 1964.
[34] J. Iizuka, Prog. Theor. Phys. Suppl. **37**, 21 (1966).
[35] N. Brambilla *et al.*, Eur. Phys. J. C **71**, 1534 (2011).
[36] N. Brambilla *et al.*, Eur. Phys. J. C **74**, 2981 (2014).
[37] S. L. Olsen, Front. Phys. (Beijing) **10**, 121 (2015).
[38] P. G. Ortega, J. Segovia, D. R. Entem, and F. Fernandez, Eur. Phys. J. C **80**, 223 (2020).
[39] J. Vijande, F. Fernandez, and A. Valcarce, J. Phys. G **31**, 481 (2005).
[40] J. Segovia, D. R. Entem, F. Fernandez, and E. Hernandez, Int. J. Mod. Phys. E **22**, 1330026 (2013).
[41] P. G. Ortega, D. R. Entem, and F. Fernandez, J. Phys. G **40**, 065107 (2013).
[42] J. Segovia, A. M. Yasser, D. R. Entem, and F. Fernandez, Phys. Rev. D **78**, 114033 (2008).
[43] J. Segovia, D. R. Entem, and F. Fernandez, J. Phys. G **37**, 075010 (2010).
[44] J. Segovia, D. R. Entem, and F. Fernandez, Phys. Rev. D **91**, 094020 (2015).
[45] J. Segovia, P. G. Ortega, D. R. Entem, and F. Fernández, Phys. Rev. D **93**, 074027 (2016).
[46] G. Yang, J. Ping, P. G. Ortega, and J. Segovia, Chin. Phys. C **44**, 023102 (2020).
[47] J. Segovia, A. M. Yasser, D. R. Entem, and F. Fernandez, Phys. Rev. D **80**, 054017 (2009).
[48] E. van Beveren, G. Rupp, and J. Segovia, Phys. Rev. Lett. **105**, 102001 (2010).
[49] J. Segovia, D. R. Entem, and F. Fernandez, Phys. Rev. D **83**, 114018 (2011).
[50] J. Segovia, D. R. Entem, and F. Fernández, Phys. Lett. B **715**, 322 (2012).
[51] J. Segovia, D. R. Entem, and F. Fernandez, Nucl. Phys. **A915**, 125 (2013).
[52] P. G. Ortega, J. Segovia, D. R. Entem, and F. Fernandez, Phys. Rev. D **81**, 054023 (2010).
[53] P. G. Ortega, J. Segovia, D. R. Entem, and F. Fernández, Phys. Rev. D **94**, 114018 (2016).
[54] P. G. Ortega, J. Segovia, D. R. Entem, and F. Fernández, Phys. Lett. B **778**, 1 (2018).
[55] P. G. Ortega, J. Segovia, D. R. Entem, and F. Fernández, Eur. Phys. J. C **79**, 78 (2019).
[56] P. G. Ortega, J. Segovia, and F. Fernandez, Phys. Rev. D **104**, 094004 (2021).
[57] G. Yang and J. Ping, Phys. Rev. D **95**, 014010 (2017).
[58] G. Yang, J. Ping, and J. Segovia, Phys. Rev. D **99**, 014035 (2019).
[59] G. Yang, J. Ping, and J. Segovia, Phys. Rev. D **102**, 054023 (2020).
[60] G. Yang, J. Ping, and J. Segovia, Phys. Rev. D **101**, 074030 (2020).
[61] G. Yang, J. Ping, and J. Segovia, Phys. Rev. D **103**, 074011 (2021).
[62] J. Segovia, S. Steinbeißer, and A. Vairo, Phys. Rev. D **99**, 074011 (2019).
[63] A. Pineda and J. Segovia, Phys. Rev. D **87**, 074024 (2013).
[64] J. J. Dudek, R. G. Edwards, and D. G. Richards, Phys. Rev. D **73**, 074507 (2006).
[65] D. Becirevic and F. Sanfilippo, J. High Energy Phys. 01 (2013) 028.
[66] K. Gottfried, Phys. Rev. Lett. **40**, 598 (1978).
[67] G. Bhanot, W. Fischler, and S. Rudaz, Nucl. Phys. **B155**, 208 (1979).
[68] M. E. Peskin, Nucl. Phys. **B156**, 365 (1979).
[69] G. Bhanot and M. E. Peskin, Nucl. Phys. **B156**, 391 (1979).
[70] M. Voloshin, Nucl. Phys. **B154**, 365 (1979).
[71] M. B. Voloshin and V. I. Zakharov, Phys. Rev. Lett. **45**, 688 (1980).
[72] T.-M. Yan, Phys. Rev. D **22**, 1652 (1980).
[73] Y.-P. Kuang and T.-M. Yan, Phys. Rev. D **24**, 2874 (1981).
[74] A. Pineda and J. Tarrús Castellà, Phys. Rev. D **100**, 054021 (2019).
[75] J. Tarrús Castellà and E. Passemar, Phys. Rev. D **104**, 034019 (2021).
[76] Y.-P. Kuang, Front. Phys. China **1**, 19 (2006).
[77] J. Segovia, D. R. Entem, and F. Fernandez, Phys. Lett. B **662**, 33 (2008).
[78] G. S. Bali, Phys. Rep. **343**, 1 (2001).
[79] G. S. Bali, K. Schilling, and A. Wachter, Phys. Rev. D **56**, 2566 (1997).
[80] C. E. Detar, O. Kaczmarek, F. Karsch, and E. Laermann, Phys. Rev. D **59**, 031501 (1999).

[81] P. de Forcrand and O. Philipsen, Phys. Lett. B **475**, 280 (2000).
[82] G. S. Bali, H. Neff, T. Duessel, T. Lippert, and K. Schilling (SESAM Collaboration), Phys. Rev. D **71**, 114513 (2005).
[83] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T.-M. Yan, Phys. Rev. D **17**, 3090 (1978); **21**, 313(E) (1980).
[84] B.-Q. Li, C. Meng, and K.-T. Chao, Phys. Rev. D **80**, 014012 (2009).
[85] E. S. Swanson, J. Phys. G **31**, 845 (2005).
[86] M. Brisudová, L. Burakovsky, and T. Goldman, Phys. Lett. B **460**, 1 (1999).
[87] C.-Y. Wong, Phys. Rev. C **69**, 055202 (2004).
[88] J. Vijande, P. Gonzalez, H. Garcilazo, and A. Valcarce, Phys. Rev. D **69**, 074019 (2004).
[89] E. Hiyama, Y. Kino, and M. Kamimura, Prog. Part. Nucl. Phys. **51**, 223 (2003).
[90] M. Berwein, N. Brambilla, J. Tarrús Castellà, and A. Vairo, Phys. Rev. D **92**, 114019 (2015).
[91] N. Brambilla, W. K. Lai, J. Segovia, J. Tarrús Castellà, and A. Vairo, Phys. Rev. D **99**, 014017 (2019); **101**, 099902(E) (2020).
[92] N. Brambilla, W. K. Lai, J. Segovia, and J. Tarrús Castellà, Phys. Rev. D **101**, 054040 (2020).
[93] L. Liu, G. Moir, M. Peardon, S. M. Ryan, C. E. Thomas, P. Vilaseca, J. J. Dudek, R. G. Edwards, B. Joo, and D. G. Richards (Hadron Spectrum Collaboration), J. High Energy Phys. 07 (2012) 126.
[94] G. K. C. Cheung, C. O'Hara, G. Moir, M. Peardon, S. M. Ryan, C. E. Thomas, and D. Tims (Hadron Spectrum Collaboration), J. High Energy Phys. 12 (2016) 089.
[95] S. M. Ryan and D. J. Wilson (Hadron Spectrum), J. High Energy Phys. 02 (2021) 214.
[96] J. Segovia, D. R. Entem, and F. Fernández, Phys. Rev. D **91**, 014002 (2015).
[97] J. Segovia, F. Fernandez, and D. R. Entem, Few Body Syst. **57**, 275 (2016).
[98] L. S. Brown and R. N. Cahn, Phys. Rev. Lett. **35**, 1 (1975).
[99] S. H. H. Tye, Phys. Rev. D **13**, 3416 (1976).
[100] R. Giles and S. H. H. Tye, Phys. Rev. D **16**, 1079 (1977).
[101] W. Buchmuller and S. H. H. Tye, Phys. Rev. Lett. **44**, 850 (1980).
[102] P. A. Zyla *et al.* (Particle Data Group), Prog. Theor. Exp. Phys. **2020**, 083C01 (2020).