Polyakov loop, gluon mass, gluon condensate, and its asymmetry near deconfinement

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We consider a BRST-invariant generalization of the "massive background Landau gauge," resembling the original Curci-Ferrari model that saw a revived interest due to its phenomenological success in modeling infrared Yang-Mills dynamics, including that of the phase transition. Unlike the Curci-Ferrari model, however, the mass parameter is no longer a phenomenological input, but it enters as a result of dimensional transmutation via a BRST-invariant dimension-2 gluon condensate. The associated renormalization constant is dealt with using Zimmermann's reduction of constants program, which fixes the value of the mass parameter to values close to those obtained within the Curci-Ferrari approach. Using a self-consistent background field, we can include the Polyakov loop and probe the deconfinement transition, including its interplay with the condensate and its electric–magnetic asymmetry. We report a continuous phase transition at $T_c \approx 0.230$ GeV in the SU(2) case and a first-order one at $T_c \approx 0.164$ GeV in the SU(3) case, values which are again rather close to those obtained within the Curci-Ferrari model at one-loop order.

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I. INTRODUCTION

It is well accepted from nonperturbative Monte Carlo lattice simulations that $SU(N)$ Yang-Mills gauge theories in the absence of fundamental matter fields undergo a deconfining phase transition at a certain critical temperature [1,2]. This transition corresponds to the breaking of a global $\mathbb{Z}_N$ center symmetry when the Euclidean temporal direction is compactified on a circle, with circumference proportional to the inverse temperature; see e.g., Refs. [3,4]. The vacuum expectation value of the Polyakov loop [5] serves as an order parameter for this symmetry and has as such inspired an ongoing research activity into its dynamics; see, e.g., Refs. [6–9]. Even in the presence of dynamical quark degrees of freedom, in which case the center symmetry is broken explicitly, the Polyakov loop remains the best observable to capture the crossover transition; see Refs. [10,11] for ruling lattice QCD estimates. Since the transition temperature is of the order of the scale at which the considered gauge theories, including QCD, become strongly coupled, it is a highly challenging endeavor to get reliable estimates for the Polyakov loop correlators, including its vacuum expectation value, analytically. This is further complicated by the nonlocal nature of the loop. These features highlight the sheer importance of lattice gauge theories to allow for a fully nonperturbative computational framework. Nonetheless, analytical takes are still desirable to offer a complementary view at the same physics, in particular as lattice simulations do also face difficulties when the physically relevant small quark mass limit must be taken, next to the issue of potentially catastrophic sign oscillations at finite density [12,13].

Over the last two decades, tremendous effort has been put into the development and application of functional methods to QCD, including the respective hierarchies of Dyson-Schwinger equations (DSE) and the functional renormalization group (FRG) equations [14–32] as well a variational approaches based on the Hamiltonian formulation or on N-particle-irreducible effective actions [33–39]. These methods are quite successful in describing vacuum properties of the theory as well as finite temperature/density aspects. They all rely, in one way or another, on the decoupling behavior of gluons in the Landau gauge, as dictated by results from lattice simulations [40–47]. More recently, a more phenomenological approach has been put forward based on the use

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of the Curci-Ferrari (CF) model [48–50]. The rationale behind the latter is that the standard Faddeev-Popov Landau gauge action, although well grounded in the ultraviolet, is incomplete in the infrared due to the presence of Gribov copies and therefore, needs to be extended. The hypothesis put forward in Ref. [48] and put on more rigorous footing in Ref. [51] (see also Ref. [52] and references therein) is that a dominant contribution to this (to date unknown) gauge-fixed action is provided by a gluon mass term, which relates to the decoupling behavior of the Landau gauge gluon propagator on the lattice. One of the attractive features of the Curci-Ferrari model is that it is perturbative in nature, at least in its applications to pure Yang-Mills (YM) theories. In fact, with just one additional parameter to adjust, it has allowed one to retrieve many of the Euclidean properties of these theories in the vacuum and at finite temperature [50]. In its applications to QCD, the perturbative nature of the pure glue sector allows one, in combination with an expansion in the inverse number of colors, to devise a systematic expansion scheme controlled by two small parameters and whose first orders are computionally tractable [53,54].

The surprising ability of the Curci-Ferrari model in reproducing well-known properties of pure YM theories has lead to the question of whether it could be derived (in its present form or with some amendments) from a proper account of the Gribov copies [55–57]. Here, we would like to investigate another possibility following the work of Ref. [58] and based on the dynamical generation of dimension-2 gluon condensates within the strict Faddeev-Popov setup. The idea here is that, upon generation of a dynamical gluon mass, the Gribov copies will be accounted for, at least partially. So, more than a consequence of taking into account the Gribov copies, the gluon mass will appear here as a self-generated cure for the Gribov problem within the Faddeev-Popov framework. In what follows, we would like to investigate these ideas, in particular how they allow one to describe salient features of YM theory such as the deconfinement transition. Since the Curci-Ferrari model has taught us that, once a mass is generated, certain features become accessible to perturbation theory, we shall consider a simple one-loop calculation as a start.

Because the decoupling behavior as observed on the lattice extends beyond the Landau gauge to linear covariant gauges [59,60], it will be important to make sure that the dynamical mass generation mechanism applies independently of the gauge-fixing parameter. Moreover, for the dynamically generated mass to carry a physical significance, it should be associated to a BRST-invariant gluon condensate. A central notion to achieve this is that of a BRST-invariant gluon field [61,62], which we discuss in Sec. II, together with its extension in the presence of a background gauge field, required for the study of the Polyakov loop. We also show that the BRST-invariant gluon field can be replaced by the original gluon field in the limit of a vanishing gauge-fixing parameter, which will later facilitate the computations. In Sec. III, we introduce the BRST-invariant dimension-2 gluon condensate, together with its BRST-invariant asymmetry at finite temperature. This asymmetry was proposed in past Landau gauge-fixed lattice QCD work [63] to constitute yet another probe of the deconfinement transition. More generally, we expect an interesting interplay between the condensate, and thus the mass, and the Polyakov loop at finite temperature. In Sec. IV, we evaluate the effective potential for the background field (related to the Polyakov loop), the BRST-invariant condensate (related to the mass), and the asymmetry in this BRST-invariant condensate. Our results for the three observables across the deconfinement transition are gathered in Sec. V together with a discussion relating to the Curci-Ferrari model.

II. BRST-INVARIANT GLUON FIELD A^h

To set the stage, we will first briefly introduce our construction at zero temperature and without background gauge fields, summarizing a larger paper in preparation [64] based on earlier work [58], before extending it in the presence of the Polyakov loop via the background field method.

A. Case of linear covariant gauges

We start from the Yang-Mills action in a linear covariant (LC) gauge and in d Euclidean space dimensions,

$$S_{\rm LC} = \int d^dx\left(\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{\alpha}{2}b^ab^a + ib^a\partial_\mu A^a_\mu + \bar{c}^a\partial_\mu D^{ab}_\mu c^b\right), \tag{1}$$

where c and $\bar{c}$ are the ghost and antighost fields, b is the Nakanishi-Lautrup field enforcing the gauge condition, and α is the gauge parameter. As we are eventually interested in the dimension-2 gluon condensate $\langle A^2_\mu\rangle$ while preserving BRST invariance, we need a BRST-invariant version of the A^a_μ field. To construct this, we insert into the corresponding path integral the unity [64,65]

$$1 = \mathcal{N}\int[\mathcal{D}\xi\mathcal{D}\tau\mathcal{D}\bar{\eta}\mathcal{D}\eta]e^{-S_h}, \tag{2a}$$

$$S_h = \int d^dx(i\tau^a\partial_\mu(A^h)^a_\mu + \bar{\eta}^a\partial_\mu(D^h)^{ab}_\mu\eta^b), \tag{2b}$$

where $\mathcal{N}$ is a normalization and $(D^h)^{ab}_\mu$ is the covariant derivative containing only the composite field $(A^h)^a_\mu$. This local but nonpolynomial composite field object is defined as

$$(A^h)_\mu = h^\dagger A_\mu h + \frac{i}{g}h^\dagger\partial_\mu h, \tag{2c}$$

$$h = e^{ig\xi} = e^{ig\xi^aT^a}, \tag{2d}$$

where the T^a are the generators of the gauge group $\mathrm{SU}(N)$. The ξ^a are similar to Stueckelberg fields, while η^a and $\bar{\eta}^a$ are additional (Grassmanian) ghost and antighost fields. They serve to account for the Jacobian arising from the functional integration over τ^a to give a Dirac delta functional of the type $\delta(\partial_\mu (A^h)^a_\mu)$. That Jacobian is similar to the one of the Faddeev-Popov operator and is supposed to be positive, which amounts to removing a large class of infinitesimal Gribov copies; see Ref. [66]. Here, positivity can be checked *a posteriori* by means of the ghost propagator, the (expectation value of the) inverse Faddeev-Popov operator. Notice that in mere perturbation theory, this is not the case, but the dynamical mass to be discussed will be large enough to ensure it dynamically, leading to a kind of "self-cured" Gribov ambiguity. This is a different approach than the Gribov-Zwanziger one, in which case positivity is imposed *a priori* [67,68], albeit with similar end result. In fact, this strategy of having a positive ghost propagator is also the one employed in, e.g., the functional Dyson-Schwinger approach [23].[1]

Expanding (2c), one finds an infinite series of local terms:

$$(A^h)^a_\mu = A^a_\mu - \partial_\mu \xi^a - g f^{abc} A^b_\mu \xi^c - \frac{g}{2} f^{abc} \xi^b \partial_\mu \xi^c + \cdots. \quad (3)$$

The unity (2a) can be used to stay within a local setup for an on-shell nonlocal quantity $(A^h)^a_\mu$ that can be added to the action. Notice that the multiplier τ^a implements $\partial_\mu (A^h)^a_\mu = 0$, which, when solved iteratively for ξ^a,

$$\xi_* = \frac{1}{\partial^2} \partial_\mu A_\mu + ig \frac{1}{\partial^2} \left[\partial_\mu A_\mu, \frac{1}{\partial^2} \partial_\nu A_\nu \right] + \cdots, \quad (4a)$$

gives the (transversal) on-shell expression

$$(A^h)_\mu = \left(\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) \left(A_\nu + ig \left[A_\nu, \frac{1}{\partial^2} \partial_\lambda A_\lambda \right] + \cdots \right), \quad (4b)$$

clearly showing the nonlocalities in terms of the inverse Laplacian. One can see that $A^h \to A$ when A^a_μ is in the Landau gauge $\partial_\mu A^a_\mu = 0$. We refer to, e.g., Refs. [64–66,69,70] for more details. It can be shown that A^h is gauge invariant order per order, which is sufficient to establish BRST invariance. We will have nothing to say about large gauge transformations.

Mark that $(A^h)^a_\mu$ is formally the value of A^a_μ that (absolutely) minimizes the functional

$$\int d^d x A^a_\mu A^a_\mu \quad (5)$$

under (infinitesimal) gauge transformations $\delta A^a_\mu = D^{ab}_\mu \omega^b$; see, e.g., Refs. [66,69,70]. As such,

$$\int d^d x (A^h)^a_\mu (A^h)^a_\mu = \min_{\text{gauge orbit}} \int d^d x A^a_\mu A^a_\mu. \quad (6)$$

In practice, we are only (locally) minimizing the functional via a power series expansion (3) coming from infinitesimal gauge variations around the original gauge field A^a_μ, whereas the extremum being a minimum is accounted for if the Faddeev-Popov operator (second-order variation, that is) is positive. This is related to the Gribov copy problem and will be ignored here in the definition of our $(A^h)^a_\mu$ or the unity. We will come back to why this is *a posteriori* allowed.

We will later on generalize this construction in the presence of a background gauge field, including the proof that, for expectation values of gauge-invariant operators, the nonlocal A^h_μ can be replaced by the local A_μ when using the Landau gauge, corresponding to the $\alpha \to 0$ case of the linear covariant gauges. The positivity of the Faddeev-Popov operator will also play a role here. But summarizing, at the level of expectation values of gauge-invariant operators, the original action (1) and the one given by

$$S_{\mathrm{LC}} + S_h = \int d^d x \left(\frac{1}{4} F^2_{\mu\nu} + \frac{\alpha}{2} b^2 + i b^a \partial_\mu A^a_\mu + \bar{c}^a \partial_\mu D^{ab}_\mu c^b + i \tau^a \partial_\mu (A^h)^a_\mu + \bar{\eta}^a \partial_\mu (D^h)^{ab}_\mu \eta^a \right) \quad (7)$$

are perturbatively fully equivalent. The renormalizability analysis for generic α can be found in, e.g., Ref. [61]. For completeness, the BRST invariance is generated by the operator s defined as

$$s A^a_\mu = -D^{ab}_\mu c^b, \qquad s c^a = \frac{1}{2} g f^{abc} c^b c^c, \qquad s \bar{c}^a = -i b^a, \quad (8)$$

and all other transformations are zero.

B. Including the Polyakov loop

Our aim is to investigate the confinement/deconfinement phase transition of Yang-Mills theory. The standard way to achieve this goal is by probing the Polyakov loop order parameter,

$$\mathcal{P} = \frac{1}{N} \mathrm{tr} \langle P e^{ig \int_0^\beta dt A_0(t,x)} \rangle, \quad (9)$$

with P denoting path ordering, needed in the non-Abelian case to ensure the gauge invariance of $\mathcal{P}$. In analytical studies of the phase transition involving the Polyakov loop,

[1]Related to this discussion, we note that (2a) is *a priori* an approximation since it ignores the Gribov copies. In the presence of a dynamically generated mass, the contribution of some of these copies is suppressed, in particular those outside the Gribov region, in a fashion similar (but not equivalent) to the Gribov-Zwanziger approach.

one usually imposes the so-called Polyakov gauge on the gauge field, in which case the time component A_0 becomes diagonal and independent of (imaginary) time: $\langle A_\mu(x)\rangle = \langle A_0\rangle\delta_{\mu 0}$, with $\langle A_0\rangle$ belonging to the Cartan subalgebra of the gauge group. In the SU(2) case, for instance, the Cartan subalgebra is one dimensional and can be chosen to be generated by $t^3 \equiv \sigma^3/2$ so that $\langle A_0^a\rangle = \delta^{a3}\langle A_0^3\rangle \equiv \delta^{a3}\langle A_0\rangle$. More details on Polyakov gauge can be found in Refs. [6,71,72]. Besides the trivial simplification of the Polyakov loop, when imposing the Polyakov gauge, it turns out that the quantity $\langle A_0\rangle$ becomes a good alternative choice for the order parameter instead of $\mathcal{P}$; see Ref. [71] for an argument using Jensen's inequality for convex functions, and see also Refs. [73–75]. For other arguments based on the use of Weyl chambers and within other gauges (see below), see Refs. [76–78].

As explained in Refs. [71,73,79], in the SU(2) case at leading order, we then simply find, using the properties of the Pauli matrices,

$$\mathcal{P} = \cos\frac{r}{2}, \tag{10}$$

where we defined

$$r = g\beta\langle A_0\rangle, \tag{11}$$

with β the inverse temperature. This way, $r = \pi$ corresponds to the "unbroken symmetry phase" (confined or disordered phase), equivalent to $\langle\mathcal{P}\rangle = 0$, while $0 < r < \pi$ corresponds to the "broken symmetry phase" (deconfined or ordered phase), equivalent to $\langle\mathcal{P}\rangle \neq 0$. Since $\mathcal{P} \propto e^{-F/T}$ with T the temperature and F the free energy of a heavy quark, it becomes clear that in the unbroken phase (because it is in the unbroken phase where the center symmetry is manifest: $\langle\mathcal{P}\rangle = 0$) an infinite amount of energy would be required to actually get a free quark. The broken/restored symmetry referred to is the $\mathbb{Z}_N$ center symmetry of a pure gauge theory (no dynamical matter in the fundamental representation). With a slight abuse of language, we will refer to the quantity r as the Polyakov loop hereafter.

It is however, a highly nontrivial job to actually compute r. An interesting way around was worked out in Refs. [71,73,79], in which it was shown that similar considerations apply within Landau-DeWitt gauges, a generalization of the Landau gauge in the presence of a background (see the next section for more details). The background needs to be seen as a field of gauge-fixing parameters and, as such, can be chosen at will *a priori*. However, specific choices turn out to be computationally more tractable while allowing one to unveil more easily the center-symmetry breaking mechanism. In particular, for the particular choice of *self-consistent backgrounds* which are designed to coincide with the thermal gluon average at each temperature, it could be shown that the background becomes an order parameter for center symmetry as it derives from a center-symmetric background effective potential (see below).

Moreover, nonperturbative physics was parametrized by a phenomenological mass parameter, akin to using a Curci-Ferrari version of the background Landau gauge [79,80]. This was based on earlier successful attempts to model $T = 0$ Yang-Mills propagators and vertices; see Refs. [48,49] for the initial works and Ref. [50] for a recent overview. The Curci-Ferrari mass was fixed from a dedicated fit to zero-temperature lattice gluon and ghost propagator data in absence of a background (see, e.g., Ref. [81]), but despite its nice consequences and quite good results compared to other nonperturbative approaches, it remains a bit uncomfortable that one needs to introduce a mass scale by hand. If we could recover a dynamical gluon mass from a first principles setup, this would reduce the dependence on external parameters or input. Of course, this does not necessarily entail we will end up with the exact Curci-Ferrari model or the background version of Ref. [79], but this is evidently of no concern, since the Curci-Ferrari was always supposed to be an effective way of modeling gauge fixing beyond standard perturbation theory.

That a proper mass scale can emerge from the Yang-Mills dynamics can already be appreciated from earlier works like Ref. [58], based on the introduction of the nonlocal but gauge-invariant gluon condensate $\langle A^2\rangle_{\min}$, which reduces to $\langle A^2\rangle$ in the Landau gauge in Refs. [82–85]. In fact, $\langle A^2\rangle_{\min} = \langle A^h A^h\rangle$; see the discussion below (5).

Other approaches in which (dynamical) gluon mass scales played a role are, for example, Refs. [19,23,30,64,86–96].

C. BRST-invariant gluon field in presence of a background

We are thus ultimately interested in investigating the spontaneous generation of a gluon mass. In the presence of a background and in the Landau-DeWitt gauge, renormalization (see Appendix A) imposes that this mass (and an asymmetry in this mass, for which see Sec. III B) should couple only to the quantum fields, i.e., the full field minus the background value. This is because quantum fields and background renormalize differently.

To implement this, the formalism of Sec. II A needs to be slightly adapted. Assume a background $\bar{A}_\mu^a$ such that the full gluon field a_μ^a can be written as

$$a_\mu^a = \bar{A}_\mu^a + A_\mu^a, \tag{12}$$

where A_μ^a now denotes the quantum part only. As there is a background, it is convenient to use the Landau-DeWitt (LDW) gauge-fixing condition or Landau background gauge

$$\bar{D}_\mu^{ab}(a_\mu^b - \bar{A}_\mu^b) = 0, \tag{13}$$

where $\bar{D}^{ab}_\mu = \delta^{ab}\partial_\mu - gf^{abc}\bar{A}^c_\mu$ is the background covariant derivative. The Landau-DeWitt gauge can be defined as corresponding to the (local) minima of the functional

$$\int d^d x (a^a_\mu - \bar{A}^a_\mu)^2 = \int d^d x A^a_\mu A^a_\mu \tag{14}$$

under infinitesimal gauge transformations $\delta a^a_\mu = \delta A^a_\mu = D^{ab}_\mu \omega^b$. We refer to Refs. [97,98] for more details. Also here, the extremum will be a minimum upon having a positive Faddeev-Popov operator $\bar{D}^{ab}_\mu D^{bc}_\mu$.

Mark that the background does not transform here; the entire gauge transformation is associated to the quantum part. In principle, the gauge transformation can be distributed over the quantum and classical part, but the choice we make is the most natural one and relates best to the BRST operator to be introduced later [see (20) and Appendix A], at vanishing external sources. The BRST operator then also leaves the background field untouched.

Mark further that invariance under gauge transformations of the background (under which the quantum part transforms as a matter field) is a separate issue and is not a problem in our case (unlike in Ref. [99], for example); see Sec. III D and Appendix A.

Finally, we note that, similarly to the case in the absence of background, we can extend the Landau-DeWitt gauge into to a linear covariant version of it, which we refer to as linear background covariant gauge,

$$S_{\mathrm{bLC}} = \int d^4 x \left(\frac{1}{4} (F^a_{\mu\nu})^2 + \frac{\alpha}{2} b^a b^a + i b^a \bar{D}_\mu (a^a_\mu - \bar{A}^a_\mu) + \bar{c}^a \bar{D}_\mu D_\mu c^a \right). \tag{15}$$

We now need to construct the field $(a^a_\mu)^h$ obeying the Landau-DeWitt gauge

$$\bar{D}^{ab}_\mu ((a^b_\mu)^h - \bar{A}^b_\mu) = 0. \tag{16}$$

To do this, we will perform an expansion in the quantum fields.[2] In this paper, we only aim to do one-loop computations, such that first order in the quantum fields will suffice.

We write the necessary gauge transform as $\mathbb{1} + h_1 + \cdots$, where $h_1 = ig\xi^a_1 t^a$ is assumed first order in the quantum fields and the dots contain higher-order terms. Up to first order, we have

$$a^h_\mu = \bar{A}_\mu + A_\mu + \frac{i}{g} \bar{D}_\mu h_1 + \cdots, \tag{17}$$

where we used that $h_1^\dagger = -h_1$ (which is a consequence of the unitarity of $\mathbb{1} + h_1 + \cdots$ at first order). Imposing the gauge condition (16) yields

$$\bar{D}_\mu A_\mu + \frac{i}{g} \bar{D}^2 h_1 + \cdots = 0 \quad \Rightarrow \quad \frac{i}{g} h_1 = -\frac{1}{\bar{D}^2} \bar{D}_\mu A_\mu. \tag{18}$$

As such, we get

$$a^h_\mu - \bar{A}_\mu = \left(\delta_{\mu\nu} - \bar{D}_\mu \frac{1}{\bar{D}^2} \bar{D}_\nu \right) A_\nu + \cdots. \tag{19}$$

At first order in the quantum fields, $(a^a_\mu)^h$ is indeed invariant under $\delta A^a_\mu = D^{ab}_\mu \omega^b = \bar{D}^{ab}_\mu \omega^b + \cdots$. After gauge fixing with the Faddeev-Popov procedure, this will translate into invariance under.

$$s a^a_\mu = -D^{ab}_\mu c^b, \qquad s c^a = \frac{1}{2} g f^{abc} c^b c^c,$$
$$s \bar{c}^a = -i b^a, \qquad s(\mathrm{rest}) = 0, \tag{20}$$

which is actually the very same BRST operator as defined in (8), as a_μ is now the complete field.

Lastly, the steps leading to (7) are now easily generalized. We can introduce a rather complicated unity,

$$1 = \mathcal{N} \int [\mathcal{D}\xi \mathcal{D}\tau \mathcal{D}\bar{\eta} \mathcal{D}\eta] e^{-S_h}, \tag{21a}$$

$$S_h = \int d^d x (i\tau^a \bar{D}^{ab}_\mu (a^{h,b}_\mu - \bar{A}^b_\mu) + \bar{\eta}^a \bar{D}^{ab}_\mu D^{bc}_\mu [a^h] \eta^c), \tag{21b}$$

and replace action (15) with

$$S \equiv S_{\mathrm{bLC}} + S_h = \int d^d x \left(\frac{1}{4} (F^a_{\mu\nu})^2 + \frac{\alpha}{2} b^a b^a + i b^a \bar{D}_\mu (a^a_\mu - \bar{A}^a_\mu) + \bar{c}^a \bar{D}_\mu D_\mu c^a + i\tau^a \bar{D}^{ab}_\mu (a^{h,b}_\mu - \bar{A}^b_\mu) + \bar{\eta}^a \bar{D}^{ab}_\mu D^{bc}_\mu [a^h] \eta^c \right). \tag{22}$$

For the record, we refrain here from a full-blown all-order algebraic analysis of the renormalizability of the theory defined by (22) and of the background mass operator (see the next subsection); this could be done by combining the technology of Appendix A, Refs. [100,92], even for a more general class of background gauge fixings as in Ref. [101].

D. $a^h_\mu \to a_\mu$ in the path integral in the Landau-DeWitt gauge

Analogously to the $\bar{A} = 0$ case, averages computed with either the action (22) or the original Yang-Mills one will not change anything at the level of physical observables,

[2] In this paper, we will always remain in the perturbative formalism.

defined via the BRST cohomology at zero ghost charge,[3] as one is free to choose any gauge to work with in practice, and employing the Landau-DeWitt gauge, we may effectively replace a_μ^h with a_μ. A formal way to show that this substitution is valid for expectation values of gauge-invariant operators $O_i(x_i)$ goes as follows. Consider then the action S as given in (22), with $\alpha \to 0$, that is $S = S_{\rm LDW} + S_h$. As before, we will use $A_\mu \coloneqq a_\mu - \bar{A}_\mu$ to denote the quantum fluctuation, that is, the field integrated over. To avoid clutter in the notation, we will strip all fields of indices and introduce the shorthand $\bar{D}D^h \coloneqq \bar{D}_\mu^{ab} D_\mu^{bc}[a^h]$. Φ collects all quantum fields,

$$\langle O_1(x_1)\ldots O_n(x_n)\rangle_S = \frac{\int[\mathcal{D}\Phi] O_1(x_1)\ldots O_n(x_n) e^{-S}}{\int[\mathcal{D}\Phi]e^{-S}}. \qquad (23)$$

Mark that the gauge invariance of O_i means that $O_i[a] = O_i[a^h]$. Integration over $b, \tau, \bar{\eta}, \eta$ leads to

$$\delta(\bar{D}A)\delta(\bar{D}(a^h - \bar{A}))\det(-\bar{D}D^h). \qquad (24)$$

Using the perturbative solution ξ_* of the constraint $\bar{D}(a^h - \bar{A}) = 0$ (see Sec. II C for the explicit solution at leading order), we may rewrite the second Dirac delta as

$$\delta(\bar{D}(a^h - \bar{A})) = \delta(\xi - \xi_*)\frac{1}{|\det \delta_\xi(-\bar{D}D^h)|_{\xi=\xi_*}} \qquad (25)$$

to facilitate the ξ integration.

We also note that

$$\delta_\xi \det(-\bar{D}D^h)|_{\xi=\xi_*} = \det(-\bar{D}D + \mathcal{O}(\bar{D}A)), \qquad (26)$$

where $\mathcal{O}(\bar{D}A)$ is a formal power series in $\bar{D}A$, starting at order $\bar{D}A$. We already used here that ξ_* itself is also such power series. The other Dirac delta constraint then leads to a factor

$$\frac{\det(-\bar{D}D)}{|\det(-\bar{D}D)|} = 1, \qquad (27)$$

so the integration over $\tau, \eta, \bar{\eta}$ effectively constitutes a unity and effectively replaces a^h with a, at least if the last step is valid. This is the case if the Faddeev-Popov operator is positive, which is equivalent to stating that the second derivative of the functional (14) is positive, i.e., that we end up in a (local) minimum. This positivity requirement is equivalent to removing infinitesimal Gribov gauge copies in the Landau-DeWitt gauge, which as we already discussed for the $\bar{A} = 0$ case can be *a posteriori* checked by means of the ghost propagator and its positivity. As before, this is not the case when using perturbation theory around the perturbative vacuum, but it is the case when a sufficiently large dynamical mass is generated; see Ref. [102] for an explicit one-loop verification. For other work on Gribov copies in presence of a background; see, for instance, Refs. [97,99,103–107].

Returning to the discussion below (6), the *a posteriori* generation of a sufficiently large dynamical mass thus (partially) tames the Gribov problem, [4] and the *a priori* assumption of ignoring the Gribov problem in defining the (unique) perturbative series solution in the minimization of (14) makes sense, and this with or without background field $\bar{A}$. This also makes (25) valid, as otherwise we would need to include all possible solutions here.

Needless to say, the quantum effective potential of a BRST-invariant operator is an example where the substitution $a_\mu^h \to a_\mu$ applies. Notice that in combination with the results of Appendix A this also implies that the effective action for BRST-invariant operators derived from the action (22) will enjoy the background gauge invariance, as follows from the Ward identity (A25). This gives an exact argument, complementing the one already provided below (43).

III. BRST-INVARIANT MASS AND ASYMMETRY

This section presents a short review of the local composite operator (LCO) formalism as proposed in Ref. [58] modified in the presence of a background field. The case without background is a special case hereof and will be discussed in greater detail in Ref. [64].

A. $d = 2$ gluon condensate

As we want to work with a background field, it is more appropriate to use the Landau background gauge [108] $\bar{D}_\mu^{ab}(A_\mu^b)^h = 0$ instead of the usual Landau gauge prescription. For other works in the Landau background gauge, see, for example, Refs. [109–112].

A BRST analysis (for BRST in the background gauge, see, for example, Refs. [100,101][5]) shows that, for the LCO formalism to stay renormalizable, the dimension-2 operator to be used is

$$(a_\mu^h - \bar{A}_\mu)^2. \qquad (28)$$

First, the source terms

[3]Which are the classically gauge-invariant operators built from the gauge field, up to irrelevant BRST exact terms and terms with equation-of-motion contributions.

[4]We have nothing to say about "large" gauge copies.

[5]In [100], the BRST transform of the background is nonzero: $s\bar{A}_\mu^a = \Omega_\mu^a$, where Ω_μ^a is a ghost source. This source greatly simplifies the proof of renormalizability. The physical case, however, is recovered when $\Omega_\mu^a \to 0$, with BRST variation (20), under which $(a^h)_\mu^a$ is invariant.

$$\int d^d x\left(\frac{1}{2}J(a_\mu^h-\bar{A}_\mu)^2-\frac{1}{2}\zeta J^2\right) \tag{29}$$

are added to the action with J the source used to couple the operator to the theory. The term in J^2 is necessary here for renormalizability of the connected-diagram-generating functional $W(J)$ and, subsequently, of the associated one-particle irreducible (1PI) diagram generating functional Γ, also known as the effective action. Here, ζ is a new coupling constant whose determination we will discuss later. In the physical vacuum, corresponding to $J\to 0$, it should decouple again, at least if we were to do the computations exactly. At (any) finite order, ζ will be explicitly present, even in physical observables, making it necessary to choose it as wisely as possible. Notice that ζ is *not* a gauge parameter as it, in fact, couples to the BRST-invariant quantity J^2. Indeed, in a BRST-invariant theory, we expect the gauge parameter to explicitly cancel order per order from physical observables, a fact guaranteed by, e.g., the Nielsen identities [113], which are in themselves a consequence of BRST invariance [114].

Thanks to ζ, the Lagrangian remains now multiplicatively renormalizable (see Appendix A).

To actually compute the effective potential, it is computationally simplest to rely on Jackiw's background field method [115]. Before integrating over any fluctuating quantum fields, a Legendre transform is performed, so that formally $\sigma=\frac{1}{2}(a_\mu^h-\bar{A}_\mu)^2-\zeta J$. Plugging this into the Legendre transformation between Γ and W, we find that we could just as well have started from the action (22) with the following unity inserted into the path integral,[6]

$$1=\mathcal{N}\int[\mathcal{D}\sigma]\exp-\frac{1}{2\zeta}\int d^d x\left(\sigma+\frac{1}{2}(a_\mu^h-\bar{A}_\mu)^2\right)^2, \tag{30}$$

with $\mathcal{N}$ an irrelevant constant. Of course, if we could integrate the path integral exactly, this unity would not change a thing. The situation only gets interesting if the perturbative dynamics of the theory would prefer to assign a nonvanishing vacuum expectation value to σ. As such, this σ field allows one to include potential nonperturbative information through its vacuum expectation value. In the case without a background, σ does indeed condense, and a vacuum with $\langle\sigma\rangle\neq 0$ is preferred.

For the record, BRST invariance is ensured if we assign $s\sigma=-s(\frac{1}{2}(a_\mu^h-\bar{A}_\mu)^2)$, which implies off shell that $s\sigma=0$ thanks to the BRST invariance of $a_\mu^h-\bar{A}_\mu$.

[6] This is equivalent to a Hubbard-Stratonovich transformation; see, for instance, Refs. [58,64]. This also evades the interpretational issues for the energy when higher-than-linear terms in the sources are present.

For completeness, let us write down the full gauge-fixed action here (we consider the $\alpha\to 0$ limit right away, that is, the Landau-deWitt gauge):

$$\begin{aligned}S_{\text{full}}&=S_{\text{LDW}}+S_h-\frac{1}{2\zeta}\left(\sigma+\frac{1}{2}(a_\mu^h-\bar{A}_\mu)^2\right)^2\\&=\int d^4x\left(\frac{1}{4}(F_{\mu\nu}^a)^2+ib^a\bar{D}_\mu(a_\mu^a-\bar{A}_\mu^a)+\bar{c}^a\bar{D}_\mu D_\mu c^a\right.\\&\quad+i\tau^a\bar{D}_\mu^{ab}(a_\mu^{h,b}-\bar{A}_\mu^b)+\bar{\eta}^a\bar{D}_\mu^{ab}D_\mu^{bc}[a^h]\eta^c\\&\quad\left.-\frac{1}{2\zeta}\left(\sigma+\frac{1}{2}(a_\mu^h-\bar{A}_\mu)^2\right)^2\right).\end{aligned} \tag{31}$$

The outcome of Sec. II. D can be immediately generalized, leading to

$$\langle O_1(x_1)\ldots O_n(x_n)\rangle_{S_{\text{full}}}=\langle O_1(x_1)\ldots O_n(x_n)\rangle_{S_{\text{mLDW}}} \tag{32}$$

for gauge-invariant operators $O_i(x_i)$, where

$$S_{\text{mLDW}}\equiv S_{\text{LDW}}-\frac{1}{2\zeta}\int d^d x\left(\sigma+\frac{1}{2}A_\mu^2\right)^2. \tag{33}$$

Notice that on shell, or to be more precise when the τ and b equations of motion are used, we have $s\sigma=-s(\frac{1}{2}(a_\mu^h-\bar{A}_\mu)^2)\to -s(\frac{1}{2}(a_\mu-\bar{A}_\mu)^2)=-A_\mu^a sA_\mu^a\neq 0$, ensuring that (33) is still BRST invariant.

We still need to discuss the new coupling ζ. First note that, given the BRST invariance of the action, we can work in a preferred gauge, that is, the Landau-DeWitt gauge; see Sec. II D, the conclusions whereof are not effected by the inclusion of the BRST-invariant unity (21a).

It is evident that ζ can be interpreted as a genuine new coupling constant. Therefore, we now have two coupling constants, g^2 and ζ, with g^2 running as usual, that is, independently of ζ. This makes our situation suitable for the Zimmermann reduction of couplings program [116]; see also Ref. [117] for a recent overview. In this program, one coupling (ζ in our case) is reexpressed as a series in the other (here g^2) so that the running of ζ controlled by $\zeta(g^2)$ is then automatically satisfied; see also Ref. [64]. More specifically, $\zeta(g^2)$ is determined such that the generating functional of connected Green's functions, $W(J)$, obeys a standard, linear renormalization group equation [58].

This selects one consistent coupling $\zeta(g^2)$ from a whole space of allowed couplings, and it is also the unique choice compatible with multiplicative renormalizability [58]. Given that we already pointed out that ζ should, in principle, not affect physics, we can safely rely here on this special choice, already made earlier in, e.g., Ref. [58]. This choice seems also to be a natural one from the point of view of the loop expansion of the background potential to be used below.[7] In the $\overline{\mathrm{MS}}$ scheme, one finds [58,118]

$$\zeta = \frac{N^2-1}{g^2N}\left(\frac{9}{13}+\frac{g^2N}{16\pi^2}\frac{161}{52}+\mathcal{O}(g^4)\right), \quad (34a)$$

$$Z_\zeta = 1-\frac{g^2N}{16\pi^2}\frac{13}{3\epsilon}+\mathcal{O}(g^2), \quad (34b)$$

$$Z_J = 1-\frac{Ng^2}{16\pi^2}\frac{35}{6\epsilon}+\mathcal{O}(g^2), \quad (34c)$$

where Z_ζ and Z_J are the renormalization factors of ζJ^2 and J, respectively.

B. Introduction of asymmetry in the $d=2$ gluon condensate

When temperature is switched on, it is natural to consider the timelike and spacelike components of the A^hA^h condensate separately, or equivalently to introduce the BRST-invariant electric-magnetic asymmetry [63],

$$\Delta_{A^2} = \langle g^2A_0^hA_0^h\rangle - \frac{1}{3}\langle g^2A_i^hA_i^h\rangle, \quad (35)$$

where the Latin index denotes the space components and A_μ^h is a shorthand for $a_\mu^h-\bar{A}_\mu$. This asymmetry can be included in exactly the same way as the $(a_\mu^h-\bar{A}_\mu)^2$ condensate; namely, we add

$$\int d^dx\left(\frac{1}{2}K_{\mu\nu}\left((a_\mu^h-\bar{A}_\mu)(a_\nu^h-\bar{A}_\nu)-\frac{\delta_{\mu\nu}}{d}(a_\mu^h-\bar{A}_\mu)^2\right)\right.$$
$$\left.-\frac{1}{2}\omega K_{\mu\nu}K_{\mu\nu}+\frac{\omega}{2d}K_{\mu\mu}^2\right), \quad (36)$$

where the dimension-2 symmetric source $K_{\mu\nu}$ couples to the traceless operator $((a_\mu^h-\bar{A}_\mu)(a_\nu^h-\bar{A}_\nu)-\frac{\delta_{\mu\nu}}{d}(a_\mu^h-\bar{A}_\mu)^2)$ (see also Ref. [119]).

The same goal can be reached by directly adding an extra part to the action,

$$\frac{1}{2\omega}\int d^dx\left(\varphi_{\mu\nu}+\frac{1}{2}(a_\mu^h-\bar{A}_\mu)(a_\nu^h-\bar{A}_\nu)\right)^2, \quad (37)$$

with $\varphi_{\mu\nu}$ an auxiliary field analogous to σ but which we will take to be a traceless matrix and which will thus couple to the asymmetric part of the condensate. The parameter ω is the analog of ζ. As we are interested in the asymmetry, we parametrize the mass matrix as

$$\varphi_{\mu\nu} = \omega\mathbb{A}\begin{pmatrix}1 & & & \\ & -\frac{1}{d-1} & & \\ & & \ddots & \\ & & & -\frac{1}{d-1}\end{pmatrix}; \quad (38)$$

i.e., we preserve rotational invariance in the spatial part. Determining ω in the same way, we found ζ gives [119]

$$\omega = \frac{N^2-1}{g^2N}\left(\frac{1}{4}+\frac{73}{1044}\frac{g^2N}{16\pi^2}+\mathcal{O}(g^4)\right), \quad (39a)$$

$$Z_\omega = 1+\frac{g^2N}{24\pi^2}\frac{11}{\epsilon}+\mathcal{O}(g^2), \quad (39b)$$

$$Z_K = 1-\frac{g^2N}{16\pi^2}\frac{29}{6\epsilon}+\mathcal{O}(g^2) \quad (39c)$$

to one-loop order. Here, Z_ω and Z_K are the renormalization factors of $K_{\mu\nu}K_{\mu\nu}-\frac{1}{2d}K_{\mu\mu}^2$ and $K_{\mu\nu}$, respectively.

C. Background field independence of physical observables

Using the standard Landau-DeWitt gauge condition, it can be nicely shown using the (extended) Slavnov-Taylor identity that physical observables do not depend on the choice of the background $\bar{A}(x)$, which is of course expected, given that choosing $\bar{A}(x)$ corresponds to choosing a specific gauge. For a formal proof, see Ref. [101]. The crux of the matter is to extend the BRST operator s to also act on the background field via $s\bar{A}=\Omega$, $s\Omega=0$ (see also Appendix A, based on Ref. [100]), with Ω an auxiliary Grassmann background field that is to be sent to zero again eventually. This is actually an extension of the BRST method to formally prove gauge parameter independence of observables in linear gauges (see Refs. [114,120]), in which case the gauge parameter α is also made part of a BRST doublet.

We will not exactly follow this procedure here; see also the comment below (22). Indeed, we would also need to properly extend the BRST operator to auxiliary fields to maintain a full extended BRST invariance of the action. But we can follow a slightly different route to make our point, again benefitting from the "reduction" $a^h\to a$ in the Landau-DeWitt gauge.

Let us first consider observables that are not directly depending on a, such as the partition function, free energy, and related quantities. One finds, only writing the integration over the gluon degrees of freedom for simplicity,

[7]The ζ independence of expectation values not involving the σ field relies, however, on taking into account the unity in an exact fashion; see Ref. [64]. Clarifying how approximations break this ζ independence and how increasing the truncation order reduces this dependence remains an interesting question.

$$\frac{\delta}{\delta\bar{A}(x)}\int[\mathcal{D}a]e^{-S_{\mathrm{LDW}}-\frac{1}{2\zeta}\int d^dx(\sigma+\frac{1}{2}(a-\bar{A})^2)^2}$$
$$=\frac{\delta}{\delta\bar{A}(x)}\int[\mathcal{D}A]e^{-S_{\mathrm{LDW}}-\frac{1}{2\zeta}\int d^dx(\sigma+\frac{1}{2}A^2)^2}$$
$$=\int[\mathcal{D}A]s\left(\frac{\delta}{\delta\bar{A}(x)}\left(\int d^dz\bar{c}\,\bar{D}A\right)\right)$$
$$\times e^{-S_{\mathrm{LDW}}-\frac{1}{2\zeta}\int d^dx(\sigma+\frac{1}{2}A^2)^2}=0 \tag{40}$$

using the BRST symmetry of the vacuum/action, including of the last term of the action (the unity) [see the comments below (33)], and the fact that the gauge-fixing part of S_{LDW} is BRST exact. We also changed integration variable $a\to A$.

Using a slightly different argumentation, we can extend the argument to correlation functions of (renormalizable) gauge-invariant operators $O_i(x_i)$. We assume $x_i\neq x_j$ for $i\neq j$ so that the aforementioned operators are considered at separate space-time points. If this were not the case, we would have to face the renormalization of further gauge invariant operators, e.g. $O(x)=O_1(x_1)O_2(x_2)$ if $x_1=x_2=x$. We get from (41)

$$\frac{\delta}{\delta\bar{A}(x)}\langle O_1\ldots O_n\rangle_{S_{\mathrm{full}}}$$
$$=\frac{\delta}{\delta\bar{A}(x)}\int[\mathcal{D}a\mathcal{D}\sigma]O_1\ldots O_ne^{-S_{\mathrm{LDW}}-\frac{1}{2\zeta}\int d^dx(\sigma+\frac{1}{2}(a-\bar{A})^2)^2}$$
$$=\left\langle O_1\ldots O_n\frac{\delta}{\delta\bar{A}(x)}\left(s\int d^dz\bar{c}\,\bar{D}A\right)\right\rangle_{S_{\mathrm{mLDW}}}+\int[\mathcal{D}a\mathcal{D}\sigma]O_1\ldots O_n\frac{1}{2\zeta}(a-\bar{A})\left(\sigma+\frac{1}{2}(a-\bar{A})^2\right)e^{-S_{\mathrm{LDW}}-\frac{1}{2\zeta}\int d^dx(\sigma+\frac{1}{2}(a-\bar{A})^2)^2}$$
$$=\left\langle s\left(O_1\ldots O_n\frac{\delta}{\delta\bar{A}(x)}\left(\int d^dz\bar{c}\,\bar{D}A\right)\right)\right\rangle_{S_{\mathrm{mLDW}}}-\int[\mathcal{D}a\mathcal{D}\sigma]O_1\ldots O_n(a-\bar{A})\frac{\delta}{\delta\sigma}e^{-S_{\mathrm{LDW}}-\frac{1}{2\zeta}\int d^dx(\sigma+\frac{1}{2}(a-\bar{A})^2)^2}$$
$$=\int[\mathcal{D}a\mathcal{D}\sigma]\frac{\delta}{\delta\sigma}(O_1\ldots O_n)(a-\bar{A})e^{-S_{\mathrm{LDW}}-\frac{1}{2\zeta}\int d^dx(\sigma+\frac{1}{2}(a-\bar{A})^2)^2}=0, \tag{41}$$

where the second-to-last step is again based on BRST invariance (first term) and the functional version of the trivial identity $(x+y)e^{-(x+y)^2/2}=-de^{-(x+y)^2/2}/dx$ (second term). In the last step, one recognizes the (functional) integral of a (functional) total derivative, which vanishes in the absence of boundary terms. We mention that the present argumentation does not require using the unity (30) and therefore shows that the background-independence property should be relatively robust to practical expansions of (30) as used below.

D. Background gauge invariance

Besides BRST invariance, another important symmetry when working with a gluonic background field is background gauge invariance,

$$\delta\bar{A}^a_\mu=\bar{D}^{ab}_\mu\beta^b,\qquad\delta\varphi^a=-f^{abc}\beta^b\varphi^c, \tag{42}$$

where φ^a stands for all the quantum fields. The ordinary Yang-Mills action with the Faddeev-Popov ghost part is invariant under this symmetry. To see the invariance of the extra part (29), consider the expansion (19). The background on the left-hand side of that expression only appears in covariant derivatives, such that the entire expression transforms as a matter field,

$$\delta((a^h)^a_\mu-\bar{A}^a_\mu)=-f^{abc}\beta^b((a^h)^c_\mu-\bar{A}^c_\mu), \tag{43}$$

meaning the mass term (29) is invariant.

To use the background field formalism, we can now follow the arguments of Ref. [79]. Consider the quantum effective action of the gluon field computed in the presence of a background $\bar{A}^a_\mu$: $\Gamma_{\bar{A}}[a]$. The physical vacuum is found by minimizing with respect to a^a_μ,

$$\Gamma_{\bar{A}}[a^{\mathrm{cl}}_{\bar{A}}]\leq\Gamma_{\bar{A}}[a]\quad\forall\ a^a_\mu, \tag{44}$$

where $(a^{\mathrm{cl}}_{\bar{A}})^a_\mu=\langle a^a_\mu\rangle_{\bar{A}}$ is the value of a^a_μ in this minimum. Now, thanks to BRST invariance, the background is in essence a gauge parameter, such that physical quantities may not depend on it, and we can freely choose it.[8] Thence, choosing a self-consistent background $\bar{A}_s$ defined by the condition $\bar{A}_s=a^{\mathrm{cl}}_{\bar{A}_s}$, we find

[8]This property is fragile to the use of approximations or modeling (as those considered within nonperturbative approaches or the Curci-Ferrari model) leading to potential spurious effects in the results obtained using the background effective action. Recently, an alternative approach has been put forward that relies, instead, on using the standard effective action $\Gamma_{\bar{A}_c}[A]$ in a particular gauge, as defined by the choice of a center-symmetric background $\bar{A}_c$. The rationale for using this approach does not rely on the background independence of the free energy and is thus more robust to violations of the latter. In the present BRST-invariant loop expanded approach, we expect these violations to be minimal and the two approaches to be essentially equivalent. This will be investigated elsewhere.

$$\Gamma_{\bar{A}_s}[\bar{A}_s] = \Gamma_{\bar{A}_s}[a^{\rm cl}_{\bar{A}_s}] = \Gamma_{\bar{A}}[a^{\rm cl}_{\bar{A}}] \leq \Gamma_{\bar{A}}[\bar{A}], \tag{45}$$

where the first equality follows from the self-consistency condition, the second one is BRST invariance, and the third one is the minimization condition (44) for the specific value $a^a_\mu = \bar{A}^a_\mu$. In conclusion, the minimum of the quantum effective action can be found by minimizing $\Gamma_{\bar{A}}[\bar{A}]$ with respect to $\bar{A}^a_\mu$. Thanks to the remaining background gauge invariance, we also know that $\Gamma_{\bar{A}}[\bar{A}]$ will be a (background) gauge-invariant functional of $\bar{A}$. In the presence of the condensate and the asymmetry, the background effective action is also a function of the condensate and the asymmetry, variables with respect to which it also needs to be minimized.

It is important to note that $\Gamma_{\bar{A}}[\bar{A}]$ does not need to be $\bar{A}$ independent, and its explicit computation in the next section will make this dependence quite explicitly clear.[9] This is not at odds with the previous subsection. On the contrary, it is even to be expected, as $\Gamma_{\bar{A}}[\bar{A}]$ does not obey a Slavnov-Taylor identity to begin with. Indeed, to avoid misconceptions, we stress here that the functional $\Gamma_{\bar{A}}[\bar{A}]$ is not the standard quantum effective action generating 1PI graphs, which is $\Gamma_{\bar{A}}[\bar{A}]$. It is, however, still a useful functional that also appears in the so-called "background equivalence theorem" for which we refer to, e.g., Ref. [101] for more details and references. Here, we appreciate its usefulness to select self-consistent backgrounds from its minimization, leading to estimates of the ground-state free energy, based on (45).

IV. COMPUTATION OF THE BACKGROUND EFFECTIVE POTENTIAL

In what follows, we evaluate the background field effective potential whose minima give access to the self-consistent background and thus to order parameters for the confinement/deconfinement transition. We work in the Landau-DeWitt gauge, which means that we send α to 0. As we have explained, we can then consider replacing a^h by a, which considerably simplifies the calculations.

A. Warming up in the absence of asymmetry

For the sake of simplicity, let us assume first that there is no asymmetry. This approximation will turn out to be justified as the asymmetry we will find below is tiny. To evaluate the background effective potential at one-loop order, we need the terms in the action that are at most quadratic in the fields,

$$\int d^dx\left(\frac{\zeta}{2}m^4\left(1-\frac{\delta\zeta}{\zeta}\right)+\frac{1}{2}A^a_\mu\left(-\delta_{\mu\nu}\bar{D}^2_{ab}+\left(1-\frac{1}{\alpha}\right)\bar{D}^{ac}_\nu\bar{D}^{cb}_\mu+\delta^{ab}\delta_{\mu\nu}m^2\right)A^b_\nu+\bar{c}^a\bar{D}^2_{ab}c^b\right), \tag{46}$$

where the limit $\alpha\to 0$ is assumed and where we used the notation $Z_\zeta = 1-\delta\zeta/\zeta$. Renormalization factors in the part quadratic in the quantum fields are ignored, as they will not be necessary at one-loop order. We also wrote $\sigma = \zeta m^2$.

Integrating out the fields yields traces of logarithms of the operators multiplying their quadratic parts. To deal with them, we work in a space where the covariant derivative is diagonal. Let us see how this works in the SU(2) case before generalizing to $SU(N)$. We first go over to a basis in color space where ϵ^{ab3} is diagonal,

$$\epsilon^{ab3}\mathsf{e}^b_\kappa = i\kappa\mathsf{e}^a_\kappa, \tag{47}$$

with $\kappa\in\{-1,0,+1\}$. If we write $A^a_\mu = A^\kappa_\mu \mathsf{e}^\kappa_a$, then we immediately find that $\bar{D}_0A^\kappa_\mu = \partial_0 A^\kappa_\mu - ir\kappa T A^\kappa_\mu$ (no sum over κ). In Fourier space, we can therefore write

$$\bar{D}_\mu A^\kappa_\nu = iP^\kappa_\mu A^\kappa_\nu \qquad (P^\kappa_0 = p_0 - r\kappa T, \qquad P^\kappa_i = p_i). \tag{48}$$

As such, the one-loop effective potential is

$$\begin{aligned}V(r,m^2) &= \frac{\zeta}{2}m^4\left(1-\frac{\delta\zeta}{\zeta}\right)\\ &\quad+\frac{1}{2}\mathrm{tr}\,\ln\left(\delta_{\mu\nu}P^2_\kappa-\left(1-\frac{1}{\alpha}\right)P^\kappa_\mu P^\kappa_\nu+\delta_{\mu\nu}m^2\right)\\ &\quad-\mathrm{tr}\,\ln P^2_\kappa,\end{aligned} \tag{49}$$

where the trace refers to space-time indices as well as color charges and momenta. An operator of the type

$$X\delta_{\mu\nu}+Y\frac{P^\kappa_\mu P^\kappa_\nu}{P^2_\kappa} \tag{50}$$

has one eigenvector parallel to P^κ_μ with eigenvalue $X+Y$ and $d-1$ eigenvectors perpendicular to P^κ_μ with eigenvalue X. This means that

$$\mathrm{tr}\,\ln\left(X\delta_{\mu\nu}+Y\frac{P^\kappa_\mu P^\kappa_\nu}{P^2_\kappa}\right) = (d-1)\ln X + \ln(X+Y). \tag{51}$$

Using this, we arrive at

$$\begin{aligned}V(r,m^2) &= \frac{\zeta}{2}m^4\left(1-\frac{\delta\zeta}{\zeta}\right)+\frac{d-1}{2}\mathrm{tr}\,\ln(P^2_\kappa+m^2)\\ &\quad+\frac{1}{2}\mathrm{tr}\,\ln\left(\frac{P^2_\kappa}{\alpha}+m^2\right)-\mathrm{tr}\,\ln P^2_\kappa,\end{aligned} \tag{52}$$

where the trace now refers to the color charges and the momenta. In the limit $\alpha\to 0$ neglecting a trivial term, this gives

$$\begin{aligned}V(r,m^2) &= \frac{\zeta}{2}m^4\left(1-\frac{\delta\zeta}{\zeta}\right)+\frac{d-1}{2}\mathrm{tr}\,\ln(P^2_\kappa+m^2)\\ &\quad-\frac{1}{2}\mathrm{tr}\,\ln P^2_\kappa,\end{aligned} \tag{53}$$

[9] It can be shown, however, to be constant on constant backgrounds, at zero temperature [76,78].

where the last term comes from a partial cancellation between the ghost contribution and the massless longitudinal gluon mode. The analysis of the SU(2) potential can be restricted to the interval $r \in [0, 2\pi]$ and even $[0, \pi]$, the center-symmetric point corresponding to $r = \pi$.

The formula for the background potential in the $SU(N)$ case is formally the same as (53). The only change is that the $N^2 - 1$ labels κ become vectors of $\mathbb{R}^{N-1}$ whose components are denoted κ_j, with j referring to the diagonal directions of the algebra [3 and 8 in the SU(3) case for instance]. Correspondingly, the variable r also becomes a vector of $\mathbb{R}^{N-1}$ of components r_j, and we have now $P_0^\kappa = p_0 - r_j \kappa_j T$. Out of the $N^2 - 1$ labels κ, $N - 1$ are equal to 0, and the rest are the roots characterizing the associated Lie algebra. In the case of SU(3), for instance, there are two zeros and six roots $\pm(1, 0)$, $\pm(1/2, \sqrt{3}/2)$, and $\pm(1/2, -\sqrt{3}/2)$. The analysis of the SU(3) potential can be restricted to $r_3 \in [0, 2\pi]$, while charge conjugation invariance (in pure YM) imposes $r_8 = 0$ (in this range of values for r_3). The center-symmetric point corresponds in this case to $r_3 = 4\pi/3$. More on this can be found in, e.g., Ref. [78]. We shall later exploit these remarks to infer the expression for the SU(3) potential from that of the SU(2) potential.

B. Including the asymmetry

In the presence of the asymmetry, the quadratic part of the action reads

$$\int d^d x \left(\frac{\zeta}{2} m^4 \left(1 - \frac{\delta\zeta}{\zeta}\right) + \frac{\omega}{2} \mathbb{A}^2 \frac{d}{d-1} \left(1 - \frac{\delta\omega}{\omega}\right) + \frac{1}{2} A_\mu^a \left(-\delta_{\mu\nu} \bar{D}_{ab}^2 + \left(1 - \frac{1}{\alpha}\right) \bar{D}_\nu^{ac} \bar{D}_\mu^{cb} + \delta^{ab} \delta_{\mu\nu} m^2 + \delta^{ab} M_{\mu\nu} \right) A_\nu^b + \bar{c}^a \bar{D}_{ab}^2 c^b \right), \tag{54}$$

where, again, the limit $\alpha \to 0$ is assumed and where we used the notation Z_ω analogous to $Z_\zeta = 1 - \frac{\delta\zeta}{\zeta}$. We also wrote $\varphi_{\mu\nu} = \omega M_{\mu\nu}$.

Integrating out the fields, we arrive now at the SU(N) one-loop effective potential

$$V(r, m^2, A) = \frac{\zeta}{2} m^4 \left(1 - \frac{\delta\zeta}{\zeta}\right) + \frac{\omega}{2} \mathbb{A}^2 \frac{d}{d-1} \left(1 - \frac{\delta\omega}{\omega}\right) + \frac{1}{2} \mathrm{tr}\, \ln\left(\delta_{\mu\nu} P_\kappa^2 - \left(1 - \frac{1}{\alpha}\right) P_\mu^\kappa P_\nu^\kappa + \delta_{\mu\nu} m^2 + M_{\mu\nu} \right) - \mathrm{tr}\, \ln P_\kappa^2, \tag{55}$$

where, as before, a summation over κ is implied. To more easily handle the $d \times d$ matrix coming from the gluon fields, let us, following Ref. [119], separate out the part without the mass matrix $M_{\mu\nu}$:

$$\begin{aligned} &\mathrm{tr}\, \ln\left(\delta_{\mu\nu} P_\kappa^2 - \left(1 - \frac{1}{\alpha}\right) P_\mu^\kappa P_\nu^\kappa + \delta_{\mu\nu} m^2 + M_{\mu\nu} \right) \\ &= \mathrm{tr}\, \ln\left(\delta_{\mu\nu} P_\kappa^2 - \left(1 - \frac{1}{\alpha}\right) P_\mu^\kappa P_\nu^\kappa + \delta_{\mu\nu} m^2 \right) \\ &\quad + \mathrm{tr}\, \ln\left(\delta_{\mu\nu} + \frac{1}{P_\kappa^2 + m^2} \left(\delta_{\mu\lambda} - (1 - \alpha) \frac{P_\mu^\kappa P_\lambda^\kappa}{P_\kappa^2 + \alpha m^2} \right) M_{\lambda\nu} \right). \end{aligned} \tag{56}$$

In the limit $\alpha \to 0$, this gives

$$(d - 1) \mathrm{tr}\, \ln(P_\kappa^2 + m^2) + \mathrm{tr}\, \ln P_\kappa^2 + \mathrm{tr}\, \ln\left(\delta_{\mu\nu} + \frac{1}{P_\kappa^2 + m^2} \left(\delta_{\mu\lambda} - \frac{P_\mu^\kappa P_\lambda^\kappa}{P_\kappa^2} \right) M_{\lambda\nu} \right) \tag{57}$$

plus an irrelevant constant term.

If we now consider the operator in the last term, we can write it in an orthonormal basis consisting of the vectors:

(i) the unit vector pointing in the direction of P_μ^κ,
(ii) the vector obtained by replacing the timelike component of P_μ^κ by $-p_i^2/P_0^\kappa$ (so as to make a vector perpendicular to P_μ^κ) followed by norming this vector,
(iii) an orthonormal basis of spacelike vectors perpendicular to p_i.

In this basis, the operator under consideration is lower triangular, such that its determinant is the product of its diagonal elements. These diagonal elements are found to be:

(i) 1,
(ii) $1 + \frac{\mathbb{A}}{P_\kappa^2 + m^2} (1 - \frac{d}{d-1} \frac{(P_0^\kappa)^2}{P_\kappa^2})$,
(iii) $1 - \frac{\mathbb{A}}{P_\kappa^2 + m^2} \frac{1}{d-1}$ (with multiplicity $d - 2$).

Gathering all these results, we find that the effective potential at one loop is equal to

$$\begin{aligned} V(r, m^2, A) = {}& \frac{\zeta}{2} m^4 \left(1 - \frac{\delta\zeta}{\zeta}\right) + \frac{\omega}{2} \mathbb{A}^2 \frac{d}{d-1} \left(1 - \frac{\delta\omega}{\omega}\right) \\ & - \mathrm{tr}\, \ln P_\kappa^2 + \frac{1}{2} \mathrm{tr}\, \ln\left(P_\kappa^2 (P_\kappa^2 + m^2) + \mathbb{A}\left(P_\kappa^2 - \frac{d}{d-1} (P_0^\kappa)^2 \right) \right) \\ & + \frac{d-2}{2} \mathrm{tr}\, \ln\left(P_\kappa^2 + m^2 - \frac{\mathbb{A}}{d-1} \right). \end{aligned} \tag{58}$$

As a cross-check of this formula, in Appendix B, we provide an alternative derivation within the Nakanishi-Lautrup formalism. We also notice that, upon taking the limit $A \to 0$, one retrieves Eq. (53).

C. Evaluation of the (sum) integrals

The formula (53) and its generalization (58) involve various (sum integrals), which we now evaluate. We consider first the SU(2) case for simplicity and then use it to infer the corresponding SU(3) formulas.

The expression in the last logarithm of (58) expands as $(p_0 - r\kappa T)^2 + \vec{p}^2 + m^2 - \frac{\mathbb{A}}{d-1}$, and we can immediately apply formula (C8) from Appendix C to find that

$$\begin{aligned}
&\frac{d-2}{2}\mathrm{tr}\,\ln\left(P_\kappa^2 + m^2 - \frac{\mathbb{A}}{d-1}\right)\\
&= 3\frac{d-2}{2}\mathrm{tr}_{T=0}\ln\left(p^2 + m^2 - \frac{\mathbb{A}}{d-1}\right)\\
&\quad + 2T\int\frac{d^3p}{(2\pi)^3}\ln(1 - 2e^{-\frac{\sqrt{\vec{p}^2+m^2-\frac{\mathbb{A}}{3}}}{T}}\cos r\\
&\quad + e^{-2\frac{\sqrt{\vec{p}^2+m^2-\frac{\mathbb{A}}{3}}}{T}}) + T\int\frac{d^3p}{(2\pi)^3}\ln(1 - e^{-\frac{\sqrt{\vec{p}^2+m^2-\frac{\mathbb{A}}{3}}}{T}})^2. \quad (59)
\end{aligned}$$

Here, we put $d = 4$ in the (finite) zero-temperature correction part, and we summed over κ. In the limit $\mathbb{A} \to 0$, this formula can be adapted to obtain the integral needed in (53). Moreover, when both $\mathbb{A} = 0$ and $m = 0$, it can be adapted to obtain

$$\begin{aligned}
-\mathrm{tr}\,\ln P_\kappa^2 &= -3\mathrm{tr}_{T=0}\ln p^2 - 2T\\
&\quad\times\int\frac{d^3p}{(2\pi)^3}\ln(1 - 2e^{-\frac{\sqrt{\vec{p}^2}}{T}}\cos r + e^{-2\frac{\sqrt{\vec{p}^2}}{T}})\\
&\quad - T\int\frac{d^3p}{(2\pi)^3}\ln(1 - e^{-\frac{\sqrt{\vec{p}^2}}{T}})^2, \quad (60)
\end{aligned}$$

which also appears in (58). Finally, when it comes to the expression in the second-to-last logarithm of (58), it is written $(p_0 - r\kappa T)^4 + (p_0 - r\kappa T)^2(2\vec{p}^2 + m^2 - \frac{\mathbb{A}}{d-1}) + \vec{p}^2(\vec{p}^2 + m^2 + \mathbb{A})$. With the definitions

$$\alpha = 2\vec{p}^2 + m^2 - \frac{\mathbb{A}}{d-1}, \qquad \beta = \vec{p}^2(\vec{p}^2 + m^2 + \mathbb{A}), \quad (61a)$$

$$z_\pm = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}, \quad (61b)$$

this becomes $((p_0 - r\kappa T)^2 + z_+)((p_0 - r\kappa T)^2 + z_-)$. As such, we find

$$\begin{aligned}
&\frac{1}{2}\mathrm{tr}\,\ln\left(P_\kappa^2 + m^2 + \mathbb{A}\left(1 - \frac{d}{d-1}\frac{(P_0^\kappa)^2}{P_\kappa^2}\right)\right) + \frac{1}{2}\mathrm{tr}\,\ln P_\kappa^2\\
&= \frac{3}{2}\mathrm{tr}_{T=0}\ln\left(p^2 + m^2 + \mathbb{A}\left(1 - \frac{d}{d-1}\frac{p_0^2}{p^2}\right)\right) + \frac{3}{2}\mathrm{tr}_{T=0}\ln p^2\\
&\quad + T\int\frac{d^3p}{(2\pi)^3}\ln\left(1 - 2e^{-\frac{\sqrt{z_+}}{T}}\cos r + e^{-2\frac{\sqrt{z_+}}{T}}\right) + \frac{T}{2}\int\frac{d^3p}{(2\pi)^3}\ln\left(1 - e^{-\frac{\sqrt{z_+}}{T}}\right)^2\\
&\quad + T\int\frac{d^3p}{(2\pi)^3}\ln\left(1 - 2e^{-\frac{\sqrt{z_-}}{T}}\cos r + e^{-2\frac{\sqrt{z_-}}{T}}\right) + \frac{T}{2}\int\frac{d^3p}{(2\pi)^3}\ln\left(1 - e^{-\frac{\sqrt{z_-}}{T}}\right)^2. \quad (62)
\end{aligned}$$

Putting all of this together, we find that the one-loop effective potential at finite temperature is

$$\begin{aligned}
V(r, m^2, \mathbb{A}) &= V_{T=0} + T\int\frac{d^3p}{(2\pi)^3}\ln\left(1 - 2e^{-\frac{\sqrt{z_+}}{T}}\cos r + e^{-2\frac{\sqrt{z_+}}{T}}\right) + \frac{T}{2}\int\frac{d^3p}{(2\pi)^3}\ln\left(1 - e^{-\frac{\sqrt{z_+}}{T}}\right)^2\\
&\quad + T\int\frac{d^3p}{(2\pi)^3}\ln\left(1 - 2e^{-\frac{\sqrt{z_-}}{T}}\cos r + e^{-2\frac{\sqrt{z_-}}{T}}\right) + \frac{T}{2}\int\frac{d^3p}{(2\pi)^3}\ln\left(1 - e^{-\frac{\sqrt{z_-}}{T}}\right)^2\\
&\quad + 2T\int\frac{d^3p}{(2\pi)^3}\ln\left(1 - 2e^{-\frac{\sqrt{\vec{p}^2+m^2-\frac{\mathbb{A}}{3}}}{T}}\cos r + e^{-2\frac{\sqrt{\vec{p}^2+m^2-\frac{\mathbb{A}}{3}}}{T}}\right) + T\int\frac{d^3p}{(2\pi)^3}\ln\left(1 - e^{-\frac{\sqrt{\vec{p}^2+m^2-\frac{\mathbb{A}}{3}}}{T}}\right)^2\\
&\quad - 2T\int\frac{d^3p}{(2\pi)^3}\ln\left(1 - 2e^{-\frac{\sqrt{\vec{p}^2}}{T}}\cos r + e^{-2\frac{\sqrt{\vec{p}^2}}{T}}\right) - T\int\frac{d^3p}{(2\pi)^3}\ln\left(1 - e^{-\frac{\sqrt{\vec{p}^2}}{T}}\right)^2. \quad (63)
\end{aligned}$$

The zero-temperature contribution does not depend on r and is therefore the same as what was found in Ref. [119], namely,

$$V_{T=0} = \frac{N^2-1}{2(4\pi)^2}\left\{\frac{1}{18}\ln\left(\frac{m^2-\mathbb{A}/3}{\bar{\mu}^2}\right)[7\mathbb{A}^2+27m^4] + \left[-\frac{155}{522}\mathbb{A}^2+\frac{11}{12}\mathbb{A}m^2-\frac{87}{26}m^4+\frac{1}{4}\frac{m^6}{\mathbb{A}}\right]\right.$$
$$+\frac{1}{18}[5\mathbb{A}^2+12\mathbb{A}m^2+9m^4]\left[\ln\left(\frac{\mathbb{A}}{\mathbb{A}-3m^2}\right)+\ln\left(1+\sqrt{\frac{m^2+\mathbb{A}}{m^2-\frac{\mathbb{A}}{3}}}\right)-\ln\left(1-\sqrt{\frac{m^2+\mathbb{A}}{m^2-\frac{\mathbb{A}}{3}}}\right)\right]$$
$$\left.-\frac{(m^2-\frac{\mathbb{A}}{3})}{12\mathbb{A}}(6\mathbb{A}^2+11\mathbb{A}m^2+3m^4)\sqrt{\frac{m^2+\mathbb{A}}{m^2-\frac{\mathbb{A}}{3}}}+\frac{9}{13}\frac{(4\pi)^2}{g^2N}m^4+\frac{1}{3}\frac{(4\pi)^2}{g^2N}\mathbb{A}^2\right\}, \tag{64}$$

with $N = 2$.

The extension to SU(3) is straightforward. Given that we can restrict to $r_8 = 0$ and that the label κ spans two zeros rather than one, together with the SU(3) roots given above, with respective projections along the direction 3 being ± 1 and twice $\pm 1/2$, the SU(3) formula for the potential as a function of $r \equiv r_3$ can be obtained from (63) by (1) using $V_{T=0}$ with $N = 3$, (2) duplicating the integrals that do not depend on r, (3) keeping the integrals that depend on r, and (4) adding twice these integrals with $r \to r/2$.

V. NUMERICAL RESULTS AND DISCUSSION

A. Zero-temperature limit and parameter setting

Using Eq. (64), it can be checked that one has $\mathbb{A} = 0$ at $T = 0$; see Ref. [119]. Then, solving $\partial V_{T=0}/\partial m^2 |_{m^2=m^2_{\min,T=0}} = 0$ with the renormalization group optimized choice $\bar{\mu} = m_{\min,T=0}$, one finds

$$\frac{g^2(m^2_{\min,T=0})N}{16\pi^2} = \frac{36}{187}. \tag{65}$$

Using the one-loop β function and $\Lambda_{\overline{\rm MS}} \approx 0.752\sqrt{\sigma}$ for $N = 2$ [121], that is, $\Lambda^{N=2}_{\overline{\rm MS}} \approx 0.331$ GeV using $\sqrt{\sigma} = 0.44$ GeV for the scale setting, one finds the solution

$$m^2_{\min,T=0} = e^{17/12}\Lambda^{N=2}_{\overline{\rm MS}} \approx 0.451\ \text{GeV}^2. \tag{66}$$

In the SU(3) case, one finds instead

$$m^2_{\min,T=0} = e^{17/12}\Lambda^{N=3}_{\overline{\rm MS}} \approx 0.207\ \text{GeV}^2 \tag{67}$$

based on $\Lambda^{N=3}_{\overline{\rm MS}} \approx 0.224$ GeV [122].

Interestingly enough, these correspond to values of the mass parameter equal to 672 and 455 MeV, respectively, pretty close to those obtained when fitting Landau gauge propagators using the Curci-Ferrari model; Ref. [79] used 710 and 510 MeV, respectively. We will see below that the similarities with the CF model do not end here.

With the parameters fixed at $T = 0$, we can now study finite temperature effects and their impact on both the Polyakov loop and the asymmetry.

B. Without asymmetry

Following the structure of the previous section, let us first assume that there is no asymmetry.

For each temperature, we find the values $r_{\min}(T)$ and $m^2_{\min}(T)$ of r and m^2 that minimize the potential $V(r, m^2)$. We notice that the minimization might be tricky since the potential is defined only over the semiaxis $m^2 \geq 0$. In particular, the absolute minimum of the potential could be located at $m^2 = 0$ without corresponding to a stationary point. Of course, we should follow the deepest stationary minimum as this corresponds to the limit of zero sources. The results of following this minimum are shown in Fig. 1. As explained earlier, the fact that $r_{\min}$ is moving away from its center-symmetric value, $r = \pi$ in the SU(2) case and $r = 4\pi/3$ in the SU(3) case, indicates deconfinement. The transition is continuous in the SU(2) case and first order in the SU(3) case as expected. This is further illustrated in Fig. 2, in which we show the potential as a function of r for temperatures just below T_c, at T_c, and just above T_c. More precisely, what is shown in this figure is the potential $V(r, m^2_{\min})$, where m^2 has been adjusted to $m^2_{\min}$ at each temperature. Although convenient in the SU(2) case, this is not the most efficient way to illustrate the transition in the SU(3) case because $m^2_{\min}$ has a jump at T_c, which complicates the interpretation. This is illustrated in the right plot of Fig. 2 and the corresponding caption. A more convenient quantity in this case is the reduced potential $V(r) \equiv V(r, m^2(r))$, where $m^2(r)$ is obtained by minimizing with respect to m^2 at fixed r (by this, we mean again locating the deepest stationary minimum). The reduced potential is also shown in Fig. 2. We see that we recover the usual interpretation of the transition in the SU(3) case. As for the SU(2) case, the interpretation is essentially the same whether we use $V(r, m^2_{\min})$ or $V(r, m^2(r))$.

Using the zero-temperature parameters given above, the transition temperatures are found to be $T_c = 0.230$ GeV in the SU(2) case and $T_c = 0.1635$ GeV in the SU(3) case. Surprisingly, these values are quite close to those reported in Ref. [79] within the one-loop CF model. This can be understood in a simple way as follows. First, we already noticed above that the values for $m^2_{\min}$ at $T = 0$ are rather

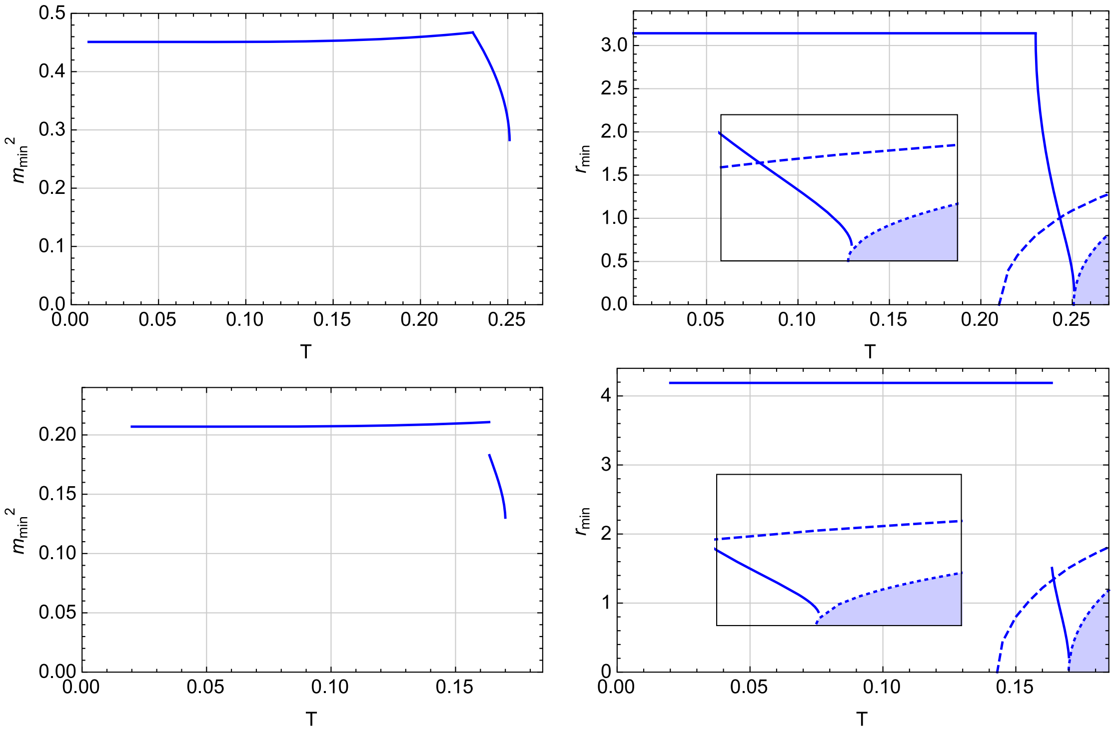

FIG. 1. The condensate $m^2_{\rm min}$ (left) and background $r_{\rm min}$ (right) at the minimum of the potential for the SU(2) (top) and SU(3) (bottom) theories, under the assumption that $\mathbb{A}=0$. The dashed lines in the right plots show the region below which the absolute minimum of the potential is no longer a stationary point, although there is still a local stationary minimum. The dotted lines show the region below which the stationary minimum is lost. The inset zooms in on the region above the transition where the solution is lost; see the text below for more details. Units in GeV.

close to those of the CF mass parameter, both in the SU(2) case and in the SU(3) case. Moreover, we note from Fig. 1 that, below T_c, $m^2_{\rm min}$ changes marginally and thus remains close to its value at $T=0$. We have thus found that, at least at the present level of approximation, the dynamically generated condensate basically reproduces (from a BRST-invariant setup) the one-loop CF model in the low-temperature phase (which usually features a constant mass). It is then not a surprise that the obtained transition temperatures are close to those in the CF model.

The marginal variation of $m^2_{\rm min}$ with T in the low-temperature phase can be further understood as follows. The gap equation that determines $m^2_{\rm min}$ is

$$0=\frac{\partial V}{\partial m^2}\bigg|_{m^2_{\rm min}}=\zeta m^2_{\rm min}\left(1-\frac{\delta\zeta}{\zeta}\right)+\frac{d-1}{2}\mathrm{tr}\frac{1}{P^2_\kappa+m^2_{\rm min}},\tag{68}$$

where we note that only massive integrals contribute. In the low-temperature phase, r should be taken equal to π [we consider the SU(2) case first for simplicity but the result generalizes to $SU(N)$]. In the presence of this confining background, the tadpole integrals corresponding to $\kappa=\pm1$ become fermionic tadpole integrals. We can then separate the $T=0$ part from the thermal part and write

$$0=\frac{\partial V_{T=0}}{\partial m^2}\bigg|_{m^2_{\rm min}}+\frac{3}{4\pi^2}\int_0^\infty dq\frac{q^2}{\sqrt{q^2+m^2_{\rm min}}}\times\left(\frac{1}{e^{\sqrt{q^2+m^2_{\rm min}}/T}-1}-\frac{2}{e^{\sqrt{q^2+m^2_{\rm min}}/T}+1}\right).\tag{69}$$

To study the behavior of $m^2_{\rm min}$ as $T\to0$, we write $m^2_{\rm min}=m^2_{\rm min,T=0}+\delta m^2_{\rm min}$ with $\delta m^2_{\rm min}$ small and expand the previous formula to first nontrivial order. Because

$$0=\frac{\partial V_{T=0}}{\partial m^2}\bigg|_{m^2_{\rm min,T=0}},\tag{70}$$

this first nontrivial order yields

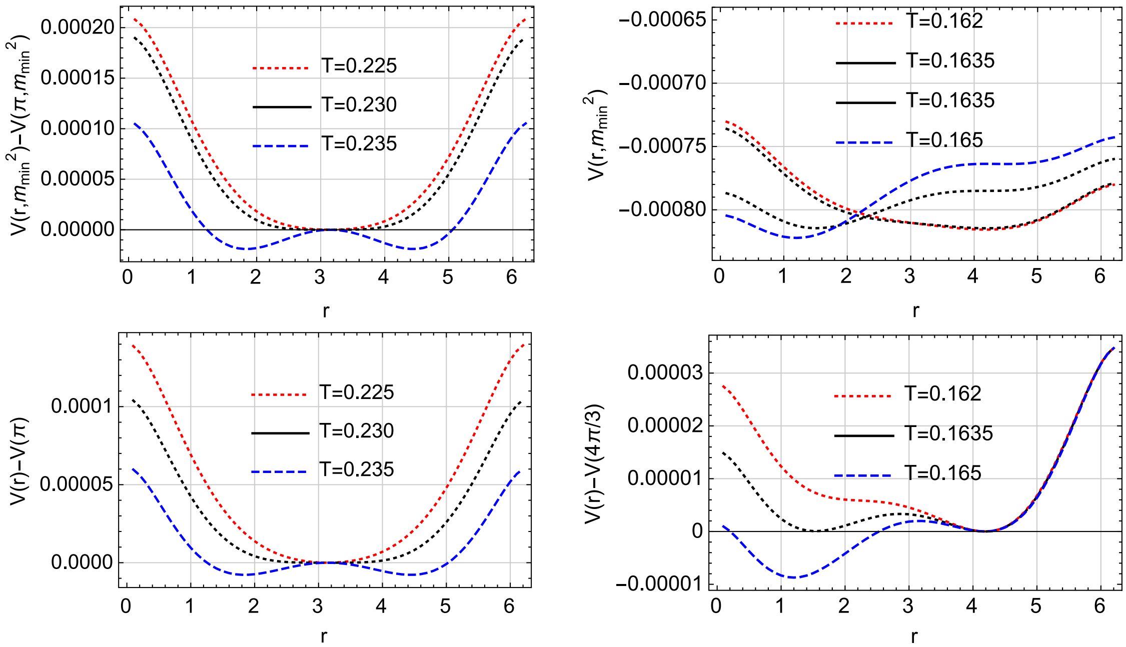

FIG. 2. Top: the potential $V(r, m^2_{\rm min})$ for different temperatures, in the SU(2) case (left) and in the SU(3) case (right). In this latter case, because $m^2_{\rm min}$ jumps at the transition, $V(r, m^2_{\rm min})$ does not provide the usual picture of a transition with degenerate minima. What happens is that just below T_c the minimum is located at $4\pi/3$, corresponding to a certain value of $m^2_{\rm min}(T < T_c)$. Similarly, above T_c, the minimum is located away from $4\pi/3$, corresponding to a certain value of $m^2_{\rm min}(T > T_c)$, and one has $m^2_{\rm min}(T_c^-) \neq m^2_{\rm min}(T_c^+)$. When approaching T_c from below, $V(r, m^2_{\rm min}(T < T_c))$ never develops a broken symmetry form, and similarly when approaching T_c from above, $V(r, m^2_{\rm min}(T > T_c))$ always keeps its broken form. What happens at $T = T_c$ is that the symmetric form of $V(r, m^2_{\rm min}(T < T_c))$ is replaced by the broken form of $V(r, m^2_{\rm min}(T > T_c))$. Bottom: the reduced potential $V(r) = V(r, m^2(r))$ for different temperatures, in the SU(2) case (left) and in the SU(3) case (right). Units in GeV.

$$\delta m^2_{\rm min} \sim \frac{3}{4\pi^2} \frac{\int_0^\infty dq \frac{q^2}{\sqrt{q^2+m^2_{{\rm min},T=0}}} \left(\frac{2}{e^{\sqrt{q^2+m^2_{{\rm min},T=0}}/T}+1} - \frac{1}{e^{\sqrt{q^2+m^2_{{\rm min},T=0}}/T}-1} \right)}{\partial^2 V_{T=0}/\partial(m^2)^2|_{m^2_{{\rm min},T=0}}}. \tag{71}$$

The numerator can be approximated using low-temperature expansions for the tadpole integrals, whereas the denominator can be computed explicitly. We arrive at

$$\frac{\delta m^2_{\rm min}}{m^2_{{\rm min},T=0}} \sim \frac{1}{3\pi} \left(\frac{m_{{\rm min},T=0}}{2\pi T} \right)^{1/2} e^{-m_{{\rm min},T=0}/T}, \tag{72}$$

so that $m^2_{\rm min}$ approaches its $T = 0$ value exponentially. In fact, since $m_{{\rm min},T=0} \simeq 3T_c$, the exponential factor remains tiny over the whole confined phase, which, in turn, explains why the mass changes marginally in this phase. Above the transition, the background r departs from π and becomes a function of T that introduces a new source of T dependence in the right-hand side of Eq. (69). This explains why $m^2_{\rm min}$ can have a stronger variation in the deconfined phase; see Fig. 1.

So far, we have been concerned with the physical solution of the gap equation as given by the minimum of $V(r, m^2)$ and which corresponds to a nonzero $m^2_{\rm min}$ at $T = 0$. At $T = 0$, there is another solution, $m^2 = 0$, corresponding to a maximum of $V_{T=0}$. Its fate when $T > 0$ is important for it controls what happens with the physical solution for $T > T_c$ as we now argue. First, let us have a look at the m^2 derivative of $V(r_{\rm min}, m^2)$ at $m^2 = 0$. This derivative reads

$$\left.\frac{\partial V}{\partial m^2}\right|_{m^2=0} = \frac{3}{2} {\rm tr} \frac{1}{P^2_\kappa} = \frac{3T^2}{4} \left[B_2(0) + 2B_2\left(\frac{r_{\rm min}}{2\pi}\right) \right], \tag{73}$$

where we have used that, in the presence of a background, the massless tadpole integral can be written in terms of the Bernouilli polynomial $B_2(x) = x^2 - x + 1/6$. Now, as long as $T < T_c$, we have $r_{\rm min} = \pi$ [again, we consider the SU(2) case, but the proof generalizes to $SU(N)$], and the term between brackets writes

$$B_2(0) + 2B_2\left(\frac{1}{2}\right) = \frac{1}{6} + 2\left(\frac{1}{4} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{6} + 2\left(-\frac{1}{12}\right) = 0, \tag{74}$$

which is nothing but the well-known cancellation between the bosonic and fermionic tadpole integrals appearing in (69) in the massless case. This means that, as long as $T < T_c$, in addition to the physical minimum at $m^2_{\rm min}$, the potential $V(r_{\rm min}, m^2)$ has a maximum at $m^2 = 0$. In contrast, whenever $T > T_c$, $r = \pi + 2\pi x$, and we find

$$B_2(0) + 2B_2\left(\frac{1}{2} + x\right) = \frac{1}{6} + 2\left(\frac{1}{4} + x + x^2 - \frac{1}{2} - x + \frac{1}{6}\right) = 2x^2 > 0. \tag{75}$$

This means that, for $T > T_c$, the maximum of $V(r_{\rm min}, m^2)$ is pushed toward $m^2 > 0$ values. As we continue increasing the temperature further, we find that this maximum eventually merges with the physical minimum, thus causing the loss of both extrema above some temperature.

To clarify this feature further, we observe that, above some temperature ($T > T_1 > T_c$) and below a certain background ($r < r_1(T)$), the function $V(r, m^2)$ has no extrema with respect to m^2. The value $r_1(T)$ corresponds to the appearance of a spinodal located at $m^2_1(T)$ at which the minimum and the maximum in m^2 merge. The functions $r_1(T)$ and $m^2_1(T)$ can be obtained by solving the coupled equations $\partial V/\partial m^2 = 0$ and $\partial^2 V/\partial(m^2)^2 = 0$ for each temperature above T_1. The function $r_1(T)$ is represented by a dotted curve in the right plots of Fig. 1. The inset in the plot shows that the curve $r_{\rm min}(T)$ eventually meets the curve $r_1(T)$ at a certain temperature $T_{\rm loss} > T_1$ beyond which the solution is lost. We find $T_{\rm loss} \simeq 0.2504$ GeV in the SU(2) case and $T_{\rm loss} \simeq 0.1715$ GeV in the SU(3) case. In conclusion, the deconfined phase can only be explored within a tiny range of temperatures above T_c. Let us also mention that the existence of $r_1(T)$ implies that, for $T > T_1$, the reduced potential introduced above is defined only for $r \in [r_1(T), 2\pi - r_1(T)]$ [we consider the SU(2) case for illustration].

Finally, for completeness, we mention that there is another function $r_2(T)$ worth mentioning for $T > T_2$ (with $T_2 < T_1$). For $r < r_2(T)$, the absolute minimum of $V(m^2, r)$ in the direction of m^2 is not a stationary point anymore (in the sense of a vanishing derivative). Rather, it is located at $m^2 = 0$ where it has a nonvanishing derivative. This means that when $r_{\rm min}(T)$ goes below that line (represented by a dashed curve in the right plots of Fig. 1) we are not following the absolute minimum of the potential anymore but rather the deepest stationary minimum. This should be fine, however, since this is the point that should correspond to the limit of zero sources.

C. With asymmetry

When we allow for the asymmetry, we have to minimize the potential with respect to three parameters: r, m^2, and $\mathbb{A}$. Let us consider, for simplicity, the SU(2) case. Up to $T = 0.234$ GeV, we find minima at $r_{\rm min} = \pi$, indicating that we are in the confined phase. The minimizing values for $\mathbb{A}$ and m^2 for $T < 0.234$ GeV are given in Fig. 3. Above $T = 0.234$ GeV, the numerics become less trustworthy. It should be mentioned here that the minimization is complicated by the fact that the arguments of the potential obey certain constraints. For instance, for a given m^2, $\mathbb{A}$ is bounded from above by $3m^2$.

What is certain is that, above T_c, $r = \pi$ is no longer a minimum. However, the values for the minimizing parameters that we find numerically at $T > T_c$, taking, for example, $T = 0.235$ GeV, do not give the exact minimum of V. To cure this, we can, in principle, perform some fine-tuning in the following way: we take the values of r and $\mathbb{A}$ that we found and plug them into the potential, and we minimize the potential for m^2 and find a new value of m^2. We then do the same but keeping m^2 and $\mathbb{A}$ constant at their last-found values to find a new r which minimizes the potential. Finally we do the same by keeping r and m^2 constant and minimizing for $\mathbb{A}$. We then go back to the first step, and we go on until we have found a stable minimum. As it turns out, however, successively minimizing in this way leads to a loss of the stationary minimum of $V(m^2)$, as one can see in Fig. 4, in which we have plotted $V(m^2)$ for several iterations of the fine-tuning procedure. It thus appears that the minimum we found for temperatures in the confining phase simply disappears above the transition. Also, the absolute minimum will now lie at $m^2 = 0$, where the effective potential becomes complex. Something similar was also observed in Ref. [123]; see also Ref. [124].

The previous difficulties can be illustrated further as follows. One checks that, for any given r (or at least in the vicinity of $r = \pi$), the potential admits a minimum in the plane $(m^2, \mathbb{A})$. This allows one to define $m^2(r)$ and $\mathbb{A}(r)$ and then the reduced potential $V(r) \equiv V(r, m^2(r), \mathbb{A}(r))$, similarly to what we did in the previous section. The reduced potential is shown in Fig. 4 for various temperatures. We see that at some temperature below T_c the maximum at $r = 0$ (or $r = 2\pi$) starts moving into the interval $]0, 2\pi[$. Eventually, it fuses with the minimum at

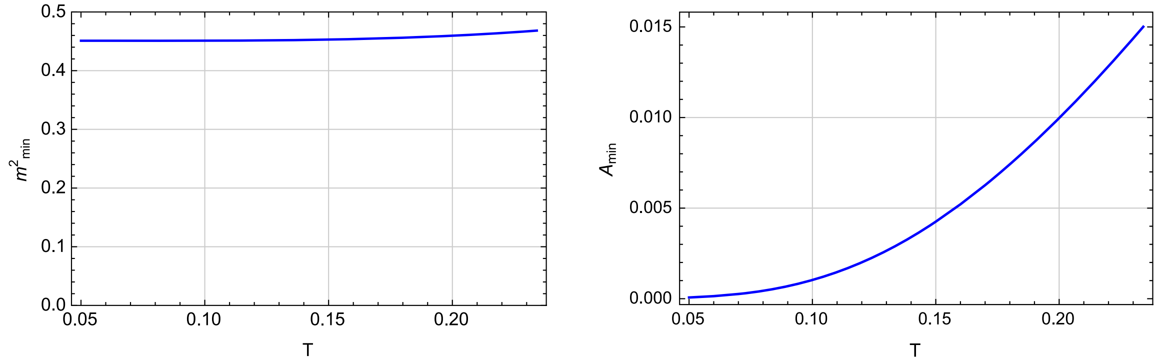

FIG. 3. The parameters m^2_{min} (top) and $\mathbb{A}_{\mathrm{min}}$ (bottom) for the SU(2) theory, shown for temperatures where $r_{\mathrm{min}} = \pi$. Units in GeV.

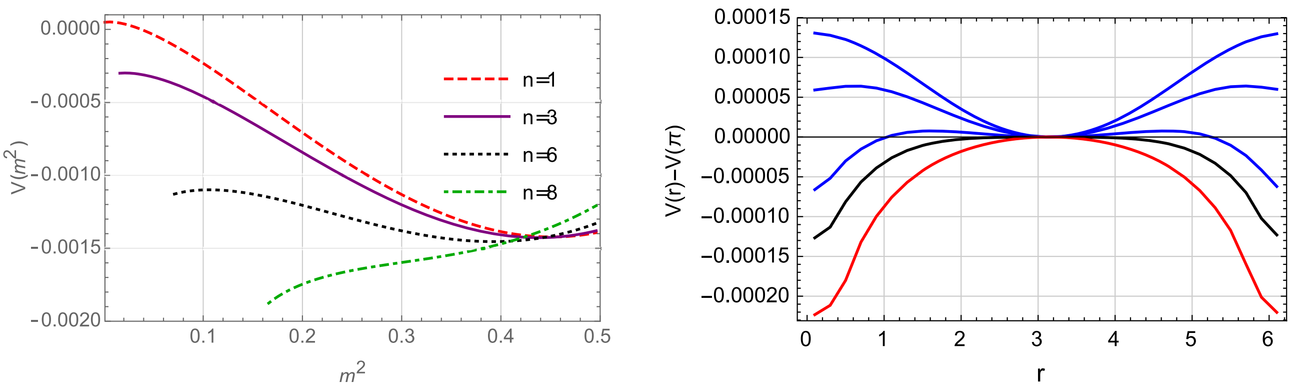

FIG. 4. Left: $V(m^2)$ where r and $\mathbb{A}$ are constants found by minimizing the potential in the fine-tuning process as described in the text, given for different numbers of iterative fine-tuning n. Right: the reduced potential $V(r) = V(r, m^2(r), \mathbb{A}(r))$ and the loss of solution at T_c. All values are in GeV.

$r = \pi$. At this spinodal, the curvature is 0, which serves as an identification of T_c.[10] At the same time, however, the minimum disappears, and one cannot follow it into the deconfined phase. Our conclusion is then that, in the presence of asymmetry, we have no access to the deconfined phase, at least not within our current level of treatment.

It is commonly known that at high(er) temperatures resummations are in order to save the perturbative expansion; see, e.g., Refs. [125,126] or the reviews [127,128]. We will not attempt this here, as our main focus was on determining the deconfinement transition and its interplay with the dimension-2 condensates. Our findings are, however, clear: the critical deconfinement estimate, $T_c \approx 0.231$ GeV, is very close to the one-loop estimate reported in Ref. [79] using the $T = 0$ Curci-Ferrari mass fit parameter, namely, $T_c \approx 0.238$ GeV. We repeat here we did not use any external (lattice) input, except for the estimate of $\Lambda_{\overline{\mathrm{MS}}}$ of course. *A posteriori*, this is not such a surprise; the used value for the (temperature independent) gluon mass in Ref. [79] was $m(T) \approx 0.710$ GeV, while here we find that the dynamical $m(T)$ indeed varies only little from its $m(T = 0) \approx 0.670$ GeV value [cf. Eq. (66)], next to a pretty small asymmetry. For the record, functional approaches as in Ref. [39] arrived at $T_c = 0.230 \pm 0.023$ GeV, i.e., all values in the same ballpark. This extends to the variational estimate of Ref. [129], $T_c \approx 0.239$ GeV or the Coulomb gauge variational estimate $T_c \approx 0.275$–0.290 GeV [130]. For comparison, lattice estimates for the SU(2) transition temperate are $T_c \approx 0.295$ GeV [2] or $T_c \approx 0.312$ GeV [1,129].

We find similar difficulties for the SU(3) case.

Let us spend a few more words on the asymmetry $\mathbb{A}$. Next to the pioneering Landau gauge lattice study of Ref. [63] and later efforts as in Refs. [131,132], the only analytical investigation of it so far is Ref. [123] by one of us; see also Ref. [119]. As the Polyakov loop was not part of that approach, nothing could be said about the sensitivity of the asymmetry to the phase transition. Now, we do have such evidence by working in the Landau-DeWitt gauge, albeit the findings are not exactly numerically comparable, not only because we do not have results in the deconfined

[10] It could seem from Fig. 4 that, after the maximum at $r = 0$ enters the $r > 0$ region, there is a stationary minimum at $r = 0$ that can become the absolute one before the destabilization of the one at $r = \pi$; this would then lead to a first-order transition. We have checked that, before this happens, the reduced potential in the vicinity of $r = 0$ is, however, no longer well-defined because of the inability to find a stationary $(m(r), \mathbb{A}(r))$.

phase, but also the magnitude of the asymmetry is considerably smaller than that reported on the lattice.

VI. CONCLUSIONS

Let us end by pointing out a few possible routes toward a further development of the framework proposed here. It has been conjectured that the electric and magnetic Landau gauge propagators at zero momentum, corresponding to the respective screening masses, are sensitive to the phase transition (and even its order) in Ref. [8].[11] This scenario is, however, far from clear in the Landau gauge [133–135]. In fact, it has been argued in Refs. [136,137] that there is no actual reason for the Landau gauge propagators to be sensitive to the transition because the average gluon field is not an order parameter in this case. Still, in Refs. [136,137], a particular background Landau gauge (referred to as *center symmetric*) has been put forward in which the background takes a center-invariant configuration in any phase. In this gauge, the average gluon field becomes an order parameter, and, correspondingly, the electric propagator shows distinctive features at the transition [137]. It would be interesting to revisit the present considerations in this particular gauge. Evidently, one expects the asymmetric gluon condensate to influence exactly the aforementioned quantities.

The center-symmetric Landau gauge is closely related to the formalism used in the present work and based on the background effective potential. However, if the former relies on the use of a standard Legendre transform, the justification of the latter (and the equivalence with the former) relies on the background independence of the free energy [78]. This property is not necessarily met identically within an approximated setting or in the presence of modeling.[12] For instance, in the case of the Curci-Ferrari model, the use of the center-symmetric Legendre transform leads to improved predictions as compared to those obtained using the background effective potential. In the present, BRST-based approach, the background independence of the observables, and in particular of the free energy, is protected by the combined use of the BRST symmetry of the action, the BRST exactness of the Faddeev-Popov action, and the fact that the background dependence in the σ sector cancels identically upon exact integration of σ. So, even though, as we have seen, the dynamical condensate mimics the Curci-Ferrari model in the low-temperature phase, we expect a better account of the background independence of the observables and therefore a better agreement between the present approach and that based on the center-symmetric Landau gauge. Results in this direction will be reported elsewhere.

Overall, the results presented are eventually rather similar to the ones of the phenomenological massive Curci-Ferrari-Landau-DeWitt model [50,79,80], the big step forward being that the (crucial) nonperturbative mass scales now have a dynamical origin and that BRST is maintained. *A posteriori*, our setup grants credit to the aforementioned model, even at the quantitative level as we have discussed. At least at one-loop order, we do not expect much will change for what concerns the other thermodynamical observables such as pressure, entropy, trace anomaly, etc. when compared to Ref. [80], but a more thorough discussion of this is relegated to future work. Furthermore, although our results are nonperturbative in nature, they do arise from an effective potential computed in a loop expansion. It remains to be investigated how stable the results are against adding the two-loop corrections, which should still be analytically tractable since the most complicated pieces have already been computed [80]. Notice that from two loops onward, more changes might occur relative to the Curci-Ferrari-Landau-DeWitt model, as the other vertices containing the σ and φ fields [arising from expanding the action of the unity (21) around the would-be condensates] will enter the game.

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APPENDIX A: RENORMALIZABILITY OF DIMENSION-2 LOCAL COMPOSITE OPERATORS IN THE BACKGROUND GAUGE

In this Appendix, we show the renormalizability to all orders in an algebraic setting of the LCO formalism in the presence of a classical gauge background field $\bar{A}^a_\mu$,

$$a^a_\mu = A^a_\mu + \bar{A}^a_\mu, \tag{A1}$$

where A^a_μ represents the quantum part of the gauge field. The Landau gauge condition $\partial_\mu A^a_\mu = 0$ is now replaced by[13]

[11]Relatedly, the integrated difference between electric and magnetic propagator in relation to the transition was studied recently in Ref. [132].

[12]By modeling, we refer to possible Ansätze for the vertex functions in DSE/FRG approaches, the phenomenological addition of operators beyond the incomplete Faddeev-Popov action (as is done for instance within the Curci-Ferrari model), etc.

[13]Mark that, if $\bar{A}^a_\mu$ is chosen to satisfy the Landau gauge, many of the expressions in this section will simplify considerably. Nevertheless, even if one plans to choose the Landau gauge for the quantum fluctuations, it is for now more opportune to leave $\bar{A}^a_\mu$ more general, as it will allow us to take functional derivatives with respect to $\bar{A}^a_\mu$ and Ω^a_μ more freely.

$$\bar{D}_\mu A^a_\mu = \bar{D}_\mu(a^a_\mu - \bar{A}^a_\mu) = 0, \tag{A2}$$

where $\bar{D}_\mu$ is the covariant derivative containing only the background field $\bar{A}^a_\mu$. In this gauge, the ghost action is changed accordingly to

$$\mathcal{L}_{\rm gh} = \int d^4x \bar{c}^a \bar{D}_\mu D_\mu c^a. \tag{A3}$$

The condensate $\langle A^2_\mu \rangle$ we want to compute the vacuum expectation value of will, of course, also need to be modified. It turns out that, if we demand renormalizability of the action, the operator $A^2_\mu = (a^a_\mu - \bar{A}^a_\mu)^2$ is to be considered. Let us now prove that this is indeed the only possible choice. For this, we use the algebraic renormalization formalism, and the computations outlined below are parallel to those done by one of us and coworkers in the linear covariant gauges [138]. The algebraic analysis of the background gauge has already been explored in Ref. [100], and their approach is used in the following.

1. Classical action

We start from the action of pure Yang-Mills theory in the Landau background gauge:

$$S_{\rm YM+gf} = \int d^4x \left(\frac{1}{4}(F^a_{\mu\nu})^2 + ib^a \bar{D}_\mu(a^a_\mu - \bar{A}^a_\mu) + \bar{c}^a \bar{D}_\mu D_\mu c^a \right). \tag{A4}$$

Here, we introduced the Nakanishi-Lautrup field b^a, which is a Lagrange multiplier for the gauge fixing condition. A second part of the action consists of the source field J coupled to the operator we are considering, which we leave more general for now,

$$S_J = \int d^4x \left(\frac{1}{2} J(a^2_\mu + \alpha a^a_\mu \bar{A}^a_\mu + \beta \bar{A}^2_\mu) + \frac{\zeta}{2} J^2 \right), \tag{A5}$$

where the term in J^2 has been added to absorb the quadratic divergences in the source field. The numbers α and β will be determined by demanding renormalizability. The parameter ζ has to be introduced here, and just as in the case without a background field, it will have to be determined using other considerations; cf. the main body of this paper. Finally, we introduce classical source fields Δ^*, A^{*a}_μ, and c^{*a} coupling to the nonlinear BRST variations of the fields and operators under consideration,

$$S_{\rm ext} = \int d^4x \left(\Delta^* \left(\left(a^a_\mu + \frac{\alpha}{2} \bar{A}^a_\mu \right) D_\mu c^a - \left(\frac{\alpha}{2} a^a_\mu + \beta \bar{A}^a_\mu \right) \Omega^a_\mu \right) - A^{*a}_\mu (D_\mu c^a + \Omega^a_\mu) + \frac{1}{2} g f^{abc} c^{*a} c^b c^c \right), \tag{A6}$$

where we have introduced the ghost field Ω^a_μ, which is the BRST transformation of $\bar{A}^a_\mu$. The term in $A^{*a}_\mu \Omega^a_\mu$ has been added in order to allow to absorb the counterterms later on. If we add one final term

$$\int d^4x \bar{c}^a D_\mu \Omega^a_\mu \tag{A7}$$

necessary to cancel some spurious terms coming from the BRST variation of $\bar{A}^a_\mu$, the total action is invariant under the nilpotent BRST transformation s defined by

$$\begin{aligned} s a^a_\mu &= -D_\mu c^a, \qquad s c^a = \frac{1}{2} g f^{abc} c^b c^c, \\ s\bar{c}^a &= ib^a, \qquad s\bar{A}^a_\mu = \Omega^a_\mu, \qquad s\Delta^* = J, \\ s b^a &= sJ = s\Omega^a_\mu = sA^{*a}_\mu = sc^{*a} = 0. \end{aligned} \tag{A8}$$

Notice that the possibility of introducing the background field $\bar{A}_\mu$ as part of a BRST doublet is almost immediately leading to the independence on $\bar{A}_\mu$ of observables, defined as elements of the BRST cohomology with zero ghost charge [101,120]. Indeed, doublets are trivial elements of the cohomology [120].

The full action can be rewritten in the form

$$S = \frac{1}{4} \int d^4x (F^a_{\mu\nu})^2 + s \int d^4x \left(\bar{c}^a \bar{D}_\mu (a^a_\mu - \bar{A}^a_\mu) + \frac{1}{2} \Delta^* (a^2_\mu + \alpha a^a_\mu \bar{A}^a_\mu + \beta \bar{A}^2_\mu) - A^{*a}_\mu (a^a_\mu - \bar{A}^a_\mu) + c^{*a} c^a + \frac{\zeta}{2} \Delta^* J \right). \tag{A9}$$

From this form, the BRST invariance is easy to see: working with s on the first term will give a mere gauge transformation with c^a as the gauge function, and working on the second part will give zero, as s is nilpotent by definition.

At the classical level, the theory is characterized by some powerful identities. We have the Slavnov-Taylor identity

$$\mathcal{S}(S) = \int d^4x \left(\frac{\delta S}{\delta a^a_\mu} \frac{\delta S}{\delta A^{*a}_\mu} + \frac{\delta S}{\delta c^a} \frac{\delta S}{\delta c^{*a}} + ib^a \frac{\delta S}{\delta \bar{c}^a} + \Omega^a_\mu \frac{\delta S}{\delta \bar{A}^a_\mu} + J \frac{\delta S}{\delta \Delta^*} \right) = 0, \tag{A10a}$$

which is nothing but a reexpression of the BRST invariance of the action. Then we have the equation for the Nakanishi-Lautrup field:

$$\frac{\delta S}{\delta b^a} = i\bar{D}_\mu(a_\mu^a - \bar{A}_\mu^a). \tag{A10b}$$

Next we have the antighost equation:

$$\frac{\delta S}{\delta \bar{c}^a} + \bar{D}_\mu \frac{\delta S}{\delta A^{*a}_\mu} = D_\mu \Omega_\mu^a, \tag{A10c}$$

which can be straightforwardly found by taking the derivative with respect to the antighost field $\bar{c}^a$ and rewriting the composite operator $D_\mu c^a$ as a derivative with respect to A^{*a}_μ. And finally we have the ghost Ward identity:

$$\begin{aligned}&\frac{\delta S}{\delta c^a} + \bar{D}_\mu \frac{\delta S}{\delta \Omega_\mu^a} - igf^{abc}\bar{c}^b \frac{\delta S}{\delta b^c}\\ &= \left(1+\frac{\alpha}{2}\right)\partial_\mu(\Delta^* a_\mu^a) + \left(\frac{\alpha}{2}+\beta\right)\partial_\mu(\Delta^*\bar{A}_\mu^a)\\ &\quad - D_\mu A^{*a}_\mu + gf^{abc}c^b c^{*c}.\end{aligned} \tag{A10d}$$

This last identity can be found by first taking the derivative of the action with respect to the ghost field c^a:

$$\begin{aligned}\frac{\delta S}{\delta c^a} &= -D_\mu \bar{D}_\mu \bar{c}^a + \partial_\mu(\Delta^* a_\mu^a) + \frac{\alpha}{2}D_\mu(\Delta^*\bar{A}_\mu^a)\\ &\quad - D_\mu A^{*a}_\mu + gf^{abc}c^b c^{*c}.\end{aligned} \tag{A11}$$

Then, we note that

$$[\bar{D}_\mu, D_\mu]^{ab} = -gf^{abc}\bar{D}_\mu(a_\mu^c - \bar{A}_\mu^c) = igf^{abc}\frac{\delta S}{\delta b^c}, \tag{A12}$$

which gives

$$\begin{aligned}\frac{\delta S}{\delta c^a} - igf^{abc}\bar{c}^b\frac{\delta S}{\delta b^c} &= -\bar{D}_\mu D_\mu \bar{c}^a + \partial_\mu(\Delta^* a_\mu^a)\\ &\quad + \frac{\alpha}{2}D_\mu(\Delta^*\bar{A}_\mu^a) - D_\mu A^{*a}_\mu + gf^{abc}c^b c^{*c}.\end{aligned} \tag{A13}$$

To get rid of the composite operator term with $D_\mu \bar{c}^a$, we consider

$$\frac{\delta S}{\delta \Omega_\mu^a} = \frac{\alpha}{2}\Delta^* a_\mu^a + \beta\Delta^*\bar{A}_\mu^a + D_\mu \bar{c}^a. \tag{A14}$$

Using this, we immediately find (A10d).

2. Most general counterterm

When doing perturbation theory, counterterms have to be added to the classical theory. If we write this as $S + \epsilon S^{\rm ct}$, where ϵ is the perturbation parameter, then we can demand the full action to obey the same set of identities (A10) up to leading order in ϵ. For the counterterm, this translates to the conditions

$$\mathcal{B}_S S^{\rm ct} = 0, \tag{A15a}$$

$$\frac{\delta S^{\rm ct}}{\delta b^a} = 0, \tag{A15b}$$

$$\frac{\delta S^{\rm ct}}{\delta \bar{c}^a} + \bar{D}_\mu \frac{\delta S^{\rm ct}}{\delta A^{*a}_\mu} = 0, \tag{A15c}$$

$$\frac{\delta S^{\rm ct}}{\delta c^a} + \bar{D}_\mu \frac{\delta S^{\rm ct}}{\delta \Omega_\mu^a} = 0, \tag{A15d}$$

where $\mathcal{B}_S$ is the linearized operator

$$\begin{aligned}\mathcal{B}_S = \int d^4x \bigg(&\frac{\delta S}{\delta a_\mu^a}\frac{\delta}{\delta A^{*a}_\mu} + \frac{\delta S}{\delta A^{*a}_\mu}\frac{\delta}{\delta a_\mu^a} + \frac{\delta S}{\delta c^a}\frac{\delta}{\delta c^{*a}} + \frac{\delta S}{\delta c^{*a}}\frac{\delta}{\delta c^a}\\ &+ ib^a\frac{\delta}{\delta \bar{c}^a} + \Omega_\mu^a \frac{\delta}{\delta \bar{A}_\mu^a} + J\frac{\delta}{\delta \Delta^*}\bigg),\end{aligned} \tag{A15e}$$

which is again nilpotent. Now, it follows from the general theory concerning algebraic renormalization that the most general invariant local counterterm can be parametrized as

$$S^{\rm ct} = \frac{p_1}{4}\int d^4x (F_{\mu\nu}^a)^2 + \mathcal{B}_S \int d^4x \Xi, \tag{A16}$$

where Ξ is the most general local polynomial with dimension 4 and ghost number -1. To write this down, we need the dimensions and ghost numbers of the fields and sources:

	a_μ^a	$\bar{A}_\mu^a$	c^a	$\bar{c}^a$	b^a	J	Ω_μ^a	A^{*a}_μ	c^{*a}	Δ^*
Dimension	1	1	0	2	2	2	1	3	4	2
Ghost number	0	0	1	−1	0	0	1	−1	−2	−1

With this, we can write down the most general form for Ξ:

$$\begin{aligned}\Xi = &\, p_2 a_\mu^a A^{*a}_\mu + p_3 \bar{A}_\mu^a A^{*a}_\mu + p_4 c^a c^{*a} + p_5 a_\mu^a \partial_\mu \bar{c}^a\\ &+ p_6 \bar{A}_\mu^a \partial_\mu \bar{c}^a + p_7 gf^{abc}\bar{A}_\mu^a a_\mu^b \bar{c}^c + p_8 gf^{abc}\bar{c}^a\bar{c}^b c^c\\ &+ p_9 b^a \bar{c}^a + p_{10}\Delta^* a_\mu^2 + p_{11}\Delta^*\bar{A}_\mu^a a_\mu^a + p_{12}\Delta^*\bar{A}_\mu^2\\ &+ p_{13}\Delta^*\bar{c}^a c^a + p_{14}\Delta^* J.\end{aligned} \tag{A17}$$

The p_i, $i = 1, \ldots, 14$, are arbitrary parameters. With this form, the constraint (A15a) is automatically fulfilled. Equation (A15b) gives

$$\begin{aligned}&i\bar{D}_\mu(p_2 a_\mu^a + p_3\bar{A}_\mu^a) - ip_5\partial_\mu a_\mu^a - ip_6\partial_\mu \bar{A}_\mu^a + ip_7 gf^{abc}\bar{A}_\mu^b a_\mu^c\\ &\quad + 2ip_8 gf^{abc}\bar{c}^b c^c + 2ip_9 b^a - ip_{13}\Delta^* c^a = 0,\end{aligned} \tag{A18}$$

from which we find

$$p_2 = p_5 = -p_7, \quad p_3 = p_6, \quad p_8 = p_9 = p_{13} = 0. \tag{A19}$$

The constraint (A15c) is now already satisfied. The ghost Ward identity (A15d) gives

$$\begin{aligned}
&(p_2+p_3)\partial_\mu \bar{D}_\mu \bar{c}^a + p_4 D_\mu \bar{D}_\mu \bar{c}^a - p_4 g f^{abc} c^b c^{*c} \\
&\quad + (p_2+p_3+p_4)\partial_\mu A^{*a}_\mu + p_4 D_\mu A^{*a}_\mu \\
&\quad + \left(p_2\left(1+\frac{\alpha}{2}\right) - p_4 + 2p_{10} + p_{11}\right)\partial_\mu(\Delta^* a^a_\mu) \\
&\quad + \left(p_3\left(1+\frac{\alpha}{2}\right) - p_4\frac{\alpha}{2} + p_{11} + 2p_{12}\right)\partial_\mu(\Delta^* \bar{A}^a_\mu) \\
&\quad + p_4\frac{\alpha}{2} g f^{abc}\Delta^* \bar{A}^b_\mu a^c_\mu = 0,
\end{aligned} \tag{A20}$$

which yields

$$\begin{aligned}
&p_2 = -p_3, \qquad p_{11} = -p_2\left(1+\frac{\alpha}{2}\right) - 2p_{10}, \\
&p_{12} = p_2\left(1+\frac{\alpha}{2}\right) + p_{10}, \qquad p_4 = 0.
\end{aligned} \tag{A21}$$

Finally, we find for Ξ

$$\begin{aligned}
\Xi = {}& p_2(a^a_\mu - \bar{A}^a_\mu)\left(A^{*a}_\mu + \bar{D}_\mu \bar{c}^a - \left(1+\frac{\alpha}{2}\right)\Delta^* \bar{A}^a_\mu\right) \\
&+ p_{10}\Delta^*(a^a_\mu - \bar{A}^a_\mu)^2 + p_{14}\Delta^* J.
\end{aligned} \tag{A22}$$

3. Absorbing the counterterm

From Eq. (A22), we can write down the most general counterterm consistent with the symmetries of the theory:

$$\begin{aligned}
S^{\rm ct} = \int d^4x\Bigg(&\left(\frac{p_1}{4} - p_2\right)(F^a_{\mu\nu})^2 + p_2(\partial_\mu a^a_\nu)F^a_{\mu\nu} + p_2(D_\mu \bar{A}^a_\nu)F^a_{\mu\nu} - p_2\bar{c}^a\bar{D}^2 c^a + (p_2+p_{10})J(a^a_\mu - \bar{A}^a_\mu)^2 + p_{14}J^2 \\
&+ \Delta^*(a^a_\mu - \bar{A}^a_\mu)\left((p_2+2p_{10})\partial_\mu c^a + \left(p_2\frac{\alpha}{2} - 2p_{10}\right) g f^{abc} c^b \bar{A}^c_\mu + \left(-p_2\frac{\alpha}{2} + 2p_{10}\right)\Omega^a_\mu\right) \\
&- p_2\left(1+\frac{\alpha}{2}\right)\Delta^*(\bar{A}^a_\mu D_\mu c^a - \Omega^a_\mu(a^a_\mu - 2\bar{A}^a_\mu)) + p_2 A^{*a}_\mu(\bar{D}_\mu c^a + \Omega^a_\mu) - p_2\Omega^a_\mu \bar{D}_\mu \bar{c}^a\Bigg).
\end{aligned} \tag{A23}$$

Now, it is clear that, in order to reabsorb this counterterm into the classical action, we need to have $\alpha = -2$ and $\beta = 1$. Then, we can absorb the counterterm with multiplicative renormalization. If we write the bare fields as $\Phi_0 = Z_\Phi^{1/2}\Phi$ for the fields $A^a_\mu = a^a_\mu - \bar{A}^a_\mu$, c^a, $\bar{c}^a$, and b^a, then we find

$$\begin{aligned}
&Z_A^{1/2} = Z_b^{-1/2} = 1 + \epsilon\left(\frac{p_1}{2} - p_2\right), \\
&Z_c^{1/2} = Z_{\bar{c}}^{1/2} = 1 - \epsilon\frac{p_2}{2}.
\end{aligned} \tag{A24a}$$

For the parameters, we write $g_0 = Z_g g$ and $\zeta_0 = Z_\zeta \zeta$, and we find

$$Z_g = 1 - \epsilon\frac{p_1}{2}, \quad Z_\zeta = 1 + \epsilon\left(2p_1 - 4p_{10} + \frac{2}{\zeta}p_{14}\right). \tag{A24b}$$

For the sources J, Δ^*, A^{*a}_μ, and c^{*a}, we write $\Phi_0 = Z_\Phi \Phi$:

$$\begin{aligned}
&Z_J = 1 + \epsilon(-p_1 + 2p_{10}), \quad Z_{\Delta^*} = 1 + \epsilon\left(-\frac{p_1}{2} + \frac{p_2}{2} + 2p_{10}\right), \\
&Z_{A^*} = Z_c^{1/2}, \quad Z_{c^*} = Z_A^{1/2}.
\end{aligned} \tag{A24c}$$

For the classical fields $\bar{A}^a_\mu$ and Ω^a_μ, we write the bare fields as $\Phi_0 = Z_\Phi^{1/2}\Phi$, and we find

$$Z_{\bar{A}}^{1/2} = Z_g^{-1}, \qquad Z_\Omega = Z_c^{-1/2}. \tag{A24d}$$

We mark that A^a_μ and $\bar{A}^a_\mu$ renormalize separately. For this reason, one must consider the local composite operator $A^2_\mu \coloneqq (a_\mu - \bar{A}_\mu)^2$ instead of a^2_μ, which would not be multiplicatively renormalizable.

As we are working in a different gauge, one could expect the ζ parameter to be modified. However, this will not be the case for dimensional reasons. In the limit $\bar{A}^a_\mu \to 0$, the Landau background gauge reduces to the ordinary Landau gauge, and so the value for ζ should be equal to the backgroundless value in that limit. Introducing a background field cannot modify it, as there are no other dimensionful quantities present to make a dimensionless function.[14] This argument also carries through for the renormalization group parameters. We can conclude that the values in Eq. (34) are valid in the Landau background gauge as well.

[14] We work in mass-independent renormalization schemes.

A final interesting few words can be said about the special values $\alpha = -2$, $\beta = 1$. These are also the unique values for which the action enjoys an extra Ward identity, namely,

$$\left(-D_\mu^{ab}\frac{\delta}{\delta a_\mu^b} - \bar{D}_\mu^{ab}\frac{\delta}{\delta \bar{A}_\mu^b} - \sum_\Phi g f^{abc}\Phi^b \frac{\delta}{\delta \Phi^c}\right) S = 0, \quad \text{(A25)}$$

where Φ runs over all other fields/sources with an adjoint color index. This identity encodes nothing but the background gauge invariance. The quantum stability of the specific dimension-2 operator A_μ^2 can thus also be appreciated from background gauge invariance. For the record, as noted in Ref. [100], the identity (A25) follows from the anticommutator of the ghost Eq. (A10d) and the Slavnov-Taylor identity (A10), at least for the identified values of α, β.

4. Inclusion of the asymmetry

We are skipping details here, as the discussion is very similar to the one of Ref. [119]. In a nutshell, once the renormalizability of the theory with coupling to it of $(a_\mu - \bar{A}_\mu)^2$ is handled, the introduction of another BRST doublet of sources, $s\eta_{\mu\nu} = K_{\mu\nu}$, $sK_{\mu\nu} = 0$, allows us to couple (the traceless part of) $(a_\mu - \bar{A}_\mu)(a_\nu - \bar{A}_\nu)$ to the theory in a BRST-invariant fashion without hampering the other identities, leading yet again to the quantum stability upon inclusion of a pure vacuum term quadratic in the new source $K_{\mu\nu}$. Just as in Ref. [119], there will be no mixing between the two sources J and $K_{\mu\nu}$. Intuitively, renormalizability is expected, as at $T = 0$ one does not expect a nonvanishing asymmetry, while no new UV divergences should emerge at $T > 0$.

APPENDIX B: EVALUATION OF THE POTENTIAL FROM THE NAKANISHI-LAUTRUP FRAMEWORK

In the Nakanishi-Lautrup framework, one implements the $\alpha \to 0$ limit explicitly at the level of the action. The quadratic part reads

$$\int d^dx \frac{1}{2}(A_\mu^a(-\delta_{\mu\nu}\bar{D}_{ab}^2 + \bar{D}_\nu^{ac}\bar{D}_\mu^{cb} + \delta^{ab}\delta_{\mu\nu}m^2 + \delta^{ab}M_{\mu\nu})A_\nu^b + \bar{c}^a\bar{D}_{ab}^2 c^b + ib^a\bar{D}_\mu^{ab}A_\mu^b), \quad \text{(B1)}$$

where h^a is the Nakanishi-Lautrup field. In Fourier space, and in the color diagonal basis, in the A, h sector, this corresponds to the matrix

$$\begin{pmatrix} (P_\kappa^2 + m^2)\delta_{\mu\nu} + M_{\mu\nu} - P_\mu^\kappa P_\nu^\kappa & P_\mu^\kappa \\ -P_\nu^\kappa & 0 \end{pmatrix}. \quad \text{(B2)}$$

To evaluate the determinant, we can always choose a frame where $P_1^\kappa = |\vec{p}|$ and $P_{\mu>1}^\kappa = 0$. We find

$$\det\begin{pmatrix} p^2+m^2+\mathbb{A} & -P_0^\kappa p & 0 & & 0 & P_0^\kappa \\ -P_0^\kappa p & (P_0^\kappa)^2+m^2-\frac{\mathbb{A}}{d-1} & 0 & & 0 & p \\ 0 & 0 & P_\kappa^2+m^2-\frac{\mathbb{A}}{d-1} & & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & & P_\kappa^2+m^2-\frac{\mathbb{A}}{d-1} & 0 \\ -P_0^\kappa & -p & 0 & & 0 & 0 \end{pmatrix}. \quad \text{(B3)}$$

Upon exchanging the third and last lines and the third and last columns, this becomes

$$\det\begin{pmatrix} p^2+m^2+\mathbb{A} & -P_0^\kappa p & P_0^\kappa & 0 & & 0 \\ -P_0^\kappa p & (P_0^\kappa)^2+m^2-\frac{\mathbb{A}}{d-1} & p & 0 & & 0 \\ -P_0^\kappa & -p & 0 & 0 & & 0 \\ 0 & 0 & 0 & P_\kappa^2+m^2-\frac{\mathbb{A}}{d-1} & & 0 \\ & & & & \ddots & \\ 0 & 0 & 0 & 0 & & P_\kappa^2+m^2-\frac{\mathbb{A}}{d-1} \end{pmatrix}, \quad \text{(B4)}$$

which is then easily computed to be

$$\left(P_\kappa^2+m^2-\frac{\mathbb{A}}{d-1}\right)^{d-2}\left\{(P_0^\kappa)^2\left(P_\kappa^2+m^2-\frac{\mathbb{A}}{d-1}\right)+p^2(P_\kappa^2+m^2+\mathbb{A})\right\}$$
$$=\left(P_\kappa^2+m^2-\frac{\mathbb{A}}{d-1}\right)^{d-2}\left\{P_\kappa^2(P_\kappa^2+m^2)+\mathbb{A}\left(P_\kappa^2-\frac{d}{d-1}(P_0^\kappa)^2\right)\right\}, \tag{B5}$$

which leads to Eq. (58) upon inclusion of the ghost contribution.

APPENDIX C: SUMS AT FINITE TEMPERATURE

At finite temperature, the imaginary time dimension is compactified with a circumference of $1/T$. This results in a discretization of the spectrum. To compute traces, the following replacement has to be made for bosons [including the (anti)ghosts]

$$\int\frac{dk_0}{2\pi}f(k_0)\to T\sum_{n=-\infty}^{+\infty}f(2\pi nT). \tag{C1}$$

To do computations, we will also need to compute sums of particle propagators. An example from which we can derive the formulas necessary in the main text is

$$\sum_{n=-\infty}^{+\infty}\frac{1}{4\pi^2T^2n^2+4\alpha\pi Tn+\beta}. \tag{C2}$$

We can rewrite this sum as a contour integral,

$$\frac{1}{2\pi i}\oint\frac{\pi\cot\pi z}{4\pi^2T^2z^2+4\alpha\pi Tz+\beta}dz, \tag{C3}$$

where the contour contains all the poles of the cotangent. The residue theorem ensures that the integral will evaluate to the sum (C2). Now, we can deform the contour and turn it inside out, which will result in it containing all poles *except* for the ones of the cotangent:

$$-\sum_{z_0}\mathcal{R}es_{z=z_0}\frac{\pi\cot\pi z}{4\pi^2T^2z^2+4\alpha\pi Tz+\beta}. \tag{C4}$$

The sum is now over the zeros of the polynomial in the denominator. Evaluating the residues leads to

$$-\frac{\cot\frac{\alpha+i\sqrt{\beta-\alpha^2}}{2T}-\cot\frac{\alpha-i\sqrt{\beta-\alpha^2}}{2T}}{4iT\sqrt{\beta-\alpha^2}}$$
$$=\frac{\sinh\frac{\sqrt{\beta-\alpha^2}}{T}}{4T\sqrt{\beta-\alpha^2}(\sin^2\frac{\alpha}{2T}+\sinh^2\frac{\sqrt{\beta-\alpha^2}}{2T})}. \tag{C5}$$

It can easily be verified that this has the correct zero-temperature limit. In exactly the same fashion, one also finds that

$$\sum_{n=-\infty}^{+\infty}\frac{4\pi Tn}{4\pi^2T^2n^2+4\alpha\pi Tn+\beta}$$
$$=\frac{-\alpha\sinh\frac{\sqrt{\beta^2-\alpha^2}}{T}+\sqrt{\beta^2-\alpha^2}\sin\frac{\alpha}{T}}{2T\sqrt{\beta-\alpha^2}(\sin^2\frac{\alpha}{2T}+\sinh^2\frac{\sqrt{\beta-\alpha^2}}{2T})}. \tag{C6}$$

From these equations, one can furthermore deduce that

$$\sum_{n=-\infty}^{+\infty}\ln(4\pi^2T^2n^2+4\alpha\pi Tn+\beta)$$
$$=\ln4\left(\sin^2\frac{\alpha}{2T}+\sinh^2\frac{\sqrt{\beta-\alpha^2}}{2T}\right), \tag{C7}$$

as this is the only expression having correct α and β derivatives in addition to having the right zero-temperature limit.

The most useful formula is found by separating out the zero-temperature limit, which yields

$$T\sum_{n=-\infty}^{+\infty}\ln(4\pi^2T^2n^2+4\alpha\pi Tn+\beta)$$
$$=\int\frac{dk_0}{2\pi}\ln(k_0^2+2\alpha k_0+\beta)$$
$$+T\ln\left(1-2e^{-\frac{\sqrt{\beta-\alpha^2}}{T}}\cos\frac{\alpha}{T}+e^{-2\frac{\sqrt{\beta-\alpha^2}}{T}}\right). \tag{C8}$$

[1] B. Lucini, M. Teper, and U. Wenger, The high temperature phase transition in SU(N) gauge theories, J. High Energy Phys. 01 (2004) 061.
[2] B. Lucini and M. Panero, SU(N) gauge theories at large N, Phys. Rep. **526**, 93 (2013).
[3] B. Svetitsky, Symmetry aspects of finite temperature confinement transitions, Phys. Rep. **132**, 1 (1986).
[4] J. Greensite, The confinement problem in lattice gauge theory, Prog. Part. Nucl. Phys. **51**, 1 (2003), and references therein.
[5] A. M. Polyakov, Thermal properties of gauge fields and quark liberation, Phys. Lett. **72B**, 477 (1978).
[6] K. Fukushima, Chiral effective model with the Polyakov loop, Phys. Lett. B **591**, 277 (2004).
[7] B.-J. Schaefer, J. M. Pawlowski, and J. Wambach, The phase structure of the Polyakov–Quark-Meson model, Phys. Rev. D **76**, 074023 (2007).
[8] A. Maas, J. M. Pawlowski, L. von Smekal, and D. Spielmann, The gluon propagator close to criticality, Phys. Rev. D **85**, 034037 (2012).
[9] C. S. Fischer and J. Luecker, Propagators and phase structure of $Nf = 2$ and $Nf = 2 + 1$ QCD, Phys. Lett. B **718**, 1036 (2013).
[10] S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabo (Wuppertal-Budapest Collaboration), Is there still any T_c mystery in lattice QCD? Results with physical masses in the continuum limit III, J. High Energy Phys. 09 (2010) 073.
[11] A. Bazavov *et al.*, The chiral and deconfinement aspects of the QCD transition, Phys. Rev. D **85**, 054503 (2012).
[12] K. Fukushima and T. Hatsuda, The phase diagram of dense QCD, Rep. Prog. Phys. **74**, 014001 (2011).
[13] P. de Forcrand and O. Philipsen, The QCD phase diagram for small densities from imaginary chemical potential, Nucl. Phys. **B642**, 290 (2002).
[14] L. von Smekal, R. Alkofer, and A. Hauck, The Infrared Behavior of Gluon and Ghost Propagators in Landau Gauge QCD, Phys. Rev. Lett. **79**, 3591 (1997).
[15] R. Alkofer and L. von Smekal, The infrared behavior of QCD Green's functions: Confinement dynamical symmetry breaking, and hadrons as relativistic bound states, Phys. Rep. **353**, 281 (2001).
[16] D. Zwanziger, Nonperturbative Landau gauge and infrared critical exponents in QCD, Phys. Rev. D **65**, 094039 (2002).
[17] C. S. Fischer and R. Alkofer, Nonperturbative propagators, running coupling and dynamical quark mass of Landau gauge QCD, Phys. Rev. D **67**, 094020 (2003).
[18] J. C. R. Bloch, Two loop improved truncation of the ghost gluon Dyson-Schwinger equations: Multiplicatively renormalizable propagators and nonperturbative running coupling, Few-Body Syst. **33**, 111 (2003).
[19] A. C. Aguilar and A. A. Natale, A dynamical gluon mass solution in a coupled system of the Schwinger-Dyson equations, J. High Energy Phys. 08 (2004) 057.
[20] P. Boucaud, T. Bruntjen, J. P. Leroy, A. Le Yaouanc, A. Y. Lokhov, J. Micheli, O. Pene, and J. Rodriguez-Quintero, Is the QCD ghost dressing function finite at zero momentum?, J. High Energy Phys. 06 (2006) 001.
[21] A. C. Aguilar and J. Papavassiliou, Power-law running of the effective gluon mass, Eur. Phys. J. A **35**, 189 (2008).
[22] P. Boucaud, J. P. Leroy, A. Le Yaouanc, J. Micheli, O. Pene, and J. Rodriguez-Quintero, On the IR behaviour of the Landau-gauge ghost propagator, J. High Energy Phys. 06 (2008) 099.
[23] C. S. Fischer, A. Maas, and J. M. Pawlowski, On the infrared behavior of Landau gauge Yang-Mills theory, Ann. Phys. (Amsterdam) **324**, 2408 (2009).
[24] J. Rodriguez-Quintero, On the massive gluon propagator, the PT-BFM scheme and the low-momentum behaviour of decoupling and scaling DSE solutions, J. High Energy Phys. 01 (2011) 105.
[25] C. Wetterich, Exact evolution equation for the effective potential, Phys. Lett. B **301**, 90 (1993).
[26] J. Berges, N. Tetradis, and C. Wetterich, Nonperturbative renormalization flow in quantum field theory and statistical physics, Phys. Rep. **363**, 223 (2002).
[27] J. M. Pawlowski, D. F. Litim, S. Nedelko, and L. von Smekal, Infrared Behavior and Fixed Points in Landau Gauge QCD, Phys. Rev. Lett. **93**, 152002 (2004).
[28] C. S. Fischer and H. Gies, Renormalization flow of Yang-Mills propagators, J. High Energy Phys. 10 (2004) 048.
[29] J. M. Pawlowski, Aspects of the functional renormalisation group, Ann. Phys. (Amsterdam) **322**, 2831 (2007).
[30] A. K. Cyrol, L. Fister, M. Mitter, J. M. Pawlowski, and N. Strodthoff, Landau gauge Yang-Mills correlation functions, Phys. Rev. D **94**, 054005 (2016).
[31] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J. M. Pawlowski, M. Tissier, and N. Wschebor, The nonperturbative functional renormalization group and its applications, Phys. Rep. **910**, 1 (2021).
[32] M. Q. Huber, Correlation functions of Landau gauge Yang–Mills theory, Phys. Rev. D **101**, 114009 (2020).
[33] W. Schleifenbaum, M. Leder, and H. Reinhardt, Infrared analysis of propagators and vertices of Yang-Mills theory in Landau and Coulomb gauge, Phys. Rev. D **73**, 125019 (2006).
[34] M. Quandt, H. Reinhardt, and J. Heffner, Covariant variational approach to Yang-Mills theory, Phys. Rev. D **89**, 065037 (2014).
[35] M. Quandt and H. Reinhardt, A covariant variational approach to Yang-Mills Theory at finite temperatures, Phys. Rev. D **92**, 025051 (2015).
[36] M. E. Carrington and E. Kovalchuk, Leading order QED electrical conductivity from the 3PI effective action, Phys. Rev. D **77**, 025015 (2008).
[37] R. Alkofer, C. S. Fischer, F. J. Llanes-Estrada, and K. Schwenzer, The quark-gluon vertex in Landau gauge QCD: Its role in dynamical chiral symmetry breaking and quark confinement, Ann. Phys. (Amsterdam) **324**, 106 (2009).
[38] M. C. A. York, G. D. Moore, and M. Tassler, 3-loop 3PI effective action for 3D SU(3) QCD, J. High Energy Phys. 06 (2012) 077.
[39] L. Fister and J. M. Pawlowski, Confinement from correlation functions, Phys. Rev. D **88**, 045010 (2013).

[40] A. Cucchieri and T. Mendes, Constraints on the IR Behavior of the Gluon Propagator in Yang-Mills Theories, Phys. Rev. Lett. **100**, 241601 (2008).

[41] A. Cucchieri and T. Mendes, Constraints on the IR behavior of the ghost propagator in Yang-Mills theories, Phys. Rev. D **78**, 094503 (2008).

[42] V. G. Bornyakov, V. K. Mitrjushkin, and M. Muller-Preussker, Infrared behavior and Gribov ambiguity in SU(2) lattice gauge theory, Phys. Rev. D **79**, 074504 (2009).

[43] A. Cucchieri and T. Mendes, Landau-gauge propagators in Yang-Mills theories at $\beta = 0$: Massive solution versus conformal scaling, Phys. Rev. D **81**, 016005 (2010).

[44] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck, Lattice gluodynamics computation of Landau gauge Green's functions in the deep infrared, Phys. Lett. B **676**, 69 (2009).

[45] V. G. Bornyakov, V. K. Mitrjushkin, and M. Muller-Preussker, SU(2) lattice gluon propagator: Continuum limit, finite-volume effects and infrared mass scale m (IR), Phys. Rev. D **81**, 054503 (2010).

[46] D. Dudal, O. Oliveira, and N. Vandersickel, Indirect lattice evidence for the Refined Gribov-Zwanziger formalism and the gluon condensate $\langle A^2 \rangle$ in the Landau gauge, Phys. Rev. D **81**, 074505 (2010).

[47] A. G. Duarte, O. Oliveira, and P. J. Silva, Lattice gluon and ghost propagators, and the strong coupling in pure SU(3) Yang-Mills theory: Finite lattice spacing and volume effects, Phys. Rev. D **94**, 014502 (2016).

[48] M. Tissier and N. Wschebor, Infrared propagators of Yang-Mills theory from perturbation theory, Phys. Rev. D **82**, 101701 (2010).

[49] M. Tissier and N. Wschebor, An infrared safe perturbative approach to Yang-Mills correlators, Phys. Rev. D **84**, 045018 (2011).

[50] M. Peláez, U. Reinosa, J. Serreau, M. Tissier, and N. Wschebor, A window on infrared QCD with small expansion parameters, Rep. Prog. Phys. **84**, 124202 (2021).

[51] F. Gao, S.-X. Qin, C. D. Roberts, and J. Rodriguez-Quintero, Locating the Gribov horizon, Phys. Rev. D **97**, 034010 (2018).

[52] M. Q. Huber, Nonperturbative properties of Yang–Mills theories, Phys. Rep. **879**, 1 (2020).

[53] M. Peláez, U. Reinosa, J. Serreau, M. Tissier, and N. Wschebor, Small parameters in infrared quantum chromodynamics, Phys. Rev. D **96**, 114011 (2017).

[54] M. Peláez, U. Reinosa, J. Serreau, M. Tissier, and N. Wschebor, Spontaneous chiral symmetry breaking in the massive Landau gauge: Realistic running coupling, Phys. Rev. D **103**, 094035 (2021).

[55] J. Serreau and M. Tissier, Lifting the Gribov ambiguity in Yang-Mills theories, Phys. Lett. B **712**, 97 (2012).

[56] M. Tissier, Gribov copies, avalanches and dynamic generation of a gluon mass, Phys. Lett. B **784**, 146 (2018).

[57] U. Reinosa, J. Serreau, R. C. Terin, and M. Tissier, Symmetry restoration and the gluon mass in the Landau gauge, SciPost Phys. **10**, 035 (2021).

[58] H. Verschelde, K. Knecht, K. Van Acoleyen, and M. Vanderkelen, The nonperturbative groundstate of QCD and the local composite operator A_μ^2, Phys. Lett. B **516**, 307 (2001).

[59] P. Bicudo, D. Binosi, N. Cardoso, O. Oliveira, and P. J. Silva, Lattice gluon propagator in renormalizable ξ gauges, Phys. Rev. D **92**, 114514 (2015).

[60] M. Napetschnig, R. Alkofer, M. Q. Huber, and J. M. Pawlowski, Yang-Mills propagators in linear covariant gauges from Nielsen identities, Phys. Rev. D **104**, 054003 (2021).

[61] M. A. L. Capri, D. Fiorentini, M. S. Guimaraes, B. W. Mintz, L. F. Palhares, and S. P. Sorella, Local and renormalizable framework for the gauge-invariant operator $A^2_{\rm min}$ in Euclidean Yang-Mills theories in linear covariant gauges, Phys. Rev. D **94**, 065009 (2016).

[62] M. A. L. Capri, D. Dudal, A. D. Pereira, D. Fiorentini, M. S. Guimaraes, B. W. Mintz, L. F. Palhares, and S. P. Sorella, Nonperturbative aspects of Euclidean Yang-Mills theories in linear covariant gauges: Nielsen identities and a BRST-invariant two-point correlation function, Phys. Rev. D **95**, 045011 (2017).

[63] M. Chernodub and E.-M. Ilgenfritz, Electric-magnetic asymmetry of the A^2 condensate and the phases of Yang–Mills theory, Phys. Rev. D **78**, 034036 (2008).

[64] G. Comitini, T. De Meerleer, D. Dudal, U. Reinosa, and S. P. Sorella (to be published).

[65] M. A. L. Capri, D. Dudal, M. S. Guimaraes, A. D. Pereira, B. W. Mintz, L. F. Palhares, and S. P. Sorella, The universal character of Zwanziger's horizon function in Euclidean Yang–Mills theories, Phys. Lett. B **781**, 48 (2018).

[66] M. A. L. Capri, D. Dudal, D. Fiorentini, M. S. Guimarães, I. F. Justo, A. D. Pereira, B. W. Mintz, L. F. Palhares, R. F. Sobreiro, and S. P. Sorella, Exact nilpotent nonperturbative BRST symmetry for the Gribov-Zwanziger action in the linear covariant gauge, Phys. Rev. D **92**, 045039 (2015).

[67] V. N. Gribov, Quantization of nonabelian gauge theories, Nucl. Phys. **B139**, 1 (1978).

[68] D. Zwanziger, Local and renormalizable action from the Gribov horizon, Nucl. Phys. **B323**, 513 (1989).

[69] G. Dell'Antonio and D. Zwanziger, Every gauge orbit passes inside the Gribov horizon, Commun. Math. Phys. **138**, 291 (1991).

[70] M. Lavelle and D. McMullan, Constituent quarks from QCD, Phys. Rep. **279**, 1 (1997).

[71] F. Marhauser and J. M. Pawlowski, Confinement in Polyakov gauge, arXiv:0812.1144.

[72] C. Ratti, M. A. Thaler, and W. Weise, Phases of QCD: Lattice thermodynamics and a field theoretical model, Phys. Rev. D **73**, 014019 (2006).

[73] J. Braun, H. Gies, and J. M. Pawlowski, Quark confinement from color confinement, Phys. Lett. B **684**, 262 (2010).

[74] H. Reinhardt and J. Heffner, The effective potential of the confinement order parameter in the Hamilton approach, Phys. Lett. B **718**, 672 (2012).

[75] H. Reinhardt and J. Heffner, Effective potential of the confinement order parameter in the Hamiltonian approach, Phys. Rev. D **88**, 045024 (2013).

[76] U. Reinosa, J. Serreau, M. Tissier, and N. Wschebor, Two-loop study of the deconfinement transition in Yang-Mills

theories: SU(3) and beyond, Phys. Rev. D **93**, 105002 (2016).
[77] T. K. Herbst, J. Luecker, and J. M. Pawlowski, Confinement order parameters and fluctuations, arXiv:1510.03830.
[78] U. Reinosa, Perturbative aspects of the deconfinement transition—Physics beyond the Faddeev-Popov model, Habilitation thesis, Sorbonne Université, 2020.
[79] U. Reinosa, J. Serreau, M. Tissier, and N. Wschebor, Deconfinement transition in SU(N) theories from perturbation theory, Phys. Lett. B **742**, 61 (2015).
[80] U. Reinosa, J. Serreau, M. Tissier, and N. Wschebor, Deconfinement transition in SU(2) Yang-Mills theory: A two-loop study, Phys. Rev. D **91**, 045035 (2015).
[81] J. A. Gracey, M. Peláez, U. Reinosa, and M. Tissier, Two loop calculation of Yang-Mills propagators in the Curci-Ferrari model, Phys. Rev. D **100**, 034023 (2019).
[82] F. Gubarev, L. Stodolsky, and V. Zakharov, On the Significance of the Vector Potential Squared, Phys. Rev. Lett. **86**, 2220 (2001).
[83] F. Gubarev and V. Zakharov, On the emerging phenomenology of $\langle A^2_{\rm min}\rangle$, Phys. Lett. B **501**, 28 (2001).
[84] P. Boucaud *et al.*, Lattice calculation of $1/p^2$ corrections to alpha(s) and of Lambda(QCD) in the MOM scheme, J. High Energy Phys. 04 (2000) 006.
[85] P. Boucaud, A. Le Yaouanc, J. Leroy, J. Micheli, O. Pene, and J. Rodriguez-Quintero, Testing the Landau gauge operator product expansion on the lattice with a $\langle A^2\rangle$ condensate, Phys. Rev. D **63**, 114003 (2001).
[86] A. C. Aguilar and J. Papavassiliou, Gluon mass generation in the PT-BFM scheme, J. High Energy Phys. 12 (2006) 012.
[87] D. Dudal, J. A. Gracey, S. P. Sorella, N. Vandersickel, and H. Verschelde, A refinement of the Gribov-Zwanziger approach in the Landau gauge: Infrared propagators in harmony with the lattice results, Phys. Rev. D **78**, 065047 (2008).
[88] A. C. Aguilar, D. Binosi, and J. Papavassiliou, Gluon and ghost propagators in the Landau gauge: Deriving lattice results from Schwinger-Dyson equations, Phys. Rev. D **78**, 025010 (2008).
[89] A. C. Aguilar, D. Binosi, and J. Papavassiliou, The dynamical equation of the effective gluon mass, Phys. Rev. D **84**, 085026 (2011).
[90] D. Dudal, S. P. Sorella, and N. Vandersickel, The dynamical origin of the refinement of the Gribov-Zwanziger theory, Phys. Rev. D **84**, 065039 (2011).
[91] M. A. L. Capri, D. Fiorentini, A. D. Pereira, R. F. Sobreiro, S. P. Sorella, and R. C. Terin, Aspects of the refined Gribov-Zwanziger action in linear covariant gauges, Ann. Phys. (Amsterdam) **376**, 40 (2017).
[92] M. A. L. Capri, D. M. van Egmond, G. Peruzzo, M. S. Guimaraes, O. Holanda, S. P. Sorella, R. C. Terin, and H. C. Toledo, On a renormalizable class of gauge fixings for the gauge invariant operator $A^2_{\rm min}$, Ann. Phys. (Amsterdam) **390**, 214 (2018).
[93] A. K. Cyrol, M. Mitter, J. M. Pawlowski, and N. Strodthoff, Nonperturbative quark, gluon, and meson correlators of unquenched QCD, Phys. Rev. D **97**, 054006 (2018).
[94] G. Comitini and F. Siringo, Variational study of mass generation and deconfinement in Yang-Mills theory, Phys. Rev. D **97**, 056013 (2018).
[95] F. Siringo and G. Comitini, Gluon propagator in linear covariant Rξ gauges, Phys. Rev. D **98**, 034023 (2018).
[96] J. Horak, F. Ihssen, J. Papavassiliou, J. M. Pawlowski, A. Weber, and C. Wetterich, Gluon condensates and effective gluon mass, arXiv:2201.09747.
[97] D. Zwanziger, Nonperturbative modification of the Faddeev-Popov formula and banishment of the naive vacuum, Nucl. Phys. **B209**, 336 (1982).
[98] A. Cucchieri and T. Mendes, The minimal Landau background gauge on the lattice, Phys. Rev. D **86**, 071503 (2012).
[99] D. Dudal and D. Vercauteren, Gauge copies in the Landau–DeWitt gauge: A background invariant restriction, Phys. Lett. B **779**, 275 (2018).
[100] P. Grassi, T. Hurth, and A. Quadri, Landau background gauge fixing and the IR properties of Yang–Mills Green functions, Phys. Rev. D **70**, 105014 (2004).
[101] R. Ferrari, M. Picariello, and A. Quadri, Algebraic aspects of the background field method, Ann. Phys. (Amsterdam) **294**, 165 (2001).
[102] U. Reinosa, J. Serreau, M. Tissier, and A. Tresmontant, Yang-Mills correlators across the deconfinement phase transition, Phys. Rev. D **95**, 045014 (2017).
[103] F. E. Canfora, D. Dudal, I. F. Justo, P. Pais, L. Rosa, and D. Vercauteren, Effect of the Gribov horizon on the Polyakov loop and vice versa, Eur. Phys. J. C **75**, 326 (2015).
[104] F. Canfora, D. Hidalgo, and P. Pais, The Gribov problem in presence of background field for $SU(2)$ Yang–Mills theory, Phys. Lett. B **763**, 94 (2016); **772**, 880(E) (2017).
[105] D. Kroff and U. Reinosa, Gribov-Zwanziger type model action invariant under background gauge transformations, Phys. Rev. D **98**, 034029 (2018).
[106] O. C. Junqueira, I. F. Justo, D. S. Montes, A. D. Pereira, and R. F. Sobreiro, Gauge copies and the fate of background independence in Yang-Mills theories: A leading order analysis, Phys. Rev. D **102**, 074029 (2020).
[107] I. F. Justo, A. D. Pereira, and R. F. Sobreiro, Towards background field independence within the Gribov horizon, Phys. Rev. D **106**, 025015 (2022).
[108] L. Abbott, Introduction to the background field method, Acta Phys. Pol. B **13**, 33 (1982), https://inspirehep.net/literature/166273.
[109] D. Binosi and J. Papavassiliou, Pinch technique: Theory and applications, Phys. Rep. **479**, 1 (2009).
[110] D. Binosi and A. Quadri, Slavnov-Taylor constraints for nontrivial backgrounds, Phys. Rev. D **84**, 065017 (2011).
[111] D. Binosi and A. Quadri, Canonical transformations and renormalization group invariance in the presence of nontrivial backgrounds, Phys. Rev. D **85**, 085020 (2012).
[112] H. Gies, D. Gkiatas, and L. Zambelli, Background effective action with nonlinear massive gauge fixing, arXiv:2205.06707.
[113] N. K. Nielsen, On the gauge dependence of spontaneous symmetry breaking in gauge theories, Nucl. Phys. **B101**, 173 (1975).
[114] O. Piguet and K. Sibold, Gauge independence in ordinary Yang-Mills theories, Nucl. Phys. **B253**, 517 (1985).

[115] R. Jackiw, Functional evaluation of the effective potential, Phys. Rev. D **9**, 1686 (1974).
[116] W. Zimmermann, Reduction in the number of coupling parameters, Commun. Math. Phys. **97**, 211 (1985).
[117] S. Heinemeyer, M. Mondragón, N. Tracas, and G. Zoupanos, Reduction of couplings and its application in particle physics, Phys. Rep. **814**, 1 (2019).
[118] J. A. Gracey, Three loop MS-bar renormalization of the Curci-Ferrari model and the dimension two BRST invariant composite operator in QCD, Phys. Lett. B **552**, 101 (2003).
[119] D. Dudal, J. A. Gracey, N. Vandersickel, D. Vercauteren, and H. Verschelde, The asymmetry of the dimension 2 gluon condensate: The zero temperature case, Phys. Rev. D **80**, 065017 (2009).
[120] O. Piguet and S. Sorella, *Algebraic Renormalization*, Lecture Notes in Physics No. M28 (Springer, New York, 1995).
[121] B. Lucini and G. Moraitis, The running of the coupling in SU(N) pure gauge theories, Phys. Lett. B **668**, 226 (2008).
[122] P. Boucaud, F. De Soto, J. P. Leroy, A. Le Yaouanc, J. Micheli, O. Pene, and J. Rodriguez-Quintero, Ghost-gluon running coupling, power corrections and the determination of Lambda(MS-bar), Phys. Rev. D **79**, 014508 (2009).
[123] D. Vercauteren and H. Verschelde, The asymmetry of the dimension 2 gluon condensate: The finite temperature case, Phys. Rev. D **82**, 085026 (2010).
[124] G. Markó, U. Reinosa, and Z. Szép, O(N) model within the Φ-derivable expansion to order λ^2: On the existence and UV/IR sensitivity of the solutions to self-consistent equations, Phys. Rev. D **92**, 125035 (2015).
[125] E. Braaten and R. D. Pisarski, Soft amplitudes in hot gauge theories: A general analysis, Nucl. Phys. **B337**, 569 (1990).
[126] S. Chiku and T. Hatsuda, Optimized perturbation theory at finite temperature, Phys. Rev. D **58**, 076001 (1998).
[127] J. O. Andersen and M. Strickland, Resummation in hot field theories, Ann. Phys. (Amsterdam) **317**, 281 (2005).
[128] M. Laine and A. Vuorinen, Basics of thermal field theory: A tutorial on perturbative computations, Lect. Notes Phys. **925**, 1 (2016).
[129] M. Quandt and H. Reinhardt, Covariant variational approach to Yang-Mills theory: Effective potential of the Polyakov loop, Phys. Rev. D **94**, 065015 (2016).
[130] J. Heffner, H. Reinhardt, and D. R. Campagnari, The deconfinement phase transition in the Hamiltonian approach to Yang–Mills theory in Coulomb gauge, Phys. Rev. D **85**, 125029 (2012).
[131] V. G. Bornyakov, V. K. Mitrjushkin, and R. N. Rogalyov, $\langle A^2 \rangle$ asymmetry in lattice $SU(2)$ gluodynamics at $T > T_c$, Phys. Rev. D **100**, 094505 (2019).
[132] V. G. Bornyakov, N. V. Gerasimeniuk, V. A. Goy, and R. N. Rogalyov, The difference between the longitudinal and transverse gluon propagators as an indicator of the postconfinement domain, arXiv:2201.04613.
[133] A. Cucchieri and T. Mendes, Electric and magnetic screening masses around the deconfinement transition, Proc. Sci. LATTICE2011 (**2011**) 206 [arXiv:1201.6086].
[134] P. J. Silva, O. Oliveira, P. Bicudo, and N. Cardoso, Gluon screening mass at finite temperature from the Landau gauge gluon propagator in lattice QCD, Phys. Rev. D **89**, 074503 (2014).
[135] P. J. Silva and O. Oliveira, Gluon dynamics, center symmetry and the deconfinement phase transition in SU(3) pure Yang-Mills theory, Phys. Rev. D **93**, 114509 (2016).
[136] D. M. van Egmond, U. Reinosa, J. Serreau, and M. Tissier, A novel background field approach to the confinement-deconfinement transition, SciPost Phys. **12**, 087 (2022).
[137] D. M. van Egmond and U. Reinosa, Signatures of the Yang-Mills deconfinement transition from the gluon two-point correlator, arXiv:2206.03841.
[138] D. Dudal, H. Verschelde, V. Lemes, M. Sarandy, R. Sobreiro, S. Sorella, and J. Gracey, Renormalizability of the local composite operator A_μ^2 in linear covariant gauges, Phys. Lett. B **574**, 325 (2003).