

Higher symmetries of 5D orbifold SCFTs

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We determine the higher symmetries of 5D SCFTs engineered from M-theory on a \mathbb{C}^3/Γ background for Γ a finite subgroup of $SU(3)$. This resolves a longstanding question as to how to extract this data when the resulting singularity is nontoric (when Γ is non-Abelian) and/or not isolated (when the action of Γ has fixed loci). The Bogomol'nyi-Prasad-Sommerfield (BPS) states of the theory are encoded in a 1D quiver quantum mechanics gauge theory which determines the possible 1-form and 2-form symmetries. We also show that this same data can also be extracted by a direct computation of the corresponding defect group associated with the orbifold singularity. Both methods agree, and these computations do not rely on the existence of a resolution of the singularity. We also observe that when the geometry faithfully captures the global 0-form symmetry, the abelianization of Γ detects a 2-group structure (when present). As such, this establishes that all of this data is indeed intrinsic to the superconformal fixed point rather than being an emergent property of an IR gauge theory phase.

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I. INTRODUCTION

Higher-form symmetries [1] provide a powerful way to constrain the nonperturbative data of a quantum field theory [2–12]. This is especially valuable in the case of $d > 4$ superconformal field theories since all known examples are intrinsically strongly coupled. Indeed, the main method to construct such examples proceeds by taking a singular limit of a string / M-theory / F-theory compactification. With this in mind, it is important to extract the corresponding data of higher-form symmetries for such systems directly from the singular geometry of a string compactification [13–46].

In this paper we determine the higher-form symmetries for 5D superconformal field theories (SCFTs) which originate from an orbifold singularity $\mathbf{X}_\Gamma = \mathbb{C}^3/\Gamma$ for Γ a finite

subgroup of $SU(3)$. We denote the resulting 5D SCFTs as $\mathcal{T}_{\mathbf{X}_\Gamma}$. There is a full classification of finite subgroups of Γ (including their group actions) which result in Gorenstein Calabi-Yau threefold singularities [47] (see also [48]). It also gives rise to a large class of well-known 5D SCFTs. For example, the trinion theory T_N with flavor symmetry algebra $\mathfrak{su}(N)^3$ arises from the singularity $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$ (see [49]). Recently the physics and geometry of many such singularities were studied in references [50,51]. For further discussion of higher-form symmetries in 5D SCFTs, see in particular [17,18,33,52,53]. For additional background on geometric engineering and 5D SCFTs, see [54–65] as well as [66–74].

The higher symmetries of 5D SCFTs that have gauge theory phases can be determined directly from the corresponding Lagrangian description, exploiting standard techniques [1]—with the subtlety that it can happen that 5D instantons are charged with respect to the center symmetry in the presence of a nonzero CS level. There are, however, many 5D SCFTs which do not have a gauge theory phase, and instead are defined purely by singular geometry. A pivotal example of this type is the famous E_0 theory [54], which is realized as the singular limit of the local Calabi-Yau threefold $O(-3) \rightarrow \mathbb{P}^2$, namely the orbifold $\mathbb{C}^3/\mathbb{Z}_3$ [57]. For these theories an alternative route to compute the corresponding higher form symmetries is given by exploiting the defect group of M-theory on the corresponding singularity [13,17,18]. For instance,

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proceeding in this way one can show that for the E_0 theory, the defect group is

$$\mathbb{D}(E_0) \supset (\mathbb{Z}_3)_e^{(1)} \oplus (\mathbb{Z}_3)_m^{(2)}, \quad (1.1)$$

where the subscripts and the superscripts refer to the fact that we have an electric 1-form symmetry and a magnetic 2-form symmetry. The one-form electric symmetry arises from M2-branes wrapped on two-cycles, and the two-form magnetic symmetry similarly arises from wrapped M5-branes on four-cycles. The two are related to different choices of global structures for the E_0 theory [17].

So long as the singularity is isolated, it is straightforward to read off the corresponding electric one-form symmetry via the abelianization $\text{Ab}[\pi_1(\partial\mathbb{C}^3/\Gamma)]$, much as was done in the case of the 6d defect group in [13]. If, however, the group action Γ results in a nonisolated singularity, then the boundary $\partial\mathbb{C}^3/\Gamma$ will also have singularities. For toric singularities, this problem was resolved in [17]. For more general orbifold singularities, however, it is still an open question as to how to read off the resulting higher-form symmetries directly from the singularity.

Our aim in this paper will be to present two complementary solutions to the computation of higher-form symmetries for such 5D orbifold SCFTs. First of all, there is a well-defined notion of the fundamental group $\pi_1(\partial\mathbb{C}^3/\Gamma) = \pi_1(S^5/\Gamma)$ even when the group action by Γ has fixed points. We use this to directly extract the electric one-form symmetry of such theories.

Second of all, we can directly exploit the fact that the higher-form symmetries are closely related to extended defects of the 5D SCFT and that the 1-form and 2-form symmetries above, upon circle reduction, give both rise to 1-form symmetries for the corresponding 4d KK theory. The defect group of the 4d KK theory is then captured by the screening of the latter by BPS particles, which is in turn specified by the BPS quiver of the 5D SCFT [75], the supersymmetric quantum mechanics (SQM) which encodes the dynamics on the worldline of the BPS particles of the 4d KK theory. Indeed, since compactification on a further circle takes us to type IIA on the same singularity, the resulting quiver is just the one obtained from a D0-brane probing \mathbb{C}^3/Γ . From the 5D BPS quiver analysis, we expect that the one-form symmetry part of the defect group of the 4d KK theory $D_{S^1}\mathcal{T}_{X_r}$ has the form

$$\mathbb{D}(D_{S^1}\mathcal{T}_{X_r})^{(1)} = \mathbb{G}^{(1)} \oplus \mathbb{G}^{(1)} \quad (1.2)$$

where

$$\mathbb{G} \simeq \bigoplus_{\ell=1}^r \mathbb{Z}_{n_\ell}. \quad (1.3)$$

In Eq. (1.2) there are two identical factors of \mathbb{G} that denote respectively the possible electric and magnetic 1-form symmetries that are controlled by a choice of global

structure for the 4d KK theory. The positive integers n_ℓ can be completely determined via a standard 't Hooft screening argument [76]—see, e.g., [77]. Moreover, the quiver also captures the Weyl pairing [77] (or linking pairing [16]) from which the resulting Heisenberg algebra of noncommuting fluxes [78,79] that governs the global structure of the theories [80,81] can be reconstructed [82]. Knowing the 1-form defect group of the 4d KK theory, it is easy to recover the corresponding factors of the defect group of the associated 5D SCFT:

$$\mathbb{D}(\mathcal{T}_{X_r}) \supseteq \mathbb{G}_e^{(1)} \oplus \mathbb{G}_m^{(2)}. \quad (1.4)$$

Whenever the 5D SCFT has a global structure which allows for a 1-form symmetry as well as a 0-form symmetry, the two can mix, and this can result in a nontrivial global 2-group symmetry—see, e.g., [2,6,7].¹ As a further result in this note, we begin exploring the 2-group symmetries of some orbifold 5D SCFTs with a Lagrangian description [33], reproducing the known features of such systems in terms of the abelianization of the orbifolding group Γ . Our result indicates that the 2-group structure is indeed a feature of the 5D SCFT rather than an emergent IR artifact.

The result of this paper is organized as follows. In Sec. II, after a brief review of the defect group and its use in determining the higher-form symmetries of a 5D SCFT, we give a general prescription for computing the higher-form symmetries of the 5D SCFT, both via a direct analysis of $\pi_1(S^5/\Gamma)$, and via the corresponding 5D BPS quiver. In Sec. III we turn to a collection of examples, illustrating how our method works in practice. In Sec. IV we turn to a preliminary analysis of 2-group structures in such theories, and in particular its (conjectural) relation to the abelianization of Γ . We present our conclusions and potential future directions in Sec. V. The appendices contain some additional technical details as well as instructions for reproducing the relevant quiver and group theory computations.

II. DEFECT GROUPS AND HIGHER SYMMETRIES IN 5D

In this section we discuss the interplay between the defect group and higher-form symmetries, with a particular emphasis on 5D theories. Recall that the defect group is a general way to capture the spectrum of defects with charges which cannot be screened by dynamical states of the theory. This notion was first introduced in Ref. [13] in the context of 6D SCFTs, but it has far wider applicability, especially when combined with flux noncommutativity [78,79], as exploited, for example in references [16–18]. It is especially helpful in the context of higher-dimensional

¹See also [83–87] for foundational work on higher group gauge symmetries.

quantum field theories specified by a compactification of string theory, and we will mainly focus on this case in what follows.

In the context of string compactification, we obtain supersymmetric defects by wrapping branes on noncompact cycles of a local geometry. Branes of the same codimension which are wrapped on compact cycles amount to dynamical degrees of freedom which can screen the charges associated with these defects. Indeed, in many quantum field theories, the corresponding collection of defects needs to be supplemented by a choice of global structure which restricts the spectrum of extended objects [1,13,88,89]. This can happen whenever the corresponding torsional fluxes do not commute [13,14,78–80,90–92]. Our conventions and treatment will follow that presented in [17], to which we refer the interested reader for further details.

In any geometric engineering setup, the BPS spectrum of the resulting quantum field theory is captured by branes of various dimensions that are wrapping on shrinking cycles of a noncompact geometry \mathbf{X} , which in our case is a Calabi-Yau threefold singularity CY_3 . When a p -brane wraps a compact k -cycle, it describes a $p - k + 1$ dimensional BPS excitation. While, p -branes wrapped on noncompact k -cycles describe $p - k + 1$ dimensional defects operators. Branes are charged with respect to flux operators, which can be used to construct the corresponding quasi-topological symmetry defects that describe the charges of the extended objects. Of course, we can have a generalized 't Hooft screening, due to the possibility of defects to end on dynamical BPS objects, which breaks the associated higher-form symmetry. The remaining symmetry is captured by the defect group:

$$\mathbb{D} := \bigoplus_n \mathbb{D}^{(n)} \quad \text{where}$$

$$\mathbb{D}^{(n)} = \bigoplus_{p\text{-branes}} \left(\bigoplus_{k \text{ s.t. } p-k+1=n} \left(\frac{H_k(\mathbf{X}, \partial\mathbf{X})}{H_k(\mathbf{X})} \right) \right) \quad (2.1)$$

In other words, the defect group \mathbb{D} is the group of charges of higher symmetries acting on defects modulo screening. Moreover, together with the corresponding Heisenberg algebra of noncommuting fluxes, it encodes the quantum data of the Hilbert space at the boundary of the noncompact geometry. This construction captures all possible global structures realized by the geometry of the string compactification. We note that in principle, there could be additional emergent higher-form symmetries in the deep infrared of such a system, which would in turn signal the existence of additional defects. Our operating assumption—which is well-supported in practice—is that such subtleties will not arise in the analysis to follow.

In this paper our focus is on the 1-form and the 2-form symmetry parts of the defect group in the context of a

geometric engineering of M-theory on a Calabi-Yau singularity \mathbf{X}_Γ

$$\mathbb{D}(M/\mathbf{X}_\Gamma) \supset \mathbb{D}_{M2}^{(1)} \oplus \mathbb{D}_{M5}^{(2)} \quad (2.2)$$

where the subscripts denote the associated branes, and the superscript indicates that the M2-branes are associated with a one-form and the M5-branes with a two-form generalized symmetry.

Since M2s and M5s are mutually nonlocal and in general the singularity \mathbf{X}_Γ might have some torsional flux, one naturally expects to find examples with nontrivial higher symmetries and a nontrivial global structure. To see this, consider a geometry $\mathcal{M}_{11} = \mathcal{M}_5 \times \mathbf{X}_\Gamma$ where for ease of exposition we take \mathcal{M}_5 compact and torsion free. The geometry \mathcal{M}_{11} has a boundary at infinity given by $\partial\mathcal{M}_{11} = \mathcal{M}_5 \times \partial\mathbf{X}_\Gamma$, to which we associate an Hilbert space $\mathcal{H}(\partial\mathcal{M}_{11})$. The resulting Hilbert space has selection sectors that can be thought of as states in a quantum mechanics where the role of operators is played by torsional fluxes, which are expected to organize themselves by an *M-theory generalized cohomology group*, $\mathbb{E}(\mathcal{M}_{11})$ (see, for example, Sec. 1 of [17] for its specific physics-motivated properties). The presence of a nontrivial torsion for the generalized cohomology,

$$\text{Tor } \mathbb{E}(\mathcal{M}_{11}) = \bigoplus_i H^{i+1}(\mathcal{M}_5) \otimes \text{Tor } \mathbb{D}^i, \quad (2.3)$$

might cause the flux operators to form a noncommutative algebra [78,79]. Indeed, fluxes corresponding to the M2 and M5-branes that contribute to the 1-form symmetry and the 2-form symmetry part of the defect groups satisfy the following relation

$$\Psi_2 \Phi_5 = \exp \left(2\pi i L(l_1, l_2) \int_{\mathcal{M}_5} \omega_1 \wedge \omega_2 \right) \Phi_5 \Psi_2, \quad (2.4)$$

where the term in the exponential is a pairing of cocycles in $\text{Tor } \mathbb{E}(\mathcal{M}_{11})$, $\omega_{1,2}$ represents the forms dual to the cycle where the extended objects have supports, $l_{1,2}$ are elements of \mathbb{D}^i and $L(\cdot, \cdot)$ is the linking form on $\partial\mathbf{X}_\Gamma$. To fully specify the quantum system, we need to select a maximal set of mutually commuting fluxes as a base for our Hilbert space.

This construction can be made more rigorous and general [17]. In particular, it is known how to compute the defect group from exact sequences in homology [13,17,18]. In the next section we will review this result and use it to compute the defect group of orbifold singularities. Moreover, we will confirm the same result exploiting the corresponding 5D BPS quivers, building on [32,82].

The rest of this section is organized as follows. Again specializing to the case of 5D SCFTs obtained from M-theory on an orbifold singularity $\mathbf{X}_\Gamma = \mathbb{C}^3/\Gamma$, we show how to extract the defect group directly from the fundamental group of S^5/Γ , which we refer to as the

“algebraic topology approach”. After this, we turn to a physical realization of the same data in terms of the 5D BPS quiver defined by the 5D SCFT. The physical interpretation of the quiver in terms of the Dirac pairing for BPS particles of the 4d KK theory provides a complementary method for extracting the same data on higher-form symmetries. We turn to examples later in Sec. III.

A. Algebraic topology approach

Let us now turn to a computation of the defect group directly via the corresponding singular geometry specified by the orbifold group \mathbb{C}^3/Γ . We start by considering M-theory on $\mathcal{M}_5 \times \mathbf{X}_\Gamma$. In order to capture the defect group one has to consider the long exact sequence of relative homology of $(\mathbf{X}_\Gamma, S^5/\Gamma)$:

$$\begin{aligned} \dots \rightarrow H_2(S^5/\Gamma) \xrightarrow{I_2} H_2(\mathbf{X}_\Gamma) \xrightarrow{J_2} H_2(\mathbf{X}_\Gamma, S^5/\Gamma) \\ \xrightarrow{\partial_2} H_1(S^5/\Gamma) \xrightarrow{I_1} \underbrace{H_1(\mathbf{X}_\Gamma)}_{=0} \rightarrow \dots \end{aligned} \quad (2.5)$$

Strictly speaking, some of the quantities in the above exact sequence may not involve smooth spaces, for example if Γ has fixed points. In the present context, we can always assume the existence of a crepant resolution, and work in terms of the resolved geometry. Since, however, our answer will be independent of a given choice of a resolution, there is a precise sense in which these objects ought to make sense even without an explicit blowup, consistently with the remark of [18] that the higher form symmetries are independent from flop transitions in the resolved geometry. Indeed, we will shortly give a precise expression of $H_1(S^5/\Gamma)$ as the abelianization of $\pi_1(S^5/\Gamma)$, even when Γ has a fixed point locus on the S^5 .

Now, we expect $H_1(\mathbf{X}_\Gamma)$ to vanish due to the fact that \mathbf{X}_Γ is a Calabi-Yau space, and, moreover, by definition [17,18]:

$$\begin{aligned} \mathbb{D}^{(1)} &\equiv \frac{H_2(\mathbf{X}_\Gamma, S^5/\Gamma)}{J_2(H_2(\mathbf{X}_\Gamma))} \\ &\simeq H_1(S^5/\Gamma) \\ &\simeq \text{Ab}[\pi_1(S^5/\Gamma)] \end{aligned} \quad (2.6)$$

Using both Poincaré duality and the universal coefficient theorem it can be shown that $\mathbb{D}_{M2}^{(1)} \simeq \mathbb{D}_{M5}^{(2)}$ whenever $H_3(\mathbf{X}_\Gamma)$ vanishes, which is indeed the case for the orbifold singularities we are considering.

The defect group is then fully captured by $\pi_1(S^5/\Gamma)$, since the abelianization of this group is just $H_1(S^5/\Gamma)$. An important subtlety here is that in general, Γ may have a fixed point locus which complicates the analysis. In the special case where there are no fixed points, we have $\pi_1(S^5/\Gamma_{\text{no-fixed}}) = \Gamma_{\text{no-fixed}}$. To extend this to the more

general case which can include fixed points, we use a result proved by Armstrong in 1967 [93]:

Let Γ be a discontinuous group of homeomorphisms of a path connected, simply connected, locally compact metric space X , and let H be the normal subgroup of Γ generated by those elements which have fixed points. Then the fundamental group of the orbit space X/Γ is isomorphic to the factor group Γ/H .

In other words, to compute $\pi_1(S^5/\Gamma)$, we just need to enumerate the generators of Γ which might have a fixed point locus on S^5 . Specifying the particular group action induced via $f_\Gamma: \Gamma \rightarrow \text{Homeo}(\mathbb{C}^3)$, we denote by $H_{\Gamma, f_\Gamma} \trianglelefteq \Gamma$ the resulting normal subgroup of Γ . The end result is that $\pi_1(S^5/\Gamma) = \Gamma/H_{\Gamma, f_\Gamma}$, so the one-form symmetry part of the defect group is just:

$$\mathbb{D}^{(1)} = \text{Ab}[\Gamma/H_{\Gamma, f_\Gamma}]. \quad (2.7)$$

This also shows that the higher-form symmetry is independent of a choice of resolution, and moreover, provides a systematic way to determine this data without specifying a blowup. We give explicit examples of this procedure in Sec. III.

B. Quiver approach

A complementary way to extract the same information on the higher-form symmetry is to determine the corresponding quiver associated with a given orbifold singularity. In physical terms, this arises from the 5D BPS quiver of the theory [75]. 5D BPS quivers are the quiver of supersymmetric quantum mechanics that capture the BPS spectrum of particles of the 4d KK theory $D_{S^1}\mathcal{T}_{\mathbf{X}_\Gamma}$. Exploiting the Kaluza-Klein (KK) circle as an M-theory circle, it is clear that the 4d KK theory associated to \mathbf{X}_Γ is obtained from type IIA on the same Calabi-Yau threefold [94]. The 5D BPS quivers are therefore captured from the BPS quivers of IIA on \mathbf{X}_Γ [75] (see also [95]). For the case at hand, the structure of the quivers can be reproduced from the D0-brane probe of this singularity. The resulting supersymmetric quiver quantum mechanics follows from the general prescription of Douglas and Moore [96]. The resulting 3d McKay quivers were obtained in [97,98]. For additional details on how to implement this procedure, see Appendix C. In many of our quiver figures, we present an explicit indexing of the nodes, and in particular this does not refer to the rank of each gauge group.

The crucial part needed for our analysis is the physical interpretation of the BPS quivers (see, e.g., [99–105]). The nodes of the BPS quivers are in one-to-one correspondence with a basis of generators of the charge lattice of the theory. Let us denote the corresponding charges $\gamma_1, \dots, \gamma_N$. The states corresponding to the charges of the generators are viewed as a collection of elementary constituents, out of the bound states of which the whole spectrum of the theory can be reconstructed. Since all the states in the spectrum are

formed by these bound states, their charges are integer multiples of the charges of the elementary constituents, and we can completely determine the 't Hooft screening in terms of the latter [77]. The charges of the line defects are in turn valued in a dual lattice of charges, where the duality is determined by the Dirac pairing [88,106]. For Lagrangian theories, the relevant Dirac pairing is determined by a simple computation in the Coulomb phase of the theory, for non-Lagrangian theories, however, geometry is needed. Here another aspect of the structure of the BPS quiver quantum mechanics is crucial, namely that the adjacency matrix which determines the structure of the BPS quiver quantum mechanics, is indeed captured by the Dirac pairing among the charges of the elementary constituents

$$\langle \gamma_i, \gamma_j \rangle_D = B_{ij}. \quad (2.8)$$

For this reason the relevant quotient, which captures the defect group from the IR [82], is also reproduced by the cokernel of the Dirac pairing. The 4d KK theory is obtained from type IIA on the same orbifold singularity \mathbf{X}_Γ and the torsional generators of the defect group are:

$$\text{Tor } \mathbb{D}^{(1)}(\text{IIA}/\mathbf{X}_\Gamma) = \text{Tor}(\text{coker}(B)) = \mathbb{G}_e^{(1)} \oplus \mathbb{G}_m^{(1)} \quad (2.9)$$

where

$$\mathbb{G}_e^{(1)} \simeq \mathbb{G}_m^{(1)} \simeq \bigoplus_{\ell=1}^r \mathbb{Z}_{n_\ell}. \quad (2.10)$$

By construction the integers n_ℓ can be recovered by the Smith normal form of the matrix B [77]. The fact that B is antisymmetric entails that one gets the two identical electric and magnetic factors in Eq. (2.9).

Whenever we choose a global form for the 5D theory $\mathcal{T}_{\mathbf{X}_\Gamma}$ with a magnetic 2-form symmetry, the latter gives rise to a magnetic 1-form symmetry for the 4d KK theory, by wrapping the corresponding surface defects on the KK circle. For this reason we identify

$$\mathbb{D}^{(2)}(M/\mathbf{X}_\Gamma) \simeq \mathbb{G}_m^{(1)} \quad (2.11)$$

above. This strategy gives rise to interesting consistency checks for the entire construction.

Mathematically, our analysis above boils down to evaluating the defect group $\mathbb{D}^{(1)} = \frac{H_2(\mathbf{X}_\Gamma, S^5/\Gamma)}{J_2(H_2(\mathbf{X}_\Gamma))}$ in two different ways. A more geometrical formulation of the quiver approach in terms of $\langle H_2, H_4 \rangle$ intersection forms can be found in Sec. 3 of [17] and Sec. 3 of [18], therefore implying the equivalence of the two approaches (as a mathematical consequence).

III. EXAMPLES

In Sec. II we presented a general prescription for how to extract the higher-form symmetries from 5D SCFTs

defined by M-theory on the background \mathbb{C}^3/Γ . Our plan in this section will be to show how this works in practice, illustrating with a number of examples that both methods produce the same result, and agree with previously established results available in the literature, including v1 and v2 of [50].

To frame the discussion to follow, we first divide the subgroups of $SU(3)$ into three main families:

- (i) Family 1: The Abelian subgroups;
- (ii) Family 2: The subgroups of $SU(3)$ induced from finite non-Abelian subgroups of $U(2)$;
- (iii) Family 3: The complement of families 1 and 2.

We proceed by way of example, illustrating how our method works in each of these cases. In the case of finite Abelian subgroups of $SU(3)$, we find agreement with the results of [17,18] as well as those of [50] which involved a direct analysis of the resolved geometry. The case of family 2 does not appear to have been treated in the existing literature, but again we find examples which contain nontrivial higher-form symmetries. In the case of family 3, all examples we considered have too many elements in Γ whose action on \mathbb{C}^3 contains a fixed point locus. The resulting normal subgroup generated by such elements is so large that $\text{Ab}[\pi_1(S^5/\Gamma)]$ is trivial, and as such they all produce a trivial higher-form symmetry. This is also in agreement with the results of v1 and v2 of [50].

We refer the reader to Appendix C for a detailed review for 3D McKay correspondence and BPS quivers.

A. Abelian subgroups of $SU(3)$

We now turn to the case of Γ a finite Abelian subgroup of $SU(3)$. In this case, we have a diagonal group action on the holomorphic coordinates (x, y, z) of \mathbb{C}^3 . Since the maximal torus of $SU(3)$ is just $U(1)^2$, the most general orbifold action is given by

$$(x, y, z) \mapsto (\omega^{a_1} \xi^{b_1} x, \omega^{a_2} \xi^{b_2} y, \omega^{a_3} \xi^{b_3} z), \quad (3.1)$$

with ω and ξ primitive m^{th} and n^{th} roots of unity, and integers a_i and b_i with $\sum_i a_i = 0 \pmod m$ and $\sum_i b_i = 0 \pmod n$.

We now consider two specific set of actions that highlight the main properties of these orbifolds. More general actions can be considered, but a complete analysis is left for future work.

The first case we consider is when $n = 1$, thus $\Gamma = \mathbb{Z}_m$. Here, the generators are of the form $\frac{1}{m}(1, a_2, a_3)$, with $1 + a_2 + a_3 = 0 \pmod m$. In this case the choice of group action dictates the fixed point locus. For these examples we can state a general rule:

$$H_{\Gamma, f_\Gamma} = \mathbb{Z}_{gcd(m, a_2)} \times \mathbb{Z}_{gcd(m, a_3)} \quad (3.2)$$

Let us check this formula on few examples:

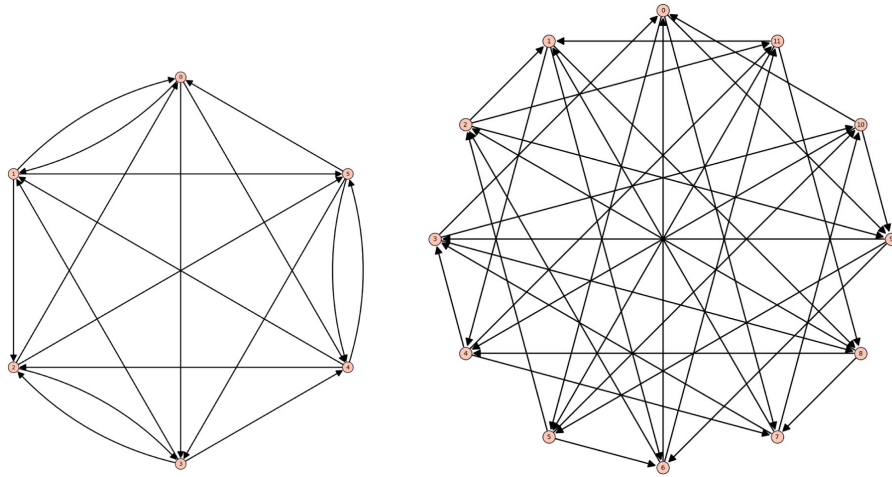


FIG. 1. Left: the orbifold quiver for the $\mathbb{C}^3/\mathbb{Z}_6$ theory generated by $g = \frac{1}{6}(1, 2, 3)$. Right: the orbifold quiver for the $\mathbb{C}^3/\mathbb{Z}_{12}$ theory generated by $g = \frac{1}{12}(1, 2, 9)$.

(i) One generator for all fixed points.

The subgroup of fixed points is generated by some unique power of the generator of the full group, i.e., g^k has fixed points, with g the generator of $\Gamma = \mathbb{Z}_m$. In these cases, $H_{\Gamma, f_\Gamma} = \mathbb{Z}_{m/k}$, and thus

$$\mathbb{D}_{M_2}^{(1)} = \text{Ab}[\pi_1(S^5/\Gamma)] = \mathbb{Z}_m/\mathbb{Z}_{m/k} \cong \mathbb{Z}_k \quad (3.3)$$

Example 1: For $g = \frac{1}{10}(1, 1, 8)$, only $k = 5$ gives $g^k = \frac{1}{2}(1, 1, 0)$ with fixed loci $|z| = 1$. So we have $\mathbb{D}_{M_2}^{(1)} = \mathbb{Z}_5$.²

(ii) More generators for all fixed points.

If there are more powers of the generator of the orbifold that lead to a fixed point, the normal subgroup is just the direct product of those factors.

Example 2: For $\Gamma = \mathbb{Z}_6$ generated by $g = \frac{1}{6}(1, 2, 3)$, both $k = 2$ and $k = 3$ have fixed points. $k = 2$ leads to a \mathbb{Z}_3 subgroup, while $k = 3$ to a \mathbb{Z}_2 . So $H_{\Gamma, f_\Gamma} = \mathbb{Z}_3 \times \mathbb{Z}_2 = \Gamma$, and thus $\mathbb{D}_{M_2}^{(1)} = 0$. We have also directly verified this by constructing the corresponding 5D BPS quiver—see Fig. 1 (left).

Example 3: For $\Gamma = \mathbb{Z}_{12}$ generated by $g = \frac{1}{12}(1, 2, 9)$, g^4 generates a \mathbb{Z}_3 subgroup and g^6 a \mathbb{Z}_2 one. So $H_{\Gamma, f_\Gamma} = \mathbb{Z}_2 \times \mathbb{Z}_3 \subset \mathbb{Z}_{12} = \Gamma$, and thus $\mathbb{D}_{M_2}^{(1)} = \mathbb{Z}_2$. The corresponding 5D BPS quiver can be found in Fig. 1 (right).

The second case we consider is $\Gamma = \mathbb{Z}_m \times \mathbb{Z}_n$, where we mainly consider the family of generators $\frac{1}{m}(1, m-1, 0)$ for \mathbb{Z}_m and $\frac{1}{n}(0, 1, n-1)$ for \mathbb{Z}_n that has been treated in [50].

²In particular, $g = \frac{1}{10}(1, 2, 7)$ is also such that $\mathbb{D}_{M_2}^{(1)} = \mathbb{Z}_5$. We have confirmed by quiver computation that the statement of “ $\Lambda_{\text{cl.}} = \mathbb{Z}_2$ ” in v1, v2 and v3 of [50] is a typo.

In this case, we can rule out any nontrivial higher symmetry using the algebraic approach. Each of these manifestly have a fixed circle ($|z| = 1$ and resp., $|x| = 1$). So they both have $H_{\Gamma, f_\Gamma} = \Gamma$ and so Armstrong’s theorem tells us that $\pi_1(S^5/\Gamma) = 0$. This result can also be established using BPS quivers. For example, for $n = m$, the 5D BPS quiver corresponding to models in this class all have a box-product form [105]³

$$\hat{A}(n, 0) \boxtimes \hat{A}(n, 0) \quad (3.4)$$

where $\hat{A}(n, 0)$ is the quiver corresponding to a loop with n arrows oriented clockwise and superpotential given by $\text{tr}(\prod_i \psi_i)$. A direct case by case analysis for $0 \leq n \leq 30$ reveals that the cokernel is trivial in all these cases. Similar considerations hold for $n \neq m$, though in this case we do not have a single concise expression as in line (3.4), and the quivers have to be extracted from the general procedure summarized in Appendix C.

In addition, we want to point out that there are more possible actions of $\mathbb{Z}_m \times \mathbb{Z}_n$ than those considered above. Here we give an example of such action with a nontrivial 1-form symmetry.

³Given two acyclic quivers Q_1 and Q_2 with adjacency matrices $B_i = S_i^t - S_i$ where S_i are upper triangular $2d/4d$ S -matrices and $i = 1, 2$, the quiver $Q_1 \boxtimes Q_2$ is a quiver with adjacency matrix $B_{\boxtimes} = (S_1 \otimes S_2)^t - (S_1 \otimes S_2)$. If the quivers are not acyclic, which is the case in Eq. (3.4), one can still define a \boxtimes operation at the level of the corresponding path algebras: the path algebra of the quiver $Q_1 \boxtimes Q_2$ is the tensor product of $\mathbb{C}Q_1$ and $\mathbb{C}Q_2$ with extra lagrange multipliers which implements the commutativity relation on the corresponding squares—the quiver $Q_1 \boxtimes Q_2$ also comes with a superpotential, which to a first approximation is a superposition of the superpotentials of Q_1 and Q_2 together with extra Lagrange multipliers which implement the commutativity of the tensor product operation on the path algebra—see, e.g., [101] for a review of the \boxtimes operation for Q_1 and Q_2 acyclic.

Example 4: For $\omega = \frac{1}{9}(1, 1, 7)$ and $\xi = \frac{1}{3}(1, 2, 0)$. The fixed loci are generated by 3ω and $3\omega + \xi$, spanning a group of $H_{\Gamma, f\Gamma} = \mathbb{Z}_3 \times \mathbb{Z}_3$. So we have $D_{M2}^{(1)} = \mathbb{Z}_3$.

B. Examples induced from subgroups of $U(2)$

Let us now turn to subgroups of $SU(3)$ induced from finite non-Abelian subgroups of $U(2)$. These groups are obtained by taking some finite $\hat{\Gamma} \subset U(2)$ and mapping the elements $\hat{g} \in \hat{\Gamma}$ to

$$g = \begin{pmatrix} \hat{g} & 0 \\ 0 & (\det \hat{g})^{-1} \end{pmatrix}. \tag{3.5}$$

The finite subgroups of $U(2)$ are well known (see [107], for example) and given by certain cyclic extensions of finite $SU(2)$ subgroups. In principle, this gives us an exhaustive method for generating such finite $SU(3)$ subgroups.

The first set of $U(2)$ derived groups we consider are those found in [48,50]. These groups form a special subclass of $SU(3)$ groups in the sense that their invariant subrings are complete intersection rings. Using the generators provided in [50], we easily see that many elements have fixed points regardless of the group. In fact, in all of these cases we get that $H \cong \Gamma$ and hence

$$\pi_1(S^5/\Gamma) = 0. \tag{3.6}$$

We list our findings in Table I.

However, our approach means we can consider more general subgroups of $SU(3)$ derived from $U(2)$. An important class which we can consider are those derived from *small*⁴ subgroups of $U(2)$ [47]. The ‘‘smallness’’ condition restricts the number of elements with fixed points in Γ , and as such can potentially have a larger defect group when compared with a ‘‘larger’’ subgroup of $SU(3)$.

Our findings in Table II can be confirmed again exploiting 5D BPS quivers. A more systematic study of the properties of the BPS categories of 5D orbifold SCFTs will appear elsewhere. For all these cases, we have a nontrivial defect group owing to the fact that they are built from small $U(2)$ subgroups. The analysis of these cases is uniform, and we carried out many consistency checks in this family. For the sake of brevity we report here only few salient examples. We refer to Appendix B for our conventions about the representations we exploit in the analysis.

1. The $D_{5,3}$ orbifold SCFT

Let us consider the lowest rank theory we find with nontrivial defect group—this happens to be the $D_{5,3}$

⁴A group $G \subset GL(n, \mathbb{C})$ is small if there are no elements $g \in G$ with exactly $n - 1$ many eigenvalues equal to 1. In other words, G contains no reflections.

TABLE I. Data for orbifold theories derived from subgroups of $U(2)$ which have complete intersection invariant subrings [48].

Γ	$ \Gamma $	$\mathbb{D}_{M2}^{(1)}$	$\text{Ab}[\Gamma]$
G_m	$8m$	0	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2(2 m); \mathbb{Z}_2 \times \mathbb{Z}_2(2 m)$
$G_{p,q}$	$8pq^2$	0	$\mathbb{Z}_2^2 \times \mathbb{Z}_{2q}(2 p); \mathbb{Z}_2 \times \mathbb{Z}_{2q}(2 p)$
G'_m	$8m$	0	$\mathbb{Z}_2 \times \mathbb{Z}_2(2 m); \mathbb{Z}_4 \times \mathbb{Z}_2(2 m)$
$E^{(1)}$	72	0	$\mathbb{Z}_3 \times \mathbb{Z}_3$
$E^{(2)}$	24	0	\mathbb{Z}_3
$E^{(3)}$	96	0	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$E^{(4)}$	48	0	\mathbb{Z}_2
$E^{(5)}$	96	0	\mathbb{Z}_4
$E^{(6)}$	48	0	$\mathbb{Z}_2 \times \mathbb{Z}_3$
$E^{(7)}$	144	0	$\mathbb{Z}_2 \times \mathbb{Z}_3$
$E^{(8)}$	192	0	$\mathbb{Z}_2 \times \mathbb{Z}_4$
$E^{(9)}$	240	0	\mathbb{Z}_2
$E^{(10)}$	360	0	\mathbb{Z}_3
$E^{(11)}$	600	0	\mathbb{Z}_5

TABLE II. Data for orbifold theories derived from small subgroups of $U(2)$. Note that the entries in this table depend are quite sensitive to the divisibility properties of m, n and q , as discussed in Appendix B. To get a sense of the size of the normal subgroup with a fixed point locus, we have also listed the abelianization of Γ .

Γ	$ \Gamma $	$\mathbb{D}_{M2}^{(1)}$	$\text{Ab}[\Gamma]$
T_m	$24m$	$\mathbb{Z}_{3m}(3 m)\mathbb{Z}_m(3 m)$	\mathbb{Z}_{3m}
O_m	$48m$	\mathbb{Z}_m	$\mathbb{Z}_2 \times \mathbb{Z}_m$
I_m	$120m$	\mathbb{Z}_m	\mathbb{Z}_m
$D_{n,q}$	$4qm(m=n-q)$	$\mathbb{Z}_{2m}(2 m), \mathbb{Z}_m(2 m)$	$\mathbb{Z}_{4m}(2 q); \mathbb{Z}_{2m} \times \mathbb{Z}_2(2 q)$

orbifold SCFT, a rank $r = 4$ SCFT with flavor symmetry of rank $f = 3$. The group $D_{5,3}$ is generated by

$$D_{5,3} = \left\langle \left(\begin{pmatrix} \zeta_6 & 0 & 0 \\ 0 & \zeta_6^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \zeta_8 & 0 & 0 \\ 0 & \zeta_8 & 0 \\ 0 & 0 & \zeta_8^{-2} \end{pmatrix} \right) \right\rangle. \tag{3.7}$$

We can now use the age grading of [108] to understand the gauge and flavor ranks of the theory. Explicitly, we find that the number of age-1 (or ‘‘junior’’) classes is 7, while the number of age-2 classes is 4. This then gives us

$$r = b_4(X) = 4, \quad f = b_2(X) - b_4(X) = 3, \tag{3.8}$$

where X is the crepant resolution of $\mathbb{C}^3/D_{5,3}$, the $b_i(X)$ are its Betti numbers. As such, we are looking for a quiver of size $2r + f + 1 = 12$ with an intersection pairing

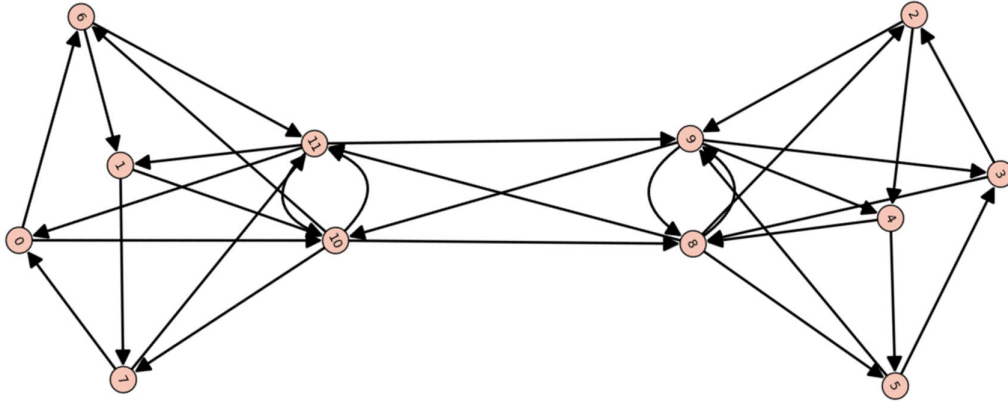


FIG. 2. The orbifold quiver for the $\mathbb{C}^3/D_{5,3}$ theory.

possessing a kernel of dimension 4. Indeed, we find the quiver in Fig. 2 which possess these properties.

The B -matrix we obtain is

$$\begin{pmatrix}
 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\
 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\
 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\
 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\
 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
 -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\
 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\
 -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\
 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0
 \end{pmatrix} \tag{3.9}$$

From which it is easy to check the defect group for the 4d KK theory is indeed $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ as expected.

2. T_3 orbifold SCFT

The T_3 group can be given as

$$\left\langle \left(\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \zeta_{18} \cdot \begin{pmatrix} (1+i)/2 & (-1+i)/2 & 0 \\ (1+i)/2 & (1-i)/2 & 0 \\ 0 & 0 & \zeta_{18}^{-3} \end{pmatrix} \right) \right\rangle,$$

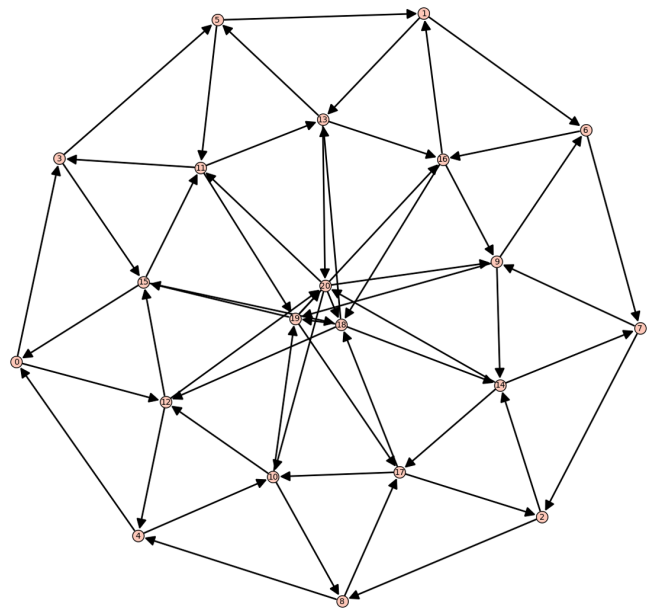


FIG. 3. The quiver for the \mathbb{C}^3/T_3 orbifold theory.

where $\zeta_{18} = e^{\pi i/9}$. From this we see that $|T_3| = 72$ and that there are 21 conjugacy classes.⁵ Furthermore, the age grading gives us

$$r = b_4(X) = 9, \quad f = b_2(X) - b_4(X) = 2, \tag{3.10}$$

where X is the crepant resolution of \mathbb{C}^3/T_3 , r is the rank of the theory and f is the rank of the flavor group. Note that these match with the constraint $2r + f + 1 = \chi(X) = 21$. Interpreting this as quiver data, this means that we should find a quiver with 21 nodes whose intersection pairing has a kernel of dimension 3. Indeed, using our program to find the quiver, we obtain the quiver in Fig. 3 which satisfies these conditions.

⁵By the 3d McKay correspondence, this is equal to the Euler characteristic of the resolved \mathbb{C}^3/T_3 orbifold.

The corresponding B -matrix is

$$\begin{pmatrix}
 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
 -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & -1 \\
 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 \\
 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & -1 \\
 -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\
 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\
 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\
 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 1 & -1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & -1 & 0
 \end{pmatrix} \tag{3.11}$$

Taking the Smith normal form we find that

$$\text{coker}(B) = \mathbb{Z}^3 \oplus \mathbb{Z}_9^e \oplus \mathbb{Z}_9^m, \tag{3.12}$$

giving us a \mathbb{Z}_9 electric one-form symmetry.

3. T_5 and T_7 orbifold SCFTs

These cases can be analyzed in a similar way as above, and we again reproduce the defect groups $\mathbb{Z}_5^{(1)}$ for the T_5 orbifold SCFT and $\mathbb{Z}_7^{(1)}$ for the T_7 orbifold SCFT. We draw the corresponding quivers in Fig. 4, in a slightly different

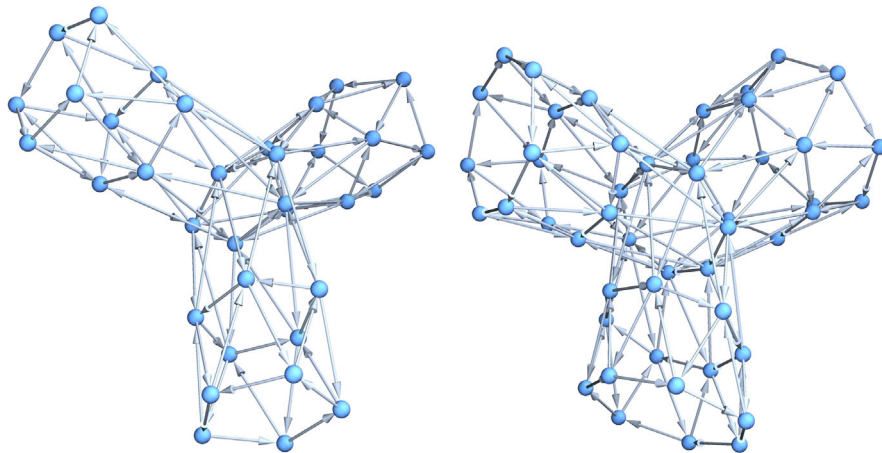


FIG. 4. Left: the BPS quiver corresponding to the orbifold group T_5 ; Right: the BPS quiver corresponding to the orbifold group T_7 . Notice that we can recognize a box product-like structure in the quiver with an affine \hat{E}_6 structure.

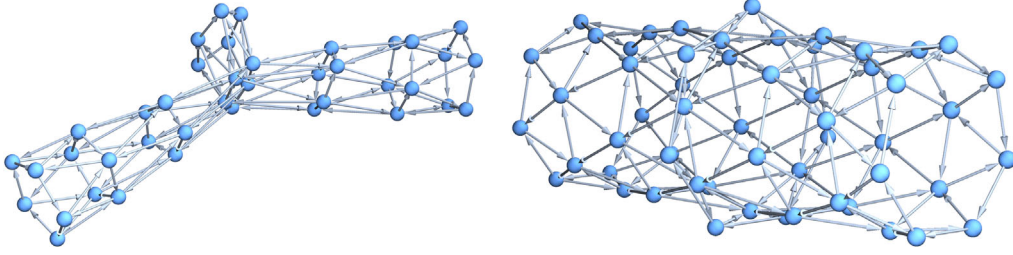


FIG. 5. Left: the BPS quiver corresponding to the orbifold group O_5 ; Right: the BPS quiver corresponding to the orbifold group O_7 . Notice that we can recognize a box product-like structure in the quiver with an affine \hat{E}_7 structure.

format to illustrate that these have a box product form. We report the corresponding B matrices in Appendix D.

4. O_5 and O_7 orbifold SCFTs

Also for these examples we report the relevant B -matrices in Appendix D, which can be used to reproduce the results in Table II also in these cases. We draw the corresponding quivers in Fig. 5 to illustrate that also these examples have quivers in a box product form.

5. General form of quivers for TOI orbifold theories

All the other cases of orbifold 5D SCFTs with TOI orbifold groups can be analyzed similarly by extracting the corresponding quiver using the procedure of Appendix C. Based on these examples and further checks for the orbifold groups T_m , O_m and I_m , we conjecture that, for suitable values of m , the quivers for many of the theories in this class can take the form of square tensor products too, leading to diagrams of the form

$$\hat{A}(m,0) \boxtimes \hat{E}_6, \quad \hat{A}(m,0) \boxtimes \hat{E}_7, \quad \hat{A}(m,0) \boxtimes \hat{E}_8, \quad (3.13)$$

respectively for 5D orbifold theories of type T_m , O_m and I_m . We stress this will not be the case for all values of m : for instance, in the case of T_m theories with $3|m$, we expect to obtain quivers similar to Fig. 3 consisting of an inner ring of m -many nodes surrounded by two rings of $3m$ -many nodes connected appropriately. Moreover, these constructions will depend on choosing a suitable representative in the mutation class of the $\hat{E}_{6,7,8}$ type. Of course, for all the examples we checked, the 5D BPS quiver reproduces the result of Table II.

C. Larger subgroups

Finally, let us briefly discuss the case of “larger subgroups,” namely transitive finite subgroups $\Gamma \subset SU(3)$ which are not Abelian, and which are also not induced from a subgroup of $U(2)$. In these cases, we expect that the larger size of the group correlates with a larger fixed point set in terms of a group action on S^5 . In fact, in all examples which we have checked, we find that the resulting defect group is trivial, simply because the normal subgroup of Γ

generated by the elements which have a fixed point is simply all of Γ ! We have directly checked the adjacency matrix of the corresponding BPS quiver in these cases as well, and again confirm this result, which is in accord with the statements of [50]. See Table III for an explicit list of these examples.

IV. $\text{Ab}[\Gamma]$ AND 2-GROUP SYMMETRIES

In the previous sections we saw that the group $\text{Ab}[\Gamma/H] = \mathbb{D}_{M2}^{(1)}$. Based on this, it is natural to ask whether the abelianization of Γ itself has any role to play in the 5D SCFT. Indeed, this structure directly appears in the related context of 6d SCFTs. Recall that in the F-theory realization of 6d SCFTs, one considers a canonical singularity of a noncompact elliptically fibered threefold $X \rightarrow B$. As shown in [109], the base B is always of the form $\mathbb{C}^2/\Gamma_{U(2)}$ for

TABLE III. Data for orbifold theories derived from transitive subgroups of $SU(3)$. We determined the higher-form symmetry by direct computation of $\text{Ab}[\pi_1(S^5/\Gamma)]$ and from an analysis of the corresponding 5D BPS quiver. In all cases, none of these theories exhibit any one-form symmetry. For completeness, we have also included the abelianization of all these groups. Here J and K follows the notation of Yau and Yu [47], while we have followed the notation of [50] in the remaining entries. See Appendix A for the definitions of all of these groups.

Γ	$ \Gamma $	$\mathbb{D}_{M2}^{(1)}$	$\text{Ab}[\Gamma]$
$\Delta(3n^2)$	$3n^2$	0	$\mathbb{Z}_3 \times \mathbb{Z}_3(3 n); \mathbb{Z}_3(3 n)$
$\Delta(6n^2)$	$6n^2$	0	\mathbb{Z}_2
$C_{3 l}^{(1)}(3 l)$	$9l^2$	0	$\mathbb{Z}_3 \times \mathbb{Z}_3$
$C_{7 l}^{(2)}$	$21l^2$	0	$\mathbb{Z}_3 \times \mathbb{Z}_3(3 n); \mathbb{Z}_3(3 n)$
$D_{3 l}^{(1)}(2 l)$	$18l^2$	0	$\mathbb{Z}_2 \times \mathbb{Z}_3$
H_{36}	108	0	\mathbb{Z}_4
H_{72}	216	0	$\mathbb{Z}_2 \times \mathbb{Z}_2$
H_{216}	648	0	\mathbb{Z}_3
H_{60}	60	0	1
H_{168}	168	0	1
H_{360}	1080	0	1
J	180	0	\mathbb{Z}_3
K	504	0	\mathbb{Z}_3

$\Gamma_{U(2)}$ a particular set of finite subgroups of $U(2)$, and in all these cases, $\partial B = S^3/\Gamma$. In this case, the corresponding defect group is associated with a two-form symmetry, as specified by string-like defects of the 6d SCFT [13] (see also [16,22,23,30,41,44]). While we leave a more complete analysis for future work, in this section we observe that in situations where the geometry faithfully reproduces the 0-form symmetry of the system, $\text{Ab}[\Gamma]$ is closely correlated with the 2-group symmetry of the 5D SCFT. Our plan in this section will be to first explain some basic aspects of 2-group symmetries, following [110], and then to turn to an analysis of a 5D SCFT where we can geometrically detect the 0-form symmetry. For further discussion on aspects of 2-group symmetries in 5D SCFTs, see, e.g., [33].

In order to investigate the potential role of $\text{Ab}[\Gamma]$ for 5D orbifold SCFTs, we begin with the following two remarks:

- (1) Whenever Γ acts without fixed points on S^5 , $\mathbb{D}_{M_2}^{(1)} \simeq \text{Ab}[\Gamma]$.
- (2) Whenever Γ acts with fixed points on S^5 , the theory $\mathcal{T}_{\mathbf{X}_\Gamma}$ typically has interesting 0-form symmetries, and $\mathbb{D}_{M_2}^{(1)}$ is a subgroup of $\text{Ab}[\Gamma]$.

For a theory which has both 0-form symmetries and 1-form symmetries, the two can form a more interesting global categorical symmetry, that can organize into a 2-group. 2-groups are characterized by a 4-plet, consisting of a 0-form symmetry group $\mathbb{F}^{(0)}$, a 1-form symmetry group $\mathbb{G}^{(1)}$, a representation $\rho: \mathbb{F}^{(0)} \rightarrow \text{Aut}(\mathbb{G}^{(1)})$ and an element $\beta \in H^3(B\mathbb{F}^{(0)}, \mathbb{G}^{(1)})$, which characterizes the obstruction to switching on nontrivial backgrounds for $\mathbb{F}^{(0)}$ independently from backgrounds for $\mathbb{G}^{(1)}$. We can think of β as if it is determining the 2-group structure constants, namely the extent to which the two higher symmetries mix with one another.

When Γ acts with fixed points on S^5 , the 5D SCFT typically has some nontrivial (non-Abelian) 0-form symmetry, which can be characterized by the structure of the noncompact singularities in \mathbf{X}_Γ . There is a “naive” answer dictated by lifting each simple Lie algebra factor to a simply connected Lie group, but this can in principle be quotiented to reach the true 0-form symmetry. In such situations, $\mathbb{D}_{M_2}^{(1)}$ is a strict subgroup of $\text{Ab}[\Gamma]$, and often the quotient can be understood in terms of the discrepancy between the naive 0-form symmetry and its quotiented counterpart. For example, in the case of the 5D T_N theory we have an orbifold with structure $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$. This geometry arises at the common intersection of three lines of singularities of the form $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_N$. In this case we have that $\mathbb{D}_{M_2}^{(1)}$ is trivial and $\text{Ab}[\Gamma] = \mathbb{Z}_N \times \mathbb{Z}_N$ which in turn can be interpreted as the subgroup of the “naive” flavor symmetry $SU(N)^3$ by which we would quotient to reach the true global 0-form symmetry given by $SU(N)^3/\mathbb{Z}_N \times \mathbb{Z}_N$ (see, e.g., [111] for a discussion of the global symmetries in the closely related 4d T_N theories). It is

therefore tempting to claim that we have a nontrivial Postnikov class β whenever the exact sequence⁶

$$1 \rightarrow \mathbb{D}_{M_2}^{(1)} \rightarrow \text{Ab}[\Gamma] \rightarrow \text{Ab}[\Gamma]/\mathbb{D}_{M_2}^{(1)} \rightarrow 1 \quad (4.1)$$

is nonsplit.

In order to check these general expectations, we seek a family of 5D orbifold SCFTs that have a Lagrangian interpretation and a nontrivial 1-form and 0-form symmetry. Since we also require that the geometry faithfully encodes the 0-form symmetry, an ideal class of examples in this case is provided by the Abelian orbifolds $\mathbb{C}^3/\mathbb{Z}_{2n}$ where \mathbb{Z}_{2n} acts as $\frac{1}{2n}(1, 1, 2n - 2)$. The latter give rise to 5D SCFTs with a gauge theory phase $SU(n)_n$. The 2-group structures have already been determined in [33], and for n even one finds that, choosing the electric global form of the theory, the 0-form symmetry of these models form a two group with the $\mathbb{Z}_n^{(1)}$ electric one-form symmetry. In our case since the group action is Abelian we find that the sequence (4.1) reduces to

$$1 \rightarrow \mathbb{Z}_n^{(1)} \rightarrow \mathbb{Z}_{2n} \rightarrow \mathbb{Z}_2 \rightarrow 1. \quad (4.2)$$

This sequence is nonsplit precisely when n is even, which exactly reproduces the result of [33] obtained via other methods.

We find this remarkable, and it is natural to ask how this extends to other situations where geometry faithfully encodes the 0-form symmetry. As a further remark we stress here that if we were to choose the magnetic form of the theory, the 2-group structure disappears: this suggests that there is an interconnection between the Heisenberg algebra of noncommuting fluxes and the 2-group structure constants.

V. CONCLUSIONS

In this paper we have presented a general prescription for extracting the higher 1-form and 2-form symmetries of 5D SCFTs obtained from M-theory on the orbifold \mathbb{C}^3/Γ with Γ a finite subgroup of $SU(3)$. We presented two complementary methods for extracting this data. First, building on [16–18], we showed how to extract it from the defining exact sequence for the defect group and the structure of

⁶According to (A2), $\mathbb{D}_{M_2}^{(1)}$ is a subgroup of $\text{Ab}[\Gamma]$. Noticing that $\text{Ab}[\Gamma]$ is Abelian, we know that it contains a certain subgroup isomorphic to $\mathbb{D}_{M_2}^{(1)}$, which however does not give a canonical embedding. Therefore, it's more natural to consider the Pontryagin dual $G^\vee \equiv \text{Hom}(G, U(1))$ for each term of the sequence:

$$1 \rightarrow (\mathbb{D}_{M_2}^{(1)})^\vee \rightarrow (\text{Ab}[\Gamma])^\vee \rightarrow (\text{Ab}[\Gamma]/\mathbb{D}_{M_2}^{(1)})^\vee \rightarrow 1,$$

so that the first term can be canonically embedded into the second term.

$\pi_1(S^5/\Gamma)$, generalizing the analysis to the case where Γ has fixed points. Second, we showed that the same data can also be read off from the 5D BPS quiver of the corresponding SCFT, thus giving a nice consistency check to the method. We also provided some hints that the abelianization of Γ detects the presence of a 2-group structure in such 5D SCFTs. We also remarked that the interplay with the Heisenberg algebra of noncommuting fluxes with the global form of the 5D SCFT must affect the 2-group structure in a nontrivial way. In the remainder of this section we discuss some avenues for further analysis.

Much of our analysis has focused on the computation of higher-form symmetries in these 5D SCFTs. We also saw hints of how the 2-group structure in these systems descends from the abelianization of Γ . It would be quite interesting to elucidate this structure. In particular, it would be desirable to extract the Postnikov class β directly from the geometry of a string compactification. We think that in order to clarify this interplay it will be very fruitful to look at the symmetry TQFT [41] for these orbifold singularities, which arises from the reduction of the topological Chern-Simons term of M-theory on the horizon S^5/Γ .

Further compactification of these 5D SCFTs will give rise to a rich class of lower-dimensional systems. The same geometric methods presented here can also be used to read off the corresponding higher-form symmetries of these systems. For example, compactification of our 5D SCFTs on a circle will give rise to 4d $\mathcal{N} = 2$ SCFTs, and compactification on a T^2 will result in 3d $\mathcal{N} = 4$ SCFTs. Perhaps the most interesting case to study is the reduction of these 5D SCFTs on a Riemann surface Σ_g as pioneered in [112] to produce 3d $\mathcal{N} = 2$ theories. In these cases the global structures we find in this paper can give rise to more interesting effects, along the lines explored for 6d (2,0) theories in [90].

Although it is notoriously difficult to engineer stable nonsupersymmetric backgrounds in M-theory, it is nevertheless natural to consider more general orbifold group actions $\Gamma \subset SU(4)$.⁷ In this case, we can still read off a corresponding nonsupersymmetric quiver gauge theory, though in this case the adjacency matrix for the bosonic and fermionic degrees of freedom will be different. This serves to define two separate notions of “defect group” depending on the boson/fermion number of the quantity in question. It would be interesting to see whether the presence of a nontrivial defect group could be used as a way of constraining the resulting nonsupersymmetric dynamics.

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⁷For some recent investigations into 5D nonsupersymmetric CFTs, see, e.g., Refs. [113–115].

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Note added.—While this paper was in preparation and after our results were announced during various online seminars in November and December 2021, we learned that v3 of [50] (posted Dec. 27, 2021) contains some new material which has some overlap with the present paper. To a large extent, the results are consistent.

APPENDIX A: PROPERTIES AND COMPUTATION OF $\text{Ab}[\Gamma]$ AND $\text{Ab}[\Gamma/H]$

In this Appendix, we present some computations used in the main body concerning the abelianization of $\Gamma \subset SU(3)$, as well as its quotient Γ/H by H the normal subgroup generated by those elements of Γ which have a fixed point locus on $S^5 = \partial\mathbb{C}^3$.

The Appendix is organized as follows. First, in Appendix A 1 we give the definition of the abelianization $\text{Ab}[\Gamma]$ of a finite group Γ together with its properties. In Appendix A 2 we give the algorithm of computing the abelianization via SAGE [116]. The code requires as input the presentation of the finite group, which we give for all finite subgroups of $SU(3)$ in Appendix A 3. Finally in Appendix A 4, we explain the algorithm for computing $\text{Ab}[\Gamma/H]$.

1. General aspects of Abelianization

Recall that for a finite group Γ , the *abelianization* $\text{Ab}[\Gamma]$ is defined as

$$\text{Ab}[\Gamma] = \frac{\Gamma}{[\Gamma, \Gamma]}, \quad (\text{A1})$$

where $[\Gamma, \Gamma]$ is the commutator subgroup, or derived subgroup, of Γ . A group that satisfies $\text{Ab}[G] = \mathbb{1}$ is called *perfect* while an Abelian group clearly satisfies $\text{Ab}[\Gamma] = \Gamma$.

A useful fact about $\text{Ab}[\Gamma]$ is that any homomorphism φ from Γ to an Abelian group A factors through $\text{Ab}[\Gamma]$, which is often referred to as the “universal property of abelianizations.” By this we mean that there exists a unique homomorphism $\psi: \text{Ab}[\Gamma] \rightarrow A$ such that following diagram commutes

$$\begin{array}{ccc}
 \Gamma & \xrightarrow{\varphi} & A \\
 \text{quot.} \downarrow & \searrow \psi & \\
 \text{Ab}[\Gamma] & &
 \end{array}$$

Using this property, we can set up the following diagram

$$\begin{array}{ccccc}
 \Gamma & \xrightarrow{q_1} & \Gamma/H & \xrightarrow{q_2} & \text{Ab}[\Gamma/H] \\
 \downarrow p & & & \nearrow u & \\
 \text{Ab}[\Gamma] & & & &
 \end{array}$$

where N is a normal subgroup of Γ and p, q_1, q_2 are all the natural quotient maps. From this it follows that u must be surjective and, by the first isomorphism theorem, we get

$$\text{Ab}[\Gamma/H] \cong \frac{\text{Ab}[\Gamma]}{\ker u}. \quad (\text{A2})$$

Since quotients of finite Abelian groups are isomorphic to subgroups of that same group, we get that $\text{Ab}[\Gamma/H]$ is isomorphic to a subgroup of $\text{Ab}[\Gamma]$.

2. Algorithm

If we can find a presentation of these groups, then SAGE computes for us the abelianization (via `G.abelian_invariants()`), where G is a `FreeGroup` quotiented by equivalence relations. We can check that this works for $SU(2)$ subgroups where both the presentations and $\text{Ab}[\Gamma]$ are known.

For non-Abelian subgroups of $SU(3)$, the presentation is not known explicitly. We take the following approach when finding them.

- (i) We start from the sets of matrix generators as given in [50], and use them to construct a free group. We then find as many relations among them (as can be checked explicitly) as possible that we mod out by.
- (ii) If we do not yet have a complete set of defining relations, then the group we get $\hat{\Gamma}$ (either finite or infinite) would have Γ as a nontrivial subgroup. In this case we go back to the previous step. If, on the other hand, we find a finite group with the correct order, then we have obtained a correct presentation of this $\Gamma \subset SU(3)$.

In the end, we use a SAGE function to compute the abelianization of the group.

An alternative method of computing $\text{Ab}[\Gamma]$ is to input the groups as a matrix group and then use the `as_permutation_group()` and `as_finitely_presented_group()` functions to convert the matrix groups into a form for which Sage can compute the abelianization readily. The advantage of this method is that we no longer need to find a presentation of the group. However, the running time of the `as_permutation_group()` function

grows significantly as $|\Gamma|$ increases, so we have primarily used the previously listed method.

3. Presentations of $SU(3)$ subgroups

In this Appendix we give the explicit presentations of various finite subgroups of $SU(3)$ used in the main text.

a. Discrete subgroups of $U(2)$

This case is organized using rather different notation in [50] when compared to [117]. We will start by doing most of the cases with the [50] notation, while shifting to [117] notation when looking at ‘‘sporadic subgroups.’’

- (i) G_m $a^{2m} = b^2 = c^2 = aba^{-1}b^{-1} = (ac)^2 = (bc)^2 a^m = 1$, where $a = M_1, b = M_2, c = M_3$ as in (3.19) of [50].
- (ii) $G_{p,q}$ $a^{2pq} = b^{2q} = c^2 = aba^{-1}b^{-1} = (ac)^2 = (bc)^{4q} = (bc)^2 a^p b^{-2} = 1$, where $a = M_1, b = M_2, c = M_3$ as in (3.26) of [50].
- (iii) G'_m
 - (a) m even $a^4 = b^{4m} = (ab)^2 = a^2 b^{2m}$, where $a = M_1, b = M_2$ as in (3.34) of [50]
 - (b) m odd $a^4 = b^{4m} = c^4 = (ab)^2 = a^2 b a^{-2} b^{-1} = a^2 b^{2m} = b c b^{-1} c^{-1} = (ac)^4 = (bc)^m$, with $a = M_1, b = M_2, c = M_3$ as in (3.37) of [50]
- (iv) ‘‘Sporadic cases’’
 - (a) $E^{(1)}$: $a^3 b^{-3} = b^3 c^{-2} = abc^{-1} = d^3 a^4 = ada^{-1} d^{-1} = bdb^{-1} d^{-1} = cdc^{-1} d^{-1} = 1$
 - (b) $E^{(2)}$: $a^4 = b^2 = aba^{-1} b^{-1} = c^3 = cac^{-1} a^{-1} = cbc^{-1} b^{-1} = 1$
 - (c) $E^{(3)}$: $a^4 b^{-3} = b^3 c^{-2} = abc^{-1} = d^2 a^4 = ada^{-1} d^{-1} = bdb^{-1} d^{-1} = cdc^{-1} d^{-1} = 1$
 - (d) $E^{(4)}$: $c^3 = bc^{-1} a^{-1} c = (da)^2 = c^{-1} bca^{-1} b = d^{-1} b a^{-2} d^{-1} = c^{-1} a^{-1} d^{-1} b^{-1} c^{-1} d^{-1} = 1$
 - (e) $E^{(5)}$: $a^3 b^{-3} = b^3 c^{-2} = abc^{-1} = d^2 c^3 = ada^{-1} d^{-1} = bdb^{-1} d^{-1} = cdc^{-1} d^{-1} = 1$
 - (f) $E^{(6)}$: $a^3 b^{-3} = b^3 c^{-2} = abc^{-1} = d^2 a^3 = ada^{-1} d^{-1} = bdb^{-1} d^{-1} = cdc^{-1} d^{-1} = 1$
 - (g) $E^{(7)}$: $a^4 b^{-3} = b^3 c^{-2} = abc^{-1} = d^3 a^4 = ada^{-1} d^{-1} = bdb^{-1} d^{-1} = cdc^{-1} d^{-1} = 1$
 - (h) $E^{(8)}$: $a^4 b^{-3} = b^3 c^{-2} = abc^{-1} = d^4 a^4 = ada^{-1} d^{-1} = bdb^{-1} d^{-1} = cdc^{-1} d^{-1} = 1$
 - (i) $E^{(9)}$: $a^5 b^{-3} = b^3 c^{-2} = abc^{-1} = d^2 c^5 = ada^{-1} d^{-1} = bdb^{-1} d^{-1} = cdc^{-1} d^{-1} = 1$
 - (j) $E^{(10)}$: $a^5 b^{-3} = b^3 c^{-2} = abc^{-1} = d^3 c^5 = ada^{-1} d^{-1} = bdb^{-1} d^{-1} = cdc^{-1} d^{-1} = 1$
 - (k) $E^{(11)}$: $a^5 b^{-3} = b^3 c^{-2} = abc^{-1} = d^5 c^5 = ada^{-1} d^{-1} = bdb^{-1} d^{-1} = cdc^{-1} d^{-1} = 1$

(v) $D_{n,q}$ cases

There are all small subgroups of $U(2)$ whose presentation and abelianization have been presented in (4.26)–(4.27) of [13].

(vi) T_m, O_m, I_m cases

We notice that $O_m = O \times \mathbb{Z}_m$ for all allowed m (namely $(m, 6 = 1)$), $I_m = I \times \mathbb{Z}_m$ for all allowed m (namely $(m, 30 = 1)$), and $T_m = T \otimes \mathbb{Z}_m$ for

$m \equiv 1, 5 \pmod{6}$, where $T, O, I \subset SU(2)$ are the tetrahedral, octahedral and isocahedral groups, respectively. So for these cases, the abelianization follows from those of $SU(2)$ finite subgroups, and for the latter we refer to [13].

For T_m where $m = 6k + 3$, the above direct product expression no longer holds. Instead, the presentation is given by $bab^{-1}a = a^4 = ab^{-2}a = acbc^{-1} = bcabc^{-1} = c^{3m} = 1$.

b. Transitive subgroups of $SU(3)$

- (i) Δ series
 - (a) $\Delta(3n^2)a^n = b^n = aba^{-1}b^{-1} = afb^{-1}f^{-1} = (af)^3$, where $a = L_n$, $f = E$ as in (3.56) of [50], and $b = \text{diag}\{1, \omega_n, \omega_n^{-1}\}$.
 - (b) $\Delta(6n^2)$ generated with a, b, f as in $\Delta(3n^2)$ with the above relations, and $h = I$ as in (3.62) of [50] such that $h^2 = (hf)^2 = hahb = 1$.
- (ii) $C_{n,l}^{(k)}$
 - (a) $(r, k, l) = (3, 1, l)$, $3|l$ defined by $a^{3l} = b^l = f^3 = afa^{-1}f^{-1} = (af)^3 = faf^{-1}a^{-1}b^{-1} = 1$, where $a = B_{9,1}$, $b = G_{7,1}$, $f = E$ as in (3.83) of [50].
 - (b) $(r, k, l) = (7, 2, l)$ defined by $a^{7l} = b^l = f^3 = afa^{-1}f^{-1} = (af)^3 = afa^{-1}f^{-1}ab^{-1} = 1$, where $a = B_{7,2}$, $b = G_{7,7}$, $f = E$ as in (3.90) of [50].
- (iii) $D_{3l,l}^{(1)}$, $2|l$ generated with $a = B_{3l,l}$, $b = G_{7,7}$, $f = E$, $h = I$ as in (3.62) and (3.83) such that $a^{3l} = b^l = f^3 = h^2 = aba^{-1}b^{-1} = (af)^3 = faf^{-1}a^{-1}b^{-1} = hahba^2 = (fh)^2 = afba^2f^{-1} = (ba^2f)^3 = 1$
- (iv) Exceptional subgroups
 - (a) H_{36} : $a^3 = b^3 = f^3 = z^4 = aba^{-1}b^{-1} = afb^{-1}f^{-1}b^{-1}a = azafz^{-1}f^{-1}a^{-1}b = azfz^{-1} = 1$, where $a = M_1$, $f = M_2$, $z = M_3$, $b = \text{diag}\{\omega_3, \omega_3^2, 1\}$
 - (b) H_{72} : $a^3 = b^3 = f^3 = z^4 = w^4 = aba^{-1}b^{-1} = afb^{-1}f^{-1}b^{-1}a = azfz^{-1} = z^2aw^2 = z^2f^2zw^3z \times fb^{-1}aw^{-1} = 1$, where a, b, f, z as defined in H_{36} , and $w = M_4$ as in (3.125) of [50].
 - (c) H_{216} : defined with a, b, f, w, z as in H_{72} satisfying all the above relations, and $y = M_4$ in (3.137) of [50] with the extra relation that $y^9 = yay^{-1}a^{-1} = (yz)^3 = wyz^{-1}y^{-1} = 1$
 - (d) H_{60} : $a^5 = b^2 = c^2 = (ab)^3 = (ac)^2 = (abc)^5 = (bc)^2 = 1$, where $a = H_1$, $b = H_2$, $c = H_3$ according to Sec. 3.4.1 of [117].
 - (e) H_{360} Defined with a, b, c as in H_{60} with the above relations, and $d = M_4$ in (3.145) of [50] with extra relations $d^2 = (ad)^3 = (cd)^2 = (adbdc)^3 = (aba^2cad)^3 = 1$
 - (f) H_{168} $a^7 = b^7 = aba^{-1}b^{-1} = E^3 = aEb^{-1}E^{-1} = Q^2 = (Qa)^4 = (Qb)^7 = (QE)^2 = (Qab)^3 = 1$, where $a = M_1$, $E = M_2$, $Q = M_3$ as in (3.132)

of [50], and $b = \text{diag}\{\omega_7^4, \omega_7, \omega_7^2\}$ is a suitable permutation of the diagonal entries in a .

- (g) J This is a case with order 180 that was filled in by Yau and Yu [47], which is isomorphic to $H_{60} \times \mathbb{Z}_3$, so $\text{Ab}[J] = \text{Ab}[H_{60}] \times \mathbb{Z}_3$.
- (h) K Similar to J , this is a case with order 504 such that $K = H_{168} \times \mathbb{Z}_3$, so $\text{Ab}[K] = \text{Ab}[H_{168}] \times \mathbb{Z}_3$.

4. Computation of $\text{Ab}[\Gamma/H]$

Given an explicit quotient singularity \mathbb{C}^3/Γ specified by a discrete group Γ and its explicit action on \mathbb{C}^3 , we now give a SAGE algorithm to compute the normal subgroup H generated by all elements inside Γ whose action has fixed points, and eventually $\text{Ab}[\Gamma/H]$.

An action of Γ on \mathbb{C}^3 is determined by a representation $\rho: g \rightarrow GL(3, \mathbb{C})$ that assign to an element $g \in \Gamma$ a 3-by-3 matrix $\rho(g)$ with complex entries. $\rho(g)$ will have a fixed element $\mathbf{v} \in \mathbb{C}^3$ if and only if

$$\exists \mathbf{v} \in \mathbb{C}^3 \setminus \{0\} \text{ s.t. } \rho(g)\mathbf{v} = \mathbf{v} \Leftrightarrow |\rho(g) - I| = 0, \quad (\text{A3})$$

where I is the 3-by-3 identity matrix. So our task is to ask SAGE to compute $|\rho(g) - I|$ for all $g \in \Gamma$, and then determine the normal subgroup $H \trianglelefteq \Gamma$ that these elements generate.

Computing H . Technically, one can simplify this by noticing that $\forall h \in \Gamma, |\rho(hgh^{-1} - I)| = |\rho(h)||\rho(g) - I||\rho^{-1}(h)| = |\rho(g) - I|$, so $|\rho(g) - I|$ only depends on the conjugacy class that g sits in. So we need these steps to compute H :

- (i) Determine the list of conjugacy classes of Γ
- (ii) Find an element g for each such conjugacy class and compute its $|\rho(g) - I|$, and thus determining $|\rho(g) - I|$ for this entire conjugacy class
- (iii) Take the union of all conjugacy classes that has $|\rho(g) - I| = 0$, and compute the subgroup H which they generate.

Computing $\text{Ab}[\Gamma/H]$. Having both Γ and H explicitly, we can then use SAGE to compute Γ/H as well as $\text{Ab}[\Gamma/H]$.

APPENDIX B: GROUP THEORY OF $D_{n,q}, T_m, O_m$, AND I_m

In this Appendix we present the explicit generators for the finite subgroups of $SU(3)$ induced by finite subgroups of $U(2)$ specified by $D_{n,q}, T_m, O_m$, and I_m . These are essentially just twists by an additional cyclic subgroup of the familiar D and E -series finite subgroups of $SU(2)$. Our discussion follows that in Ref. [47].

1. $D_{n,q}$

This group is built from a small representation of an extension of the binary dihedral group \mathbf{BD}_n . To ensure the smallness of this representation, we require $1 < q < n$ and

$(n, q) = 1$. Furthermore, we must split this case into two subclasses.

Taking $m = n - q$ to be odd, we can generate $D_{n,q}$ by

$$D_{n,q} = \left\langle \left(\begin{array}{ccc} \zeta_{2q} & 0 & 0 \\ 0 & \zeta_{2q}^{-1} & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} \zeta_{2m} & 0 & 0 \\ 0 & \zeta_{2m} & 0 \\ 0 & 0 & \zeta_{2m}^{-2} \end{array} \right) \right\rangle. \quad (\text{B1})$$

where $\zeta_k = e^{2\pi i/k}$. Now taking $m = n - q$ even, we generate the group as

$$D_{n,q} = \left\langle \left(\begin{array}{ccc} \zeta_{2q} & 0 & 0 \\ 0 & \zeta_{2q}^{-1} & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} \zeta_{4m} & 0 & 0 \\ 0 & \zeta_{4m} & 0 \\ 0 & 0 & \zeta_{4m}^{-2} \end{array} \right) \right\rangle. \quad (\text{B2})$$

Note that in the even case, there are only two generators.

2. T_m

This group is built from a small representation of an extension of the binary tetrahedral group **BT**. The smallness condition in this case is that m must be odd. Again, we must split this into two further cases.

Taking $m = 1$ or $5 \pmod{6}$, we generate the group as

$$\left\langle \left(\begin{array}{ccc} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} (1+i)/2 & (-1+i)/2 & 0 \\ (1+i)/2 & (1-i)/2 & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} \zeta_{2m} & 0 & 0 \\ 0 & \zeta_{2m} & 0 \\ 0 & 0 & \zeta_{2m}^{-2} \end{array} \right) \right\rangle. \quad (\text{B3})$$

Taking $m = 3 \pmod{6}$, we generate the group by

$$\left\langle \left(\begin{array}{ccc} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{array} \right), \zeta_{6m} \cdot \left(\begin{array}{ccc} (1+i)/2 & (-1+i)/2 & 0 \\ (1+i)/2 & (1-i)/2 & 0 \\ 0 & 0 & \zeta_{6m}^{-3} \end{array} \right) \right\rangle. \quad (\text{B4})$$

3. O_m

This group is built from a small representation of an extension of the binary octahedral group **BO**. To ensure smallness, we impose $(m, 6) = 1$. This is enough to give one set of generators for any valid m . The group is therefore always generated by

$$\left\langle \left(\begin{array}{ccc} \zeta_8 & 0 & 0 \\ 0 & \zeta_8^{-1} & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} (1+i)/2 & (-1+i)/2 & 0 \\ (1+i)/2 & (1-i)/2 & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} \zeta_{2m} & 0 & 0 \\ 0 & \zeta_{2m} & 0 \\ 0 & 0 & \zeta_{2m}^{-2} \end{array} \right) \right\rangle. \quad (\text{B5})$$

4. I_m

This group is built from a small representation of an extension of the binary icosahedral group **BI**. The smallness condition is given by $(m, 30) = 1$. The group can always be generated as

$$\left\langle \left(\begin{array}{ccc} \zeta_{2m} & 0 & 0 \\ 0 & \zeta_{2m} & 0 \\ 0 & 0 & \zeta_{2m}^{-2} \end{array} \right), \left(\begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} \zeta_5^3 & 0 & 0 \\ 0 & \zeta_5^2 & 0 \\ 0 & 0 & 1 \end{array} \right), \frac{1}{\sqrt{5}} \left(\begin{array}{ccc} \zeta_5^4 - \zeta_5 & \zeta_5^2 - \zeta_5^3 & 0 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 & 0 \\ 0 & 0 & \sqrt{5} \end{array} \right) \right\rangle. \quad (\text{B6})$$

APPENDIX C: 3D MCKAY CORRESPONDENCE

In this Appendix we review the construction of the BPS quiver obtained from D-branes probing an orbifold singularity. In the main body of the text, this is used to compute the BPS quiver of the 5D SCFT, and is equivalent to extracting the world volume theory of a probe D0-brane of the orbifold singularity. With this in mind, it suffices to

consider a T-dual description as obtained from 4d $\mathcal{N} = 4$ Super-Yang Mills theory with gauge group $U(n)$. We then can apply the general orbifold prescription of [96] as described in Refs. [97,98].

To begin, recall that the matter content of 4d $\mathcal{N} = 4$ SYM is given by adjoint valued fields in the singlet, fundamental and two-index antisymmetric representation

of the R-symmetry $SU(4)$, respectively describing the vector bosons, fermions, and scalars of the theory:

	$U(n)$ gauge	$SU(4)$ R-symmetry
A_{IJ}	Adj.	1
ψ_{IJ}^α	Adj.	4
Φ_{IJ}^α	Adj.	6

To construct the orbifolded theory, we can consider Γ , a finite subgroup of $SU(3)$, which is in turned embedded in the $SU(4)_R$ symmetry so that the fundamental decomposes as $\mathbf{4} \rightarrow \mathbf{3} \oplus \mathbf{1}$. Under the orbifold action, the indices of the vector boson break into various representation of Γ , γ_i , such that

$$U(N) \rightarrow \prod_i U(N_i), \quad (\text{C1})$$

where $N_i = n \dim(\gamma_i)$.

Indeed, we can write adjoint fields of $U(N)$ as $\text{Hom}(\mathbb{C}^N, \mathbb{C}^N)$. When we take the orbifold quotient, we keep only the invariant homomorphisms under the action of irreducible representations (irreps) of Γ

$$(\text{Hom}(\mathbb{C}^N, \mathbb{C}^N))^\Gamma = \bigoplus_i \text{Hom}(\mathbb{C}^{N_i}, \mathbb{C}^{N_i}). \quad (\text{C2})$$

Since we are dealing with a brane probe theory which preserves $4d \mathcal{N} = 1$ supersymmetry, it suffices to consider the fermions, which will be paired with scalar degrees of freedom.

The ψ_{IJ}^α transforms as $\mathbf{4}_R \otimes \text{Hom}(\mathbb{C}^N, \mathbb{C}^N)$. When we quotient we get:

$$(\mathbf{4}_R \otimes \text{Hom}(\mathbb{C}^N, \mathbb{C}^N))^\Gamma = \bigoplus_{i,j} a_{ij}^4 \text{Hom}(\mathbb{C}^{N_i}, \mathbb{C}^{N_j}). \quad (\text{C3})$$

The fermions are now bifundamentals charged under the various $U(N_i)$, and the matrix a_{ij}^4 gives the adjacency matrix for the quiver describing the theory. To be precise, a_{ij}^4 gives the number of arrows from node i to node j in the quiver. To compute a_{ij}^4 , we use the following decomposition:

$$\mathbf{4}_R \otimes \gamma_i = \bigoplus_j a_{ij}^4 \gamma_j. \quad (\text{C4})$$

We now trace this relation to have a relation between characters of irreducible representations:

$$\chi(\mathbf{4}_R)^\alpha \chi(\gamma_i)^\alpha = \sum_j a_{ij}^4 \chi(\gamma_j)^\alpha, \quad (\text{C5})$$

where α indicates the conjugacy class. Using the orthogonality of the characters we can express a_{ij}^4 as

$$a_{ij}^4 = \frac{1}{|\Gamma|} \sum_\alpha r_\alpha \chi(\mathbf{4}_R)^\alpha \chi(\gamma_i)^\alpha \overline{\chi(\gamma_j)^\alpha}, \quad (\text{C6})$$

where r_α counts the dimension of the α conjugacy class and the bar means complex conjugate.

What we need to specify now is $\chi(\mathbf{4}_R)^\alpha$. Using the decomposition $\mathbf{4}_R \rightarrow \mathbf{3} \oplus \mathbf{1}$, for the fundamental of $SU(4)_R$ into $SU(3)$, the character becomes

$$\chi(\mathbf{4}_R)^\alpha = \chi(\mathbf{1})^\alpha + \chi(\mathbf{3})^\alpha = 1 + \chi(\mathbf{3})^\alpha. \quad (\text{C7})$$

This tells us that Γ acts on fermions with a three dimensional representation, which needs not to be irreducible. We have the following possible decompositions for three dimensional representations:

$$\begin{aligned} \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{1}''' &\rightarrow \chi(\mathbf{1}')^\alpha + \chi(\mathbf{1}'')^\alpha + \chi(\mathbf{1}''')^\alpha \\ \mathbf{1}' \oplus \mathbf{2} &\rightarrow \chi(\mathbf{1}')^\alpha + \chi(\mathbf{2})^\alpha \\ \mathbf{3} &\rightarrow \chi(\mathbf{3})^\alpha, \end{aligned} \quad (\text{C8})$$

where $\mathbf{1}'$ is a one-dimensional, possibly nontrivial, irreducible representation.

In order to choose a consistent decomposition of a three-dimensional representation in term of irreducible representations, we require that the product of the determinant of all irreducible representations be unity. This means that the one-dimensional representation must be chosen such that

$$\prod_\alpha \chi(\mathbf{1}')^\alpha \chi(\mathbf{1}'')^\alpha \chi(\mathbf{1}''')^\alpha = \mathbf{1} \quad (\text{C9})$$

since the character table of one dimensional irreps corresponds with the representations itself. For the $\mathbf{1}' \oplus \mathbf{2}$ we can also work out the determinant of the two-dimensional irreducible representations using the Adams operations.

1. Computing the defect group

As briefly discussed in Sec. II, from the matrix a_{ij}^4 we can compute the Dirac pairing

$$B_{ij} = a_{ij}^4 - a_{ji}^4, \quad (\text{C10})$$

and using the results of [77] (see also [17,32]), we can extract the defect group of the theory.

Since B is a $n \times n$ matrix with entries in \mathbb{Z} , one can decompose it into the Smith normal form (SNF). This amounts to finding invertible matrices S and T over \mathbb{Z} such that $B = SB_{\text{SNF}}T$, this is a change of base of B . In the new frame, $B_{\text{SNF}} = \text{diag}\{a_1, a_2, a_3, \dots, a_m, 0, \dots, 0\}$, such that a_i are integers and a_i divides a_{i+1} for each $i < m$.

The matrices B and B_{SNF} have the same cokernel given by

$$\begin{aligned} \text{coker}(B) &= \text{coker}(B_{\text{SNF}}) \\ &= \mathbb{Z}^l \oplus \mathbb{Z}^m / (a_1 \mathbb{Z} \oplus a_2 \mathbb{Z} \oplus \dots \oplus a_m \mathbb{Z}), \end{aligned} \quad (\text{C11})$$

with l the number of zero diagonal elements of B_{SNF} , corresponding to vectors which lie in $\ker(Q)$. We also have that the a_i comes in equal pair, corresponding to electric and magnetic defect charges.

As discussed in [17], the cokernel of B gives the defect group of the theory. In particular, we have that

$$\begin{aligned} \text{Tor } \mathbb{D}^{(1)} &= \text{Tor}(\text{coker}(B)) \\ &= (\mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_1}) \oplus \dots \oplus (\mathbb{Z}_{a_{m/2}} \oplus \mathbb{Z}_{a_{m/2}}), \end{aligned} \quad (\text{C12})$$

APPENDIX E: INSTRUCTION FOR USING THE SAGE/JUPYTER NOTEBOOK FILES

Attached to this submission is the `SU3_defect_groups_FINAL.ipynb` file which contains all the major code we wrote for use in this project. In this appendix we will briefly describe the major functions of the notebook for the reader's use.

1. Group database

The first set of functions are simply functions to construct the finite subgroups of $SU(3)$. Many of the groups listed are given primarily as a finitely presented group. However, by calling `mat = True` in the constructors, a genuine $SU(3)$ matrix group presentation is typically given. The exceptions to this are listed in the file, but notably the groups D_{nq} , T_m , O_m and I_m are currently only given as matrix groups.⁸

2. Quiver and defect group computation

The most major function in the notebook is the `orbifold_defect_group()` which takes a group G and a representation R and computes the defect group of the resulting orbifold theory by constructing the McKay quiver relative to R . Note that the labeling of representations is given by the character table produced by Sage. As such, the desired representation should be specified as `[i]` where i

⁸If need be, SAGE can obtain finite presentations for these groups by first turning these into permutation groups and then into finitely presented groups.

denotes the representation given in row i of the character table. For direct sums of representations, simply specify more rows of the table as `[i, j, ...]`.

If one wishes to avoid specifying a representation like this, the function `orbifold_mat_grp()` will instead take a matrix group, decompose the given matrices into the desired representations and then calculate the appropriate McKay quiver and defect group.

Should one want to plot the quiver, the function `quiver_out()` wraps the previous two functions together and outputs a SAGE digraph plot. This can be accessed with either finitely presented groups or matrix groups, with the condition that `mat=True` should be called for the latter. It is worth noting that the placement of the nodes in this plot is done automatically by the in-built 'spring' layout. This can be changed in the final line of the function, should the presentation look sub-optimal.

3. Check on the order of the fundamental group

The final important function is `pi1_order()`. This function takes a matrix group and calculates the order of the fundamental group using Armstrong's theorem. To do so, it inspects each conjugacy class of the given group and singles out the classes with eigenvalue 1 and forms a group out of them. In order to keep computation time down, the option `batch_size` is included. If this option is specified, the algorithm will only take the specified number of elements from each fixed point conjugacy class to form the normal subgroup generated by fixed point elements. This can then be incrementally increased in order to check the size of $\pi_1(S^5/\Gamma)$.

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