Quantum thin shell surrounding a black hole

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In a previous work we obtained exact solutions for the proper time quantum mechanics of a thin dust shell, collapsing in a vacuum. We extend these results to the quantum collapse of a dust shell surrounding a preexisting black hole. In lieu of exact solutions, which have so far proved difficult to obtain for this system, we establish the essential features of the quantum shell through a Wentzel-Kramers-Brillouin approximation, which is valid only when the mass of the shell is much greater than the Planck mass. There are many similarities with the vacuum collapse: only bound states exist and the proper energy spectrum of the shell is unaffected by the presence of the central black hole to this order. There are no peculiar or distinguishing features of the wave function near the black hole horizon. It vanishes at the center and oscillates between the origin and the classically forbidden region, beyond which it decays exponentially.

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I. INTRODUCTION

Thin shells of matter, collapsing in a variety of environments, have been used extensively as simplified systems with which to model the final stages of gravitational collapse under many different conditions [1–10]. This is because thin shell collapse captures many of the features of more realistic collapse models while avoiding some of their technical difficulties. Moreover, the quantum mechanics of the shell is exactly solvable in some cases, which helps to shine a light on some of the problems of quantum gravity. For example, "time" has different meanings in classical general relativity and in the quantum theory. All choices of the time function yield the same local geometries, but quantum theories built on different time parameters are not unitarily equivalent. In [11], we showed that exact quantizations, based on different time variables, of a shell that is collapsing in a vacuum yield incompatible descrptions. When the shell quantization is based on coordinate time, solutions exist only when its mass is *less* than the Planck mass [12], but when it is based on proper time, solutions exist only when its mass is *greater* than the Planck mass, which is more in keeping with what is observed.

Among other important issues that one would like to understand from the point of view of quantum gravity is the Hawking effect [13] and the information loss paradox in black hole physics. Most discussions of the Hawking effect examine particle production in a scalar field propagating in the classical background geometry of a collapsing body from the point of view of the asymptotic observer. This is an effective field theory approach from which black hole evaporation and information loss are inferred. The geometry excites scalar field quanta, which then propagate to infinity as thermal, nearly thermal, or unitary radiation (recently, it was argued in [14,15] from this point of view that the radiation from a thin shell during its collapse is unitary). The result is that the black hole appears to evaporate over time as energy is drawn from it by the excited field quanta. Reasoning that this effect should have a counterpart in a time-dependent quantum gravitational collapse and from the point of view of a comoving observer (the black hole either evaporates or it does not), we constructed a midisuperspace quantization of a nonrotating dust ball (the simplest form of collapse) [16,17] using the equivalent of Kuchař variables [18] in the LeMaître-Tolman-Bondi (LTB) frame [19]. We were able to build exact diffeomorphism invariant states on a lattice, thereby treating the dust ball as a series of shells labeled by their LTB radial coordinate, and showed that matching the shell wave functions across the apparent horizon requires ingoing modes in the exterior to be matched to outgoing modes in the interior and, vice versa, ingoing modes in the interior are matched to outgoing modes in the exterior [20,21]. In each case, the relative amplitude of the outgoing wave is suppressed by the square root of the Boltzman factor at the Hawking temperature determined by the total Misner-Sharp mass contained within the shell.

There are two independent solutions. In one, exterior, infalling waves representing the collapsing shells of dust are accompanied by interior, outgoing waves. These interior waves, which are of quantum origin, represent an interior "Unruh" radiation. In the other solution, waves move away from the apparent horizon on both sides of it. Interior, infalling waves representing the continued collapse of the dust shells across the apparent horizon are accompanied by exterior, outgoing waves. These latter

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ougoing waves represent the exterior Unruh radiation, which is thermal. Continued collapse across the apparent horizon from an initial diffuse state can be achieved by combining the two solutions and requiring the net flux to vanish at the apparent horizon. The effect is that the collapse ends in a central singularity and is accompanied by thermal Unruh radiation in the exterior. The net effect in the proper time quantum theory is therefore a quasi-classical tunneling of particles as described in [22,23], but only if a continued collapse beyond the apparent horizon is assumed.

There are some ambiguities involved in the midisuperspace quantization program that are avoided in the quantization of a thin shell. Therefore, in this paper, we address the quantum mechanics of single thin dust shell that is collapsing in the background of a preexisting black hole from the comoving observer's point of view. The purpose is to understand what differences the background brings about in the shell wave function, in particular in its behavior at the event horizon of the black hole and at infinity, i.e., is there a similar tunneling effect as described above for the dust ball and if so does the shell evaporate thermally as suggested by Hawking?

On the classical level, the shell has just one degree of freedom and is completely described by its radius, $R(t)$ and its conjugate momentum, $P(t)$. We are unable to find exact wave functions, as we did in the case of a shell collapsing in a vacuum, but a Wentzel-Kramers-Brillouin (WKB) approximation is sufficient to extract many of their key features. We find several differences between the behavior of shells in a dust ball and the single shell, all of them traceable to the fact that, unlike the shells of a dust ball, the single shell possesses a self-interaction that is inversely proportional to its area radius. As a consequence the single shell is classically bound. We will show that there is no tunneling across the horizon and determine the energy spectrum of the shell. Its WKB wave function extends from the origin (where it vanishes), is well behaved at the black hole horizon, and falls off exponentially beyond the classically allowed region.

In Sec. II we derive the proper time dynamics of a classical thin shell collapsing onto a preexisting mass. The classical dynamics are obtained by an application of the Israel-Darmois-Lanczos (IDL) formalism [24–26], which yields a first integral of the motion involving the preexisting mass at the center, M^-, the Arnowitt-Deser-Misner (ADM) mass at infinity, M^+, and the proper mass, m, of the shell. Of these, the mass at the center and the proper mass of the shell are nondynamical parameters, constant over the entire phase space. The ADM mass is a dynamical variable that represents the energy of the system. There are three time variables, *viz.,* the coordinate times in the interior and the exterior, and the shell proper time. What is not clear is the time variable in which the ADM mass generates the evolution. We follow Hajiček, Kay, and Kuchař [12] and assume that it generates the evolution in the time coordinate inside the shell. This allows us to construct Lagrangians and Hamiltonian evolutions for the shell in the other time variables, in particular in the shell proper time. Here we also show that the Hamiltonian obtained in this way is structurally similar to the proper time Hamiltonian derived for the full Einstein-dust system in the LTB dust ball models, which lends confidence in the choice of [12].

In Sec. III, we quantize the classical model. The Wheeler-DeWitt equation is elliptic with a positive semi-definite inner product for energies less than the shell's proper mass. We find the WKB approximation of the wave function and, in Sec. IV, analyze its $U(1)$ current. Requiring that a lowest energy state exists and that the $U(1)$ current is finite and well behaved everywhere, a complete set of bound states exists and we determine its spectrum. By comparing the solutions with the shell collapsing in a vacuum, we conclude that the approximation is valid only so long as the proper mass of the shell is much greater than the Planck mass. We conclude in Sec. V with a brief summary of our results and tie them in with a previously suggested model for quantum black holes.

II. THE CLASSICAL SHELL MODEL

The equation of motion of a spherical, thin, massive shell is obtained by applying the Israel-Darmois-Lanczos conditions on the timelike surface $\Sigma = \mathbb{R} \times \mathbb{S}^2$ that represents its world sheet. The world sheet forms the three-dimensional boundary between an internal spacetime, $\mathcal{M}^-$, and an external spacetime, $\mathcal{M}^+$. $\mathcal{M}^\mp$ are described in coordinates $x^\mu_\mp$ by metrics $g^\mp_{\mu\nu}$ that solve Einstein's equations. Let ξ^a be a set of intrinsic coordinates on the surface of the shell and differentiable functions of $x^\mu_\mp$, then $e^{\mp\mu}{}_a = \partial x^\mu_\mp/\partial\xi^a$ are the components of the three basis vectors on this surface and $h^\mp_{ab} = g^\mp_{\mu\nu} e^{\mp\mu}{}_a e^{\mp\nu}{}_b$ is the induced metric on the shell on the two sides of it. The first junction condition requires the shell to have a well-defined metric, i.e., $h^-_{ab} = h^+_{ab}$ or $[h_{ab}] = 0$.

The second junction condition, which follows from Einstein's equations, says that the surface stress-energy tensor, S_{ab}, of the shell is given by

$$S_{ab} = -\frac{\varepsilon}{8\pi}([K_{ab}] - [K]h_{ab}), \tag{1}$$

where K_{ab} is the extrinsic curvature of the boundary, $K = K^a_a$ and $\varepsilon = +1$ for a timelike shell. If $\mathcal{M}^\mp$ are taken to be vacuum spacetimes, then spherical symmetry implies that $g^\mp_{\mu\nu}$ are Schwarzschild metrics, with mass parameters $M^\mp$ respectively, and M^+ represents the total mass of the system. We may write the respective line elements as

$$ds^2_\mp = -g^\mp_{\mu\nu} dx^\mu_\mp dx^\nu_\mp = B^\mp dt^2_\mp - \frac{1}{B^\mp} dr^2_\mp - r^2_\mp d\Omega^2, \tag{2}$$

where $B^{\mp} = 1–2GM^{\mp}/r_{\mp}$ and we have assumed that the interior and exterior share the same spherical coordinates, θ and ϕ. The shell is described by the parametric equations $r_{\mp} = r = R(\tau)$, $t_{\mp} = t_{\mp}(\tau)$, where τ is the proper time for comoving observers and the interior and exterior time coordinates are related to the shell proper time (and indirectly to each other) by

$$\frac{dt_{\mp}}{d\tau} = \frac{\sqrt{B^{\mp} + R_{\tau}^2}}{B^{\mp}}, \tag{3}$$

where the subscript indicates a derivative with respect to τ. Choosing the intrinsic coordinates of the shell to be $\xi^a = \{\tau, \theta, \phi\}$, the induced metric is

$$ds_{\Sigma}^2 = d\tau^2 - R^2(\tau)d\Omega^2, \tag{4}$$

while the nonvanishing components of the extrinsic curvature are

$$K^{\theta}_{\mp\theta} = K^{\phi}_{\mp\phi} = \frac{\beta^{\mp}}{R}, \qquad K^{\tau}_{\mp\tau} = \frac{\beta^{\mp}_{\tau}}{\dot{R}_{\tau}}, \tag{5}$$

where

$$\beta^{\mp} = \sqrt{B^{\mp} + \dot{R}_{\tau}^2}. \tag{6}$$

Therefore, according to (1),

$$\begin{aligned} S^{\tau}{}_{\tau} &= \frac{\beta^+ - \beta^-}{4\pi GR} = -\sigma, \\ S^{\theta}{}_{\theta} = S^{\phi}{}_{\phi} &= \frac{\beta^+ - \beta^-}{8\pi GR} + \frac{\beta^+_{\tau} - \beta^-_{\tau}}{8\pi GR_{\tau}} = p, \end{aligned} \tag{7}$$

where we have set $S^a{}_b = \mathrm{diag}(-\sigma, p, p)$.

The mass density of the shell is "σ" and "p" is its tangential pressure, which, for dust shells, we take to be zero. Integrating the second equation in (7),

$$\beta^+ - \beta^- = -\frac{Gm}{R}, \tag{8}$$

where m is a constant of the integration, which represents the rest mass of the shell, as is seen by inserting this solution into the first. Equation (8) may be put in the form

$$M^+ - M^- = \Delta M = m\sqrt{B^- + R_{\tau}^2} - \frac{Gm^2}{2R}. \tag{9}$$

It is reasonable to think of the above as a first integral of the motion and associate ΔM with the total energy, E, of the shell. When expressed in terms of the momentum conjugate to $R(\tau)$, (9) will represent the Hamiltonian of the system. It is, however, given in terms of the velocities, which are dependent variables in the canonical theory and, to determine the momentum, it becomes necessary to know in which of the three time coordinates the Hamiltonian is evolving the system.

Within the thin shell construction, there is no *à priori* way to determine a canonical Hamiltonian because the constraint equation has been derived from the IDL conditions and not a fundamental action principle. One approach would be to compare the thin shell Hamiltonians with a similar system for which a canonical theory has been derived from an action principle. Our goal will be to recover a proper time Hamiltonian that is compatible with the midisuperspace Hamiltonian [16,17] obtained for the spherically symmetric Einstein-dust action by an application of a canonical chart analogous to that employed by Kuchař in [18,27]. Because a dust ball can be thought of as a sequence of noninteracting shells, the proper time Hamiltonian for a single shell should be of the same form apart from any self-interaction terms peculiar to the thin shell itself. Thus, for example, if the evolution is taken to be in the shell proper time and the energy is taken to be ΔM, we have

$$R_{\tau} = \frac{\partial H_I}{\partial p} \tag{10}$$

and the Hamiltonian is [28,29]

$$H_I = m\sqrt{B}\cosh\frac{p}{m} - \frac{Gm^2}{2R}. \tag{11}$$

It does not have the same form as the Hamiltonian derived for the dust ball in [16,17].

As mentioned in the Introduction, we will show that the choice of [12], taking ΔM to evolve the system in the coordinate time of the interior, yields a compatible proper time Hamiltonian. Because the right-hand side of (9) involves only the interior we drop the superscripts $\pm$ and, employing (3), we can rewrite it as

$$\Delta M = \frac{mB^{3/2}}{\sqrt{B^2 - R_t^2}} - \frac{Gm^2}{2R}. \tag{12}$$

Then $R_t = \partial H_{II}/\partial p$ gives

$$H_{II} = -P_{(t)} = \sqrt{m^2 B + B^2 p^2} - \frac{Gm^2}{2R}, \tag{13}$$

and

$$p = \frac{mR_t}{\sqrt{B}\sqrt{B^2 - R_t^2}}. \tag{14}$$

The action for the shell may now be given as a Legendre transform of H_{II},

$$S = \int dt\left[-m\sqrt{B - \frac{R_t^2}{B}} + \frac{Gm^2}{2R}\right] \tag{15}$$

and then transformed into an action in proper time, once again with the help of (3). One finds

$$S = \int d\tau\left[-m + \frac{Gm^2}{2R}\frac{\sqrt{B+R_\tau^2}}{B}\right] \tag{16}$$

and the proper time Hamiltonian

$$\mathcal{H} = -P_{(\tau)} = m - \sqrt{\frac{f^2}{B} - BP^2}, \tag{17}$$

where we have set $f(R) = Gm^2/2R$ and the momentum, P, conjugate to R, is now given by

$$P = \frac{fR_\tau}{B\sqrt{B+R_\tau^2}}. \tag{18}$$

This proper time Hamiltonian is bounded from above by the mass of the shell and the shell momentum is bounded from above by f/B. From the comoving observer's point of view, the shell is always bound to the center.

The equations of motion that follow from (17) are derivable from the super-Hamiltonian

$$h_{(\tau)} = (P_{(\tau)} + m)^2 + BP^2 - \frac{f^2}{B} = 0. \tag{19}$$

In this form, the Hamiltonian structure of the shell is identical to that of the dust ball, as derived in the Einstein-dust system, with one important exception: the shells in a dust ball do not possess a self-interaction that depends on their area radius: for the shells in a dust ball, $f(R)$ gets replaced by the Misner-Sharp mass density, $F'(r)$, where r is the LTB shell label coordinate and $F(r)$ represents the mass of the dust ball up to r. The dependence of the self-interaction term, $f(R)$, on the area radius is responsible for the fact that the shell is classically bound in the proper time description. It will also play an important role in the matching conditions at the horizon.

III. THE QUANTUM SHELL

The structure of the super-Hamiltonian in (19) indicates that the DeWitt metric is

$$\gamma_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1/B \end{pmatrix}, \tag{20}$$

so we choose a factor ordering that is symmetric with respect to the measure "$dR/\sqrt{B}$." Raising the momenta to operator status following Dirac we get the Wheeler-DeWitt equation

$$\hat{h}_{(\tau)}\Psi(\tau,R) = \left[\left(-i\hbar\frac{\partial}{\partial\tau} + m\right)^2 - \hbar^2\sqrt{B}\frac{\partial}{\partial_R}\sqrt{B}\frac{\partial}{\partial_R} - \frac{f^2}{B}\right] \times \Psi(\tau,R) = 0. \tag{21}$$

We have been unable to find exact solutions to this equation, but the WKB approximation suffices to yield a general picture of the quantum shell. With $\Psi(\tau,R) = e^{iW(\tau,R)/\hbar}$, (21) reads

$$-i\hbar\frac{\partial^2 W}{\partial\tau^2} + \left(\frac{\partial W}{\partial\tau} + m\right)^2 - i\hbar B\frac{\partial^2 W}{\partial R^2} - \frac{i\hbar}{2}B'\frac{\partial W}{\partial R} + B\left(\frac{\partial W}{\partial R}\right)^2 - \frac{f^2}{B} = 0, \tag{22}$$

where the prime indicates a derivative with respect to R, and taking

$$W = -\mathcal{E}\tau + S_0(R) + \frac{\hbar}{i}\ln A(R), \tag{23}$$

where $\mathcal{E}$ is the shell proper energy, we find up to $\mathcal{O}(\hbar)$,

$$BS_0'^2 + (m-\mathcal{E})^2 - \frac{f^2}{B} = 0,$$
$$\sqrt{B}(\sqrt{B}S_0')' + \frac{2B}{A}A'S_0' = 0. \tag{24}$$

The first equation is solved by the Hamilton-Jacobi function,

$$S_0(R) = \pm\int\frac{dR}{B}\sqrt{f^2 - (m-\mathcal{E})^2 B}, \tag{25}$$

and the second gives

$$A = \frac{C}{|B|^{1/4}\sqrt{|S_0'|}}. \tag{26}$$

Let us now show that the classical limit of this solution yields the classical dynamical equations that follow from (17). To order $\hbar^0$, $W(\tau,R)$ is just the Hamilton-Jacobi function, so the function $R(\tau)$ defined by the principle of constructive interference,

$$\frac{\partial S_0}{\partial\mathcal{E}} = 0 = -\tau \pm \int\frac{(m-\mathcal{E})dR}{\sqrt{f^2 - (m-\mathcal{E})^2 B}} \tag{27}$$

and

$$P(\tau) = S_0' = \pm\frac{1}{B}\sqrt{f^2 - (m-\mathcal{E})^2 B} \tag{28}$$

should satisfy the Hamiltonian equations based on (17). Taking a derivative of (27),

$$1 = \pm \frac{R_\tau (m-\mathcal{E})}{\sqrt{f^2 - B(m-\mathcal{E})^2}} \tag{29}$$

and therefore

$$m - \mathcal{E} = \frac{f}{\sqrt{B + R_\tau^2}}. \tag{30}$$

Inserting this into (28) shows that

$$BP = \sqrt{f^2 - B(m-\mathcal{E})^2} = \frac{f R_\tau}{\sqrt{B+R_\tau^2}} \tag{31}$$

or

$$R_\tau = \frac{BP}{\sqrt{\frac{f^2}{B} - BP^2}} = \{R, \mathcal{H}\}. \tag{32}$$

Again, taking the derivative of P in (28) and using (32) we find

$$P_\tau = \frac{2ff' - (f^2/B + BP^2)B'}{B\sqrt{\frac{f^2}{B} - BP^2}} = \{P, \mathcal{H}\}. \tag{33}$$

It follows that the trajectories implied by the principle of constructive interference in (27) are identical to those determined by the Hamiltonian equations of motion that follow from (17).

The WKB solutions may now be given as

$$\Psi_\pm(\tau, R) = \frac{C_\pm e^{-i\mathcal{E}\tau}}{|B^{1/4}\sqrt{S_0'}} \exp\left[\pm i \int \frac{dR}{B}\sqrt{f^2 - (m-\mathcal{E})^2 B}\right]. \tag{34}$$

Inside the black hole horizon, because $B < 0$, the WKB wave function is always oscillatory, but the situation is different outside. The phase S_0 is real in the classically allowed region, for which $f^2 - (m-\mathcal{E})^2 B > 0$, and imaginary in the classically forbidden region. The wave function thus falls off exponentially when $f^2 - (m-\mathcal{E})^2 B < 0$ *i.e.,* when

$$R > R_+ = \frac{1}{2}\left(\kappa + \sqrt{\kappa^2 + \frac{\mu^4}{(m-\mathcal{E})^2}}\right), \tag{35}$$

where $\kappa = 2GM^-$ and $\mu^2 = Gm^2 = (m/m_p)^2$. We will henceforth distinguish between the "interior" region ($R < \kappa$) and the "exterior" region ($R > \kappa$), which itself consists of the classically allowed region ($\kappa < R < R_+$) and the classically forbidden region ($R_+ < R$). In the interior we write

$$\Psi(\tau, R) = \frac{e^{-i\mathcal{E}\tau}}{|B|^{1/4}\sqrt{|S_0'|}}\left\{F_1 \exp\left[+i\int \frac{dR}{B}\sqrt{f^2 - (m-\mathcal{E})^2 B}\right] + F_2 \exp\left[-i\int \frac{dR}{B}\sqrt{f^2-(m-\mathcal{E})^2 B}\right]\right\}, \qquad 0 < R < \kappa \tag{36}$$

and in the exterior,

$$\Psi(\tau, R) = \begin{cases} \frac{e^{-i\mathcal{E}\tau}}{|B|^{1/4}\sqrt{|S_0'|}}\left\{D_1 \exp\left[+i\int \frac{dR}{B}\sqrt{f^2-(m-\mathcal{E})^2 B}\right] + D_2 \exp\left[-i\int \frac{dR}{B}\sqrt{f^2-(m-\mathcal{E})^2 B}\right]\right\}, & \kappa < R < R_+ \\ \frac{e^{-i\mathcal{E}\tau}}{|B|^{1/4}\sqrt{|S_0'|}}\left\{D_3 \exp\left[-\int \frac{dR}{B}\sqrt{(m-\mathcal{E})^2 B - f^2}\right] + D_4 \exp\left[+\int \frac{dR}{B}\sqrt{(m-\mathcal{E})^2 B - f^2}\right]\right\}, & R > R_+ \end{cases} \tag{37}$$

where D_j and F_j are constants. The wave functions in the exterior, i.e., in the classically allowed and forbidden regions, can be matched in the standard way by invoking the asymptotic forms of the Airy functions far from the boundary between the regions. One readily finds the connection rules,

$$D_1 = \left(D_3 - \frac{i}{2}D_4\right)e^{i\pi/4}, \qquad D_2 = \left(D_3 + \frac{i}{2}D_4\right)e^{-i\pi/4}. \tag{38}$$

Since the classically forbidden region extends to infinity we take $D_4 = 0$, which implies that $D_1 e^{-i\pi/4} = D_2 e^{i\pi/4} = D_3$. It remains to match the interior and exterior solutions at the horizon, where the integral defining the phase has an essential singularity of order one.

We define the integral by analytically continuing to the complex plane, deforming the path so as to go around the pole at $R = \kappa$ in an infinitesimal circle of radius ε. Let C_ε denote the deformed path, S_ε the semicircle of radius ε about $R = \kappa$ in the complex R plane, then we *define*

$$\int^R \frac{dR}{\sqrt{B}}\sqrt{\frac{f^2}{B}-(m-\mathcal{E})^2} \overset{\text{def}}{=} \lim_{\varepsilon\to 0}\int^R_{(C_\varepsilon)} \frac{dR}{\sqrt{B}}\sqrt{\frac{f^2}{B}-(m-\mathcal{E})^2} \tag{39}$$

and choose the orientation of the semicircle as a boundary condition. Performing the integration from left to right for $R=\kappa+\varepsilon$

$$\int^{\kappa+\varepsilon}_{(C_\varepsilon)} \frac{dR}{B}\sqrt{f^2-(m-\mathcal{E})^2 B} = \int^{\kappa-\varepsilon}_{(C_\varepsilon)} \frac{dR}{B}\sqrt{f^2-(m-\mathcal{E})^2 B} + \int_{(S_\varepsilon)} \frac{dR}{B}\sqrt{f^2-(m-\mathcal{E})^2 B}. \tag{40}$$

Since ε is small, we perform a near horizon approximation of the integrand in the second integral,

$$\int_{(S_\varepsilon)} \frac{dR}{B}\sqrt{f^2-(m-\mathcal{E})^2 B} \approx \int_{S_\varepsilon} \frac{dRRf}{(R-\kappa)} = \pm\frac{i\pi\mu^2}{2}, \tag{41}$$

where the positive sign occurs if the path is deformed in the lower half complex plane, and the negative sign occurs when the path is deformed in the upper half complex plane. In the present situation, there appears to be no good reason to choose one over the other. Each choice amounts to the identifications

$$F_1 = D_1 e^{\mp\pi\mu^2/2} = D_3 e^{i\pi/4} e^{\mp\pi\mu^2/2},$$
$$F_2 = D_2 e^{\pm\pi\mu^2/2} = D_3 e^{-i\pi/4} e^{\pm\pi\mu^2/2}. \tag{42}$$

Owing to the sign change in B across the horizon, outgoing waves in the exterior are matched to infalling waves in the interior and, vice versa, infalling waves in the exterior are matched to outgoing waves in the interior, and the complete wave function is

$$\Psi(\tau,R) = \begin{cases} \frac{D_3 e^{-i\mathcal{E}\tau}}{|B|^{1/4}\sqrt{|S_0'|}}\left[e^{\mp\pi\mu^2/2} e^{\frac{i\pi}{4}} e^{iS_0} + e^{\pm\pi\mu^2/2} e^{-\frac{i\pi}{4}} e^{-iS_0}\right] & 0<R<\kappa \\ \frac{2D_3 e^{-i\mathcal{E}\tau}}{|B|^{1/4}\sqrt{|S_0'|}} \cos\left[S_0+\frac{\pi}{4}\right], & \kappa<R<R_+ \\ \frac{D_3 e^{-i\mathcal{E}\tau}}{|B|^{1/4}\sqrt{|S_0'|}} e^{-\int|S_0'|dR}, & R>R_+ \end{cases} \tag{43}$$

where $S_0(R)$ is defined in (25).

IV. THE ENERGY SPECTRUM

For any two solutions of the wave equation in (21) there is a conserved bilinear current density given by

$$J_i = -\frac{i}{2}\Phi^* \overset{\leftrightarrow}{\nabla}_i \Psi + m\delta_{i\tau}\Phi^*\Psi \tag{44}$$

the time component of which determines a physical inner product

$$\langle\Phi,\Psi\rangle = \int \frac{dR}{\sqrt{B}}\left[-\frac{i}{2}\Phi^*\overset{\leftrightarrow}{\nabla}_\tau\Psi + m\Phi^*\Psi\right]. \tag{45}$$

Consider two stationary states,

$$\Psi_{\mathcal{E}}(\tau,R) = e^{-i\mathcal{E}\tau}\psi_{\mathcal{E}}(R), \quad \Phi_{\mathcal{E}'}(\tau,R) = e^{-i\mathcal{E}'\tau}\psi_{\mathcal{E}'}(R) \tag{46}$$

of energies $\mathcal{E}$ and $\mathcal{E}'$, then

$$\langle\Phi,\Psi\rangle = \left[m-\frac{1}{2}(\mathcal{E}+\mathcal{E}')\right] e^{-i(\mathcal{E}+\mathcal{E}')\tau}\int\frac{dR}{\sqrt{B}}\phi^*_{\mathcal{E}'}\psi_{\mathcal{E}} \tag{47}$$

is positive semidefinite so long as $\mathcal{E}<m$. By the wave equation we get

$$\phi^*_{\mathcal{E}'}\sqrt{B}\partial_R\sqrt{B}\partial_R\psi_{\mathcal{E}} = \left[(m-\mathcal{E})^2-\frac{f^2}{B}\right]\phi^*_{\mathcal{E}'}\psi_{\mathcal{E}},$$
$$\psi_{\mathcal{E}}\sqrt{B}\partial_R\sqrt{B}\partial_R\phi^*_{\mathcal{E}'} = \left[(m-\mathcal{E}')^2-\frac{f^2}{B}\right]\phi^*_{\mathcal{E}'}\psi_{\mathcal{E}}.$$

Subtracting the second from the first,

$$\sqrt{B}\partial_R(\sqrt{B}\phi^*_{\mathcal{E}'}\overset{\leftrightarrow}{\partial}_R\psi_{\mathcal{E}}) = (\mathcal{E}-\mathcal{E}')(\mathcal{E}+\mathcal{E}'-2m)\phi^*_{\mathcal{E}'}\psi_{\mathcal{E}} \tag{48}$$

showing that the inner product in (47) is just a surface term which, mindful of the three regions, we give as

$$\langle\Phi,\Psi\rangle = -\frac{i}{(\mathcal{E}-\mathcal{E}')}\left[\int_0^\kappa dR\partial_R\sqrt{B}J_R + \int_\kappa^{R_+} dR\partial_R\sqrt{B}J_R + \int_{R_+}^\infty dR\partial_R\sqrt{B}J_R\right]. \tag{49}$$

To guarantee orthonormality of the wave functions, we must require that the inner product vanishes whenever $\mathcal{E}'\neq\mathcal{E}$. Therefore, calling $\Omega=\sqrt{B}J_R$, we seek the conditions under which

$$\lim_{R\to\infty}\Omega - \lim_{R\to R_+^+}\Omega + \lim_{R\to R_+^-}\Omega - \lim_{R\to\kappa^+}\Omega + \lim_{R\to\kappa^-}\Omega - \lim_{R\to 0}\Omega = 0, \tag{50}$$

where the superscripts indicate the left/right limits. The first term vanishes because the wave function vanishes exponentially at infinity. Direct computation also shows that the second and third terms cancel and the last term vanishes but the fourth and fifth terms, which must be evaluated at the black hole horizon, neither separately vanish nor cancel one another. This occurs because the black hole horizon is an essential singularity of the phase integral. One finds

$$-\lim_{R\to\kappa^+}\Omega + \lim_{R\to\kappa^-}\Omega \sim \sin\left[\frac{\mu^2}{2}\ln\left(\frac{m-\mathcal{E}}{m-\mathcal{E}'}\right)\right], \tag{51}$$

and therefore

$$\frac{m-\mathcal{E}}{m-\mathcal{E}'} = e^{\frac{2n\pi}{\mu^2}} \tag{52}$$

for integer values of n. Assuming the existence of a ground state, it implies the energy spectrum,

$$\mathcal{E}_n = m(1 - e^{-2n\pi/\mu^2}), \tag{53}$$

where n is a whole number. This is identical to the spectrum of the shell collapsing in a vacuum when $\mu \gg 1$, i.e., when the proper mass of the shell is much larger than the Planck mass. The presence of the external black hole does not disturb the spectrum to this order.

V. CONCLUSIONS

The purpose of this work was to understand the similarities and differences in the quantum mechanics of a single thin shell and the midisuperspace quantization of the shells in a collapsing dust ball. We examined the WKB approximation to the proper time quantum mechanics of the thin dust shell when it surrounds a preexisting black hole. We have shown that although the constructions of the Hamiltonians governing the evolution of the two systems have very different origins (the dynamics of the thin shell are obtained via an application of the IDL conditions whereas the dynamics of the dust ball are fully derived from the Einstein-dust system), the Hamiltonians one ends up with are structurally similar with a crucial exception: the thin shell possesses a self-interaction that depends on its area radius, but the shells of a dust ball do not. This self-interaction causes the thin shell to always stay bound to the center, regardless of whether the interior of the shell is a vacuum or a black hole. On the contrary, the shells of a dust ball may be unbound. Again as a consequence of the self-interaction, the matching of the wave function at the horizon of the black hole is accomplished with essentially no information about the horizon length, only the proper mass of the shell plays a role. Neither does the black hole play any role in the energy spectrum of the shell, which we have shown is identical to the spectrum of a shell collapsing in a vacuum to this order.

What is most surprising is that bound state solutions exist for shell proper masses greater than the Planck mass and the shell does *not* collapse into the central singularity, suggesting that quantum uncertainty plays a role in the collapse over distance scales determined by the size of the black hole's event horizon and larger than previously suspected. The wave function vanishes at the center, extends out to the turning point, which, depending on the energy of the shell, may lie close to the horizon but always outside it, and falls off exponentially beyond this point. This is reminiscent of a gravitational atom and supports another solution of the quantum dust ball collapse, which does not involve continued collapse as described in the Introduction: if the collapse does not continue past the apparent horizon, the solution is described by the first of the two solutions given in the Introduction and the shells will coalesce on the apparent horizon. No event horizon will form and the collapse will end in an ultracompact star instead of a black hole [30,31].

In general, proper time quantization seems to enjoy several advantages over coordinate time quantizations. For one, the proper time quantum theory exists for shells of mass greater than the Planck mass, unlike the quantum mechanics that is based on coordinate time. But the most important is that it satisfies a basic requirement of the quantum theory, i.e., observer independence of the time parameter. It is therefore "democratic" in regard to all foliations of spacetime: all coordinate time variables would be functions of the phase space [in the simple case of the shell these are given by (3)] as are the spatial coordinates. The same would be true of the metric components. Thus they would all be operator valued and we would be able to speak of time intervals and spatial distances only in terms of averages. In the proper time formulation, these averages can be calculated and fluctuations about them quantified because the Wheeler-DeWitt equation yields a conserved, positive, semidefinite inner product.

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