

# Nonrelativistic transport theory from vorticity dependent quantum kinetic equation

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We study the three-dimensional transport theory of massive spin-1/2 fermions resulting from the vorticity dependent quantum kinetic equation. This quantum kinetic equation has been introduced to take account of noninertial properties of rotating coordinate frames. We show that it is the appropriate relativistic kinetic equation which provides the vorticity dependent semiclassical transport equations of the three-dimensional Wigner function components. We establish the semiclassical kinetic equations of a linearly independent set of components. By means of them, kinetic equations of the chiral scalar distribution functions are derived. They furnish the 3D kinetic theory which permits us to study the vector and axial vector current densities by focusing on the mass corrections to the chiral vortical and separation effects.

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## I. INTRODUCTION

Transport of Dirac fermions in the presence of external electromagnetic fields can be studied by means of the covariant Wigner function which obeys the quantum kinetic equation (QKE) [1,2]. The Wigner function can be decomposed into some covariant fields whose equations of motion follow from the QKE. Founded on these field equations, one derives relativistic transport theories of Dirac particles. A brief overview of the covariant Wigner function approach was given in [3] and recently it has been reviewed in details in [4]. Relativistic formalism has the advantage of being manifestly Lorentz invariant. Nevertheless, nonrelativistic transport equations are necessary for being able to start with initial distribution functions and construct solutions of transport equations [5–7]. There exist some different methods of formulating nonrelativistic kinetic theories of Dirac particles. One of these methods is to construct the four-dimensional (4D) transport equations of a set of covariant fields and then integrate them over the zeroth-component of four-momentum, so that the three-dimensional (3D) transport equations which are correlative to the 4D ones are extracted. Another method is to integrate all of the quantum kinetic equations of the covariant fields over the zeroth-component of four-momentum at the beginning and then derive the nonrelativistic transport theory from these 3D quantum kinetic equations [7,8]. This is also called the equal-time formalism. There also exists a strictly 3D

approach of acquiring a transport equation of Dirac particles which does not refer to the Wigner function [9].

Quarks of the quark-gluon plasma formed in heavy-ion collisions are treated as massless [10,11]. Thus, chiral kinetic theory (CKT) is useful to inspect their dynamical features [12–20]. The QKE of the relativistic Wigner function generates the anomalous magnetic effects as well as the vorticity effects correctly [20]. It is worth noting that the vorticity of fluid matches the angular velocity of the fluid in the comoving frame. The QKE possesses an explicit dependence on the electromagnetic fields but not on the vorticity of fluid. When the Wigner function is expressed in the Clifford algebra basis, the QKE gives a set of equations for the chiral vector fields. In solving some of these equations one introduces the frame four-vector [21–23]. It can be identified with the comoving frame velocity which also appears in equilibrium distribution functions. Derivatives of the comoving frame four-velocity generate terms depending on the vorticity. These are the sources of vorticity dependence in the relativistic QKE formulations of the massless fermions. One cannot generate noninertial forces like the Coriolis force within this formalism. However, in [24], vortical effects were derived by using the similarity between the Lorentz and the Coriolis forces. This formulation has been shown to result in a rotating coordinate frame from the first principles [25]. To build in this similarity, a modification of QKE by means of enthalpy current was introduced in [26]. The discrepancy in treating magnetic and vortical effects reflects itself drastically, especially in 3D CKT when both electromagnetic fields and vorticity are taken into account. The vorticity dependent quantum kinetic equation (VQKE) was shown to yield a 3D CKT which does not depend on the spatial coordinates [26]. It is consistent with the chiral anomaly and generates the

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chiral magnetic and vortical effects and the Coriolis force. The underlying Lagrangian formalism which yields VQKE was presented in [27].

Constituent quarks of the quark-gluon plasma created in heavy-ion collisions are approximately massless. Thus, to get a better understanding of their dynamical properties, one needs to uncover the mass corrections to chiral theories. Covariant kinetic theories of massive spin-1/2 particles have been studied in terms of QKE within two different approaches in [28,29]. In principle, nonrelativistic transport equations can be provided by integrating the 4D kinetic equations. But, for massive fermions, extracting the 3D kinetic equations which are correlative to the 4D kinetic equations can only be done under some simplifying approximations as they have been shown within the VQKE approach in [27,30]. We have already mentioned that there also exists another nonrelativistic approach which is the so-called equal-time formalism. By integrating the equations of the relativistic Wigner function components over the zeroth-component of momentum, one sets up the equations of the components of the 3D Wigner function. Then, one employs these 3D equations to derive nonrelativistic kinetic equations of Dirac particles in the presence of the external electromagnetic fields. This has been studied in [31] by developing the original formulation of [8]. In contrast to the 4D Wigner function approaches, this 3D formalism does not generate vortical effects. Because, without solving some of the equations of the 4D Wigner function components one cannot generate vorticity dependent terms. Therefore, to obtain a similar 3D approach by taking account of vorticity of the fluid, the QKE of the Wigner function should possess an explicit dependence on the fluid vorticity. The VQKE is the unique covariant formalism which has this property [32]. In this work, we study the 3D formulation of the VQKE by extending the method of [8,31], by taking care of the vortical effects only.

We briefly review the VQKE in the absence of electromagnetic fields and present the equations of the components of the covariant Wigner function in the next section. Their integration over the zeroth component of momentum in a frame adequate to study nonrelativistic dynamics lead to the 3D constraint and transport equations as reported in Sec. III. These equations which the components of the 3D Wigner function obey are studied and their semiclassical solutions are acquired in terms of a set of independent functions. In Sec. IV, kinetic equations of this set of fields are established up to the first order in the Planck constant,  $\hbar$ . Mass corrections to the chiral vortical and separation effects are studied in Sec. V. Conclusions and discussions of possible future directions are given in Sec. VI.

## II. VORTICITY DEPENDENT QUANTUM KINETIC EQUATION

The quantum kinetic equation for a fluid in the comoving frame with the four-velocity  $u_\mu$ ;  $u_\mu u^\mu = 1$ , whose linear acceleration vanishes,  $u_\nu \partial^\nu u_\mu = 0$ , is

$$\left[ \gamma_\mu \left( \pi^\mu + \frac{i\hbar}{2} \mathcal{D}^\mu \right) - m \right] W(x, p) = 0. \quad (1)$$

Here,

$$\mathcal{D}^\mu \equiv \partial^\mu - j_0(\Delta) w^{\mu\nu} \partial_{p\nu}, \quad (2)$$

$$\pi^\mu \equiv p^\mu - \frac{\hbar}{2} j_1(\Delta) w^{\mu\nu} \partial_{p\nu}, \quad (3)$$

where  $\partial^\mu \equiv \partial/\partial x_\mu$ ,  $\partial_p^\mu \equiv \partial/\partial p_\mu$ , and  $j_0, j_1$  are spherical Bessel functions in  $\Delta \equiv \frac{\hbar}{2} \partial_p \cdot \partial_x$ . The 4D space-time derivative,  $\partial_\mu$ , contained in  $\Delta$  acts on  $w^{\mu\nu}$ , but not on the Wigner function,  $W(x, p)$ . On the contrary,  $\partial_{p\nu}$  acts on the Wigner function, but not on  $w^{\mu\nu}$ . The action which generates (1) has been presented in [27]. There, it was shown that when the equations of motion of the fields presenting fluid are satisfied,  $w^{\mu\nu}$  is given with an arbitrary constant  $\kappa$  as

$$w_{\mu\nu} = (\partial_\mu h) u_\nu - (\partial_\nu h) u_\mu + \kappa h \Omega_{\mu\nu}, \quad (4)$$

where  $h = u \cdot p$  and  $\Omega_{\mu\nu} = \frac{1}{2} (\partial_\mu u_\nu - \partial_\nu u_\mu)$ . The fluid four-vorticity is defined as  $\omega_\mu = (1/2) \epsilon_{\mu\nu\alpha\beta} u^\nu \Omega^{\alpha\beta}$ .

The Wigner function can be written through the 16 generators of the Clifford algebra as

$$W = \frac{1}{4} \left( \mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5 \gamma^\mu \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} \mathcal{S}_{\mu\nu} \right), \quad (5)$$

where  $\mathcal{C}_a \equiv \{ \mathcal{F}, \mathcal{P}, \mathcal{V}_\mu, \mathcal{A}_\mu, \mathcal{S}_{\mu\nu} \}$ , respectively, are the scalar, pseudoscalar, vector, axial-vector, and antisymmetric tensor components of the 4D Wigner function. These covariant fields can be expanded in powers of the Planck constant:

$$\mathcal{C}_a = \sum_n \hbar^n \mathcal{C}_a^{(n)}. \quad (6)$$

We deal with the semiclassical approximation where only the zeroth- and first-order fields in  $\hbar$  are considered. Thus, to derive the equations which they satisfy, instead of (2), (3), we only need to deal with

$$D^\mu \equiv \partial_x^\mu - w^{\mu\nu} \partial_{p\nu}, \quad (7)$$

and  $p^\mu$ . By plugging the decomposed Wigner function, (5), into the VQKE, (1), one derives the equations satisfied by the fields  $\mathcal{C}_a$ , whose real parts are

$$p \cdot \mathcal{V} - m\mathcal{F} = 0, \quad (8)$$

$$p_\mu \mathcal{F} - \frac{\hbar}{2} D^\nu \mathcal{S}_{\nu\mu} - m\mathcal{V}_\mu = 0, \quad (9)$$

$$-\frac{\hbar}{2}D_\mu\mathcal{P} + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}p^\nu\mathcal{S}^{\alpha\beta} + m\mathcal{A}_\mu = 0, \quad (10)$$

$$\frac{\hbar}{2}D_{[\mu}\mathcal{V}_{\nu]} - \epsilon_{\mu\nu\alpha\beta}p^\alpha\mathcal{A}^\beta - m\mathcal{S}_{\mu\nu} = 0, \quad (11)$$

$$\frac{\hbar}{2}D \cdot \mathcal{A} + m\mathcal{P} = 0, \quad (12)$$

and the imaginary parts are

$$\hbar D \cdot \mathcal{V} = 0, \quad (13)$$

$$p \cdot \mathcal{A} = 0, \quad (14)$$

$$\frac{\hbar}{2}D_\mu\mathcal{F} + p^\nu\mathcal{S}_{\nu\mu} = 0, \quad (15)$$

$$p_\mu\mathcal{P} + \frac{\hbar}{4}\epsilon_{\mu\nu\alpha\beta}D^\nu\mathcal{S}^{\alpha\beta} = 0, \quad (16)$$

$$p_{[\mu}\mathcal{V}_{\nu]} + \frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}D^\alpha\mathcal{A}^\beta = 0. \quad (17)$$

It can be observed that not all of the fields  $\mathcal{C}_a$  are relevant to formulate a 4D kinetic theory. Depending on the choice of the independent set of fields, one establishes different relativistic kinetic theories [28,29]. In the subsequent sections we will refer to the 4D formulation given in [30], which was acquired following the approach of [28].

### III. 3D SEMICLASSICAL TRANSPORT AND CONSTRAINT EQUATIONS

The equal-time transport theory of Dirac particles in the presence of external electromagnetic fields which has been proposed in [5] was incomplete. In [8], it was shown that to have a complete nonrelativistic transport theory of spinor electrodynamics, one should start with the covariant QKE of [1,2]. We mainly adopt the approach of [8]. However, there is a subtle difference: Electromagnetic field strength is independent of  $p_0$  in contrast to  $w_{\mu\nu}$  which explicitly depends on it. We will show how to surmount this difficulty.

The nonrelativistic (3D) Wigner function is defined as the integral of the 4D Wigner function over the zeroth-component of momentum:

$$W_3(x, \mathbf{p}) = \int dp_0 W(x, p) \gamma_0. \quad (18)$$

Let us define the 3D components in the Clifford algebra basis as

$$W_3(x, \mathbf{p}) = \frac{1}{4} [f_0 + \gamma_5 f_1 - i\gamma_0 \gamma_5 f_2 + \gamma_0 f_3 + \gamma_5 \gamma_0 \boldsymbol{\gamma} \cdot \mathbf{g}_0 + \gamma_0 \boldsymbol{\gamma} \cdot \mathbf{g}_1 - i\boldsymbol{\gamma} \cdot \mathbf{g}_2 - \gamma_5 \boldsymbol{\gamma} \cdot \mathbf{g}_3]. \quad (19)$$

The 4D and 3D components are related as

$$\begin{aligned} f_0(x, \mathbf{p}) &= \int dp_0 \mathcal{V}_0(x, p), & f_1(x, \mathbf{p}) &= \int dp_0 \mathcal{A}_0(x, p), \\ f_2(x, \mathbf{p}) &= \int dp_0 \mathcal{P}(x, p), & f_3(x, \mathbf{p}) &= \int dp_0 \mathcal{F}(x, p), \\ \mathbf{g}_0(x, \mathbf{p}) &= \int dp_0 \mathcal{A}(x, p), & \mathbf{g}_1(x, \mathbf{p}) &= \int dp_0 \mathcal{V}(x, p), \\ \mathbf{g}_2^i(x, \mathbf{p}) &= - \int dp_0 \mathcal{S}^{0i}(x, p), & \mathbf{g}_3^i(x, \mathbf{p}) &= \frac{1}{2} \epsilon^{ijk} \int dp_0 \mathcal{S}_{jk}(x, p). \end{aligned}$$

To express  $w_{\mu\nu}$  in terms of the 3D vorticity,  $\boldsymbol{\omega}$ , which is uniform, we choose the frame

$$u^\mu = (1, \mathbf{0}), \quad \omega^\mu = (0, \boldsymbol{\omega}). \quad (20)$$

Thus, (4) yields

$$w^{0i} = -\epsilon^{ijk} p_j \omega_k, \quad w^{ij} = \kappa p_0 \epsilon^{ijk} \omega_k.$$

In contrast to the electromagnetic field strength,  $w_{ij}$  is  $p_0$  dependent. In this frame, the components of  $D_\mu = (D_t, \mathbf{D})$ , (7), are

$$D_t = \partial_t + (\mathbf{p} \times \boldsymbol{\omega}) \cdot \nabla_p, \quad (21)$$

$$\mathbf{D} = \nabla + \kappa p_0 \boldsymbol{\omega} \times \nabla_p. \quad (22)$$

While obtaining the 3D formalism by integrating the 4D transport equations over  $p_0$ , the dependence of  $\mathbf{D}$  on  $p_0$  should be handled carefully.

To establish the 3D formalism, we integrate the relativistic equations (8)–(17) over  $p_0$ . They will be separated into two groups [8] by inspecting their dependence on the time derivative  $\partial_t$ . The equations containing  $\partial_t$  yield the transport equations:

$$\hbar \left( D_t f_0 + \int dp_0 \mathbf{D} \cdot \mathbf{V} \right) = 0, \quad (23a)$$

$$\hbar \left( D_t f_1 + \int dp_0 \mathbf{D} \cdot \mathbf{A} \right) + 2mf_2 = 0, \quad (23b)$$

$$\hbar D_t f_2 + 2\mathbf{p} \cdot \mathbf{g}_3 - 2mf_1 = 0, \quad (23c)$$

$$\hbar D_t f_3 - 2\mathbf{p} \cdot \mathbf{g}_2 = 0, \quad (23d)$$

$$\hbar \left( D_t \mathbf{g}_0 + \int dp_0 \mathbf{D} A_0 \right) - 2\mathbf{p} \times \mathbf{g}_1 = 0, \quad (23e)$$

$$\hbar \left( D_t \mathbf{g}_1 + \int dp_0 \mathbf{D} V_0 \right) - 2\mathbf{p} \times \mathbf{g}_0 + 2m\mathbf{g}_2 = 0, \quad (23f)$$

$$\hbar \left( D_t \mathbf{g}_2^i - \int dp_0 D_j S^{ji} \right) + 2p^i f_3 - 2m\mathbf{g}_1^i = 0, \quad (23g)$$

$$\hbar \left( D_t \mathbf{g}_3^i + \int dp_0 \varepsilon^{ijk} D_j S_{k0} \right) - 2p^i f_2 = 0. \quad (23h)$$

The others are the constraint equations:

$$\int dp_0 p_0 V_0 - \mathbf{p} \cdot \mathbf{g}_1 - mf_3 = 0, \quad (24a)$$

$$\int dp_0 p_0 A_0 - \mathbf{p} \cdot \mathbf{g}_0 = 0, \quad (24b)$$

$$\int dp_0 p_0 P - \frac{1}{4} \hbar \int dp_0 \varepsilon_{ijk} D^i S^{jk} = 0, \quad (24c)$$

$$\int dp_0 p_0 F + \frac{1}{2} \hbar \int dp_0 D^i S_{0i} - mf_0 = 0, \quad (24d)$$

$$\int dp_0 p_0 \mathbf{A} - \mathbf{p} f_1 - \frac{\hbar}{2} \int dp_0 \mathbf{D} \times \mathbf{V} - m\mathbf{g}_3 = 0, \quad (24e)$$

$$\int dp_0 p_0 \mathbf{V} - \mathbf{p} f_0 - \frac{\hbar}{2} \int dp_0 \mathbf{D} \times \mathbf{A} = 0, \quad (24f)$$

$$\int dp_0 p_0^0 S_{0i} - \mathbf{p} \times \mathbf{g}_3 + \frac{\hbar}{2} \int dp_0 D_i F = 0, \quad (24g)$$

$$\frac{1}{2} \varepsilon_{ijk} \int dp_0 p_0^0 S^{jk} - (\mathbf{p} \times \mathbf{g}_2)_i - \frac{\hbar}{2} \int dp_0 D_i P + m\mathbf{g}_{0i} = 0. \quad (24h)$$

These equations show that not all of the 3D fields are independent. In fact, we can express them in terms of  $f_0$  and  $\mathbf{g}_0$  as it will be discussed below.

In the classical limit, (1) simplifies and yields the classical on-shell condition

$$(p^2 - m^2)W(x, p) = 0, \quad (25)$$

whose solutions are  $p_0 = \pm E_p$ , where  $E_p = \sqrt{\mathbf{p}^2 + m^2}$ . Therefore, in the classical limit, the fields can be written as the sum of positive and negative energy solutions:

$$\begin{aligned} \mathcal{C}_a^{(0)}(x, p) &= \mathcal{C}_a^{(0)+}(x, p)\delta(p_0 - E_p) \\ &+ \mathcal{C}_a^{(0)-}(x, p)\delta(p_0 + E_p). \end{aligned} \quad (26)$$

Thus, at the leading order in  $\hbar$ , the  $p_0$  integrals in (23a)–(24h) can easily be performed and all of the 3D fields can be expressed in terms of  $f_0$  and  $\mathbf{g}_0$  in the classical limit as [8]

$$f_1^{(0)\pm} = \pm \frac{\mathbf{p}}{E_p} \cdot \mathbf{g}_0^{(0)\pm}, \quad (27)$$

$$f_2^{(0)\pm} = 0, \quad (28)$$

$$f_3^{(0)\pm} = \pm \frac{m}{E_p} f_0^{(0)\pm}, \quad (29)$$

$$\mathbf{g}_1^{(0)\pm} = \pm \frac{\mathbf{p}}{E_p} f_0^{(0)\pm}, \quad (30)$$

$$\mathbf{g}_2^{(0)\pm} = \frac{\mathbf{p}}{m} \times \mathbf{g}_0^{(0)\pm}, \quad (31)$$

$$\mathbf{g}_3^{(0)\pm} = \pm \frac{E_p^2 \mathbf{g}_0^{(0)\pm} - (\mathbf{p} \cdot \mathbf{g}_0^{(0)\pm})\mathbf{p}}{mE_p}. \quad (32)$$

To solve the transport and constraint equations to determine the 3D fields in terms of  $f_0$  and  $\mathbf{g}_0$  at the  $\hbar$  order, one can attempt to add a  $\hbar$ -order term to the on-shell condition (25). However, by inspecting the relativistic semiclassical solutions of (8)–(17) given in [30], it can be observed that each field  $\mathcal{C}_a$  satisfies a different mass shell condition at the  $\hbar$  order. In [31], it was suggested to write

$$\mathcal{C}_a^\pm(x, p) = \mathcal{C}_a^{(0)\pm}(x, p)\delta(p_0 \mp E_p) + \hbar \mathcal{A}_a^\pm(p), \quad (33)$$

and define the 3D shell shifts as

$$\Delta E_a^\pm(x, \mathbf{p}) = \int dp_0 p_0 \mathcal{A}_a^\pm(p). \quad (34)$$

The operator  $\mathbf{D}$  depends on  $p_0$ , so that the related energy averages are expressed as

$$\begin{aligned} \int dp_0 \mathbf{D} \mathcal{C}_a(x, p) &= \int dp_0 (\nabla + \kappa p_0 \boldsymbol{\omega} \times \nabla_p) \\ &\times (\mathcal{C}_a^\pm(x, p)\delta(p_0 \mp E_p) + \hbar \mathcal{A}_a^\pm(p)) \\ &= \mathbf{D}_\pm^{(0)} \mathcal{C}_a^\pm(x, \mathbf{p}) + \hbar \kappa (\boldsymbol{\omega} \times \nabla_p) \Delta E_a^\pm, \end{aligned} \quad (35)$$

where

$$\begin{aligned} \mathbf{D}_{\pm}^{(0)} &= \nabla \pm \kappa E_p \boldsymbol{\omega} \times \nabla_p \pm \frac{\kappa}{E_p} \boldsymbol{\omega} \times \mathbf{p} \\ &\equiv \partial_{\pm}^{(0)} \pm \frac{\kappa}{E_p} \boldsymbol{\omega} \times \mathbf{p}. \end{aligned} \quad (36)$$

Let us now compare this formulation with the equal-time QKE approach of [8,31]. There, the energy averages  $\int dp_0 D_{\mu}^{(EM)} C_a(x, \mathbf{p}) = (D_t^{(EM)}, \mathbf{D}^{(EM)}) C_a(x, \mathbf{p})$  are given in terms of the electromagnetic fields  $\mathbf{E}$ ,  $\mathbf{B}$  as  $D_t^{(EM)} = \partial_t + \mathbf{E} \cdot \nabla_p$ , and  $\mathbf{D}^{(EM)} = \nabla + \mathbf{B} \times \nabla_p$ . We set the electric charge  $Q = 1$ . Observe that one obtains  $D_t$  given in (21) from  $D_t^{(EM)}$  by the substitution  $\mathbf{E} \rightarrow \mathbf{p} \times \boldsymbol{\omega}$ . However,  $\mathbf{D}^{(EM)}$  is quite different from (35). First of all, although  $\mathbf{D}^{(EM)}$  is independent of  $\hbar$ , in (35) there exists a term which is at the order of  $\hbar$ . Also the  $\hbar$  independent terms are not similar. Only  $\mathbf{D}^{(EM)}$  corresponds to  $\partial^{(0)}$  by  $\mathbf{B} \rightarrow \kappa E_p \boldsymbol{\omega}$ . Thus, one cannot generate our results from the ones given in [31], by substituting  $\mathbf{E}$ ,  $\mathbf{B}$  with  $\mathbf{p} \times \boldsymbol{\omega}$ ,  $\kappa E_p \boldsymbol{\omega}$ .

The transport equations at the first order in  $\hbar$  can be read from (23a)–(23h) as

$$D_t f_0^{(0)\pm} + \mathbf{D}_{\pm}^{(0)} \cdot \mathbf{g}_1^{(0)\pm} = 0, \quad (37a)$$

$$D_t f_1^{(0)\pm} + \mathbf{D}_{\pm}^{(0)} \cdot \mathbf{g}_0^{(0)\pm} + 2m f_2^{(1)\pm} = 0, \quad (37b)$$

$$D_t f_2^{(0)\pm} + \mathbf{p} \cdot \mathbf{g}_3^{(1)\pm} - 2m f_1^{(1)\pm} = 0, \quad (37c)$$

$$D_t f_3^{(0)\pm} - 2\mathbf{p} \cdot \mathbf{g}_2^{(1)\pm} = 0, \quad (37d)$$

$$D_t \mathbf{g}_0^{(0)\pm} + \mathbf{D}_{\pm}^{(0)} f_1^{(0)\pm} - 2\mathbf{p} \times \mathbf{g}_1^{(1)\pm} = 0, \quad (37e)$$

$$D_t \mathbf{g}_1^{(0)\pm} + \mathbf{D}_{\pm}^{(0)} f_0^{(0)\pm} - 2\mathbf{p} \times \mathbf{g}_0^{(1)\pm} + 2m \mathbf{g}_2^{(1)\pm} = 0, \quad (37f)$$

$$D_t \mathbf{g}_2^{(0)\pm} + \mathbf{D}_{\pm}^{(0)} \times \mathbf{g}_3^{(0)\pm} + 2\mathbf{p} f_3^{(1)\pm} - 2m \mathbf{g}_1^{(1)\pm} = 0, \quad (37g)$$

$$D_t \mathbf{g}_3^{(0)\pm} - \mathbf{D}_{\pm}^{(0)} \times \mathbf{g}_2^{(0)\pm} - 2\mathbf{p} f_2^{(1)\pm} = 0. \quad (37h)$$

By plugging (33) into (24a)–(24h) and employing the definition (34), the constraint equations at the first order in  $\hbar$  are acquired as

$$\pm E_p f_0^{(1)\pm} + \Delta E_{f_0}^{\pm} - \mathbf{p} \cdot \mathbf{g}_1^{(1)\pm} - m f_3^{(1)\pm} = 0, \quad (38a)$$

$$\pm E_p f_1^{(1)\pm} + \Delta E_{f_1}^{\pm} - \mathbf{p} \cdot \mathbf{g}_0^{(1)\pm} = 0, \quad (38b)$$

$$\pm E_p f_2^{(1)\pm} + \Delta E_{f_2}^{\pm} + \frac{1}{2} \mathbf{D}_{\pm}^{(0)} \cdot \mathbf{g}_3^{(0)\pm} = 0, \quad (38c)$$

$$\pm E_p f_3^{(1)\pm} + \Delta E_{f_3}^{\pm} - \frac{1}{2} \mathbf{D}_{\pm}^{(0)} \cdot \mathbf{g}_2^{(0)\pm} - m f_0^{(1)\pm} = 0, \quad (38d)$$

$$\pm E_p \mathbf{g}_0^{(1)\pm} + \Delta E_{\mathbf{g}_0}^{\pm} - \mathbf{p} f_1^{(1)\pm} - \frac{1}{2} \mathbf{D}_{\pm}^{(0)} \times \mathbf{g}_1^{(0)\pm} - m \mathbf{g}_3^{(1)\pm} = 0, \quad (38e)$$

$$\pm E_p \mathbf{g}_1^{(1)\pm} + \Delta E_{\mathbf{g}_1}^{\pm} - \mathbf{p} f_0^{(1)\pm} - \frac{1}{2} \mathbf{D}_{\pm}^{(0)} \times \mathbf{g}_0^{(0)\pm} = 0, \quad (38f)$$

$$\pm E_p \mathbf{g}_2^{(1)\pm} + \Delta E_{\mathbf{g}_2}^{\pm} - \mathbf{p} \times \mathbf{g}_3^{(1)\pm} + \frac{1}{2} \mathbf{D}_{\pm}^{(0)} f_3^{(0)\pm} = 0, \quad (38g)$$

$$\pm E_p \mathbf{g}_3^{(1)\pm} + \Delta E_{\mathbf{g}_3}^{\pm} + \mathbf{p} \times \mathbf{g}_2^{(1)\pm} - m \mathbf{g}_0^{(1)\pm} = 0. \quad (38h)$$

Once we are acquainted with  $\Delta E_a^{\pm}(x, \mathbf{p})$ , the constraint equations (38a)–(38h) can be solved to express the first-order components of the fields in terms of  $f_0^{(0)\pm}$  and  $\mathbf{g}_0^{(0)\pm}$ . Some of the shell shifts can be acquired by making use the covariant formalism as we have presented in Appendix. The remaining shell shifts should be determined by using the constraint and transport equations (37a)–(38h). In conclusion, we calculated the mass shell shifts as

$$\Delta E_{f_0}^{\pm} = -\frac{\kappa}{2} \boldsymbol{\omega} \cdot \mathbf{g}_0^{(0)\pm}, \quad (39)$$

$$\Delta E_{f_1}^{\pm} = \mp \frac{\kappa}{2E_p} \mathbf{p} \cdot \boldsymbol{\omega} f_0^{(0)\pm}, \quad (40)$$

$$\Delta E_{f_2}^{\pm} = \frac{(1+\kappa)}{2m} (\mathbf{p} \times \boldsymbol{\omega}) \cdot \mathbf{g}_0^{(0)\pm}, \quad (41)$$

$$\begin{aligned} \Delta E_{f_3}^{\pm} &= \mp \frac{1}{2mE_p} (\mathbf{p} \times \boldsymbol{\omega}) \cdot (\mathbf{p} \times \mathbf{g}_0^{(0)\pm}) \\ &\mp \kappa \left( \frac{E_p^2 \mathbf{g}_0^{(0)\pm} - (\mathbf{p} \cdot \mathbf{g}_0^{(0)\pm}) \mathbf{p}}{2mE_p} \right) \cdot \boldsymbol{\omega}, \end{aligned} \quad (42)$$

$$\Delta E_{\mathbf{g}_0}^{\pm} = -\left( \frac{\kappa}{2} \boldsymbol{\omega} + \frac{\boldsymbol{\omega} p^2 - \mathbf{p}(\boldsymbol{\omega} \cdot \mathbf{p})}{2E_p^2} \right) f_0^{(0)\pm}, \quad (43)$$

$$\Delta E_{\mathbf{g}_1}^{\pm} = \mp \frac{1}{2E_p} (\mathbf{p} \times \boldsymbol{\omega}) \times \mathbf{g}_0^{(0)\pm} \mp \frac{\kappa}{2E_p} (\mathbf{p} \cdot \mathbf{g}_0^{(0)\pm}) \boldsymbol{\omega}, \quad (44)$$

$$\Delta E_{\mathbf{g}_2}^{\pm} = \frac{m\mathbf{p} \times \boldsymbol{\omega}}{2E_p^2} f_0^{(0)\pm}, \quad (45)$$

$$\Delta E_{\mathbf{g}_3}^{\pm} = \mp \frac{m\kappa}{2E_p} \boldsymbol{\omega} f_0^{(0)\pm}. \quad (46)$$

It is a curious fact that although  $\mathbf{D}^{(0)}$  given in (36) is different from its electromagnetic analog  $\mathbf{D}^{(EM)}$ , we still get the correspondence between the shell shifts given in [31] and the ones calculated here as in (39)–(46), by the substitution  $\mathbf{E} \rightarrow \mathbf{p} \times \boldsymbol{\omega}$  and  $\mathbf{B} \rightarrow \kappa E_p \boldsymbol{\omega}$ .

Now, the constraint equations (38a)–(38h) are employed to determine the field components at the first order in  $\hbar$ , in terms of  $f_0^{(0)\pm}$  and  $\mathbf{g}_0^{(0)\pm}$  as follows:

$$f_1^{(1)\pm} = \frac{\kappa}{2E_p^2} \mathbf{p} \cdot \boldsymbol{\omega} f_0^{(0)\pm} \pm \frac{1}{E_p} \mathbf{p} \cdot \mathbf{g}_0^{(1)\pm}, \quad (47)$$

$$f_2^{(1)\pm} = \mp \frac{(1-\kappa)}{2mE_p} (\mathbf{p} \times \boldsymbol{\omega}) \cdot \mathbf{g}_0^{(0)\pm} - \frac{1}{2m} \mathbf{D}_\pm \cdot \mathbf{g}_0^{(0)\pm} + \frac{1}{2mE_p^2} \mathbf{p} \cdot (\mathbf{p} \cdot \partial_\pm^{(0)}) \mathbf{g}_0^{(0)\pm}, \quad (48)$$

$$f_3^{(1)\pm} = \pm \frac{m}{E_p} f_0^{(1)\pm} \mp \frac{(\mathbf{p} \times \mathbf{D}_\pm^{(0)}) \cdot \mathbf{g}_0^{(0)\pm}}{2mE_p} + \frac{1}{2mE_p^2} (\mathbf{p} \times \boldsymbol{\omega}) \cdot (\mathbf{p} \times \mathbf{g}_0^{(0)\pm}) - \frac{\kappa}{2m} \boldsymbol{\omega} \cdot \mathbf{g}_0^{(0)\pm} - \kappa \frac{(\mathbf{p} \cdot \mathbf{g}_0^{(0)\pm}) \mathbf{p} \cdot \boldsymbol{\omega}}{2mE_p^2}, \quad (49)$$

$$\mathbf{g}_1^{(1)\pm} = \pm \frac{1}{E_p} \mathbf{p} f_0^{(1)\pm} + \frac{1}{2E_p^2} (\mathbf{p} \times \boldsymbol{\omega}) \times \mathbf{g}_0^{(0)\pm} + \frac{\kappa}{2E_p^2} (\mathbf{p} \cdot \mathbf{g}_0^{(0)\pm}) \boldsymbol{\omega} \pm \frac{1}{2E_p} \mathbf{D}_\pm^{(0)} \times \mathbf{g}_0^{(0)\pm}, \quad (50)$$

$$\mathbf{g}_2^{(1)\pm} = \frac{1}{m} \mathbf{p} \times \mathbf{g}_0^{(1)\pm} + \frac{\mathbf{p} (\mathbf{p} \cdot \partial_\pm^{(0)}) f_0^{(0)\pm}}{2mE_p^2} \mp \frac{1}{2mE_p} (\mathbf{p} \times \boldsymbol{\omega}) f_0^{(0)\pm} - \frac{1}{2m} \mathbf{D}_\pm^{(0)} f_0^{(0)\pm}, \quad (51)$$

$$\mathbf{g}_3^{(1)\pm} = \pm \frac{E_p}{m} \mathbf{g}_0^{(1)\pm} \mp \frac{1}{mE_p} \mathbf{p} (\mathbf{p} \cdot \mathbf{g}_0^{(1)\pm}) + \frac{1}{2mE_p^2} \mathbf{p} \times (\mathbf{p} \times \boldsymbol{\omega}) f_0^{(0)\pm} + \frac{m\kappa}{2E_p^2} \boldsymbol{\omega} f_0^{(0)\pm} \pm \frac{1}{2mE_p} \mathbf{p} \times \mathbf{D}_\pm^{(0)} f_0^{(0)\pm}. \quad (52)$$

We determined all of the 3D field components in terms of  $f_0$  and  $\mathbf{g}_0$  up to the first order in  $\hbar$ . In the next section we will derive their semiclassical kinetic equations.

#### IV. SEMICLASSICAL KINETIC EQUATIONS OF $f_0$ AND $\mathbf{g}_0$

Kinetic equation of the particle number density  $f_0$  at the zeroth order in  $\hbar$  can be easily derived from (30) and (37a) as

$$\left( D_t \pm \frac{\mathbf{p}}{E_p} \cdot \partial_\pm^{(0)} \right) f_0^{(0)\pm} = 0. \quad (53)$$

By employing (37b) and (37h) we can get kinetic equation of the spin density  $\mathbf{g}_0$  at the zeroth order in  $\hbar$  as follows:

$$\left( D_t \pm \frac{\mathbf{p}}{E_p} \cdot \partial_\pm^{(0)} \right) \mathbf{g}_0^{(0)\pm} = \frac{1}{E_p^2} (\mathbf{p} \times \boldsymbol{\omega}) (\mathbf{p} \cdot \mathbf{g}_0^{(0)\pm}) - \kappa \boldsymbol{\omega} \times \mathbf{g}_0^{(0)\pm}. \quad (54)$$

Let us now derive the kinetic equations of  $f_0$  and  $\mathbf{g}_0$  at next-to-leading order in  $\hbar$ . To carry out our calculations the transport equations at the second order in  $\hbar$  are needed. They can be acquired by making use of (35) in (23a)–(23h):

$$D_t f_0^{(1)\pm} + \mathbf{D}_\pm^{(0)} \cdot \mathbf{g}_1^{(1)\pm} + \frac{\kappa \boldsymbol{\omega}}{2E_p} \mathbf{p} \cdot (\boldsymbol{\omega} \times \nabla_p) \mathbf{g}_0^{(0)\pm} = 0, \quad (55a)$$

$$D_t f_1^{(1)\pm} + \mathbf{D}_\pm^{(0)} \cdot \mathbf{g}_0^{(1)\pm} + \frac{\kappa (\boldsymbol{\omega} \cdot \mathbf{p})}{2E_p^2} \mathbf{p} \cdot (\boldsymbol{\omega} \times \nabla_p) f_0^{(0)\pm} + 2m f_2^{(2)\pm} = 0, \quad (55b)$$

$$D_t f_2^{(1)\pm} + 2\mathbf{p} \cdot \mathbf{g}_3^{(2)\pm} - 2m f_1^{(2)\pm} = 0, \quad (55c)$$

$$D_t f_3^{(1)\pm} - 2\mathbf{p} \cdot \mathbf{g}_2^{(2)\pm} = 0, \quad (55d)$$

$$D_t \mathbf{g}_0^{(1)\pm} + \mathbf{D}_\pm^{(0)} f_1^{(1)\pm} \pm \frac{\kappa^2 (\boldsymbol{\omega} \cdot \mathbf{p})}{2E_p} \boldsymbol{\omega} \times \left( \frac{\mathbf{p}}{E_p^2} - \nabla_p \right) f_0^{(0)\pm} - 2\mathbf{p} \times \mathbf{g}_1^{(2)\pm} = 0, \quad (55e)$$

$$D_t \mathbf{g}_1^{(1)\pm} + \mathbf{D}_\pm^{(0)} f_0^{(1)\pm} - \frac{\kappa^2}{2} (\boldsymbol{\omega} \times \nabla_p) (\boldsymbol{\omega} \cdot \mathbf{g}_0^{(0)\pm}) - 2\mathbf{p} \times \mathbf{g}_0^{(2)\pm} + 2m \mathbf{g}_2^{(2)\pm} = 0, \quad (55f)$$

$$D_t \mathbf{g}_2^{(1)\pm} + \mathbf{D}_\pm^{(0)} \times \mathbf{g}_3^{(1)\pm} \mp \frac{m\kappa^2}{2E_p} \boldsymbol{\omega} \times \left( \boldsymbol{\omega} \times \left( \frac{\mathbf{p}}{E_p^2} - \nabla_p \right) \right) f_0^{(0)\pm} + 2\mathbf{p} f_3^{(2)\pm} - 2m \mathbf{g}_1^{(2)\pm} = 0, \quad (55g)$$

$$D_t \mathbf{g}_3^{(1)\pm} - \mathbf{D}_\pm^{(0)} \times \mathbf{g}_2^{(1)\pm} + \frac{m\kappa \boldsymbol{\omega}}{2E_p^2} \mathbf{p} \cdot (\boldsymbol{\omega} \times \nabla_p) f_0^{(0)\pm} - 2\mathbf{p} f_2^{(2)\pm} = 0. \quad (55h)$$

We would like to emphasize the fact that at this order the resemblance between  $\mathbf{D}^{(EM)}$  and (35) is completely lost due to the presence of the  $\hbar$ -order term in the latter.

By employing (44), (50), and (55a), we derived the time evolution of  $f_0^{(1)\pm}$  in terms of  $\mathbf{g}_0^{(0)\pm}$  as

$$\begin{aligned} \left( D_t \pm \frac{\mathbf{p}}{E_p} \cdot \partial_\pm^{(0)} \right) f_0^{(1)\pm} &= -\frac{\kappa}{2E_p^2} (\boldsymbol{\omega} \times \mathbf{p}) \cdot (\partial_\pm^{(0)} \times \mathbf{g}_0^{(0)\pm}) \\ &+ \frac{1}{2E_p^2} (\mathbf{p} \times \boldsymbol{\omega}) \cdot (\nabla \times \mathbf{g}_0^{(0)\pm}) \\ &- \frac{\kappa}{2E_p^2} \boldsymbol{\omega} \cdot (\mathbf{p} \cdot \nabla) \mathbf{g}_0^{(0)\pm}. \end{aligned} \quad (56)$$

After some cumbersome calculations by making use of (47), (51), (52), (53), (55e), and (55g), we obtained the dynamical evolution of  $\mathbf{g}_0^{(1)\pm}$  depending on  $f_0^{(0)\pm}$  as

$$\begin{aligned}
\left(D_t \pm \frac{\mathbf{p}}{E_p} \cdot \partial_{\pm}^{(0)}\right) \mathbf{g}_0^{(1)\pm} &= -\kappa(\boldsymbol{\omega} \times \mathbf{g}_0^{(1)\pm}) + \frac{\mathbf{p} \cdot \mathbf{g}_0^{(1)\pm}}{E_p^2} (\mathbf{p} \times \boldsymbol{\omega}) \\
&\quad - \frac{\kappa \boldsymbol{\omega}}{2E_p^2} \mathbf{p} \cdot \nabla f_0^{(0)\pm} + \frac{\mathbf{p}(\mathbf{p} \cdot \boldsymbol{\omega})}{2E_p^4} (\mathbf{p} \cdot \partial_{\pm}^{(0)}) f_0^{(0)\pm} \\
&\quad - \frac{\mathbf{p}^2(\mathbf{p} \cdot \boldsymbol{\omega})}{2E_p^4} D_{\pm}^{(0)} f_0^{(0)\pm} \mp \frac{\mathbf{p} \cdot \boldsymbol{\omega}}{2E_p^3} (\mathbf{p} \times \boldsymbol{\omega}) f_0^{(0)\pm} \\
&\quad \pm \frac{\kappa}{2E_p} \mathbf{p} \times \left( -\boldsymbol{\omega}^2 \nabla_p + \boldsymbol{\omega}(\boldsymbol{\omega} \cdot \nabla_p) + \frac{1}{E_p^2} \boldsymbol{\omega}(\mathbf{p} \cdot \boldsymbol{\omega}) \right) f_0^{(0)\pm} \\
&\quad \mp \frac{1}{2E_p^3} \mathbf{p} \times (\mathbf{p} \times \boldsymbol{\omega}) D_t f_0^{(0)\pm}. \tag{57}
\end{aligned}$$

We established the semiclassical kinetic equations of  $f_0$  and  $\mathbf{g}_0$ . It is also possible to deal with the kinetic equations of some other components of the 3D Wigner function like  $f_1$  and  $\mathbf{g}_3$ , where the latter is related to the magnetic dipole moment.

## V. KINETIC THEORIES OF THE RIGHT- AND LEFT-HANDED FERMIONS

In heavy-ion collisions, because of considering the constituent quarks of the quark-gluon plasma as massless, one expects that the collective dynamics yield the chiral vortical and separation effects due to vorticity. We would like to study how the quark mass affects this picture. To study the mass corrections, we need the kinetic equations satisfied by the right- and left-handed distribution functions  $f_R, f_L$ , defined by

$$f_{\chi} = \frac{1}{2}(f_0 + \chi f_1), \tag{58}$$

where  $\chi = \{+, -\}$ , and  $f_+ \equiv f_R$  and  $f_- \equiv f_L$ . However, the 3D kinetic equations (53), (54) and (56), and (57) are given in terms of  $f_0$  and  $\mathbf{g}_0$ . Thus, we have to specify the spin current  $\mathbf{g}_0$ , by respecting the relations (27) and (47). First, let the direction of the spin current be parallel to  $\mathbf{p}$ . Then, (27) implies that

$$\mathbf{g}_0^{(0)\pm} = \pm \frac{E_p}{p^2} \mathbf{p} f_1^{(0)\pm}. \tag{59}$$

By plugging (59) into the classical kinetic equation of the spin current given by (54), we find

$$\left(D_t \pm \frac{\mathbf{p}}{E_p} \cdot \partial_{\pm}^{(0)}\right) f_1^{(0)\pm} = 0. \tag{60}$$

Recall that it has the same form with the classical kinetic equation satisfied by  $f_0^{(0)\pm}$ , (53). Additionally, (59) allows us to write the right-hand side of (56) in terms of  $f_1^{(0)\pm}$ :

$$\begin{aligned}
\left(D_t \pm \frac{\mathbf{p}}{E_p} \cdot \partial_{\pm}^{(0)}\right) f_0^{(1)\pm} &= \pm \frac{1}{2E_p p^2} ((1 + \kappa)(\mathbf{p} \times (\mathbf{p} \times \boldsymbol{\omega})) \\
&\quad - \kappa(\boldsymbol{\omega} \cdot \mathbf{p}) \mathbf{p}) \cdot \nabla f_1^{(0)\pm} \\
&\quad + \frac{\kappa^2}{2p^2} (\mathbf{p} \cdot \boldsymbol{\omega})(\mathbf{p} \times \boldsymbol{\omega}) \cdot \nabla_p f_1^{(0)\pm}. \tag{61}
\end{aligned}$$

Now, we desire to find the kinetic equation satisfied by  $f_1^{(1)\pm}$ . For this purpose, first observe that Eq. (47) can be solved as

$$\mathbf{g}_0^{(1)\pm} = \pm \frac{E_p \mathbf{p}}{p^2} f_1^{(1)\pm} \mp \frac{\kappa \boldsymbol{\omega}}{2E_p} f_0^{(0)\pm} \pm E_p \mathbf{p} \times \mathbf{F}^{\pm}, \tag{62}$$

where  $\mathbf{F}^{\pm}$  is a free vector field which will be fixed shortly. Then, by plugging (62) into (57) and then multiplying it with  $\pm \mathbf{p}/E_p$ , we find

$$\begin{aligned}
\left(D_t \pm \frac{\mathbf{p}}{E_p} \cdot \partial_{\pm}^{(0)}\right) f_1^{(1)\pm} &= -(\mathbf{p} \times (\mathbf{p} \times \boldsymbol{\omega})) \cdot \mathbf{F}^{\pm} \mp \frac{\kappa \mathbf{p} \cdot \boldsymbol{\omega}}{2E_p^3} \mathbf{p} \cdot \nabla f_0^{(0)\pm} \\
&\quad + \frac{\kappa \boldsymbol{\omega}}{2E_p^2} \cdot \mathbf{p} \left( D_t \pm \frac{\mathbf{p}}{E_p} \cdot \partial_{\pm}^{(0)} \right) f_0^{(0)\pm}. \tag{63}
\end{aligned}$$

To have an equation compatible with (61), we choose  $\mathbf{F}^{\pm}$  to be

$$\mathbf{F}^{\pm} = \mp \frac{(\kappa + 1)}{2E_p p^2} \nabla f_0^{(0)\pm} - \frac{\kappa^2}{2p^2} \boldsymbol{\omega} \times \nabla_p f_0^{(0)\pm}. \tag{64}$$

By inserting it into (63) one gets the kinetic equation

$$\begin{aligned} \left( D_t \pm \frac{\mathbf{p}}{E_p} \cdot \partial_{\pm}^{(0)} \right) f_1^{(1)\pm} &= \pm \frac{(\kappa+1)}{2E_p p^2} (\mathbf{p} \times (\mathbf{p} \times \boldsymbol{\omega})) \cdot \nabla f_0^{(0)\pm} + \frac{(\mathbf{p} \cdot \boldsymbol{\omega}) \kappa^2}{2p^2} (\mathbf{p} \times \boldsymbol{\omega}) \cdot \nabla_p f_0^{(0)\pm} \\ &\mp \frac{\kappa \boldsymbol{\omega} \cdot \mathbf{p}}{2E_p^3} \mathbf{p} \cdot \nabla f_0^{(0)\pm} + \frac{\kappa \boldsymbol{\omega} \cdot \mathbf{p}}{2E_p^2} \left( D_t \pm \frac{\mathbf{p}}{E_p} \cdot \partial_{\pm}^{(0)} \right) f_0^{(0)\pm}. \end{aligned} \quad (65)$$

The last term can be set equal to zero due to (53). However, instead of doing that, we add a similar vanishing term to the right-hand side of (61):

$$\begin{aligned} \left( D_t \pm \frac{\mathbf{p}}{E_p} \cdot \partial_{\pm}^{(0)} \right) f_0^{(1)\pm} &= \pm \frac{(\kappa+1)}{2E_p p^2} (\mathbf{p} \times (\mathbf{p} \times \boldsymbol{\omega})) \cdot \nabla f_1^{(0)\pm} + \frac{(\mathbf{p} \cdot \boldsymbol{\omega}) \kappa^2}{2p^2} (\mathbf{p} \times \boldsymbol{\omega}) \cdot \nabla_p f_1^{(0)\pm} \\ &\mp \frac{\kappa}{2E_p} \frac{\boldsymbol{\omega} \cdot \mathbf{p}}{p^2} (\mathbf{p} \cdot \nabla) f_1^{(0)\pm} + \frac{\kappa \boldsymbol{\omega} \cdot \mathbf{p}}{2E_p^2} \left( D_t \pm \frac{\mathbf{p}}{E_p} \cdot \partial_{\pm}^{(0)} \right) f_1^{(0)\pm}. \end{aligned} \quad (66)$$

Notice that adding the last term is equivalent to a shift of  $f_0^{(1)\pm}$  with the term  $\frac{\kappa \boldsymbol{\omega} \cdot \mathbf{p}}{2E_p^2} f_1^{(0)\pm}$ . By combining (65) and (66), we find the kinetic equations

$$\begin{aligned} &\left\{ \left( 1 - \hbar \frac{\chi \kappa (\mathbf{p} \cdot \boldsymbol{\omega})}{2E_p^2} \right) \partial_t + \left[ \left( 1 - \hbar \frac{\chi \kappa (\mathbf{p} \cdot \boldsymbol{\omega})}{2E_p^2} \right) (\kappa+1) - \hbar \frac{\chi (\mathbf{p} \cdot \boldsymbol{\omega}) \kappa^2}{2p^2} \right] (\mathbf{p} \times \boldsymbol{\omega}) \cdot \nabla_p \right. \\ &\quad \left. + \left[ \pm \frac{\mathbf{p}}{E_p} \mp \hbar \frac{\chi (\kappa+1)}{2E_p p^2} \mathbf{p} \times (\mathbf{p} \times \boldsymbol{\omega}) \pm \hbar \frac{\chi \kappa m^2 (\mathbf{p} \cdot \boldsymbol{\omega})}{4E_p^3 p^2} \mathbf{p} \right] \cdot \nabla \right\} f_{\chi}^{\pm} \\ &= \pm \hbar \frac{\chi \kappa m^2 \mathbf{p} \cdot \boldsymbol{\omega}}{4E_p^3 p^2} \mathbf{p} \cdot \nabla f_{-\chi}^{\pm}. \end{aligned} \quad (67)$$

Therefore, we establish the kinetic theory

$$\begin{aligned} &[\sqrt{\eta_{\chi}^{\pm}} \partial_t + (\sqrt{\eta_{\chi}^{\pm}} \dot{\mathbf{x}})_{\chi}^{\pm} \cdot \nabla + (\sqrt{\eta_{\chi}^{\pm}} \dot{\mathbf{p}})_{\chi}^{\pm} \cdot \nabla_p] f_{\chi}^{\pm} \\ &= \pm \hbar \frac{\chi m^2 \mathbf{p} \cdot \boldsymbol{\omega}}{4E_p^3 p^2} \mathbf{p} \cdot \nabla f_{-\chi}^{\pm}, \end{aligned} \quad (68)$$

with

$$\sqrt{\eta_{\chi}^{\pm}} = 1 - \hbar \frac{\chi (\mathbf{p} \cdot \boldsymbol{\omega})}{2E_p^2}, \quad (69)$$

$$(\sqrt{\eta_{\chi}^{\pm}} \dot{\mathbf{x}})_{\chi}^{\pm} = \pm \frac{\mathbf{p}}{E_p} \pm \hbar \frac{\chi \boldsymbol{\omega}}{E_p} \mp \hbar \frac{\chi (\mathbf{p} \cdot \boldsymbol{\omega}) \mathbf{p}}{4E_p} \left( \frac{3}{p^2} + \frac{1}{E_p^2} \right), \quad (70)$$

$$(\sqrt{\eta_{\chi}^{\pm}} \dot{\mathbf{p}})_{\chi}^{\pm} = \left[ 2 \left( 1 - \hbar \frac{\chi (\mathbf{p} \cdot \boldsymbol{\omega})}{2E_p^2} \right) - \hbar \frac{\chi (\mathbf{p} \cdot \boldsymbol{\omega})}{2p^2} \right] (\mathbf{p} \times \boldsymbol{\omega}). \quad (71)$$

We set  $\kappa = 1$  for acquiring the Coriolis force correctly. The term appearing on the right-hand side of (68) shows that for the massive fermions right- and left-handed distributions cannot be decoupled.

Getting inspiration from the left- and right-handed decompositions of the distribution functions,  $f_0 = f_R + f_L$ ,  $f_1 = f_R - f_L$ , we write the shell shifts in (39) and (40) as

$$\Delta E_{f_0}^{\pm} = \Delta E_{f_{0R}}^{\pm} + \Delta E_{f_{0L}}^{\pm} = \mp \frac{E_p \mathbf{p} \cdot \boldsymbol{\omega}}{2p^2} (f_R^{(0)\pm} - f_L^{(0)\pm}), \quad (72)$$

$$\Delta E_{f_1}^{\pm} = \Delta E_{f_{1R}}^{\pm} - \Delta E_{f_{1L}}^{\pm} = \mp \frac{\mathbf{p} \cdot \boldsymbol{\omega}}{2E_p} (f_R^{(0)\pm} + f_L^{(0)\pm}). \quad (73)$$

Hence, for the left- and right-handed fermions we define

$$\Delta E_{\chi}^{\pm} = \mp \frac{\chi}{4E_p} \left( 1 + \frac{E_p^2}{p^2} \right) \mathbf{p} \cdot \boldsymbol{\omega}. \quad (74)$$

Therefore, the dispersion relations are

$$\epsilon_{p,\chi}^{\pm} = \pm E_p \mp \hbar \frac{\chi}{4E_p} \left( 1 + \frac{E_p^2}{p^2} \right) \mathbf{p} \cdot \boldsymbol{\omega}. \quad (75)$$

The particle number current density can be written in terms of the equilibrium distribution function as

$$\mathbf{j}_{\chi}^{\pm} = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} (\sqrt{\eta_{\chi}^{\pm}} \dot{\mathbf{x}})_{\chi}^{\pm} f_{\chi}^{eq\pm} (\epsilon_{p,\chi}^{\pm}). \quad (76)$$

Let the equilibrium distribution function be taken as the Fermi-Dirac distribution:

$$f_{\chi}^{eq\pm}(\epsilon_{p,\chi}^{\pm}) = \frac{1}{e^{\pm(\epsilon_{p,\chi}^{\pm} - \mu_{\chi})/T} + 1}, \quad (77)$$

where  $\mu_{\chi}$  is the chiral chemical potential,  $T$  is the temperature, and we employed the dispersion relations (75). We can expand (77) in Taylor series as

$$f_{\chi}^{eq\pm}(\epsilon_{p,\chi}^{\pm}) \approx f_{\chi}^{eq\pm}(E_p) \mp \hbar \frac{\chi}{4E_p} \left(1 + \frac{E_p^2}{\mathbf{p}^2}\right) \mathbf{p} \cdot \boldsymbol{\omega} \frac{df_{\chi}^{eq\pm}(E_p)}{dE_p}, \quad (78)$$

where

$$f_{\chi}^{eq\pm}(E_p) = \frac{1}{e^{(E_p \mp \mu_{\chi})/T} + 1}. \quad (79)$$

Notice that the equilibrium distribution function depends only on the magnitude of the momentum. Therefore, we can evaluate the angular part of the integral in (76), yielding

$$\begin{aligned} \mathbf{j}_{\chi}^{\pm} &= \hbar \chi \boldsymbol{\omega} \int \frac{d|\mathbf{p}|}{24\pi^2} \mathbf{p}^2 \left[ \left( \pm \frac{8}{E_p} \pm \frac{m^2}{E_p^3} \right) f_{\chi}^{eq\pm}(E_p) \right. \\ &\quad \left. - \left( \frac{1}{E_p^2} + \frac{1}{\mathbf{p}^2} \right) \mathbf{p}^2 \frac{df_{\chi}^{eq\pm}(E_p)}{dE_p} \right]. \end{aligned} \quad (80)$$

Since the classical terms vanish, the current densities are at the order of  $\hbar$ . Then, the vector and axial vector current densities,  $\mathbf{j}_V = \mathbf{j}_R + \mathbf{j}_L$ ,  $\mathbf{j}_A = \mathbf{j}_R - \mathbf{j}_L$ , are accomplished as

$$\begin{aligned} \mathbf{j}_{V,A} &= \sum_{\pm} \hbar \boldsymbol{\omega} \int \frac{d|\mathbf{p}|}{24\pi^2} \mathbf{p}^2 \left[ \left( \pm \frac{8}{E_p} \pm \frac{m^2}{E_p^3} \right) f_{V,A}^{eq\pm}(E_p) \right. \\ &\quad \left. - \left( \frac{1}{E_p^2} + \frac{1}{\mathbf{p}^2} \right) \mathbf{p}^2 \frac{df_{V,A}^{eq\pm}(E_p)}{dE_p} \right] \\ &\equiv \boldsymbol{\sigma}_{V,A} \boldsymbol{\omega}. \end{aligned} \quad (81)$$

We introduced

$$f_{V,A}^{\pm} = \frac{1}{e^{(E_p \mp \mu_R)/T} + 1} \mp \frac{1}{e^{(E_p \mp \mu_L)/T} + 1}. \quad (82)$$

Observe that at zero temperature, the distribution functions transform into the Heaviside step function for positive energy particles and vanish for negative energy particles. For simplicity, let us set  $\mu_R = \mu_L \equiv \mu$ . Then, the vector current vanishes and the axial vector current gives

$$\begin{aligned} \lim_{T \rightarrow 0} \boldsymbol{\sigma}_A &= \frac{\hbar}{2\pi^2} \left[ \frac{3\mu^2 - m^2}{3\mu} \sqrt{\mu^2 - m^2} \right. \\ &\quad \left. - \frac{1}{2} m^2 \ln \left( \frac{\mu + \sqrt{\mu^2 - m^2}}{m} \right) \right] \boldsymbol{\omega}(\mu - m). \end{aligned} \quad (83)$$

In the small mass limit we get

$$\lim_{T \rightarrow 0} \sigma_A^+ = \frac{\hbar}{2\pi^2} \left[ \mu \sqrt{\mu^2 - m^2} - \frac{m^2}{3} \right] \theta(\mu - m). \quad (84)$$

This result is in harmony with the field theoretic calculations performed by means of the Kubo formula in [34,35] when one ignores the last term. However, if one excludes the chemical potential terms only the  $m^2$  term survives. In fact, the latter is similar to the small mass correction obtained in curved space in [36].

Although we do not consider the electromagnetic fields, by inspecting the kinetic equations obtained in [31] one can observe that the time evolution of spatial coordinates linear in the magnetic field can be acquired from (70) by substituting  $\boldsymbol{\omega}$  with  $\mathbf{B}/E_p$ . Hence, the related axial current will produce the finite mass corrections to the kinetic coefficient of the chiral separation effect.

Let us inspect the chiral (massless) limit: First of all, (68)–(71) generate the chiral kinetic theory

$$[\sqrt{\eta} \chi^{C\pm} \partial_t + (\sqrt{\eta} \dot{\mathbf{x}})_{\chi}^{C\pm} \cdot \nabla + (\sqrt{\eta} \dot{\mathbf{p}})_{\chi}^{C\pm} \cdot \nabla_p] f_{\chi}^{\pm} = 0, \quad (85)$$

with

$$\sqrt{\eta} \chi^{C\pm} = 1 - \hbar \frac{\chi \boldsymbol{\omega} \cdot \mathbf{p}}{2\mathbf{p}^2}, \quad (86)$$

$$(\sqrt{\eta} \dot{\mathbf{x}})_{\chi}^{C\pm} = \pm \frac{\mathbf{p}}{|\mathbf{p}|} \mp \hbar \frac{\chi}{|\mathbf{p}|^3} \mathbf{p}(\mathbf{p} \cdot \boldsymbol{\omega}) \pm \hbar \frac{\chi}{|\mathbf{p}|} \boldsymbol{\omega}, \quad (87)$$

$$(\sqrt{\eta} \dot{\mathbf{p}})_{\chi}^{C\pm} = 2\mathbf{p} \times \boldsymbol{\omega} - \hbar \chi \frac{3(\mathbf{p} \cdot \boldsymbol{\omega})}{2\mathbf{p}^2} (\mathbf{p} \times \boldsymbol{\omega}). \quad (88)$$

Then, (75) gives the dispersion relation for chiral particles as

$$\epsilon_{p,\chi}^{C\pm} = \pm |\mathbf{p}| \mp \hbar \frac{\chi}{2} \hat{\mathbf{p}} \cdot \boldsymbol{\omega}. \quad (89)$$

It is consistent with the dispersion relation obtained in [9,37,38]. Moreover, the dynamical evolution of the spatial coordinate vector, (87), coincides with the one established in [26]. Let the equilibrium distribution be given by the Fermi-Dirac distribution:

$$f_{\chi}^{eq\pm}(\epsilon_{p,\chi}^{C\pm}) = \frac{1}{e^{\pm(\epsilon_{p,\chi}^{C\pm} - \mu_{\chi})/T} + 1}, \quad (90)$$

Thus, the chiral particle number current densities are acquired as

$$\mathbf{j}_{\chi}^{C\pm} = \hbar \chi \boldsymbol{\omega} \int \frac{d|\mathbf{p}|}{3\pi^2} \left( \pm |\mathbf{p}| f_{\chi}^{eq\pm}(|\mathbf{p}|) - \frac{1}{4} \mathbf{p}^2 \frac{df_{\chi}^{eq\pm}(|\mathbf{p}|)}{d|\mathbf{p}|} \right). \quad (91)$$

We can perform the integrals and obtain the vector and axial vector currents as

$$j_V = \hbar \frac{\mu\mu_A}{2\pi^2} \boldsymbol{\omega}, \quad j_A = \hbar \left( \frac{T^2}{12} + \frac{\mu^2 + \mu_A^2}{4\pi^2} \right) \boldsymbol{\omega},$$

where  $\mu = \mu_R + \mu_L$ ,  $\mu_A = \mu_R - \mu_L$ . These coincide with the results reported in [20]. Therefore, we conclude that in the massless limit the chiral vector and separation effects are generated correctly.

## VI. DISCUSSIONS

The VQKE of the Wigner function leads to the transport equations of the components of the covariant Wigner function. By integrating them over  $p_0$ , we write the equations which the components of 3D Wigner function obey. They can be separated into the transport and constraint equations. The vector component of the covariant  $D_\mu = (D_t, \mathbf{D})$  operator depends explicitly on  $p_0$  as in (22). Hence, also the transport equations depend explicitly on  $p_0$ , in contrast to the transport equations which have been defined in [8,31].  $p_0$  integrals are performed by employing the on-shell conditions of the covariant fields. Then,  $\mathbf{D}$  effectively becomes as in (35), which is very different from the  $\mathbf{D}^{(EM)}$  appearing in [8,31]. Therefore, it is not possible to generalize the method of [31] directly to our case. Nevertheless, to study the 3D transport and constraint equations, we follow the method proposed in [31] and let each component of the Wigner function satisfy a different on-shell condition at  $\hbar$  order. We presented these shell shifts and by plugging them into the constraint equations we expressed the components of the 3D Wigner function at first order in terms of  $f_0$ ,  $\mathbf{g}_0$ . We consider  $f_0$  and  $\mathbf{g}_0$  as independent components. The main objective is to establish the semiclassical kinetic equations of the fields which are chosen as the independent set of components. After some cumbersome calculations we acquired them as in (53), (54) and (56), (57).

To accomplish the mass corrections to the chiral (massless) kinetic equations, we have fixed the spin current  $\mathbf{g}_0$  in terms of  $f_0$  and  $f_1$ . Then, we derived the kinetic equations of right- and left-handed distribution functions in (67), which provide us the kinetic theories of the right- and left-handed fermions. We acquired their dispersion relations and calculated particle number current densities by choosing the equilibrium distribution functions appropriately. We have shown that the massless case generates the chiral vortical and separation effects correctly. Therefore, we succeeded in accomplishing the mass corrections to the chiral effects.

In principle we can consider a system with the nonvanishing linear acceleration  $a_\mu = u_\nu \partial^\nu u_\mu$ , by adding the term  $(a_\mu u_\nu - a_\nu u_\mu)$  to  $w_{\mu\nu}$  given in (4). The procedure which we employed here in obtaining mass shell shifts, relies on the solutions of the covariant equations reported in

[30]. Hence, to deal with nonvanishing  $a_\mu$ , one should first study solutions of the kinetic equations obeyed by the covariant Wigner function components with this altered  $w_{\mu\nu}$ , which would be complicated.

A challenging future research direction is the study of 3D transport theory of VQKE in the presence of electromagnetic fields. As far as the contributions linear in electromagnetic fields and vorticity are concerned, this can simply be achieved by gathering the results obtained here and the ones reported in [31], as we discussed after (84). However, establishing kinetic equations of  $f_0$  and  $\mathbf{g}_0$  up to the first order in  $\hbar$  in the presence of only vorticity or electromagnetic fields is already very difficult. Thus, when they are considered together, deriving the semiclassical kinetic equations of  $f_0$  and  $\mathbf{g}_0$  will be a demanding task. Covariant kinetic equations established in [30] may give some hints to solve this problem.

Kinetic equations are useful mainly when collisions are taken into account. Thus, incorporating scatterings in the 3D formulation is desired. Unfortunately, we do not know how to do it for the VQKE. In principle, this can be achieved by considering the collisions in the covariant approach first and then deal with the 3D VQKE by integrating them over  $p_0$ . This will generate collision terms on the right-hand side of (23a)–(24h). In this respect, the methods employed in [39,40] can be useful. The other method would be to introduce collisions to the kinetic equations of the independent set of fields (53), (54), (56), and (57). This is another challenging open problem.

## APPENDIX: CALCULATION OF SHELL SHIFTS FROM THE COVARIANT FORMULATION

Semiclassical solutions of the kinetic equations obeyed by the components of 4D Wigner function, (8)–(17), have been presented in [30]. By inspecting  $\hbar$ -order components of those solutions, one observes that some of them are expressed in the form

$$c_i^{(1)} = \beta_i^{(1)} \delta(p^2 - m^2) - \Delta E_i(p) \delta'(p^2 - m^2), \quad (A1)$$

where  $\beta_i^{(1)}$  are first-order fields. One can notice that the 3D mass shell shifts for these fields can be obtained as

$$\Delta E_i(p) = \int \Delta E_i(p) \delta(p^2 - m^2) dp_0. \quad (A2)$$

In this fashion, we calculated the on-shell energy shifts for the following components of 4D Wigner function.

(i) The scalar field  $\mathcal{F}$ :

$$\mathcal{F}^{(1)} = m \delta(p^2 - m^2) f_V^1 - \frac{m}{2} \delta'(p^2 - m^2) f_A^0 \Sigma_{\mu\nu}^{(0)} w^{\mu\nu}. \quad (A3)$$

$f_A^0, f_V^1$  are scalars and  $\Sigma_{\mu\nu}^{(0)} = -(1/m)\epsilon_{\mu\nu\alpha\beta}p^\alpha s^\beta$ , where  $s_\mu$  is the spin quantization direction four-vector.

$$\begin{aligned}\Delta E_{f_3}^\pm(\mathbf{p}) &= \pm \frac{m}{2} \int dp_0 \Sigma_{\mu\nu}^{(0)} w^{\mu\nu} f_A^0 \delta(p^2 - m^2) \\ &= \mp \frac{1}{2E_p} (\mathbf{p} \times \boldsymbol{\omega}) \cdot \mathbf{g}_2^{(0)\pm} - \kappa \mathbf{g}_3^{(0)\pm} \cdot \boldsymbol{\omega}. \quad (\text{A4})\end{aligned}$$

(ii) The axial-vector field  $\mathcal{A}_\mu$ :

$$\begin{aligned}\mathcal{A}_\mu^{(1)} &= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} p^\nu \Sigma^{(1)\rho\sigma} \delta(p^2 - m^2) \\ &\quad - \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} w^{\rho\sigma} p^\nu f_V^0 \delta'(p^2 - m^2). \quad (\text{A5})\end{aligned}$$

$\Sigma_{\mu\nu}^{(1)}$  is an antisymmetric tensor field and  $f_V^0$  is a scalar.

For  $\mathcal{A}_0$ ,

$$\begin{aligned}\Delta E_{f_1}^\pm(\mathbf{p}) &= \pm \frac{1}{2} \int dp_0 \epsilon_{ijk} w^{jk} p^i f_V^0 \delta(p^2 - m^2) \\ &= \mp \frac{\kappa}{2E_p} \mathbf{p} \cdot \boldsymbol{\omega} f_0^{(0)\pm}. \quad (\text{A6})\end{aligned}$$

For  $\mathcal{A}$ ,

$$\begin{aligned}\Delta E_{\mathbf{g}_0}^{i\pm}(\mathbf{p}) &= \mp \frac{1}{2} \int dp_0 \epsilon^{i\nu\alpha\beta} w_{\alpha\beta} p_\nu f_V^0 \delta(p^2 - m^2) \\ &= - \left( \frac{\kappa}{2} \boldsymbol{\omega} + \frac{\boldsymbol{\omega} p^2 - \mathbf{p}(\boldsymbol{\omega} \cdot \mathbf{p})}{2E_p^2} \right)^i f_0^{(0)\pm}. \quad (\text{A7})\end{aligned}$$

(iii) The antisymmetric tensor field  $S_{\mu\nu}$ :

$$S_{\mu\nu}^{(1)} = m \Sigma_{\mu\nu}^{(1)} \delta(p^2 - m^2) - m w_{\mu\nu} f_V^0 \delta'(p^2 - m^2). \quad (\text{A8})$$

For  $S_{0i}$ ,

$$\begin{aligned}\Delta E_{\mathbf{g}_2}^{i\pm}(\mathbf{p}) &= \pm m \int dp_0 w^{i0} f_V^0 \delta(p^2 - m^2) \\ &= \frac{m(\mathbf{p} \times \boldsymbol{\omega})^i}{2E_p^2} f_0^{(0)\pm}. \quad (\text{A9})\end{aligned}$$

For  $S_{ij}$ ,

$$\begin{aligned}\Delta E_{\mathbf{g}_3}^{i\pm}(\mathbf{p}) &= \pm \frac{m}{2} \int dp_0 \epsilon^{ijk} w_{jk} f_V^0 \delta(p^2 - m^2) \\ &= \mp \frac{m\kappa}{2E_p} \omega^i f_0^{(0)\pm}. \quad (\text{A10})\end{aligned}$$

These are the mass shell shifts which we determine from the covariant approach.

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