Generalized entropy increase at linear order in general second-order scalar-tensor gravity

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Although the Wald entropy is commonly suitable for the first law of black hole thermodynamics, whether it satisfies the second law has not been examined sufficiently. For any high-order curvature gravity, the Wald entropy has been shown not to obey the linearized second law, and the general expression of entropy of black holes that always satisfies the linearized second law is further obtained. For gravity with nonminimal coupling matter fields, however, whether the Wald entropy of black holes satisfies the second law has not been studied enough. Recently, a general second-order scalar-tensor gravitational theory has been proposed. The Lagrangian of the gravitational theory is a linear combination of four components, and the Wald entropy can also be written as four parts. We should investigate each part of the expression separately to examine whether the Wald entropy satisfies the linearized second law. For the first and fourth parts contributed by two nonminimal coupling terms in Lagrangian contained in Horndeski gravity and Gauss-Bonnet gravity, the entropy of black holes is not expressed as the Wald entropy because the entropy in Gauss-Bonnet gravity is Jacobson-Myers entropy. The third part contributed by the Lagrangian in Einstein-Hilbert action is Bekenstein-Hawking entropy, which obeys the second law automatically. Therefore, to obtain the entropy of black holes that meets the linearized second law, we only need to study the second part of the Wald entropy. According to the null energy condition and the Raychaudhuri equation, one can show that the second part of the Wald entropy with correction terms will increase monotonically constrained by the null energy condition during the perturbation process. The second part of the Wald entropy should be modified to satisfy the linearized second law, and the expression of the entropy of black holes, which always obeys the linearized second law, is obtained in the general second-order scalar-tensor gravity.

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I. INTRODUCTION

General relativity (GR) has predicted the existence of black holes as particular spacetime structures in our universe, and the boundary of black holes is a specific null hypersurface called the event horizon of black holes. Many essential properties of black holes, especially black hole thermodynamics, are reflected by the event horizon at an equilibrium or a dynamical evolution state. The area theorem of black holes was first proposed by Hawking [1], which states that the area of the event horizon of black holes never decreases with time evolution. Based on this theorem, Bekenstein [2] conjectured that the area of the event horizon of black holes could be equivalent to the entropy of an ordinary thermodynamic system. From the perspective of the quantum field theory in curved spacetime, Hawking [3] proved that black holes can be regarded as thermodynamic systems, not just pure spacetime structures, while the temperature and the entropy can be defined by the surface gravity and the area of the event horizon of black holes, respectively. The entropy of black holes is called Bekenstein-Hawking entropy, which can be written as $S_{\rm BH} = A/4$, where A is the area of the event horizon. Based on definitions of the temperature and the entropy of black holes, the black hole thermodynamics is established gradually, which is identical to the four laws of thermodynamics for a classical thermodynamic system [4–6]. Although the four laws of thermodynamics are established for black holes in classical GR, it has become clear that the four laws of classical black hole thermodynamics still hold in much more general contexts.

If a timelike Killing vector ξ^a exists in spacetime at infinity and is normal to a null hypersurface, the null hypersurface is called the Killing horizon. The event horizon of stationary black holes can be treated as a Killing horizon according to the rigidity theorem [7]. The surface gravity κ at any point on a Killing horizon is defined as [8]

$$\xi^b \nabla_b \xi^a = \kappa \xi^a. \tag{1}$$

The zeroth law of black hole thermodynamics states that κ is a constant over the event horizon of stationary black holes in

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the classical theory of gravity. The proof of this law is closely related to a specific form of the Einstein equation, and this proof process seems unlikely to be extended to other theories of gravity [5]. Furthermore, it has been shown that the zeroth law is held without the concrete expression of the field equation of gravity when the bifurcation surface exists in spacetime [9]. In addition, based on the rigidity theorem and the dominant energy condition, the zeroth law has been generally demonstrated without dependence on the field equation and other additional assumptions [10,11]. This result means that if the rigidity theorem is valid, the zeroth law of black hole thermodynamics can be generalized to other modified gravitational theories.

The black hole uniqueness theorem requires that all black hole solutions in Einstein-Maxwell gravity can be characterized by only three externally observable quantities, i.e., the mass M, the electric charge Q, and the angular momentum J [12]. The most general black hole in the gravitational theory is the Kerr-Newman (KN) black hole. The first law of black hole thermodynamics describes how the variation of mass is related to the change of area, angular momentum, and electric charge of the event horizon for the perturbation of stationary black holes, and the first law of KN black holes can be expressed as

$$\delta M = T\delta S_{\rm BH} + \Phi_H \delta Q + \Omega_H \delta J, \tag{2}$$

where the quantities T, Φ_H, and Ω_H are the temperature, the electrostatic potential, and the angular velocity of the event horizon, respectively. Since the first law describes the change of related physical quantities of KN black holes from one equilibrium state to another, the initial and final states obey the rigidity theorem. Equation (2) is the general expression of the first law of black holes in classical GR. Furthermore, whether the first law can still be suitable for much more general gravitational theory should be further investigated. Based on the Noether charge method, the first law of black hole thermodynamics in any diffeomorphism invariant theory of gravity has been investigated by Iyer and Wald [13,14]. The entropy in the general first law is called the Wald entropy rather than Bekenstein-Hawking entropy, which is defined as

$$S_{\rm W} = -2\pi \int_s d^{\rm n} y \sqrt{\gamma} \frac{\partial \mathcal{L}}{\partial R_{abcd}} \epsilon_{ab} \epsilon_{cd}, \tag{3}$$

where y is used to describe the transverse coordinates on any cross section of the event horizon, γ is the determinant of the volume element of the cross section, $\mathcal{L}$ is the Lagrangian of the gravitational theory, R_{abcd} is the Riemann curvature tensor, and ϵ_{ab} represents the binormal of any slice on the event horizon. Besides, for the general first law, the rigidity theorem is still suitable for the initial and final states in the perturbation process because the two states are both stationary.

The third law of black hole thermodynamics states that it is impossible to form a black hole with vanishing surface gravity through finite operators. If the surface gravity continues to exceed the critical value, i.e., $\kappa = 0$, the value of surface gravity will become negative. It implies that the event horizon of black holes disappears, and the naked singularity exports into spacetime. The naked singularity is forbidden to appear because it can invalidate the causality law of spacetime. To guarantee the completeness of spacetime geometry and the validity of the causality law, an assumption which is called the cosmic censorship conjecture (CCC) was proposed by Penrose [15]. The CCC can be divided into the weak cosmic censorship conjecture (WCCC) and the strong cosmic censorship conjecture (SCCC). Recently, the CCC, especially the WCCC, has been widely investigated using the perturbation method [16]. The stability condition is further introduced to examine the WCCC of black holes, which states that the spacetime geometry still belongs to the class of original solutions of spacetime geometry after the perturbation process. If black holes still exist at the final state, the WCCC of black holes is valid at the end of the perturbation. The stability condition requires that black holes in the initial and final states in the perturbation process are both stationary black holes. In other words, the rigidity theorem still holds in the initial and final states. Although this conjecture closely relates to the extreme black hole, it is not identical to the third law of black hole thermodynamics. In other words, examining the WCCC cannot be regarded as proving the third law directly. However, if the surface gravity of black holes does not approach or exceed the critical value, the third law of black hole thermodynamics is guaranteed in significant measure.

The above discussions indicate that the rigidity theorem permeates through the investigation of the zeroth, the first, and the third laws of black hole thermodynamics. However, the study of the second law of black hole thermodynamics is different from the other three laws of black hole thermodynamics because it focuses on how the entropy of black holes changes with a dynamic process. In this situation, the rigidity theorem is no longer valid during the dynamical process. Therefore, how to study the second law in more general gravitational theory has become an important research direction for black hole thermodynamics until now. Additionally, the two most profound of the four laws of black hole thermodynamics are the first and the second laws. If black holes in any gravitational theory are regarded as thermodynamic systems, the first two laws of black hole thermodynamics should first be satisfied. Since the Wald entropy generally obeys the first law, we naturally expect that the Wald entropy of black holes in any diffeomorphism invariant theory of gravity can also satisfy the second law of black hole thermodynamics.

The quantization of gravitational field is a frontline issue in the research of GR. Although many methods are used to quantize the gravitational field, an appropriate quantization scheme has not been established perfectly until now. It means that when the self-interaction of gravity or the nonminimal coupling interaction between gravity and matter fields is involved in the theory of gravity, we cannot investigate these interactions rigorously at the quantum level. Therefore, to study these interactions from the perspective of quantum gravity, the most convenient method is to construct a corresponding low-energy efficient field theory, and the results obtained in classical GR should be recovered through the efficient field theory. At present, one of the most fruitful approaches to construct the approximation of the low-energy efficient theory of gravity is that of the extended gravitational theory based on corrections and enlargements of Einstein's theory, which have become a sort of paradigm. This paradigm essentially introduces high-order curvature invariants and minimal or nonminimal coupling matter fields into the effective Lagrangian of quantum gravity theory at a low energy approximation. In other words, quantum correction terms are added to the Lagrangian of the gravitational theory when the low-energy efficient field theory is used to deal with the nonminimal coupling interactions. The quantum correction terms are the high-order curvature terms that describe the self-interaction of gravity and the nonminimal coupling terms that describes the interaction between gravity and matter fields. Besides, it is recognized that these correction terms are unavoidable to obtain the efficient effect of quantum gravity under the Planck scale [17]. The expression of the Wald entropy is sufficiently affected by the quantum correction terms in the Lagrangian according to Eq. (3). It implies that examining whether the Wald entropy of black holes in a specific gravitational theory containing quantum correction terms satisfies the second law is a prerequisite for finding a general method to check whether the Wald entropy of black holes in any diffeomorphism invariant theory of gravity obeys the second law. The Wald entropy of black holes in $f(R)$ gravity has been investigated according to the field redefinition, and the result shows that it obeys the second law of thermodynamics [18,19]. Subsequently, considering a quasistationary accretion process of black holes, the spacetime geometry of black holes will be perturbed by the accretion process. And an assumption that black holes will settle down to a stationary state at the end of the perturbation process is introduced. This assumption means that the initial and the final states under the perturbation process still satisfy the rigidity theorem in the proving process of the second law. In other words, the validity of the rigidity theorem still affects the study of the second law. From the method of matter fields perturbation, the second law in the most general second-order higher curvature theory of gravity has been examined under the first-order approximation [20]. The second law of black hole thermodynamics that holds under the linear order approximation of the perturbation is called the linearized second law. Utilizing the perturbation method, the linearized second laws of black holes in the Gauss-Bonnet gravity and the Lovelock gravity are investigated [21,22]. The results show that the entropy of black holes that obeys the linearized second law in two gravitational theories is called the Jacobson-Myers (JM) entropy rather than the Wald entropy. Furthermore, a common approach to investigate the linearized second law of black holes in the gravitational theory that contains arbitrary high-order curvature terms has been proposed by Wall [23]. The general expression of the entropy of black holes that obeys the linearized second law is given, which can be written as a linear combination of Wald entropy and some correction terms. The result illustrates that the Wald entropy should be corrected to satisfy the linearized second law when the quantum correction terms of arbitrary high-order curvature are contained in the Lagrangian of the gravitational theory.

In the previous research, from the perspective of perturbation, the situation that quantum correction terms of arbitrary high-order curvature are contained in the Lagrangian has been sufficiently investigated because the general expression of entropy of black holes satisfying the linearized second law has been obtained. As mentioned above, the quantum corrections in the Lagrangian not only involve any high-order curvature term but the nonminimal coupling term that describes the interaction between gravity and matter fields. The linearized second law of black holes in the gravitational theory with the nonminimal coupling scalar field or gauge fields has been investigated in the recent research through the construction of entropy current [24]. Although the linearized second law of black holes has been discussed extensively in the theory of gravity involving arbitrary nonminimal coupling matter fields, the general expression of entropy of black holes obeying the linearized second law has not been obtained until now. To deduce the entropy of black holes generally obeying the linearized second law in the gravitational theory with the nonminimal coupling matter fields, we should first examine whether the Wald entropy of black holes in this kind of gravitational theory always satisfies the linearized second law. If the Wald entropy does not generally obey the second law, we will need to explore the new expression of the entropy of black holes that satisfies the linearized second law during the perturbation all the time. In our previous works, the Wald entropies of black holes in Horndeski gravity and the general quadric corrected Einstein-Maxwell gravity were both demonstrated to obey the linearized second law [25,26]. For Horndeski gravity, there are no more nonminimal coupling terms between gravity and the scalar field in the Lagrangian, except the nonminimal coupling term between the Einstein tensor and the scalar field. In other words, the Wald entropy always obeys the linearized second law in general Horndeski gravity. Recently, a new general second-order scalar-tensor gravitational theory has been proposed, and the Lagrangian is a linear combination of four components. There are three types of nonminimal coupling interaction terms in the four components. These terms are the Gauss-Bonnet combination, the nonminimal coupling interaction between the Einstein tensor and the scalar field that is the same as the nonminimal coupling term in Horndeski gravity, and the nonminimal coupling term that describes the interaction between the double dual of Riemann curvature and the scalar field. The Wald entropy of black holes can be expressed as the linear combination of four parts that correspond to the four components in the Lagrangian according to the definition. To examine the second law of thermodynamics under the first-order approximation of the matter fields perturbation, we only need to investigate whether each part in the expression of the Wald entropy of black holes obeys the linearized second law respectively. As mentioned above, the linearized second law of black holes in Horndeski gravity and the Gauss-Bonnet gravity has been investigated. The results show that the Wald entropy does not satisfy the linearized second law because the entropy of black holes in Gauss-Bonnet gravity obeying the second law is JM entropy rather than the Wald entropy. It implies that the Wald entropy no longer meets the linearized second law even if the nonminimal coupling term between the double dual of Riemann curvature and the scalar field is not contained in the Lagrangian of the gravitational theory. Therefore, to explore the final expression of black holes obeying the linearized second law in general second-order scalar-tensor gravitational theory, we still start from the definition of Wald entropy to examine whether the entropy contributed by the nonminimal coupling term containing the double dual of the Riemann curvature satisfies the linearized second law. If it does not meet the requirement of the linearized second law, we should further explore the concrete expression of the entropy that always satisfies the linearized second law from this nonminimal coupling term in the Lagrangian.

The organization of the paper is as follows. In Sec. II, the general second-order scalar-tensor gravity is introduced first, and the expression of the Wald entropy, especially the specific expression contributed by the nonminimal coupling term between the double dual of the Riemann curvature and the scalar field, is given. In Sec. III, based on the null energy condition of the additional matter fields, while according to the definition of the Wald entropy, whether the entropy of black holes contributed by the nonminimal coupling term between the double dual of the Riemann curvature and the scalar field in general second-order scalar-tensor gravity still satisfies the linearized second law during the perturbation process will be examined. The paper ends with conclusions in Sec. IV.

II. THE GENERAL SECOND-ORDER SCALAR-TENSOR GRAVITATIONAL THEORY AND THE WALD ENTROPY

We consider the $(n+2)$-dimensional general second-order scalar-tensor gravitational theory [27]. The Lagrangian of the gravitational theory is a linear combination of four components. These four components are the Einstein tensor nonminimal coupling with the scalar field, the double dual of the Riemann tensor nonminimal coupling with the scalar field, the Ricci scalar that is the Lagrangian in the Einstein-Hilbert action, and the Gauss-Bonnet combination. To investigate the linearized second law of black hole thermodynamics, the quasistationary accretion process that describes additional matter fields minimally coupling to gravity outside black holes passing through the event horizon and falling into black holes should be considered. The accretion process will perturb the spacetime geometry of black holes. If the perturbation process can be regarded as a complete dynamical process, the Lagrangian of the matter fields should be added to the Lagrangian of the original gravitational theory. Therefore, the specific expression of the Lagrangian can be written as

$$\mathcal{L}_{\text{total}} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_{\text{mt}}, \tag{4}$$

where

$$\begin{aligned}
\mathcal{L}_1 &= \boldsymbol{\epsilon} V_1(\phi) G^{ab} \nabla_a \phi \nabla_b \phi, \\
\mathcal{L}_2 &= \boldsymbol{\epsilon} V_2(\phi) P^{abcd} \nabla_a \phi \nabla_c \phi \nabla_b \nabla_d \phi, \\
\mathcal{L}_3 &= \boldsymbol{\epsilon} V_3(\phi) R, \\
\mathcal{L}_4 &= \boldsymbol{\epsilon} V_4(\phi) \hat{G}.
\end{aligned} \tag{5}$$

For the four components in the Lagrangian, G_{ab} is the Einstein tensor, $P^{abcd} = \frac{1}{4}\epsilon^{abef} R_{efgh} \epsilon^{cdgh}$ is the double dual of the Riemann tensor with the Levi-Civita tensor ϵ_{abcd}, R is the Ricci scalar, $\hat{G} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$ is the Gauss-Bonnet combination, $\boldsymbol{\epsilon}$ is the volume element of the spacetime, and $\mathcal{L}_{\text{mt}}$ represents the Lagrangian of minimal coupling additional matter fields. In the following, to simplify the calculation and the expression as well as make the final result clearer, the values of coefficients $V_1(\phi)$, $V_2(\phi)$, $V_3(\phi)$, and $V_4(\phi)$ in the four components of the Lagrangian are set as $V_1(\phi) = V_2(\phi) = V_3(\phi) = V_4(\phi) = 1$ without losing generality.

It has been shown that the entropy of black holes in any diffeomorphism invariant gravity can be expressed as the Wald entropy that generally obeys the first law of black hole thermodynamics. According to the definition in Eq. (3), the Wald entropy for stationary black holes in the general second-order scalar-tensor gravitational theory can be written formally as a linear combination of four parts that corresponds to the first four components of the Lagrangian, respectively, in Eq. (4), i.e.,

$$
\begin{aligned}
S_{\mathrm{W}} &= S_{\mathrm{W}1} + S_{\mathrm{W}2} + S_{\mathrm{W}3} + S_{\mathrm{W}4} \\
&= \frac{1}{4}\int_s d^n y \sqrt{\gamma}(\rho_{\mathrm{W}1} + \rho_{\mathrm{W}2} + \rho_{\mathrm{W}3} + \rho_{\mathrm{W}4}), \qquad (6)
\end{aligned}
$$

where $\rho_{\mathrm{W}i}$ ($i = 1, 2, 3, 4$) represents the entropy density of the four parts of the Wald entropy. The first component of the Lagrangian $\mathcal{L}_1$ is the nonminimal coupling term that describes the interaction between the Einstein tensor and the scalar field, which is the only one nonminimal coupling interaction term between gravity and the scalar field contained in the Lagrangian of the general Horndeski gravity. In our previous work, we have proved that the Wald entropy in Horndeski gravity always satisfies the linearized second law during the matter fields perturbation process. The third part and the fourth part of the Lagrangian, $\mathcal{L}_3$ and $\mathcal{L}_4$, contain the Ricci scalar term and the Gauss-Bonnet combination. The entropy of black holes contributed by the third part is the Bekenstein-Hawking entropy which obeys the second law of black hole thermodynamics automatically according to the area theorem. The fourth part of the Wald entropy contributed by the fourth component of the Lagrangian does not obey the linearized second law because the entropy of black holes meets the linearized second law during the perturbation process in Gauss-Bonnet gravity and is expressed as the JM entropy. It indicates that even without considering the second part of the Lagrangian, the Wald entropy of black holes in the theory of gravity composed of the first, third, and fourth parts in the Lagrangian is no longer obeying the requirements of the linearized second law. In other words, the Wald entropy of black holes in general second-order scalar-tensor gravity does not satisfy the linearized second law. Therefore, in the following, the expression of the entropy obeying the linearized second law in the general scalar-tensor gravity should be further deduced to investigate the linearized second law in the theory of gravity. Since the entropy contributed by the first, the third, and the fourth components in the Lagrangian from the definition of the Wald entropy has been studied except the entropy comes from the second component of the Lagrangian, we only focus on the second part of the Wald entropy to investigate whether it satisfies the second law under the first-order approximation of the perturbation process. If the second part of the Wald entropy does not meet the linear second law, the second part of entropy, which comes from the second components of the Lagrangian and always obeys the second law under the linear order approximation of the matter fields perturbation, should be further calculated. From Eq. (4), the Lagrangian that is used to study the linearized second law in this gravitational theory can be simplified as

$$
\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{\mathrm{mt}} = \boldsymbol{\epsilon} P^{abcd}\nabla_a\phi\nabla_c\phi\nabla_b\nabla_d\phi + \mathcal{L}_{\mathrm{mt}}. \qquad (7)
$$

From the Lagrangian in Eq. (7), the equation of motion of the gravitational part can be formally written as

$$
H_{ab} = 8\pi T_{ab}. \qquad (8)
$$

The left-hand side of the equation of motion H_{ab} can be further written as a linear combination of $H_{ab}^{(G1)}$, $H_{ab}^{(G2)}$, $H_{ab}^{(\phi 1)}$, and $H_{ab}^{(\phi 2)}$, i.e.,

$$
H_{ab} = H_{ab}^{(G1)} + H_{ab}^{(G2)} + H_{ab}^{(\phi 1)} + H_{ab}^{(\phi 2)}. \qquad (9)
$$

The first two parts of Eq. (9), $H_{ab}^{(G1)}$ and $H_{ab}^{(G2)}$, are derived from the derivative of the Lagrangian with respect to the Riemann curvature R_{abcd}; specifically, the second part $H_{ab}^{(G2)}$ only contains all total derivative terms in the first two parts. The third part of Eq. (9), $H_{ab}^{(\phi 1)}$, comes from the derivative of the Lagrangian with respect to the first-order derivative of the scalar field $\nabla_a\phi$. The fourth part $H_{ab}^{(\phi 2)}$ is deduced from the derivative of the Lagrangian with respect to the second-order derivative of the scalar field $\nabla_a\nabla_b\phi$. The specific expression of the four parts in Eq. (9) can be written, respectively, as

$$
\begin{aligned}
H_{ab}^{(G1)} = &-\frac{1}{4}\nabla_c\phi\nabla_d\nabla_a\phi\nabla_e\phi R_b{}^{cde} + \frac{1}{4}\nabla_a\phi\nabla_d\nabla_c\phi\nabla_e\phi R_b{}^{cde} + \frac{1}{4}\nabla_c\phi\nabla_d\phi\nabla_e\nabla_a\phi R_b{}^{cde} - \frac{1}{4}\nabla_a\phi\nabla_d\phi\nabla_e\nabla_c\phi R_b{}^{cde} \\
&+ \frac{1}{4}g_{ce}\nabla_a\phi\nabla_d\phi\nabla_f\nabla^f\phi R_b{}^{cde} - \frac{1}{4}g_{ae}\nabla_c\phi\nabla_d\phi\nabla_f\nabla^f\phi R_b{}^{cde} - \frac{1}{4}g_{cd}\nabla_a\phi\nabla_e\phi\nabla_f\nabla^f\phi R_b{}^{cde} \\
&+ \frac{1}{4}g_{ad}\nabla_c\phi\nabla_e\phi\nabla_f\nabla^f\phi R_b{}^{cde} - \frac{1}{4}g_{ce}\nabla_a\phi\nabla_d\nabla_f\phi\nabla^f\phi R_b{}^{cde} + \frac{1}{4}g_{ae}\nabla_c\phi\nabla_d\nabla_f\phi\nabla^f\phi R_b{}^{cde} \\
&+ \frac{1}{4}g_{cd}\nabla_a\phi\nabla_e\nabla_f\phi\nabla^f\phi R_b{}^{cde} - \frac{1}{4}g_{ad}\nabla_c\phi\nabla_e\nabla_f\phi\nabla^f\phi R_b{}^{cde} + \frac{1}{2}g_{ce}\nabla_d\nabla_a\phi\nabla_f\phi\nabla^f\phi R_b{}^{cde} \\
&+ \frac{1}{2}g_{ad}\nabla_e\nabla_c\phi\nabla_f\phi\nabla^f\phi R_b{}^{cde} - \frac{1}{2}g_{ce}\nabla_d\phi\nabla_f\nabla_a\phi\nabla^f\phi R_b{}^{cde} + \frac{1}{2}g_{ae}\nabla_d\phi\nabla_f\nabla_c\phi\nabla^f\phi R_b{}^{cde} \\
&- \frac{1}{2}g_{ad}g_{ce}\nabla_f\phi\nabla^f\phi\nabla_g\nabla^g\phi R_b{}^{cde} + \frac{1}{2}g_{ad}g_{ce}\nabla^f\phi\nabla_g\nabla_f\phi\nabla^g\phi R_b{}^{cde}, \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
H_{ab}^{(G2)} = & -\frac{1}{2}\nabla^c\nabla^d(\nabla_c\phi\nabla_b\nabla_a\phi\nabla_d\phi) + \frac{1}{2}\nabla^c\nabla^d(\nabla_a\phi\nabla_b\nabla_c\phi\nabla_d\phi) + \frac{1}{2}\nabla^c\nabla^d(\nabla_c\phi\nabla_b\phi\nabla_d\nabla_a\phi) \\
& -\frac{1}{2}\nabla^c\nabla^d(\nabla_a\phi\nabla_b\phi\nabla_d\nabla_c\phi) + \frac{1}{2}\nabla^c\nabla^d(g_{cd}\nabla_a\phi\nabla_b\phi\nabla_e\nabla^e\phi) - \frac{1}{2}\nabla^c\nabla^d(g_{ad}\nabla_c\phi\nabla_b\phi\nabla_e\nabla^e\phi) \\
& -\frac{1}{2}\nabla^c\nabla^d(g_{cb}\nabla_a\phi\nabla_d\phi\nabla_e\nabla^e\phi) + \frac{1}{2}\nabla^c\nabla^d(g_{ab}\nabla_c\phi\nabla_d\phi\nabla_e\nabla^e\phi) - \frac{1}{2}\nabla^c\nabla^d(g_{cd}\nabla_a\phi\nabla_b\nabla_e\phi\nabla^e\phi) \\
& +\frac{1}{2}\nabla^c\nabla^d(g_{ad}\nabla_c\phi\nabla_b\nabla_e\phi\nabla^e\phi) + \frac{1}{2}\nabla^c\nabla^d(g_{cb}\nabla_a\phi\nabla_d\nabla_e\phi\nabla^e\phi) - \frac{1}{2}\nabla^c\nabla^d(g_{ab}\nabla_c\phi\nabla_d\nabla_e\phi\nabla^e\phi) \\
& +\frac{1}{2}\nabla^c\nabla^d(g_{cd}\nabla_b\nabla_a\phi\nabla_e\phi\nabla^e\phi) - \frac{1}{2}\nabla^c\nabla^d(g_{ad}\nabla_b\nabla_c\phi\nabla_e\phi\nabla^e\phi) - \frac{1}{2}\nabla^c\nabla^d(g_{cb}\nabla_d\nabla_a\phi\nabla_e\phi\nabla^e\phi) \\
& +\frac{1}{2}\nabla^c\nabla^d(g_{ab}\nabla_d\nabla_c\phi\nabla_e\phi\nabla^e\phi) - \frac{1}{2}\nabla^c\nabla^d(g_{cd}\nabla_b\phi\nabla_e\nabla_a\phi\nabla^e\phi) + \frac{1}{2}\nabla^c\nabla^d(g_{cb}\nabla_d\phi\nabla_e\nabla_a\phi\nabla^e\phi) \\
& +\frac{1}{2}\nabla^c\nabla^d(g_{ad}\nabla_b\phi\nabla_e\nabla_c\phi\nabla^e\phi) - \frac{1}{2}\nabla^c\nabla^d(g_{ab}\nabla_d\phi\nabla_e\nabla_c\phi\nabla^e\phi) + \frac{1}{2}\nabla^c\nabla^d(g_{ad}g_{cb}\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi) \\
& -\frac{1}{2}\nabla^c\nabla^d(g_{ab}g_{cd}\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi) - \frac{1}{2}\nabla^c\nabla^d(g_{ad}g_{cb}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi) + \frac{1}{2}\nabla^c\nabla^d(g_{ab}g_{cd}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi), \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
H_{ab}^{(\phi1)} = & \frac{1}{4}R\nabla^c\phi\nabla_a\nabla_c\phi\nabla_b\phi - \frac{1}{2}R_{cd}\nabla^c\phi\nabla_a\nabla^d\phi\nabla_b\phi + \frac{1}{4}R\nabla^c\phi\nabla_c\nabla_a\phi\nabla_b\phi - \frac{1}{2}R\nabla_a\phi\nabla_c\nabla^c\phi\nabla_b\phi \\
& -\frac{1}{2}R_{ad}\nabla^c\phi\nabla_c\nabla^d\phi\nabla_b\phi + R_{ac}\nabla^c\phi\nabla_d\nabla^d\phi\nabla_b\phi - \frac{1}{2}R_{cd}\nabla^c\phi\nabla^d\nabla_a\phi\nabla_b\phi - \frac{1}{2}R_{ad}\nabla^c\phi\nabla^d\nabla_c\phi\nabla_b\phi \\
& + R_{cd}\nabla_a\phi\nabla^d\nabla^c\phi\nabla_b\phi - \frac{1}{2}R_{adce}\nabla^c\phi\nabla^e\nabla^d\phi\nabla_b\phi - \frac{1}{2}R_{aecd}\nabla^c\phi\nabla^e\nabla^d\phi\nabla_b\phi, \qquad (12)
\end{aligned}
$$

and

$$
\begin{aligned}
H_{ab}^{(\phi2)} = & -\frac{1}{2}\nabla_a\phi\nabla^c(R\nabla_c\phi\nabla_b\phi) + \nabla_a\phi\nabla^c(R_{bd}\nabla_c\phi\nabla^d\phi) + \nabla_a\phi\nabla^c(R_{cd}\nabla_b\phi\nabla^d\phi) - \nabla_a\phi\nabla^c(R_{cb}\nabla_d\phi\nabla^d\phi) \\
& +\frac{1}{2}g_{cb}\nabla_a\phi\nabla^c(R\nabla_d\phi\nabla^d\phi) - g_{cb}\nabla_a\phi\nabla^c(R_{de}\nabla^d\phi\nabla^e\phi) + \nabla_a\phi\nabla^c(R_{cdbe}\nabla^d\phi\nabla^e\phi) + \frac{1}{4}\nabla_c\phi\nabla^c(R\nabla_a\phi\nabla_b\phi) \\
& -\frac{1}{2}\nabla_c\phi\nabla^c(R_{bd}\nabla_a\phi\nabla^d\phi) - \frac{1}{2}\nabla_c\phi\nabla^c(R_{ad}\nabla_b\phi\nabla^d\phi) + \frac{1}{2}\nabla_c\phi\nabla^c(R_{ab}\nabla_d\phi\nabla^d\phi) - \frac{1}{4}g_{ab}\nabla_c\phi\nabla^c(R\nabla_d\phi\nabla^d\phi) \\
& +\frac{1}{2}g_{ab}\nabla_c\phi\nabla^c(R_{de}\nabla^d\phi\nabla^e\phi) - \frac{1}{2}\nabla_c\phi\nabla^c(R_{adbe}\nabla^d\phi\nabla^e\phi) + \frac{1}{4}R\nabla_a\phi\nabla_b\phi\nabla_c\nabla^c\phi - \frac{1}{2}R_{bd}\nabla_a\phi\nabla^d\phi\nabla_c\nabla^c\phi \\
& -\frac{1}{2}R_{ad}\nabla_b\phi\nabla^d\phi\nabla_c\nabla^c\phi + \frac{1}{2}R_{ab}\nabla_d\phi\nabla^d\phi\nabla_c\nabla^c\phi - \frac{1}{4}g_{ab}R\nabla_d\phi\nabla^d\phi\nabla_c\nabla^c\phi + \frac{1}{2}g_{ab}R_{de}\nabla^d\phi\nabla^e\phi\nabla_c\nabla^c\phi \\
& -\frac{1}{2}R_{adbe}\nabla^d\phi\nabla^e\phi\nabla_c\nabla^c\phi. \qquad (13)
\end{aligned}
$$

For the right-hand side of Eq. (8), T_{ab} represents the total stress-energy tensor of the gravitational theory. Since the scalar field is involved in the nonminimal coupling terms between the double dual of the Riemann curvature and the scalar field in Eq. (7), the total stress-energy tensor only contains the stress-energy tensor of the additional matter fields, i.e.,

$$T_{ab} = T_{ab}^{\mathrm{mt}}, \qquad (14)$$

where T_{ab}^{mt} is the stress-energy tensor of the additional matter fields. From the physical perspective, a reasonable assumption should be introduced to obtain the entropy of black holes satisfying the linearized second law, which states that the stress-energy tensor of the additional matter fields should obey the null energy condition. For an arbitrary null vector field along the future direction n^a, the expression of the null energy condition can be expressed as $T_{ab}^{\mathrm{mt}}n^a n^b \geq 0$. Moreover, according to Eq. (14), the assumption also means that the total stress-energy tensor in the gravitational theory still obeys the requirement of the null energy condition, i.e.,

$$T_{ab}n^a n^b \geq 0. \qquad (15)$$

According to the definition of the Wald entropy in Eq. (3) and the expression of Eq. (6), the second part of the Wald entropy of black holes contributed by the nonminimal coupling interaction between the double dual of the Riemann curvature and the scalar field in the first term of Eq. (7) can be obtained as

$$S_{\rm W2} = \frac{1}{4}\int_s d^n y\sqrt{\gamma}\rho_{\rm W2}, \tag{16}$$

and the specific expression of the entropy density $\rho_{\rm W2}$ is given as

$$\begin{aligned}\rho_{\rm W2} = & -\frac{1}{2}\nabla_a\phi\nabla^a\phi\nabla_b\nabla^b\phi + \frac{1}{2}\nabla^a\phi\nabla_b\nabla_a\phi\nabla^b\phi - \frac{1}{2}k^a k^b l^c l^d \nabla_a\phi\nabla_b\nabla_d\phi\nabla_c\phi \\ & - k^a l^b \nabla_a\phi\nabla_b\phi\nabla_c\nabla^c\phi + \frac{1}{2}k^a l^b \nabla_a\nabla_c\phi\nabla_b\phi\nabla^c\phi + \frac{1}{2}k^a l^b \nabla_a\phi\nabla_b\nabla_c\phi\nabla^c\phi \\ & - \frac{1}{2}k^a l^b \nabla_a\nabla_b\phi\nabla_c\phi\nabla^c\phi - \frac{1}{2}k^a l^b \nabla_b\nabla_a\phi\nabla_c\phi\nabla^c\phi + \frac{1}{2}k^a l^b \nabla_b\phi\nabla_c\nabla_a\phi\nabla^c\phi \\ & + \frac{1}{2}k^a l^b \nabla_a\phi\nabla_c\nabla_b\phi\nabla^c\phi + \frac{1}{2}k^a k^b l^c l^d \nabla_b\nabla_a\phi\nabla_c\phi\nabla_d\phi - \frac{1}{2}k^a k^b l^c l^d \nabla_a\phi\nabla_c\phi\nabla_d\nabla_b\phi \\ & + \frac{1}{2}k^a k^b l^c l^d \nabla_a\phi\nabla_b\phi\nabla_d\nabla_c\phi. \end{aligned} \tag{17}$$

As mentioned above, to obtain the entropy of black holes that obeys the linearized second law of black hole thermodynamics in the general second-order scalar-tensor tensor gravitational theory, we should consider whether the second part of the Wald entropy still satisfies the linearized second law first. In other words, we only focus on the expression of Eq. (17) to check whether it obeys the second law of black hole thermodynamics under the linear order approximation of the matter fields perturbation in the following. If this expression does not meet the linearized second law, the new expression for the entropy of black holes that comes from the first term in Eq. (7) should be further explored.

III. INVESTIGATION OF THE LINEARIZED SECOND LAW OF BLACK HOLES IN GENERAL SECOND-ORDER SCALAR-TENSOR GRAVITY

The nonminimal coupling term that describes the nonminimal coupling interaction between the double dual of the Riemann curvature and the scalar field exists in the second component of the Lagrangian. According to the definition, this term will substantially contribute to the expression of the Wald entropy. To obtain the expression of the entropy of black holes in the general second-order scalar-tensor gravitational theory that satisfies the linearized second law, we should first examine whether the second part of the Wald entropy contributed by the second component of the Lagrangian still obeys the linearized second law as mentioned above. If it does not meet the linearized second law, we need to investigate further what correction terms will be added to the expression of the second part of the Wald entropy so that the modified second part of the Wald entropy always satisfies the linearized second law. A quasistationary physical accretion process should be introduced to investigate the linearized second law of black hole thermodynamics, which describes that additional matter fields minimally coupling to gravity outside black holes pass through the event horizon and fall into black holes during the dynamical accretion process. In other words, the spacetime geometry of black holes will be perturbed by the accretion process. Besides, another essential assumption should be further introduced to investigate the linearized second law, which states that black holes will settle down to a stationary state after the perturbation process.

For black holes in $(n+2)$-dimensional general second-order scalar-tensor gravitational theory, the event horizon of black holes is denoted as $\mathcal{H}$, which is $(n+1)$-dimensional null hypersurface in spacetime. If the parameter λ is chosen as an affine parameter to parametrize the event horizon, the null vector $k^a = (\partial/\partial\lambda)^a$ can be used to generate the event horizon and obeys the geodesic equation $k^b\nabla_b k^a = 0$. The coordinates with two null vectors $\{k^a, l^a, y_i^a\}$ on any cross section of the event horizon can be further constructed. In the coordinates, the null vector field k^a is tangent to the event horizon of black holes, l^a is another null vector field that is different from the null vector field k^a, and y_i^a is used to describe transverse coordinates on any cross section. The relationships between the two null vectors can be further given as

$$k^a k_a = l^a l_a = 0, \qquad k^a l_a = -1. \tag{18}$$

According to the two null vectors, the binormal on the cross section of the event horizon can be defined as $\epsilon_{ab} = 2k_{[a}l_{b]}$, while the definition of the induced metric on any slice of the future event horizon can be given as

$$\gamma_{ab} = g_{ab} + 2k_{(a}l_{b)}. \tag{19}$$

From the definition of the induced metric, the relationship between the two null vectors and the induced metric can be given as $k^a\gamma_{ab} = l^a\gamma_{ab} = 0$. Besides, for any spatial tensor $X_{a_1a_2\cdots}$, we can define the spatial derivative operator D_a as

$$D_a X_{a_1a_2\cdots} = \gamma_a{}^b \gamma_{a_1}{}^{b_1} \gamma_{a_2\cdots}{}^{b_2\cdots} \nabla_b X_{b_1b_2\cdots}, \tag{20}$$

and the spatial derivative operator is compatible with the induced metric on the cross section of the event horizon, i.e., $D_c\gamma_{ab} = 0$.

The extrinsic curvature of the event horizon can be defined as

$$B_{ab} = \gamma_a{}^c \gamma_b{}^d \nabla_c k_d, \tag{21}$$

and the evolution of the induced metric along the future direction of the event horizon is further given as

$$\gamma_a{}^c \gamma_b{}^d \mathcal{L}_k \gamma_{cd} = 2\left(\sigma_{ab} + \frac{\theta}{n}\gamma_{ab}\right) = 2B_{ab}, \tag{22}$$

where σ_{ab} and θ represent the shear and the expansion of the event horizon, respectively. Furthermore, the evolution of the extrinsic curvature along the same direction can be obtained as

$$\gamma_d{}^c \gamma_b{}^d \mathcal{L}_k B_{cd} = B_{ac} B_b{}^c - \gamma_d{}^c \gamma_b{}^d R_{ecfd} k^e k^f. \tag{23}$$

According to this result, the Raychaudhuri equation can be written as

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{n} - \sigma^{ab}\sigma_{ab} - R_{kk}, \tag{24}$$

where $R_{kk} = R_{ab}k^a k^b$. In the following, we will use the convention $X_{kk} = k^a k^b X_{ab}$ to simplify the expression for any tensor X_{ab} contracting with two null vector fields k^a.

To describe the perturbation for the spacetime geometry of black holes from the additional matter fields, the sufficient small parameter ϵ is introduced to represent the order of the approximation during the perturbation process. We assume that three quantities, i.e., the extrinsic curvature, the expansion, and the shear of the event horizon, are all treated as the quantity under the first-order approximation during the quasistationary perturbation process. Using the small parameter ϵ, the relationship of three quantities can be expressed as $B_{ab} \sim \theta \sim \sigma_{ab} \sim \mathcal{O}(\epsilon)$. In the following, the quantity under the zero-order approximation during the perturbation process is called the zero-order quantity (or the background quantity), and the quantity under the linear order approximation of the matter fields perturbation is called the first-order quantity directly. Besides, two symbols are introduced to simplify the expression of the equation and emphasize that the second law is studied only under the first-order approximation. As above, since we will investigate the linearized second law of black holes in the general second-order scalar-tensor gravity under the process of the matter fields perturbation, the symbol "$\simeq$" is introduced to represent the identity under the linear order approximation of the perturbation process. Besides, the other symbol "$\hat{}$" is introduced to represent the pure spatial index in any tensor to simplify the concrete expression of the following calculation. The extrinsic curvature and the evolution equation of the extrinsic curvature in Eqs. (21) and (23) under the first-order approximation of the perturbation process using the two symbols can be further expressed as

$$B_{\hat{a}\hat{b}} = \nabla_{\hat{a}} k_{\hat{b}}, \qquad \mathcal{L}_k B_{\hat{a}\hat{b}} \simeq -k^c k^d R_{\hat{a}c\hat{b}d}, \tag{25}$$

and the linear version of the Raychaudhuri equation is written as

$$\frac{d\theta}{d\lambda} \simeq -R_{kk}. \tag{26}$$

Analogous to the calculation for the evolution of the external curvature along the future direction of the event horizon, the evolution of the quantity $(\nabla_{\hat{a}} l_{\hat{b}})$ along the same direction on the background spacetime can be obtained as

$$\mathcal{L}_k(\nabla_{\hat{a}} l_{\hat{b}}) = -k^c l^d R_{\hat{a}c\hat{b}d}. \tag{27}$$

Since the additional matter fields are required to satisfy the null energy condition during the perturbation process, the total stress-energy tensor obeys the null energy condition naturally according to Eq. (14). From Eq. (15), while using the above convention, the null energy condition of the total stress-energy tensor in coordinates $\{k^a, l^a, y_i^a\}$ can be rewritten as

$$T_{kk} \geq 0. \tag{28}$$

So far, the two assumptions that black holes will settle down to a stationary state and that the total stress-energy tensor obeys the null energy condition have been introduced. To investigate the linearized second law of black holes in the general second-order scalar-tensor gravity, another assumption that a regular bifurcation surface exists in the background spacetime should also be introduced. The coordinates with two null vectors k^a and l^a can be constructed on any cross section of the Killing horizon as $\{k^a, l^a, y_i^a\}$, and an arbitrary vector $\bar{z}_i^a$ is involved to represent one of the two null vectors, i.e., $\bar{z}_i^a \in \{k^a, l^a\}$. The assumption indicates that when any tensor $X_{a_1\cdots a_k}$ is smooth on the whole Killing horizon $\mathcal{H}$ with bifurcation surface, all indexes of the tensor $X_{a_1\cdots a_k}$ contracting with the vectors $\bar{z}_i^a$, i.e., $X_{a_1\cdots a_k}\bar{z}_1^{a_1}\cdots\bar{z}_k^{a_k}$, is also smooth over the whole Killing horizon on the background spacetime if the number of the null vector k^a is less than or equal to the number of the null vector l^a. This result has been demonstrated in Ref. [28]. Next, we will briefly review the proof process.

For stationary black holes, the event horizon $\mathcal{H}$ can be regarded as the Killing horizon directly, and the bifurcation surface $\mathcal{B}$ exists at the beginning of the Killing horizon. We set ξ^a as a null Killing vector that is tangent to the Killing horizon, and $\mathcal{S}$ represents any cross section on the event horizon. Other coordinates with two Killing vectors, i.e., $\{\xi^a, s^a, y_i^a\}$, can also be constructed on the cross section of the event horizon, in which s^a is another null Killing vector on the event horizon $\mathcal{H}$ which is different from ξ^a, and y_i^a also represent transverse coordinates on any cross section. Two null Killing vectors and transverse coordinates in the coordinates $\{\xi^a, s^a, y_i^a\}$ satisfy the following relationships:

$$\xi^a s_a = -1, \quad \xi^a\xi_a = s^a s_a = 0, \quad y_i^a\xi_a = y_i^a s_a = 0. \tag{29}$$

After introducing any scalar field $\mathcal{C}$ on the cross section $\mathcal{S}$, the two relationships between the two Killing null vectors, ξ^a and s^a, and the other two null vectors, k^a and l^a, can be given as

$$k^a = \mathcal{C}\xi^a, \qquad l^a = \mathcal{C}^{-1}s^a. \tag{30}$$

On the other hand, for any kth-order tensor $X_{a_1\cdots a_k}$, we assume that it is smooth on the Killing horizon and satisfies the condition $\mathcal{L}_\xi X_{a_1\cdots a_k} = 0$. This condition means that the tensor is invariant along the Killing horizon. Since the coordinates $\{\xi^a, s^a, y_i^a\}$ are also invariant along the Killing horizon, and any vector z_i^a that corresponds to one of the two null Killing vectors, i.e., $z_i^a \in \{\xi^a, s^a\}$, the vector z_i^a will obey the condition $\mathcal{L}_\xi z_i^a = 0$. After contracting z_i^a with all indexes in the tensor $X_{a_1\cdots a_k}$, the quantity $X_{a_1\cdots a_k}z_i^{a_1}\cdots z_k^{a_k}$ obeys a similar condition $\mathcal{L}_\xi(X_{a_1\cdots a_k}z_1^{a_1}\cdots z_k^{a_k}) = 0$, which also means that the quantity is invariant along the Killing horizon. When the selected cross section approaches the bifurcation surface $\mathcal{S}\to\mathcal{B}$, the value of the Killing vector ξ^a will tend to zero $\xi^a\to 0$. However, the identity $s^a\xi_a = -1$ in Eq. (29) implies that the value of the Killing null vector s^a is divergent when the Killing vector ξ^a approaches zero. The value of the scalar field $\mathcal{C}$ on the Killing horizon is also divergent $\mathcal{C}\to\infty$ when $\mathcal{S}\to\mathcal{B}$ because the two null vectors k^a and l^a in the coordinates $\{k^a, l^a, y_i^a\}$ are finite on the whole Killing horizon. Besides, when the cross section $\mathcal{S}$ does not approach the bifurcation surface $\mathcal{B}$, the values of the two vectors z_i^a and $\bar{z}_i^a$ are both finite. According to Eq. (30), all indexes in the tensor $X_{a_1\cdots a_k}$ contracted with z_i^a and $\bar{z}_i^a$, respectively, can be represented by the following identity:

$$X_{a_1\cdots a_k}\bar{z}_1^{a_1}\cdots\bar{z}_k^{a_k} = C^{m-n}X_{a_1\cdots a_k}z_1^{a_1}\cdots z_k^{a_k}, \tag{31}$$

where m is the number of the Killing vector ξ^a and n is the number of the Killing vector s^a. Since the vector $\bar{z}_i^a$ is always finite, the quantity $X_{a_1\cdots a_k}\bar{z}_1^{a_1}\cdots\bar{z}_k^{a_k}$ on the left-hand side of Eq. (31) is finite over the whole Killing horizon on the background spacetime. However, since the value of the scalar field $\mathcal{C}$ is divergent $\mathcal{C}\to\infty$ when $\mathcal{S}\to\mathcal{B}$, the value of $X_{a_1\cdots a_k}z_1^{a_1}\cdots z_k^{a_k}$ must be zero to ensure the quantity on the left-hand side of Eq. (31) is finite on the background spacetime when $m > n$. In other words, when $\mathcal{S}\to\mathcal{B}$ and $m > n$, we have

$$X_{a_1\cdots a_k}\bar{z}_1^{a_1}\cdots\bar{z}_k^{a_k} = 0 \tag{32}$$

on the background spacetime. This result means that for any kth-order tensor $X_{a_1\cdots a_k}$, after contracting all its indexes with the vectors $\bar{z}_i^a$, the quantity $X_{a_1\cdots a_k}\bar{z}_1^{a_1}\cdots\bar{z}_k^{a_k}$ should be zero on the background spacetime when the number of k^a is greater than the number of l^a. It indicates that this quantity can be regarded as a first-order quantity. Otherwise, when the number of k^a is less than or equal to the number of l^a, the quantity $X_{a_1\cdots a_k}\bar{z}_1^{a_1}\cdots\bar{z}_k^{a_k}$ can be regarded as a background quantity. In the following calculation, we will directly use this result to judge the order of the relevant quantity during the perturbation process.

According to the above three assumptions, we will examine whether the second part of the Wald entropy S_{W2} always obeys the linearized second law during the matter fields perturbation. If it does not satisfy the linearized second law, we will need to find what kinds of correction terms should be added to the second part of the Wald entropy to make it meet the linearized second law. Following the similar train of thought in Ref. [22], when the additional matter fields always satisfy the null energy condition, if the second part of the Wald entropy of black holes in the general second-order scalar-tensor gravity always obeys the linearized second law during the perturbation process, the second part of the Wald entropy should meet the following relationship under the first-order approximation of the matter fields perturbation:

$$\mathcal{L}_k^2 S_{\mathrm{W2}} \simeq -\frac{1}{4}\int_s \tilde{\epsilon}H_{kk} = -2\pi\int_s \tilde{\epsilon}T_{kk} \le 0, \tag{33}$$

where the equation of motion in Eq. (8) has been used in the second step. The sign of inequality is only contributed by the null energy condition of the additional matter fields, which determines the variation trend of entropy of black holes with the perturbation process. Next, it should be clarified further why the inequality in Eq. (33) reflects the second part of the Wald entropy obeying the linearized second law. Since the assumption states that black holes should settle down to a stationary state after the perturbation process has been introduced, it implies that the rate of change of the second part of the Wald entropy will decrease to zero at the end of the process. In other words, the tendency of the rate of change of the second part can be expressed equivalently as the second-order Lie derivative of the second part of the Wald entropy and is always negative, $\mathcal{L}_k^2 S_{W2} \leq 0$, during the process. On the other hand, the second part of the Wald entropy should always increase during the process to ensure that black holes eventually evolve into a stationary state. It means that the first-order Lie derivative of the second part is positive, i.e., $\mathcal{L}_k S_{W2} \geq 0$. It indicates that the value of S_{W2} monotonously increases with the perturbation process, and the second part of the Wald entropy satisfies the second law. Therefore, according to the above discussion, if the second part of the Wald entropy satisfies the relationship in Eq. (33), it will obey the linearized second law of black hole thermodynamics. In the following, we will only focus on whether the second part of the Wald entropy satisfies Eq. (33) to check whether it obeys the second law under the first-order approximation of the matter fields perturbation.

According to Eq. (33), to investigate whether the second part of the Wald entropy satisfies the linearized second law, we should first calculate the specific expression of H_{kk} under the first-order approximation of the perturbation process. After contracting two null vectors k^a and using the convenience as above, Eq. (9) can be rewritten as

$$H_{kk} = H_{kk}^{(G1)} + H_{kk}^{(G2)} + H_{kk}^{(\phi 1)} + H_{kk}^{(\phi 2)}, \tag{34}$$

and the concrete expression of each part in H_{kk} is given as

$$\begin{aligned}
H_{kk}^{(G1)} = &-\frac{1}{4}k^a k^b \nabla^c\phi\nabla^d\nabla_a\phi\nabla^e\phi R_{bcde} + \frac{1}{4}k^a k^b \nabla_a\phi\nabla^d\nabla^c\phi\nabla^e\phi R_{bcde} + \frac{1}{4}k^a k^b \nabla^c\phi\nabla^d\phi\nabla^e\nabla_a\phi R_{bcde} \\
&-\frac{1}{4}k^a k^b \nabla_a\phi\nabla^d\phi\nabla^e\nabla^c\phi R_{bcde} + \frac{1}{4}k^a k^b \nabla_a\phi\nabla^d\phi\nabla_f\nabla^f\phi R_{bd} - \frac{1}{4}k^a k^b \nabla^c\phi\nabla^d\phi\nabla_f\nabla^f\phi R_{bcda} \\
&+\frac{1}{4}k^a k^b \nabla_a\phi\nabla^c\phi\nabla_d\nabla^d\phi R_{bc} + \frac{1}{4}k^a k^b \nabla^c\phi\nabla^e\phi\nabla_f\nabla^f\phi R_{bcae} - \frac{1}{4}k^a k^b \nabla_a\phi\nabla^c\nabla_d\phi\nabla^d\phi R_{bc} \\
&+\frac{1}{4}k^a k^b \nabla^c\phi\nabla^d\nabla_e\phi\nabla^e\phi R_{bcda} - \frac{1}{4}k^a k^b \nabla_a\phi\nabla^c\nabla^d\phi\nabla_d\phi R_{bc} - \frac{1}{4}k^a k^b \nabla^c\phi\nabla^d\nabla^e\phi\nabla_e\phi R_{bcad} \\
&+\frac{1}{4}k^a k^b \nabla^d\nabla_a\phi\nabla_f\phi\nabla^f\phi R_{bd} - \frac{1}{4}k^a k^b \nabla^d\nabla^c\phi\nabla_f\phi\nabla^f\phi R_{bcda} + \frac{1}{4}k^a k^b \nabla^e\nabla_a\phi\nabla_f\phi\nabla^f\phi R_{be} \\
&+\frac{1}{4}k^a k^b \nabla^e\nabla^c\phi\nabla_f\phi\nabla^f\phi R_{bcae} - \frac{1}{4}k^a k^b \nabla^d\phi\nabla_f\nabla_a\phi\nabla^f\phi R_{bd} - \frac{1}{4}k^a k^b \nabla^e\phi\nabla_f\nabla^a\phi\nabla^f\phi R_{be} \\
&+\frac{1}{4}k^a k^b \nabla^d\phi\nabla_f\nabla^c\phi\nabla^f\phi R_{bcda} - \frac{1}{4}k^a k^b \nabla^e\phi\nabla_f\nabla^c\phi\nabla^f\phi R_{bcae} - \frac{1}{4}k^a k^b \nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi R_{ab} \\
&-\frac{1}{4}k^a k^b \nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi R_{ab} + \frac{1}{4}k^a k^b \nabla^f\phi\nabla_g\nabla_f\phi\nabla^g\phi R_{ab} + \frac{1}{4}k^a k^b \nabla^f\phi\nabla_g\nabla_f\phi\nabla^g\phi R_{ab},
\end{aligned} \tag{35}$$

$$\begin{aligned}
H_{kk}^{(G2)} = &-\frac{1}{2}k^a k^b \nabla^c\nabla^d(\nabla_c\phi\nabla_b\nabla_a\phi\nabla_d\phi) + \frac{1}{2}k^a k^b \nabla^c\nabla^d(\nabla_a\phi\nabla_b\nabla_c\phi\nabla_d\phi) + \frac{1}{2}k^a k^b \nabla^c\nabla^d(\nabla_c\phi\nabla_b\phi\nabla_d\nabla_a\phi) \\
&-\frac{1}{2}k^a k^b \nabla^c\nabla^d(\nabla_a\phi\nabla_b\phi\nabla_d\nabla_c\phi) + \frac{1}{2}k^a k^b \nabla^c\nabla^d(g_{cd}\nabla_a\phi\nabla_b\phi\nabla_e\nabla^e\phi) - \frac{1}{2}k^a k^b \nabla^c\nabla^d(g_{ad}\nabla_c\phi\nabla_b\phi\nabla_e\nabla^e\phi) \\
&-\frac{1}{2}k^a k^b \nabla^c\nabla^d(g_{cb}\nabla_a\phi\nabla_d\phi\nabla_e\nabla^e\phi) - \frac{1}{2}k^a k^b \nabla^c\nabla^d(g_{cd}\nabla_a\phi\nabla_b\nabla_e\phi\nabla^e\phi) + \frac{1}{2}k^a k^b \nabla^c\nabla^d(g_{ad}\nabla_c\phi\nabla_b\nabla_e\phi\nabla^e\phi) \\
&+\frac{1}{2}k^a k^b \nabla^c\nabla^d(g_{cb}\nabla_a\phi\nabla_d\nabla_e\phi\nabla^e\phi) + \frac{1}{2}k^a k^b \nabla^c\nabla^d(g_{cd}\nabla_b\nabla_a\phi\nabla_e\phi\nabla^e\phi) - \frac{1}{2}k^a k^b \nabla^c\nabla^d(g_{ad}\nabla_b\nabla_c\phi\nabla_e\phi\nabla^e\phi) \\
&-\frac{1}{2}k^a k^b \nabla^c\nabla^d(g_{cb}\nabla_d\nabla_a\phi\nabla_e\phi\nabla^e\phi) - \frac{1}{2}k^a k^b \nabla^c\nabla^d(g_{cd}\nabla_b\phi\nabla_e\nabla_a\phi\nabla^e\phi) + \frac{1}{2}k^a k^b \nabla^c\nabla^d(g_{cb}\nabla_d\phi\nabla_e\nabla_a\phi\nabla^e\phi) \\
&+\frac{1}{2}k^a k^b \nabla^c\nabla^d(g_{ad}\nabla_b\phi\nabla_e\nabla_c\phi\nabla^e\phi) + \frac{1}{2}k^a k^b \nabla^c\nabla^d(g_{ad}g_{cb}\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi) \\
&-\frac{1}{2}k^a k^b \nabla^c\nabla^d(g_{ad}g_{cb}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi),
\end{aligned} \tag{36}$$

$$H_{kk}^{(\phi 1)} = \frac{1}{4}k^a k^b R \nabla^c\phi \nabla_a \nabla_c \phi \nabla_b \phi - \frac{1}{2}k^a k^b R_{cd}\nabla^c\phi\nabla_a\nabla^d\phi\nabla_b\phi + \frac{1}{4}k^a k^b R\nabla^c\phi\nabla_c\nabla_a\phi\nabla_b\phi - \frac{1}{2}k^a k^b R\nabla_a\phi\nabla_c\nabla^c\phi\nabla_b\phi$$
$$- k^a k^b R_{ad}\nabla^c\phi\nabla_c\nabla^d\phi\nabla_b\phi + k^a k^b R_{ac}\nabla^c\phi\nabla_d\nabla^d\phi\nabla_b\phi - \frac{1}{2}k^a k^b R_{cd}\nabla^c\phi\nabla^d\nabla_a\phi\nabla_b\phi + k^a k^b R_{cd}\nabla_a\phi\nabla^d\nabla^c\phi\nabla_b\phi$$
$$- \frac{1}{2}k^a k^b R_{adce}\nabla^c\phi\nabla^e\nabla^d\phi\nabla_b\phi - \frac{1}{2}k^a k^b R_{aecd}\nabla^c\phi\nabla^e\nabla^d\phi\nabla_b\phi, \tag{37}$$

and

$$H_{kk}^{(\phi 2)} = -\frac{1}{2}k^a k^b\nabla_a\phi\nabla^c(R\nabla_c\phi\nabla_b\phi) + k^a k^b\nabla_a\phi\nabla^c(R_{bd}\nabla_c\phi\nabla^d\phi) + k^a k^b\nabla_a\phi\nabla^c(R_{cd}\nabla_b\phi\nabla^d\phi)$$
$$- k^a k^b\nabla_a\phi\nabla^c(R_{cb}\nabla_d\phi\nabla^d\phi) + \frac{1}{2}k^a k^b\nabla_a\phi\nabla_b(R\nabla_d\phi\nabla^d\phi) - k^a k^b\nabla_a\phi\nabla_b(R_{de}\nabla^d\phi\nabla^e\phi)$$
$$- k^a k^b\nabla_a\phi\nabla^c(R_{cdbe}\nabla^d\phi\nabla^e\phi) + \frac{1}{4}k^a k^b\nabla_c\phi\nabla^c(R\nabla_a\phi\nabla_b\phi) - \frac{1}{2}k^a k^b\nabla_c\phi\nabla^c(R_{bd}\nabla_a\phi\nabla^d\phi)$$
$$- \frac{1}{2}k^a k^b\nabla_c\phi\nabla^c(R_{ad}\nabla_b\phi\nabla^d\phi) + \frac{1}{2}k^a k^b\nabla_c\phi\nabla^c(R_{ab}\nabla_d\phi\nabla^d\phi) - \frac{1}{2}k^a k^b\nabla_c\phi\nabla^c(R_{adbe}\nabla^d\phi\nabla^e\phi)$$
$$+ \frac{1}{4}k^a k^b R\nabla_a\phi\nabla_b\phi\nabla_c\nabla^c\phi - \frac{1}{2}k^a k^b R_{bd}\nabla_a\phi\nabla^d\phi\nabla_c\nabla^c\phi - \frac{1}{2}k^a k^b R_{ad}\nabla_b\phi\nabla^d\phi\nabla_c\nabla^c\phi$$
$$+ \frac{1}{2}k^a k^b R_{ab}\nabla_d\phi\nabla^d\phi\nabla_c\nabla^c\phi - \frac{1}{2}k^a k^b R_{adbe}\nabla^d\phi\nabla^e\phi\nabla_c\nabla^c\phi. \tag{38}$$

According to Appendix A, for the convenience of calculations, the expression of $H_{kk}^{(G1)}$ under the first-order approximation of the matter fields perturbation can finally be simplified as two parts,

$$H_{kk}^{(G1)} \simeq H_{kk}^{(G1)1} + H_{kk}^{(G1)2}, \tag{39}$$

where the first part is

$$H_{kk}^{(G1)1} = \frac{1}{2}(D^{\hat{e}}\phi)(k^a l^d\nabla_d\nabla_a\phi)(D^{\hat{f}}\phi)(k^b k^c R_{b\hat{e}c\hat{f}}) + \frac{1}{2}(D^{\hat{e}}\phi D^{\hat{f}}\phi\nabla_m\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}})$$
$$- \frac{1}{2}(D^{\hat{e}}\phi\nabla^{\hat{f}}\nabla_m\phi\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}}) - \frac{1}{2}(k^a l^d\nabla_d\nabla_a\phi\nabla_m\phi\nabla^m\phi)(R_{kk})$$
$$+ \frac{1}{2}(D^{\hat{f}}D^{\hat{e}}\phi\nabla_m\phi\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}}) - \frac{1}{2}(D^{\hat{f}}\phi\nabla_m\nabla^{\hat{e}}\phi\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}})$$
$$- \frac{1}{2}(\nabla_m\phi\nabla^m\phi\nabla_n\nabla^n\phi)(R_{kk}) + \frac{1}{2}(\nabla^m\phi\nabla_n\nabla_m\phi\nabla^n\phi)(R_{kk}), \tag{40}$$

and the second part is

$$H_{kk}^{(G1)2} = \frac{1}{2}(D^{\hat{e}}\phi)(\mathcal{L}_k^2\phi)(D^{\hat{f}}\phi)(k^c l^d R_{c\hat{e}d\hat{f}}) - \frac{1}{2}(\mathcal{L}_k^2\phi)(D^{\hat{e}}\phi D_{\hat{e}}\phi)(k^c l^d R_{cd}). \tag{41}$$

For the expression of $H_{kk}^{(G2)}$ in Eq. (36), this part only contains all total derivative terms that come from the derivation of the Lagrangian with respect to the Riemann curvature. However, we cannot directly calculate the result from the original expression of $H_{kk}^{(G2)}$ because this expression is surprisingly complex after expanding all total derivative terms. Therefore, a useful identity should be further introduced to simplify the calculative process to deduce the final expression of $H_{kk}^{(G2)}$ under the linear order approximation of the perturbation process. Considering any two-form tensor X^{ab}, one can demonstrate that the identity can be written as

$$\int_s \tilde{\epsilon}k_b\nabla_a X^{ab} = \mathcal{L}_k\int_s \tilde{\epsilon}k_a l_b X^{ab}. \tag{42}$$

Using the identity of Eq. (42) and Leibniz's law, the expression of $H_{kk}^{(G2)}$ can be decomposed into four parts, which can be formally written as

$$\int_s \tilde{\epsilon} H_{kk}^{(G2)} = -\mathcal{L}_k^2 \int_s \tilde{\epsilon} H_{kk}^{(G2)1} - \int_s \tilde{\epsilon} H_{kk}^{(G2)2} + \mathcal{L}_k \int_s \tilde{\epsilon} H_{kk}^{(G2)3} + \mathcal{L}_k \int_s \tilde{\epsilon} H_{kk}^{(G2)4}. \tag{43}$$

The first part of Eq. (43) is given as

$$\begin{aligned}
-\mathcal{L}_k^2 \int_s \tilde{\epsilon} H_{kk}^{(G2)1} = & -\mathcal{L}_k^2 \int_s \tilde{\epsilon} \Big(-\frac{1}{2}\nabla_a\phi\nabla^a\phi\nabla_b\nabla^b\phi + \frac{1}{2}\nabla^a\phi\nabla_b\nabla_a\phi\nabla^b\phi - \frac{1}{2}k^a k^b l^c l^d \nabla_a\phi\nabla_b\nabla_d\phi\nabla_c\phi - k^a l^b \nabla_a\phi\nabla_b\phi\nabla_c\nabla^c\phi \\
& + \frac{1}{2}k^a l^b \nabla_a\nabla_c\phi\nabla_b\phi\nabla^c\phi + \frac{1}{2}k^a l^b \nabla_a\phi\nabla_b\nabla_c\phi\nabla^c\phi - \frac{1}{2}k^a l^b \nabla_a\nabla_b\phi\nabla_c\phi\nabla^c\phi - \frac{1}{2}k^a l^b \nabla_b\nabla_a\phi\nabla_c\phi\nabla^c\phi \\
& + \frac{1}{2}k^a l^b \nabla_b\phi\nabla_c\nabla_a\phi\nabla^c\phi + \frac{1}{2}k^a l^b \nabla_a\phi\nabla_c\nabla_b\phi\nabla^c\phi + \frac{1}{2}k^a k^b l^c l^d \nabla_b\nabla_a\phi\nabla_c\phi\nabla_d\phi \\
& - \frac{1}{2}k^a k^b l^c l^d \nabla_a\phi\nabla_c\phi\nabla_d\nabla_b\phi + \frac{1}{2}k^a k^b l^c l^d \nabla_a\phi\nabla_b\phi\nabla_d\nabla_c\phi \Big) \\
= & -\mathcal{L}_k^2 \int_s \tilde{\epsilon} \rho_{\mathrm{W2}},
\end{aligned} \tag{44}$$

where the entropy density of the second part of the Wald entropy in Eq. (17) has been used in the last step.

The second part on the right-hand side of Eq. (43) can be written as

$$\begin{aligned}
-\int_s \tilde{\epsilon} H_{kk}^{(G2)2} = & -\frac{1}{2}\int_s \tilde{\epsilon}(k^a\nabla^c k^b)\nabla^d(-\nabla_c\phi\nabla_b\nabla_a\phi\nabla_d\phi + \nabla_a\phi\nabla_b\nabla_c\phi\nabla_d\phi + \nabla_c\phi\nabla_b\phi\nabla_d\nabla_a\phi - \nabla_a\phi\nabla_b\phi\nabla_d\nabla_c\phi \\
& + g_{cd}\nabla_a\phi\nabla_b\phi\nabla_e\nabla^e\phi - g_{ad}\nabla_c\phi\nabla_b\phi\nabla_e\nabla^e\phi - g_{cb}\nabla_a\phi\nabla_d\phi\nabla_e\nabla^e\phi + g_{ab}\nabla_c\phi\nabla_d\phi\nabla_e\nabla^e\phi \\
& - g_{cd}\nabla_a\phi\nabla_b\phi\nabla_e\nabla^e\phi + g_{ad}\nabla_c\phi\nabla_b\nabla_e\phi\nabla^e\phi + g_{cb}\nabla_a\phi\nabla_d\nabla_e\phi\nabla^e\phi - g_{ab}\nabla_c\phi\nabla_d\nabla_e\phi\nabla^e\phi \\
& + g_{cd}\nabla_b\nabla_a\phi\nabla_e\phi\nabla^e\phi - g_{ad}\nabla_b\nabla_c\phi\nabla_e\phi\nabla^e\phi - g_{cb}\nabla_d\nabla_a\phi\nabla_e\phi\nabla^e\phi + g_{ac}\nabla_d\nabla_b\phi\nabla_e\phi\nabla^e\phi \\
& - g_{cd}\nabla_b\phi\nabla_e\nabla_a\phi\nabla^e\phi + g_{cb}\nabla_d\phi\nabla_e\nabla_a\phi\nabla^e\phi + g_{ad}\nabla_b\phi\nabla_e\nabla_c\phi\nabla^e\phi - g_{ab}\nabla_d\phi\nabla_e\nabla_c\phi\nabla^e\phi \\
& + g_{ad}g_{cb}\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi - g_{ad}g_{cb}\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi - g_{ad}g_{cb}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi + g_{ab}g_{cd}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi).
\end{aligned} \tag{45}$$

According to the result in the first part of Appendix B, the integrand is equal to zero under the first-order approximation of the matter fields perturbation. Therefore, the second-part on the right-hand side of Eq. (43) does not contribute to the final result of $H_{kk}^{(G2)}$ under the linear order approximation, i.e.,

$$\int_s \tilde{\epsilon} H_{kk}^{(G2)2} \simeq 0. \tag{46}$$

The specific expression of the integrand in the third part on the right-hand side of Eq. (43) is given as

$$\begin{aligned}
H_{kk}^{(G2)3} = & \frac{1}{2}(k^b\nabla^d l^a)k^c(-\nabla_a\phi\nabla_b\nabla_c\phi\nabla_d\phi + \nabla_c\phi\nabla_b\nabla_a\phi\nabla_d\phi + \nabla_a\phi\nabla_b\phi\nabla_d\nabla_c\phi \\
& - \nabla_c\phi\nabla_b\phi\nabla_d\nabla_a\phi + g_{ad}\nabla_c\phi\nabla_b\phi\nabla_e\nabla^e\phi - g_{cd}\nabla_a\phi\nabla_b\phi\nabla_e\nabla^e\phi \\
& - g_{ab}\nabla_c\phi\nabla_d\phi\nabla_e\nabla^e\phi + g_{cb}\nabla_a\phi\nabla_d\phi\nabla_e\nabla^e\phi - g_{ad}\nabla_c\phi\nabla_b\nabla_e\phi\nabla^e\phi \\
& + g_{cd}\nabla_a\phi\nabla_b\nabla_e\phi\nabla^e\phi + g_{ab}\nabla_c\phi\nabla_d\nabla_e\phi\nabla^e\phi - g_{cb}\nabla_a\phi\nabla_d\nabla_e\phi\nabla^e\phi \\
& + g_{ad}\nabla_b\nabla_c\phi\nabla_e\phi\nabla^e\phi - g_{cd}\nabla_b\nabla_a\phi\nabla_e\phi\nabla^e\phi - g_{ab}\nabla_d\nabla_c\phi\nabla_e\phi\nabla^e\phi \\
& + g_{cb}\nabla_d\nabla_a\phi\nabla_e\phi\nabla^e\phi - g_{ad}\nabla_b\phi\nabla_e\nabla_c\phi\nabla^e\phi + g_{ab}\nabla_d\phi\nabla_e\nabla_c\phi\nabla^e\phi \\
& + g_{cd}\nabla_b\phi\nabla_e\nabla_a\phi\nabla^e\phi - g_{cb}\nabla_d\phi\nabla_e\nabla_a\phi\nabla^e\phi + g_{cd}g_{ab}\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi \\
& - g_{cb}g_{ad}\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi - g_{cd}g_{ab}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi + g_{cb}g_{ad}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi).
\end{aligned} \tag{47}$$

According to the result in the second part of Appendix B, the expression of Eq. (47) under the linear order approximation of the perturbation process can finally be simplified as

$$H_{kk}^{(G2)3} \simeq -\frac{1}{2}(\mathcal{L}_k^2\phi)(D_{\hat{c}}\phi)(D_{\hat{d}}\phi)(\nabla^{\hat{d}}l^{\hat{c}}) + \frac{1}{2}(\mathcal{L}_k^2\phi)(D_{\hat{e}}\phi D^{\hat{e}}\phi)(\nabla^{\hat{c}}l_{\hat{c}}). \tag{48}$$

The specific expression of the integrand in the fourth part on the right-hand side of Eq. (43) is expressed as

$$\begin{aligned} H_{kk}^{(G2)4} = \frac{1}{2}(k^b\nabla^d k^c)l^a(&-\nabla_a\phi\nabla_b\nabla_c\phi\nabla_d\phi + \nabla_c\phi\nabla_b\nabla_a\phi\nabla_d\phi + \nabla_a\phi\nabla_b\phi\nabla_d\nabla_c\phi \\ &- \nabla_c\phi\nabla_b\phi\nabla_d\nabla_a\phi + g_{ad}\nabla_c\phi\nabla_b\phi\nabla_e\nabla^e\phi - g_{cd}\nabla_a\phi\nabla_b\phi\nabla_e\nabla^e\phi \\ &- g_{ab}\nabla_c\phi\nabla_d\phi\nabla_e\nabla^e\phi + g_{cb}\nabla_a\phi\nabla_d\phi\nabla_e\nabla^e\phi - g_{ad}\nabla_c\phi\nabla_b\nabla_e\phi\nabla^e\phi \\ &+ g_{cd}\nabla_a\phi\nabla_b\nabla_e\phi\nabla^e\phi + g_{ab}\nabla_c\phi\nabla_d\nabla_e\phi\nabla^e\phi - g_{cb}\nabla_a\phi\nabla_d\nabla_e\phi\nabla^e\phi \\ &+ g_{ad}\nabla_b\nabla_c\phi\nabla_e\phi\nabla^e\phi - g_{cd}\nabla_b\nabla_a\phi\nabla_e\phi\nabla^e\phi - g_{ab}\nabla_d\nabla_c\phi\nabla_e\phi\nabla^e\phi \\ &+ g_{cb}\nabla_d\nabla_a\phi\nabla_e\phi\nabla^e\phi - g_{ad}\nabla_b\phi\nabla_e\nabla_c\phi\nabla^e\phi + g_{ab}\nabla_d\phi\nabla_e\nabla_c\phi\nabla^e\phi \\ &+ g_{cd}\nabla_b\phi\nabla_e\nabla_a\phi\nabla^e\phi - g_{cb}\nabla_d\phi\nabla_e\nabla_a\phi\nabla^e\phi + g_{cd}g_{ab}\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi \\ &- g_{cb}g_{ad}\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi - g_{cd}g_{ab}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi + g_{cb}g_{ad}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi). \end{aligned} \tag{49}$$

According to the final result of the third part in Appendix B, the remainder terms of $H_{kk}^{(G2)4}$ under the first-order approximation are given as

$$\begin{aligned} H_{kk}^{(G2)4} \simeq &\frac{1}{2}(k^a l^b\nabla_a\nabla_b\phi)(D_{\hat{e}}\phi D_{\hat{f}}\phi)(B^{\hat{e}\hat{f}}) + \frac{1}{2}(D_{\hat{e}}\phi D_{\hat{f}}\phi)(\nabla_m\nabla^m\phi)(B^{\hat{e}\hat{f}}) \\ &- \frac{1}{2}(D_{\hat{e}}\phi\nabla_{\hat{f}}\nabla_m\phi\nabla^m\phi)(B^{\hat{e}\hat{f}}) - \frac{1}{2}(k^a l^b\nabla_a\nabla_b\phi)(\nabla_m\phi\nabla^m\phi)(B^{\hat{e}}{}_{\hat{e}}) \\ &+ \frac{1}{2}(D_{\hat{f}}D_{\hat{e}}\phi\nabla_m\phi\nabla^m\phi)(B^{\hat{e}\hat{f}}) - \frac{1}{2}(D_{\hat{f}}\phi\nabla_m\nabla_{\hat{e}}\phi\nabla^m\phi)(B^{\hat{e}\hat{f}}) \\ &- \frac{1}{2}(\nabla^m\phi\nabla_m\phi\nabla_n\nabla^n\phi)(B^{\hat{e}}{}_{\hat{e}}) + \frac{1}{2}(\nabla^m\phi\nabla_n\nabla_m\phi\nabla^n\phi)(B^{\hat{e}}{}_{\hat{e}}). \end{aligned} \tag{50}$$

Furthermore, the evolution of the determinant of the induced metric on any cross section of the event horizon along the future direction can be calculated as

$$\mathcal{L}_k\sqrt{\gamma} = \theta\sqrt{\gamma} \sim \mathcal{O}(\epsilon). \tag{51}$$

It is shown that the evolution of the induced metric is a first-order quantity based on the assumption that the expansion does not exist in the background spacetime. From Eq. (50), utilizing Leibniz's law, while using the second identity in Eqs. (25) and (51), the expression of the fourth part on the right-hand side of Eq. (43) under the first-order approximation can be further calculated as

$$\begin{aligned} \mathcal{L}_k\int_s \tilde{\epsilon}H_{kk}^{(G2)4} &= \int_s \tilde{\epsilon}\theta H_{kk}^{(G2)4} + \int_s \tilde{\epsilon}\mathcal{L}_k H_{kk}^{(G2)4} \simeq \int_s \tilde{\epsilon}\mathcal{L}_k H_{kk}^{(G2)4} \\ &\simeq \int_s \tilde{\epsilon}\Big[-\frac{1}{2}(k^a l^b\nabla_a\nabla_b\phi)(D^{\hat{e}}\phi D^{\hat{f}}\phi)(k^b k^c R_{b\hat{e}c\hat{f}}) - \frac{1}{2}(D^{\hat{e}}\phi D^{\hat{f}}\phi)(\nabla_m\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}}) \\ &\quad + \frac{1}{2}(D^{\hat{e}}\phi D^{\hat{f}}\nabla_m\phi\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}}) + \frac{1}{2}(k^a l^b\nabla_a\nabla_b\phi)(\nabla^m\phi\nabla_m\phi)(R_{kk}) \\ &\quad - \frac{1}{2}(D_{\hat{f}}D_{\hat{e}}\phi\nabla_m\phi\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}}) + \frac{1}{2}(D^{\hat{f}}\phi\nabla_m\nabla^{\hat{e}}\phi\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}}) \\ &\quad + \frac{1}{2}(\nabla_m\phi\nabla^m\phi\nabla_n\nabla^n\phi)(R_{kk}) - \frac{1}{2}(\nabla^m\phi\nabla_n\nabla_m\phi\nabla^n\phi)(R_{kk})\Big]. \end{aligned} \tag{52}$$

Integrating Eq. (40) on the cross section of the event horizon and combining this integral with Eq. (52), one can see that these two terms will cancel each other out,

$$\int_s \tilde{\epsilon} H_{kk}^{(G1)1} + \mathcal{L}_k \int_s \tilde{\epsilon} H_{kk}^{(G2)4} \simeq 0. \tag{53}$$

Besides, from the expressions of Eqs. (41) and (48), utilizing Eqs. (27) and (51), the integral form of the second part in Eq. (39) on the cross section and the third part of Eq. (43) can be simplified as

$$\begin{aligned}\int_s \tilde{\epsilon} H_{kk}^{(G1)2} + \mathcal{L}_k \int_s \tilde{\epsilon} H_{kk}^{(G2)3} \simeq& \int_s \tilde{\epsilon}\bigg\{\frac{1}{2}(D^{\hat{e}}\phi)(\mathcal{L}_k^2\phi)(D^{\hat{f}}\phi)(k^c l^d R_{c\hat{e}d\hat{f}}) - \frac{1}{2}\mathcal{L}_k[(\mathcal{L}_k^2\phi)(D_{\hat{c}}\phi)(D_{\hat{d}}\phi)(\nabla^{\hat{d}} l^{\hat{c}})] \\ &- \frac{1}{2}(\mathcal{L}_k^2\phi)(D^{\hat{e}}\phi D_{\hat{e}}\phi)(k^c l^d R_{cd}) + \frac{1}{2}\mathcal{L}_k[(\mathcal{L}_k^2\phi)(D_{\hat{e}}\phi\nabla^{\hat{e}}\phi)(\nabla^{\hat{c}} l_{\hat{c}})]\bigg\} \\ \simeq& -\mathcal{L}_k^2 \int_s \tilde{\epsilon}\left[\frac{1}{2}(\mathcal{L}_k\phi)(D^{\hat{e}}\phi)(D^{\hat{f}}\phi)(\nabla_{\hat{f}} l_{\hat{e}}) - \frac{1}{2}(\mathcal{L}_k\phi)(D^{\hat{e}}\phi D_{\hat{e}}\phi)(\nabla^{\hat{c}} l_{\hat{c}})\right].\end{aligned} \tag{54}$$

Therefore, the integral form on any slice of the event horizon of the first two terms in Eq. (34) under the first-order approximation of the matter fields perturbation can finally be written as

$$\int_s \tilde{\epsilon}(H_{kk}^{(G1)} + H_{kk}^{(G2)}) \simeq -\mathcal{L}_k^2 \int_s \tilde{\epsilon}\left[\rho_W + \frac{1}{2}(\mathcal{L}_k\phi)(D^{\hat{e}}\phi)(D^{\hat{f}}\phi)(\nabla_{\hat{f}} l_{\hat{e}}) - \frac{1}{2}(\mathcal{L}_k\phi)(D^{\hat{e}}\phi D_{\hat{e}}\phi)(\nabla^{\hat{c}} l_{\hat{c}})\right]. \tag{55}$$

According to the final result in Appendix C, Eq. (37) is equal to zero under the first-order approximation of the matter fields perturbation, i.e.,

$$H_{kk}^{(\phi 1)} \simeq 0. \tag{56}$$

It means that the third part on the right-hand side of Eq. (34) does not affect whether the Wald entropy satisfies the linearized second law.

According to Appendix D, Eq. (38) under the linear order approximation can finally be written as

$$\begin{aligned}H_{kk}^{(\phi 2)} \simeq& \frac{1}{2}D_{\hat{f}}(R_{kk})(D^{\hat{f}}\phi)(\nabla_e\phi\nabla^e\phi) + \frac{1}{2}(R_{kk})(D^{\hat{f}}\phi)D_{\hat{f}}(\nabla_e\phi\nabla^e\phi) \\ &+ \frac{1}{2}(R_{kk})(\nabla_e\phi\nabla^e\phi)(D^{\hat{c}}D_{\hat{c}}\phi) - \frac{1}{2}(D^{\hat{c}}\phi)D_{\hat{c}}(k^a k^b R_{aebf}\nabla^e\phi\nabla^f\phi) \\ &- \frac{1}{2}(k^a k^b R_{aebf}\nabla^e\phi\nabla^f\phi)(D^{\hat{c}}D_{\hat{c}}\phi) - (R_{kk})(\nabla_e\phi\nabla^e\phi)(k^a l^d\nabla_a\nabla_d\phi) \\ &- \frac{1}{2}(\mathcal{L}_k R_{kk})(\nabla_e\phi\nabla^e\phi)(l^d\nabla_d\phi) + (k^a l^d\nabla_a\nabla_d\phi)(D^{\hat{g}}\phi)(D^{\hat{h}}\phi)(k^b k^c R_{b\hat{g}c\hat{h}}) \\ &+ \frac{1}{2}(k^a k^b k^c\nabla_c R_{a\hat{g}b\hat{h}})(l^d\nabla_d\phi)(D^{\hat{g}}\phi)(D^{\hat{h}}\phi).\end{aligned} \tag{57}$$

The first three terms of Eq. (57) can be further simplified as

$$\begin{aligned}&\frac{1}{2}D_{\hat{f}}(R_{kk})(D^{\hat{f}}\phi)(\nabla_e\phi\nabla^e\phi) + \frac{1}{2}(R_{kk})(D^{\hat{f}}\phi)D_{\hat{f}}(\nabla_e\phi\nabla^e\phi) + \frac{1}{2}(R_{kk})(\nabla_e\phi\nabla^e\phi)(D^{\hat{c}}D_{\hat{c}}\phi) \\ &= \frac{1}{2}D_{\hat{f}}[R_{kk}(\nabla^e\phi\nabla_e\phi)(D^{\hat{f}}\phi)].\end{aligned} \tag{58}$$

The fourth and fifth terms in Eq. (57) are calculated as

$$-\frac{1}{2}(D^{\hat{c}}\phi)D_{\hat{c}}(k^a k^b R_{aebf}\nabla^e\phi\nabla^f\phi) - \frac{1}{2}(k^a k^b R_{aebf}\nabla^e\phi\nabla^f\phi)(D^{\hat{c}}D_{\hat{c}}\phi) = -\frac{1}{2}D_{\hat{c}}[(k^a k^b R_{aebf}\nabla^e\phi\nabla^f\phi)(D^{\hat{c}}\phi)]. \tag{59}$$

If we assume that the event horizon of black holes in the general second-order scalar-tensor gravity is compact, the surface term of the integral on the cross section of the event horizon does not contribute to the final result. It means that two results of Eqs. (58) and (59) are both neglected directly. The sixth and seventh terms of Eq. (57) are further simplified as

$$-(R_{kk})(\nabla_e\phi\nabla^e\phi)(k^a l^d\nabla_a\nabla_d\phi)-\frac{1}{2}(\mathcal{L}_k R_{kk})(\nabla_e\phi\nabla^e\phi)(l^d\nabla_d\phi)=(\mathcal{L}_k\theta)(\nabla_e\phi\nabla^e\phi)\mathcal{L}_k(l^d\nabla_d\phi)+\frac{1}{2}(\mathcal{L}_k^2\theta)(\nabla_e\phi\nabla^e\phi)(l^d\nabla_d\phi)$$
$$\simeq\mathcal{L}_k^2\left[\frac{1}{2}\theta(\nabla_e\phi\nabla^e\phi)(l^d\nabla_d\phi)\right]. \tag{60}$$

The last two terms of Eq. (57) are

$$(k^a l^d\nabla_a\nabla_d\phi)(D^{\hat g}\phi)(D^{\hat h}\phi)(k^b k^c R_{b\hat g c\hat h})+\frac{1}{2}(k^a k^b k^c\nabla_c R_{a\hat g b\hat h})(l^d\nabla_d\phi)(D^{\hat g}\phi)(D^{\hat h}\phi)$$
$$=-(\mathcal{L}_k B_{\hat a\hat b})(D^{\hat a}\phi)(D^{\hat b}\phi)\mathcal{L}_k(l^c\nabla_c\phi)-\frac{1}{2}(\mathcal{L}_k^2 B_{\hat a\hat b})(D^{\hat a}\phi)(D^{\hat b}\phi)(l^c\nabla_c\phi)$$
$$\simeq-\mathcal{L}_k^2\left[\frac{1}{2}B_{\hat a\hat b}(D^{\hat a}\phi)(D^{\hat b}\phi)(l^c\nabla_c\phi)\right]. \tag{61}$$

According to two results in Eqs. (60) and (61), the integral of $H_{kk}^{(\phi 2)}$ on the cross section of the event horizon under the first-order approximation of the perturbation process can be written as

$$\int_s\tilde\epsilon H_{kk}^{(\phi 2)}\simeq-\mathcal{L}_k^2\int_s\tilde\epsilon\left[\frac{1}{2}B_{\hat a\hat b}(D^{\hat a}\phi)(D^{\hat b}\phi)(l^c\nabla_c\phi)-\frac{1}{2}\theta(\nabla_e\phi\nabla^e\phi)(l^d\nabla_d\phi)\right]. \tag{62}$$

Finally, combining Eqs. (55) and (56) with Eq. (62) and supplementing the coefficient $1/4$ on two sides of the equation, while according to the equation of motion in Eq. (8) and the null energy condition in Eq. (28), we have

$$\mathcal{L}_k^2(S_{\mathrm{W}2}+S_{\mathrm{ct}})\simeq-\frac{1}{4}\int_s\tilde\epsilon H_{kk}=-2\pi\int_s\tilde\epsilon T_{kk}\le 0, \tag{63}$$

where

$$S_{\mathrm{ct}}=\frac{1}{8}\int_s\tilde\epsilon[(\mathcal{L}_k\phi)(D^{\hat e}\phi)(D^{\hat f}\phi)(\nabla_{\hat f}l_{\hat e})-(\mathcal{L}_k\phi)(D^{\hat e}\phi D_{\hat e}\phi)(\nabla^{\hat c}l_{\hat c})+B_{\hat a\hat b}(D^{\hat a}\phi)(D^{\hat b}\phi)(l^c\nabla_c\phi)-\theta(\nabla_e\phi\nabla^e\phi)(l^d\nabla_d\phi)]. \tag{64}$$

Equation (63) indicates that the second-order Lie derivate of the second part of the Wald entropy with the correction terms is always negative under the first-order approximation of the matter fields perturbation, and the sign of inequality is contributed only by the null energy condition of the total stress-energy tensor. According to our discussion under Eq. (33), this result indicates that the first-order Lie derivative of the second part of the Wald entropy with the correction terms is always positive during the perturbation process. It means that the entropy of black holes that consists of the second part of the Wald entropy and the correction terms will always increase during the perturbation process. In other words, the second part of the Wald entropy does not obey the linearized second law directly, and the second part of the Wald entropy with correction terms will collectively meet the linearized second law. Besides, the second part of the Wald entropy comes from the nonminimal coupling term in the Lagrangian that describes the interaction between the double dual of Riemann curvature and the scalar field. If the nonminimal coupling term is involved in the Lagrangian of any gravitational theory, the Wald entropy of black holes in the gravitational theory must not satisfy the linearized second law. And the correction terms in Eq. (64) should be added to the expression of the Wald entropy at least so that the modified Wald entropy obeys the linearized second law during the perturbation process. On the other hand, this result also shows that when any nonminimal coupling term that describes the interaction between gravity and any matter field is involved in the Lagrangian of the gravitational theory, the Wald entropy does not generally satisfy the linearized second law. Moreover, we should further reconstruct the expression of the entropy of black holes that meets the linearized second law all the time.

IV. CONCLUSIONS

Although it has been shown that the Wald entropy always satisfies the first law of black hole thermodynamics in any diffeomorphism invariant theory of gravity, whether the Wald entropy meets the second law of black hole thermodynamics all the time during a physical dynamical process has not been generally demonstrated. Since an appropriate scheme to quantize gravity has not been proposed until now, we have not been able to study the self-interaction of gravity and the interaction between gravity and any kinds of matter fields under the quantum scale directly. Therefore, the low-energy efficient gravitational theory that corresponds to the original gravitational theory should be constructed first to investigate these two types of interactions at the quantum level. Furthermore, the properties of the two types of interactions can be studied approximately, respectively, through the corresponding efficient theory. When considering the low-energy efficient theory of a quantum gravitational theory with the self-interaction of gravity or the interaction between gravity and any matter field, some quantum correction terms corresponding to different types of interactions under the quantum scale are introduced in the expression of the Lagrangian. According to the definition of the Wald entropy, these quantum correction terms, especially those describing the self-interaction of gravity and the nonminimal coupling interaction between gravity and the matter fields, will substantially modify the expression of Wald entropy. Since the Wald entropy generally satisfies the first law of black hole thermodynamics in any diffeomorphism invariant theory of gravity, we expect the Wald entropy in any gravitational theory also meets the second law all the time. However, before finding a general method to study the second law of black hole thermodynamics, when any gravitational theory contains arbitrary types of nonminimal coupling interaction, we need to verify whether the Wald entropy of black holes in these gravitational theories still meets the second law of thermodynamics one by one.

Considering a situation where the self-interaction of gravity is contained in the gravitational theory, some high-order curvature correction terms are added to the Lagrangian in the corresponding low-energy efficient gravitational theory. However, for this situation, when considering the perturbation process by the matter fields minimally coupling to gravity, it has been generally demonstrated that the Wald entropy does not meet the linearized second law during the perturbation process. Furthermore, the general expression of the entropy of black holes, which can be written as the Wald entropy with some correction terms, always obeying the linearized second law in any high-order curvature gravitational theory, has been obtained. However, unlike the first situation, the second law of black holes in the gravitational theory with the nonminimal coupling interactions, which describe the interaction between gravity and any matter fields, has not been sufficiently investigated. Recently, a general second-order scalar-tensor gravity has been proposed, and the Lagrangian of the theory can be expressed as a linear combination of four components. In the four components, there are three types of nonminimal coupling terms. The three nonminimal coupling terms are the nonminimal coupling term that represents the interaction between the Einstein tensor and the scalar field, the nonminimal coupling term that describes the interaction between the double dual of the Riemann tensor and the scalar field, and the Gauss-Bonnet combination. According to the definition, the three nonminimal coupling terms will substantially contribute to the expression of the Wald entropy, and the Wald entropy can also be written as a linear combination of four parts because the Lagrangian is a linear combination of four components. To investigate whether the Wald entropy of black holes in the general scalar-tensor gravity obeys the linearized second law and whether each part in the Wald entropy satisfies the second law under the linear order approximation of the perturbation process need to be examined separately. If one of the four parts does not obey the linearized second law, the Wald entropy of black holes in the gravitational theory does not generally satisfy the linearized second law. And we need to reconstruct the new expression of the entropy of the black holes such that it meets the linearized second law all the time during the perturbation process. In previous work, we have demonstrated that the Wald entropy in Horndeski gravity satisfies the linearized second law. It indicates that the first part of the Wald entropy always obeys the linearized second law. Since the third part of the Wald entropy is contributed by the third component in the Lagrangian that only contains the Ricci scalar, the third part of the Wald entropy will degenerate into the Bekenstein-Hawking entropy that automatically satisfies the second law according to the area theorem. However, the fourth part of the Wald entropy contributed by the Gauss-Bonnet combination does not obey the linearized second law because the entropy of black holes in Gauss-Bonnet gravity that meets the linearized second law is JM entropy. Therefore, it implies that even without considering the second component of the Lagrangian, the entropy of black holes satisfying the linearized second law contributed by the first, the third, and the fourth components of the Lagrangian can no longer be written as the Wald entropy. Therefore, to obtain the entropy of black holes in the general second-order scalar-tensor gravity that always obeys the linearized second law during the perturbation process, we only need to investigate whether the second part of the Wald entropy satisfies the linearized second law. If it does not obey the second law, we need to figure out how to correct the second part of the Wald entropy such that the expression of the modified entropy meets the linearized second law.

First, a quasistationary accreting process should be introduced to examine whether the second part of the Wald entropy satisfies the linearized second law. This process describes that additional matter fields minimally coupling to gravity outside black holes pass through the event horizon and fall into black holes. It means that the accreting process can perturb the spacetime geometry of black holes. Moreover, we require that black holes will settle down to a stationary state after the accreting process and that the additional matter fields should always satisfy the null energy condition. Besides, to ensure that any tensor field is smooth near the Killing horizon, we also assume that a regular bifurcation surface exists in spacetime. From the three assumptions and the Raychaudhuri equation, to examine whether the second part of the Wald entropy satisfies the linearized second law, we should check whether the second-order Lie derivative of the second part of the Wald entropy is always negative during the perturbation process. The results show that the second-order Lie derivative of the second part of the Wald entropy with some correction terms is always negative, and the sign of inequality only comes from the null energy condition of the additional matter fields. In other words, the second part of the Wald entropy with correction terms increases with the perturbation process. It indicates that the second part of the Wald entropy needs to be corrected to satisfy the linearized second law. On the other hand, this result also implies that for any gravitational theory, when the nonminimal coupling term, which describes the interaction between the double dual Riemann curvature and the scalar field, is contained in the Lagrangian of the gravity, the Wald entropy should be further corrected to satisfy the linearized second law.

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APPENDIX A: CALCULATION $H_{kk}^{(G1)}$ UNDER THE LINEAR ORDER APPROXIMATION

In this appendix, we would like to calculate $H_{kk}^{(G1)}$ in Eq. (35) to obtain the expression under the first-order approximation of the matter fields perturbation. In previous research work, we have shown that on the background spacetime, two null vectors k^a and l^a in the coordinates $\{k^a, l^a, y_i^a\}$ satisfy the following two identities [26]:

$$k^b\nabla_b l^a \simeq 0, \qquad \nabla_{\hat{a}}k_b \simeq 0. \tag{A1}$$

Equation (A1) means that the two quantities $k^b\nabla_b l^a$ and $\nabla_{\hat{a}}k_b$ are both first-order quantities during the perturbation process. In the following calculation, we will directly use the results in Eq. (A1) to evaluate the expression of $H_{kk}^{(G1)}$ under the first-order approximation. To clearly represent the order of each tensor after contracting with the two null vectors k^a and l^a, we will introduce two kinds of brackets with subscript n, i.e., $()_n$ or $[]_n$ $(n = 0, 1)$, where $n = 0$ represents a background quantity in the two kinds of brackets and $n = 1$ represents a first-order quantity in the two kinds of brackets. Besides, to simplify the calculation and the expression of the equation, we first combine some terms in Eq. (35) according to the symmetry of the Riemann curvature tensor and further calculate the expression of these terms under the linear order approximation. The specific calculation process is as follows.

The first four terms in Eq. (35) can be calculated as

$$\begin{aligned}
&-\frac{1}{4}k^ak^b\nabla^c\phi\nabla^d\nabla_a\phi\nabla^e\phi R_{bcde}+\frac{1}{4}k^ak^b\nabla_a\phi\nabla^d\nabla^c\phi\nabla^e\phi R_{bcde}\\
&\quad+\frac{1}{4}k^ak^b\nabla^c\phi\nabla^d\phi\nabla^e\nabla_a\phi R_{bcde}-\frac{1}{4}k^ak^b\nabla_a\phi\nabla^d\phi\nabla^e\nabla^c\phi R_{bcde}\\
&=\frac{1}{2}k^ak^b\nabla_c\phi\nabla_d\phi\nabla_e\nabla_a\phi R_{bfgh}\gamma^{cf}\gamma^{dg}\gamma^{eh}-\frac{1}{2}k^ak^b\nabla_a\phi\nabla_c\phi\nabla_e\nabla_d\phi R_{bgfh}\gamma^{cf}\gamma^{dg}\gamma^{eh}\\
&\quad+\frac{1}{2}\nabla_e\phi\nabla_d\nabla_a\phi\nabla_f\phi k^ak^bk^cl^dR_{bgch}\gamma^{eg}\gamma^{fh}-\frac{1}{2}\nabla_a\phi\nabla_d\nabla_e\phi\nabla_f\phi k^ak^bk^cl^dR_{bgch}\gamma^{eg}\gamma^{fh}\\
&\quad-\frac{1}{2}\nabla_e\phi\nabla_d\phi\nabla_f\nabla_a\phi k^ak^bk^cl^dR_{bgch}\gamma^{eg}\gamma^{fh}+\frac{1}{2}\nabla_a\phi\nabla_d\phi\nabla_f\nabla_e\phi k^ak^bk^cl^dR_{bgch}\gamma^{eg}\gamma^{fh}\\
&\quad+\frac{1}{2}\nabla_e\phi\nabla_b\nabla_a\phi\nabla_f\phi k^ak^bk^cl^dR_{cgdh}\gamma^{eg}\gamma^{fh}-\frac{1}{2}\nabla_a\phi\nabla_b\nabla_e\phi\nabla_f\phi k^ak^bk^cl^dR_{cgdh}\gamma^{eg}\gamma^{fh}\\
&\quad-\frac{1}{2}\nabla_e\phi\nabla_b\phi\nabla_f\nabla_a\phi k^ak^bk^cl^dR_{cgdh}\gamma^{eg}\gamma^{fh}+\frac{1}{2}\nabla_a\phi\nabla_b\phi\nabla_f\nabla_e\phi k^ak^bk^cl^dR_{cgdh}\gamma^{eg}\gamma^{fh}\\
&\quad+\frac{1}{2}\nabla_g\phi\nabla_b\nabla_a\phi\nabla_e\phi k^ak^bk^ck^dl^el^fR_{cfdh}\gamma^{gh}-\frac{1}{2}\nabla_a\phi\nabla_b\nabla_g\phi\nabla_e\phi k^ak^bk^ck^dl^el^fR_{cfdh}\gamma^{gh}\\
&\quad-\frac{1}{2}\nabla_g\phi\nabla_b\phi\nabla_e\nabla_a\phi k^ak^bk^ck^dl^el^fR_{cfdh}\gamma^{gh}+\frac{1}{2}\nabla_a\phi\nabla_b\phi\nabla_e\nabla_g\phi k^ak^bk^ck^dl^el^fR_{cfdh}\gamma^{gh}.
\end{aligned} \tag{A2}$$

The first term of Eq. (A2) is

$$\frac{1}{2}k^a k^b \nabla_c\phi\nabla_d\phi\nabla_e\nabla_a\phi R_{bfgh}\gamma^{cf}\gamma^{dg}\gamma^{eh} = \frac{1}{2}(D^{\hat{c}}\phi)_0(D^{\hat{d}}\phi)_0 D^{\hat{e}}(\mathcal{L}_k\phi)_1(k^b R_{b\hat{c}\hat{d}\hat{e}})_1 \simeq 0. \tag{A3}$$

The second term of Eq. (A2) is

$$-\frac{1}{2}k^a k^b \nabla_a\phi\nabla_c\phi\nabla_e\nabla_d\phi R_{bgfh}\gamma^{cf}\gamma^{dg}\gamma^{eh} = -\frac{1}{2}(\mathcal{L}_k\phi)_1(k^b R_{b\hat{d}\hat{c}\hat{e}})_1(D^{\hat{e}}D^{\hat{d}}\phi)_0(D^{\hat{c}}\phi)_0 \simeq 0. \tag{A4}$$

The third term of Eq. (A2) is

$$\frac{1}{2}\nabla_e\phi\nabla_d\nabla_a\phi\nabla_f\phi k^a k^b k^c l^d R_{bgch}\gamma^{eg}\gamma^{fh} = \frac{1}{2}(D^{\hat{e}}\phi)_0(k^a l^d\nabla_d\nabla_a\phi)_0(D^{\hat{f}}\phi)_0(k^b k^c R_{b\hat{e}c\hat{f}})_1 \sim \mathcal{O}(\epsilon). \tag{A5}$$

The fourth term of Eq. (A2) is

$$-\frac{1}{2}\nabla_a\phi\nabla_d\nabla_e\phi\nabla_f\phi k^a k^b k^c l^d R_{bgch}\gamma^{eg}\gamma^{fh} = -\frac{1}{2}(\mathcal{L}_k\phi)_1(l^d\nabla_d\nabla^{\hat{e}}\phi)_0(D^{\hat{f}}\phi)_0(k^b k^c R_{b\hat{e}c\hat{f}})_1 \simeq 0. \tag{A6}$$

The fifth term of Eq. (A2) is

$$-\frac{1}{2}\nabla_e\phi\nabla_d\phi\nabla_f\nabla_a\phi k^a k^b k^c l^d R_{bgch}\gamma^{eg}\gamma^{fh} = -\frac{1}{2}(D^{\hat{e}}\phi)_0(l^d\nabla_d\phi)_0 D^{\hat{f}}(\mathcal{L}_k\phi)_1(k^b k^c R_{b\hat{e}c\hat{f}})_1 \simeq 0. \tag{A7}$$

The sixth term of Eq. (A2) is

$$\frac{1}{2}\nabla_a\phi\nabla_d\phi\nabla_f\nabla_e\phi k^a k^b k^c l^d R_{bgch}\gamma^{eg}\gamma^{fh} = \frac{1}{2}(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0(D^{\hat{f}}D^{\hat{e}}\phi)_0(k^b k^c R_{b\hat{e}c\hat{f}})_1 \simeq 0. \tag{A8}$$

The seventh term of Eq. (A2) is

$$\frac{1}{2}\nabla_e\phi\nabla_b\nabla_a\phi\nabla_f\phi k^a k^b k^c l^d R_{cgdh}\gamma^{eg}\gamma^{fh} = \frac{1}{2}(D^{\hat{e}}\phi)_0(\mathcal{L}_k^2\phi)_1(D^{\hat{f}}\phi)_0(k^c l^d R_{c\hat{e}d\hat{f}})_0 \sim \mathcal{O}(\epsilon). \tag{A9}$$

The eighth term of Eq. (A2) is

$$-\frac{1}{2}\nabla_a\phi\nabla_b\nabla_e\phi\nabla_f\phi k^a k^b k^c l^d R_{cgdh}\gamma^{eg}\gamma^{fh} = -\frac{1}{2}(\mathcal{L}_k\phi)_1 D^{\hat{e}}(\mathcal{L}_k\phi)_1(D^{\hat{f}}\phi)_0(k^c l^d R_{c\hat{e}d\hat{f}})_0 \simeq 0. \tag{A10}$$

The ninth term of Eq. (A2) is

$$-\frac{1}{2}\nabla_e\phi\nabla_b\phi\nabla_f\nabla_a\phi k^a k^b k^c l^d R_{cgdh}\gamma^{eg}\gamma^{fh} = -\frac{1}{2}(D^{\hat{e}}\phi)_0(\mathcal{L}_k\phi)_1 D^{\hat{f}}(\mathcal{L}_k\phi)_1(k^c l^d R_{c\hat{e}d\hat{f}})_0 \simeq 0. \tag{A11}$$

The tenth term of Eq. (A2) is

$$\frac{1}{2}\nabla_a\phi\nabla_b\phi\nabla_f\nabla_e\phi k^a k^b k^c l^d R_{cgdh}\gamma^{eg}\gamma^{fh} = \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(D^{\hat{f}}D^{\hat{e}}\phi)_0(k^c l^d R_{c\hat{e}d\hat{f}})_0 \simeq 0. \tag{A12}$$

The eleventh term of Eq. (A2) is

$$\frac{1}{2}\nabla_g\phi\nabla_b\nabla_a\phi\nabla_e\phi k^a k^b k^c k^d l^e l^f R_{cfdh}\gamma^{gh} = \frac{1}{2}(D^{\hat{g}}\phi)_0(\mathcal{L}_k^2\phi)_1(l^e\nabla_e\phi)_0(k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \tag{A13}$$

The twelfth term of Eq. (A2) is

$$-\frac{1}{2}\nabla_a\phi\nabla_b\nabla_g\phi\nabla_e\phi k^a k^b k^c k^d l^e l^f R_{cfdh}\gamma^{gh} = -\frac{1}{2}(\mathcal{L}_k\phi)_1 D^{\hat{g}}(\mathcal{L}_k\phi)_1(l^e\nabla_e\phi)_0(k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \tag{A14}$$

The thirteenth term of Eq. (A2) is

$$-\frac{1}{2}\nabla_g\phi\nabla_b\phi\nabla_e\nabla_a\phi k^a k^b k^c k^d l^e l^f R_{cfdh}\gamma^{gh} = -\frac{1}{2}(D^{\hat{g}}\phi)_0(\mathcal{L}_k\phi)_1(k^a l^e\nabla_e\nabla_a\phi)_0(k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \tag{A15}$$

The fourteenth term of Eq. (A2) is

$$\frac{1}{2}\nabla_a\phi\nabla_b\phi\nabla_e\nabla_g\phi k^a k^b k^c k^d l^e l^f R_{cfdh}\gamma^{gh} = \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^e\nabla_e\nabla^{\hat{g}}\phi)_0(k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \tag{A16}$$

Therefore, the first four terms under the first-order approximation are expressed as

$$\begin{aligned}
&-\frac{1}{4}k^a k^b\nabla^c\phi\nabla^d\nabla_a\phi\nabla^e\phi R_{bcde} + \frac{1}{4}k^a k^b\nabla_a\phi\nabla^d\nabla^c\phi\nabla^e\phi R_{bcde} + \frac{1}{4}k^a k^b\nabla^c\phi\nabla^d\phi\nabla^e\nabla_a\phi R_{bcde} - \frac{1}{4}k^a k^b\nabla_a\phi\nabla^d\phi\nabla^e\nabla^c\phi R_{bcde}\\
&\simeq\frac{1}{2}(D^{\hat{e}}\phi)(k^a l^d\nabla_d\nabla_a\phi)(D^{\hat{f}}\phi)(k^b k^c R_{b\hat{e}c\hat{f}}) + \frac{1}{2}(D^{\hat{e}}\phi)(\mathcal{L}_k^2\phi)(D^{\hat{f}}\phi)(k^c l^d R_{c\hat{e}d\hat{f}}).
\end{aligned} \tag{A17}$$

The fifth, sixth, seventh, and eighth terms of Eq. (35) are further given as

$$\begin{aligned}
&\frac{1}{4}k^a k^b\nabla_a\phi\nabla^d\phi\nabla_f\nabla^f\phi R_{bd} - \frac{1}{4}k^a k^b\nabla^c\phi\nabla^d\phi\nabla_f\nabla^f\phi R_{bcda} + \frac{1}{4}k^a k^b\nabla_a\phi\nabla^c\phi\nabla_d\nabla^d\phi R_{bc} + \frac{1}{4}k^a k^b\nabla^c\phi\nabla^e\phi\nabla_f\nabla^f\phi R_{bcae}\\
&= \frac{1}{2}k^a k^b\nabla_a\phi\nabla_c\phi\nabla_m\nabla^m\phi R_{bgdh}\gamma^{cd}\gamma^{gh} - \frac{1}{2}\nabla_a\phi\nabla_d\phi\nabla_m\nabla^m\phi k^a k^b k^c l^d R_{bgch}\gamma^{gh}\\
&\quad + \frac{1}{2}\nabla_e\phi\nabla_f\phi\nabla_m\nabla^m\phi k^b k^c R_{bgch}\gamma^{eg}\gamma^{fh} - \frac{1}{2}\nabla_a\phi\nabla_b\phi\nabla_m\nabla^m\phi k^a k^b k^c l^d R_{cgdh}\gamma^{gh}\\
&\quad + \frac{1}{2}g_{ae}\nabla_g\phi\nabla_b\phi\nabla_m\nabla^m\phi k^a k^b k^c k^d l^e l^f R_{cfdh}\gamma^{gh}.
\end{aligned} \tag{A18}$$

The first term of Eq. (A18) is

$$\frac{1}{2}k^a k^b\nabla_a\phi\nabla_c\phi\nabla_m\nabla^m\phi R_{bgdh}\gamma^{cd}\gamma^{gh} = \frac{1}{2}(\mathcal{L}_k\phi)_1(k^b R_{b\hat{c}})_1(D^{\hat{c}}\phi)_0(\nabla^m\nabla_m\phi)_0 \simeq 0. \tag{A19}$$

The second term of Eq. (A18) is

$$-\frac{1}{2}\nabla_a\phi\nabla_d\phi\nabla_m\nabla^m\phi k^a k^b k^c l^d R_{bgch}\gamma^{gh} = -\frac{1}{2}(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0(\nabla_m\nabla^m\phi)_0(R_{kk})_1 \simeq 0. \tag{A20}$$

The third term of Eq. (A18) is

$$\frac{1}{2}\nabla_e\phi\nabla_f\phi\nabla_m\nabla^m\phi k^b k^c R_{bgch}\gamma^{eg}\gamma^{fh} = \frac{1}{2}(D^{\hat{e}}\phi D^{\hat{f}}\phi\nabla_m\nabla^m\phi)_0(k^b k^c R_{b\hat{e}c\hat{f}})_1 \sim \mathcal{O}(\epsilon). \tag{A21}$$

The fourth term of Eq. (A18) is

$$-\frac{1}{2}\nabla_a\phi\nabla_b\phi\nabla_m\nabla^m\phi k^a k^b k^c l^d R_{cgdh}\gamma^{gh} = -\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(\nabla_m\nabla^m\phi)_0(k^c l^d R_{cd})_0 \simeq 0. \tag{A22}$$

The fifth term of Eq. (A18) is

$$-\frac{1}{2}\nabla_g\phi\nabla_b\phi\nabla_m\nabla^m\phi k^b k^c k^d l^f R_{cfdh}\gamma^{gh} = -\frac{1}{2}(D^{\hat{g}}\phi)_0(\mathcal{L}_k\phi)_1(\nabla_m\nabla^m\phi)_0(k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \tag{A23}$$

So the fifth, sixth, seventh, and eighth terms of Eq. (35) under the first-order approximation are written as

$$\frac{1}{4}k^a k^b \nabla_a\phi\nabla^d\phi\nabla_f\nabla^f\phi R_{bd} - \frac{1}{4}k^a k^b \nabla^c\phi\nabla^d\phi\nabla_f\nabla^f\phi R_{bcda} + \frac{1}{4}k^a k^b \nabla_a\phi\nabla^c\phi\nabla_d\nabla^d\phi R_{bc} + \frac{1}{4}k^a k^b \nabla^c\phi\nabla^e\phi\nabla_f\nabla^f\phi R_{bcae}$$
$$\simeq \frac{1}{2}(D^{\hat{e}}\phi D^{\hat{f}}\phi\nabla_m\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}}). \tag{A24}$$

The ninth, tenth, eleventh, and twelfth terms of Eq. (35) are calculated as

$$-\frac{1}{4}k^a k^b \nabla_a\phi\nabla^c\nabla_d\phi\nabla^d\phi R_{bc} + \frac{1}{4}k^a k^b \nabla^c\phi\nabla^d\nabla_e\phi\nabla^e\phi R_{bcda} - \frac{1}{4}k^a k^b \nabla_a\phi\nabla^c\nabla^d\phi\nabla_d\phi R_{bc} - \frac{1}{4}k^a k^b \nabla^c\phi\nabla^d\nabla^e\phi\nabla_e\phi R_{bcad}$$
$$= -\frac{1}{2}k^a k^b R_{be}\nabla_a\phi\nabla_c\phi\nabla_d\phi\nabla_c\phi\gamma^{de} + \frac{1}{2}\nabla_a\phi\nabla_d\nabla_m\phi\nabla^m\phi k^a k^b k^c l^d R_{bgch}\gamma^{gh} - \frac{1}{2}\nabla_e\phi\nabla_f\nabla_m\phi\nabla^m\phi k^b k^c R_{bgch}\gamma^{eg}\gamma^{fh}$$
$$+ \frac{1}{2}\nabla_a\phi\nabla_b\nabla_m\phi\nabla^m\phi k^a k^b k^c l^d R_{cd} + \frac{1}{2}\nabla_g\phi\nabla_b\nabla_m\phi\nabla^m\phi k^b k^c k^d l^f R_{cfdh}\gamma^{gh}. \tag{A25}$$

The first term of Eq. (A25) is

$$-\frac{1}{2}k^a k^b R_{be}\nabla_a\phi\nabla_c\phi\nabla_d\phi\nabla_c\phi\gamma^{de} = -\frac{1}{2}(\mathcal{L}_k\phi)_1(k^b R_{b\hat{e}})_1(\nabla^{\hat{e}}\nabla_c\phi\nabla^c\phi)_0 \simeq 0. \tag{A26}$$

The second term of Eq. (A25) is

$$\frac{1}{2}\nabla_a\phi\nabla_d\nabla_m\phi\nabla^m\phi k^a k^b k^c l^d R_{bgch}\gamma^{gh} = \frac{1}{2}(\mathcal{L}_k\phi)_1(l^d\nabla_d\nabla_m\phi\nabla^m\phi)_0(R_{kk})_1 \simeq 0. \tag{A27}$$

The third term of Eq. (A25) is

$$-\frac{1}{2}\nabla_e\phi\nabla_f\nabla_m\phi\nabla^m\phi k^b k^c R_{bgch}\gamma^{eg}\gamma^{fh} = -\frac{1}{2}(D^{\hat{e}}\phi\nabla^{\hat{f}}\nabla_m\phi\nabla^m\phi)_0(k^b k^c R_{b\hat{e}c\hat{f}})_1 \sim \mathcal{O}(\epsilon). \tag{A28}$$

The fourth term of Eq. (A25) is

$$\frac{1}{2}\nabla_a\phi\nabla_b\nabla_m\phi\nabla^m\phi k^a k^b k^c l^d R_{cd} = \frac{1}{2}(\mathcal{L}_k\phi)_1(k^b\nabla_b\nabla_m\phi\nabla^m\phi)_1(k^c l^d R_{cd})_0 \simeq 0. \tag{A29}$$

The fifth term of Eq. (A25) is

$$\frac{1}{2}\nabla_g\phi\nabla_b\nabla_m\phi\nabla^m\phi k^b k^c k^d l^f R_{cfdh}\gamma^{gh} = \frac{1}{2}(D^{\hat{g}}\phi)_0(k^b\nabla_b\nabla_m\phi\nabla^m\phi)_1(k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \tag{A30}$$

So the ninth, tenth, eleventh, and twelfth terms of Eq. (35) under the first-order approximation of the perturbation can be given as

$$-\frac{1}{4}k^a k^b \nabla_a\phi\nabla^c\nabla_d\phi\nabla^d\phi R_{bc} + \frac{1}{4}k^a k^b \nabla^c\phi\nabla^d\nabla_e\phi\nabla^e\phi R_{bcda} - \frac{1}{4}k^a k^b \nabla_a\phi\nabla^c\nabla^d\phi\nabla_d\phi R_{bc} - \frac{1}{4}k^a k^b \nabla^c\phi\nabla^d\nabla^e\phi\nabla_e\phi R_{bcad}$$
$$\simeq -\frac{1}{2}(D^{\hat{e}}\phi\nabla^{\hat{f}}\nabla_m\phi\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}}). \tag{A31}$$

The thirteenth, fourteenth, fifteenth, and sixteenth terms of Eq. (35) are

$$\frac{1}{4}k^a k^b \nabla^d\nabla_a\phi\nabla_f\phi\nabla^f\phi R_{bd} - \frac{1}{4}k^a k^b \nabla^d\nabla^c\phi\nabla_f\phi\nabla^f\phi R_{bcda} + \frac{1}{4}k^a k^b \nabla^e\nabla_a\phi\nabla_f\phi\nabla^f\phi R_{be} + \frac{1}{4}k^a k^b \nabla^e\nabla^c\phi\nabla_f\phi\nabla^f\phi R_{bcae}$$
$$= \frac{1}{2}k^a k^b R_{be}\nabla_d\nabla_a\phi\nabla_c\phi\nabla^c\phi\gamma^{de} - \frac{1}{2}\nabla_d\nabla_a\phi\nabla_m\phi\nabla^m\phi k^a k^b k^c l^d R_{bc} + \frac{1}{2}\nabla_f\nabla_e\phi\nabla_m\phi\nabla^m\phi k^b k^c R_{bgch}\gamma^{eg}\gamma^{fh}$$
$$- \frac{1}{2}\nabla_b\nabla_a\phi\nabla_m\phi\nabla^m\phi k^a k^b k^c l^d R_{cd} - \frac{1}{2}\nabla_b\nabla_g\phi\nabla_m\phi\nabla^m\phi k^b k^c k^d l^f R_{cfdh}\gamma^{gh}. \tag{A32}$$

The first term of Eq. (A32) is

$$\frac{1}{2}k^a k^b R_{be}\nabla_d\nabla_a\phi\nabla_c\phi\nabla^c\phi\gamma^{de} = \frac{1}{2}D^{\hat{d}}(\mathcal{L}_k\phi)_1(k^b R_{b\hat{d}})_1(\nabla_c\phi\nabla^c\phi)_0 \simeq 0. \tag{A33}$$

The second term of Eq. (A32) is

$$-\frac{1}{2}\nabla_d\nabla_a\phi\nabla_m\phi\nabla^m\phi k^a k^b k^c l^d R_{bc} = -\frac{1}{2}(k^a l^d\nabla_d\nabla_a\phi)_0(\nabla_m\phi\nabla^m\phi)_0(R_{kk})_1 \sim \mathcal{O}(\epsilon). \tag{A34}$$

The third term of Eq. (A32) is

$$\frac{1}{2}\nabla_f\nabla_e\phi\nabla_m\phi\nabla^m\phi k^b k^c R_{bgch}\gamma^{eg}\gamma^{fh} = \frac{1}{2}(D^{\hat{f}}D^{\hat{e}}\phi\nabla_m\phi\nabla^m\phi)_0(k^b k^c R_{b\hat{e}c\hat{f}})_1 \sim \mathcal{O}(\epsilon). \tag{A35}$$

The fourth term of Eq. (A32) is

$$\begin{aligned}-\frac{1}{2}\nabla_b\nabla_a\phi\nabla_m\phi\nabla^m\phi k^a k^b k^c l^d R_{cd} &= -\frac{1}{2}(\mathcal{L}_k^2\phi)_1(D_{\hat{e}}\phi D^{\hat{e}}\phi)_0(k^c l^d R_{cd})_0 + (\mathcal{L}_k^2\phi)_1(\mathcal{L}_k\phi)_1(l^e\nabla_e\phi)_0(k^c l^d R_{cd})_0\\ &\simeq -\frac{1}{2}(\mathcal{L}_k^2\phi)_1(D_{\hat{e}}\phi D^{\hat{e}}\phi)_0(k^c l^d R_{cd})_0 \sim \mathcal{O}(\epsilon).\end{aligned} \tag{A36}$$

The fifth term of Eq. (A32) is

$$-\frac{1}{2}\nabla_b\nabla_g\phi\nabla_m\phi\nabla^m\phi k^b k^c k^d l^f R_{cfdh}\gamma^{gh} = -\frac{1}{2}(k^b\nabla_b\nabla^{\hat{g}}\phi\nabla_m\phi\nabla^m\phi)_1(k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \tag{A37}$$

Therefore, the thirteenth, fourteenth, fifteenth, and sixteenth terms of Eq. (35) under the linear order approximation can finally be written as

$$\begin{aligned}&\frac{1}{4}k^a k^b\nabla^d\nabla_a\phi\nabla_f\phi\nabla^f\phi R_{bd} - \frac{1}{4}k^a k^b\nabla^d\nabla^c\phi\nabla_f\phi\nabla^f\phi R_{bcda} + \frac{1}{4}k^a k^b\nabla^e\nabla_a\phi\nabla_f\phi\nabla^f\phi R_{be} + \frac{1}{4}k^a k^b\nabla^e\nabla^c\phi\nabla_f\phi\nabla^f\phi R_{bcae}\\ &\simeq -\frac{1}{2}(k^a l^d\nabla_d\nabla_a\phi)(\nabla_m\phi\nabla^m\phi)(R_{kk}) + \frac{1}{2}(D^{\hat{f}}D^{\hat{e}}\phi\nabla_m\phi\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}}) - \frac{1}{2}(\mathcal{L}_k^2\phi)(D_{\hat{e}}\phi D^{\hat{e}}\phi)(k^c l^d R_{cd}).\end{aligned} \tag{A38}$$

The seventeenth, eighteenth, nineteenth, and twentieth terms of Eq. (35) are

$$\begin{aligned}&-\frac{1}{4}k^a k^b\nabla^d\phi\nabla_f\nabla_a\phi\nabla^f\phi R_{bd} - \frac{1}{4}k^a k^b\nabla^e\phi\nabla_f\nabla^a\phi\nabla^f\phi R_{be} + \frac{1}{4}k^a k^b\nabla^d\phi\nabla_f\nabla^c\phi\nabla^f\phi R_{bcda} - \frac{1}{4}k^a k^b\nabla^e\phi\nabla_f\nabla^c\phi\nabla^f\phi R_{bcae}\\ &= -\frac{1}{2}k^a k^b R_{be}\nabla_c\nabla_a\phi\nabla^c\phi\nabla_d\phi\gamma^{de} + \frac{1}{2}\nabla_d\phi\nabla_m\nabla_a\phi\nabla^m\phi k^a k^b k^c l^d R_{bc} - \frac{1}{2}\nabla_f\phi\nabla_m\nabla_e\phi\nabla^m\phi k^b k^c R_{bgch}\gamma^{eg}\gamma^{fh}\\ &\quad + \frac{1}{2}\nabla_b\phi\nabla_m\nabla_a\phi\nabla^m\phi k^a k^b k^c l^d R_{cd} - \frac{1}{2}\nabla_b\phi\nabla_m\nabla_g\phi\nabla^m\phi k^b k^c k^d l^f R_{cfdh}\gamma^{gh}.\end{aligned} \tag{A39}$$

The first term of Eq. (A39) is

$$-\frac{1}{2}k^a k^b R_{be}\nabla_c\nabla_a\phi\nabla^c\phi\nabla_d\phi\gamma^{de} = -\frac{1}{2}(k^a\nabla_c\nabla_a\phi\nabla^c\phi)_1(k^b R_{b\hat{d}})_1(D^{\hat{d}}\phi)_0 \simeq 0. \tag{A40}$$

The second term of Eq. (A39) is

$$\frac{1}{2}\nabla_d\phi\nabla_m\nabla_a\phi\nabla^m\phi k^a k^b k^c l^d R_{bc} = \frac{1}{2}(l^d\nabla_d\phi)_0(k^a\nabla_m\nabla_a\phi\nabla^m\phi)_1(R_{kk})_1 \simeq 0. \tag{A41}$$

The third term of Eq. (A39) is

$$-\frac{1}{2}\nabla_f\phi\nabla_m\nabla_e\phi\nabla^m\phi k^b k^c R_{bgch}\gamma^{eg}\gamma^{fh} = -\frac{1}{2}(D^{\hat{f}}\phi\nabla_m\nabla^{\hat{e}}\phi\nabla^m\phi)_0(k^b k^c R_{b\hat{e}c\hat{f}})_1 \sim \mathcal{O}(\epsilon). \tag{A42}$$

The fourth term of Eq. (A39) is

$$\frac{1}{2}\nabla_b\phi\nabla_m\nabla_a\phi\nabla^m\phi k^a k^b k^c l^d R_{cd} = \frac{1}{2}(\mathcal{L}_k\phi)_1(k^a\nabla_m\nabla_a\phi\nabla^m\phi)_1(k^c l^d R_{cd})_0 \simeq 0. \tag{A43}$$

The fifth term of Eq. (A39) is

$$-\frac{1}{2}\nabla_b\phi\nabla_m\nabla_g\phi\nabla^m\phi k^b k^c k^d l^f R_{cfdh}\gamma^{gh} = \frac{1}{2}(\mathcal{L}_k\phi)_1(\nabla_m\nabla^{\hat{g}}\phi\nabla^m\phi)_0(k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \tag{A44}$$

So the seventeenth, eighteenth, nineteenth, and twentieth terms of Eq. (35) under the first-order approximation can be written as

$$\begin{aligned}&-\frac{1}{4}k^a k^b\nabla^d\phi\nabla_f\nabla_a\phi\nabla^f\phi R_{bd} - \frac{1}{4}k^a k^b\nabla^e\phi\nabla_f\nabla^a\phi\nabla^f\phi R_{be} + \frac{1}{4}k^a k^b\nabla^d\phi\nabla_f\nabla^c\phi\nabla^f\phi R_{bcda} - \frac{1}{4}k^a k^b\nabla^e\phi\nabla_f\nabla^c\phi\nabla^f\phi R_{bcae}\\ &\simeq -\frac{1}{2}(D^{\hat{f}}\phi\nabla_m\nabla^{\hat{e}}\phi\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}}).\end{aligned} \tag{A45}$$

The twenty-first and twenty-second terms of Eq. (35) are

$$-\frac{1}{4}k^a k^b\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi R_{ab} - \frac{1}{4}k^a k^b\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi R_{ab} = -\frac{1}{2}(\nabla_m\phi\nabla^m\phi\nabla_n\nabla^n\phi)_0(R_{kk})_1 \sim \mathcal{O}(\epsilon). \tag{A46}$$

The twenty-third and twenty-fourth terms of Eq. (35) are

$$\frac{1}{4}k^a k^b\nabla^f\phi\nabla_g\nabla_f\phi\nabla^g\phi R_{ab} + \frac{1}{4}k^a k^b\nabla^f\phi\nabla_g\nabla_f\phi\nabla^g\phi R_{ab} = \frac{1}{2}(\nabla^m\phi\nabla_n\nabla_m\phi\nabla^n\phi)_0(R_{kk})_1 \sim \mathcal{O}(\epsilon). \tag{A47}$$

Combining with the above results, for the convenience of calculation, we will decompose the expression of $H_{kk}^{(G1)}$ under the first-order approximation of the matter fields perturbation into two parts, i.e.,

$$H_{kk}^{(G1)} \simeq H_{kk}^{(G1)1} + H_{kk}^{(G1)2}, \tag{A48}$$

where

$$\begin{aligned}H_{kk}^{(G1)1} = &\frac{1}{2}(D^{\hat{e}}\phi)(k^a l^d\nabla_d\nabla_a\phi)(D^{\hat{f}}\phi)(k^b k^c R_{b\hat{e}c\hat{f}}) + \frac{1}{2}(D^{\hat{e}}\phi D^{\hat{f}}\phi\nabla_m\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}}) - \frac{1}{2}(D^{\hat{e}}\phi\nabla^{\hat{f}}\nabla_m\phi\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}})\\ &-\frac{1}{2}(k^a l^d\nabla_d\nabla_a\phi\nabla_m\phi\nabla^m\phi)(R_{kk}) + \frac{1}{2}(D^{\hat{f}}D^{\hat{e}}\phi\nabla_m\phi\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}}) - \frac{1}{2}(D^{\hat{f}}\phi\nabla_m\nabla^{\hat{e}}\phi\nabla^m\phi)(k^b k^c R_{b\hat{e}c\hat{f}})\\ &-\frac{1}{2}(\nabla_m\phi\nabla^m\phi\nabla_n\nabla^n\phi)(R_{kk}) + \frac{1}{2}(\nabla^m\phi\nabla_n\nabla_m\phi\nabla^n\phi)(R_{kk}),\end{aligned} \tag{A49}$$

and

$$H_{kk}^{(G1)2} = \frac{1}{2}(D^{\hat{e}}\phi)(\mathcal{L}_k^2\phi)(D^{\hat{f}}\phi)(k^c l^d R_{c\hat{e}d\hat{f}}) - \frac{1}{2}(\mathcal{L}_k^2\phi)(D^{\hat{e}}\phi D_{\hat{e}}\phi)(k^c l^d R_{cd}). \tag{A50}$$

APPENDIX B: SIMPLIFICATION OF RELEVANT TERMS IN $H_{kk}^{(G2)}$ UNDER THE LINEAR ORDER APPROXIMATION

In the second appendix, we mainly focus on the three integrands, $H_{kk}^{(G2)2}$, $H_{kk}^{(G2)3}$, and $H_{kk}^{(G2)4}$, in the last three integrals of Eq. (43) to calculate the expressions of the three integrands under the first-order approximation of the perturbation process. Similar to the calculation procedure in Appendix A, we still first combine relevant terms according to the symmetry of the Riemann tensor to reduce the complexity of the calculation in this appendix. In the following, we will divide the content of this appendix into three parts to calculate the three integrands under the linear order approximation of the matter fields perturbation, respectively.

In the first part, we simplify the integrand $H_{kk}^{(G2)2}$ in the second integral of Eq. (43) to obtain the final expression under the first-order approximation of the perturbation. The specific expression of $H_{kk}^{(G2)2}$ has been given in Eq. (45). The first four terms of Eq. (45) can be calculated as

$$\begin{aligned}
&\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(\nabla_c\phi\nabla_b\nabla_a\phi\nabla_d\phi)-\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(\nabla_a\phi\nabla_b\nabla_c\phi\nabla_d\phi)\\
&\quad-\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(\nabla_c\phi\nabla_b\phi\nabla_d\nabla_a\phi)+\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(\nabla_a\phi\nabla_b\phi\nabla_d\nabla_c\phi)\\
&=-\frac{1}{2}[k^ak^bl^c\gamma_g{}^f\nabla_b(\nabla_f\nabla_c\nabla_a\phi\nabla_d\phi)](\gamma^g{}_e\nabla^e k^d)-\frac{1}{2}[k^ak^bl^c\gamma_g{}^f\nabla_c(\nabla_f\phi\nabla_b\nabla_a\phi\nabla_d\phi)](\gamma^g{}_e\nabla^e k^d)\\
&\quad+\frac{1}{2}[k^a\gamma^{ef}\gamma_g{}^d\nabla_f(\nabla_d\phi\nabla_e\nabla_a\phi\nabla_b\phi)](\gamma^g{}_c\nabla^c k^b)+\frac{1}{2}[k^ak^bl^c\gamma_g{}^f\nabla_b(\nabla_a\phi\nabla_c\nabla_f\phi\nabla_d\phi)](\gamma^g{}_e\nabla^e k^d)\\
&\quad+\frac{1}{2}[k^ak^bl^c\gamma_g{}^f\nabla_c(\nabla_a\phi\nabla_b\nabla_f\phi\nabla_d\phi)](\gamma^g{}_e\nabla^e k^d)-\frac{1}{2}[k^a\gamma^{ef}\gamma_g{}^d\nabla_f(\nabla_a\phi\nabla_e\nabla_d\phi\nabla_b\phi)](\gamma^g{}_c\nabla^c k^b)\\
&\quad+\frac{1}{2}[k^ak^bl^c\gamma_g{}^f\nabla_b(\nabla_f\phi\nabla_c\phi\nabla_d\nabla_a\phi)](\gamma^g{}_e\nabla^e k^d)+\frac{1}{2}[k^ak^bl^c\gamma_g{}^f\nabla_c(\nabla_f\phi\nabla_b\phi\nabla_d\nabla_a\phi)](\gamma^g{}_e\nabla^e k^d)\\
&\quad-\frac{1}{2}[k^a\gamma^{ef}\gamma_g{}^d\nabla_f(\nabla_d\phi\nabla_e\phi\nabla_b\nabla_a\phi)](\gamma^g{}_c\nabla^c k^b)-\frac{1}{2}[k^ak^bl^c\gamma_g{}^f\nabla_b(\nabla_a\phi\nabla_c\phi\nabla_d\nabla_f\phi)](\gamma^g{}_e\nabla^e k^d)\\
&\quad-\frac{1}{2}[k^ak^bl^c\gamma_g{}^f\nabla_c(\nabla_a\phi\nabla_b\phi\nabla_d\nabla_b\phi)](\gamma^g{}_e\nabla^e k^d)+\frac{1}{2}[k^a\gamma^{ef}\gamma_g{}^d\nabla_f(\nabla_a\phi\nabla_e\phi\nabla_b\nabla_d\phi)](\gamma^g{}_c\nabla^c k^b).
\end{aligned}\tag{B1}$$

The first term of Eq. (B1) is

$$\begin{aligned}
-\frac{1}{2}[k^ak^bl^c\gamma_g{}^f\nabla_b(\nabla_f\nabla_c\nabla_a\phi\nabla_d\phi)](\gamma^g{}_e\nabla^e k^d)&=-\frac{1}{2}(k^b\nabla_b\nabla_{\hat f}\phi)_1(k^al^c\nabla_c\nabla_a\phi)_0(\nabla_d\phi)(\nabla^{\hat f}k^d)_1\\
&\quad-\frac{1}{2}(D_{\hat f}\phi)_0(k^ak^bl^c\nabla_b\nabla_c\nabla_a\phi)_1(\nabla_d\phi)(\nabla^{\hat f}k^d)_1\\
&\quad-\frac{1}{2}(D_{\hat f}\phi)_0(k^al^c\nabla_c\nabla_a\phi)_0(k^b\nabla_b\nabla_d\phi)_1(\nabla^{\hat f}k^d)_1\simeq 0.
\end{aligned}\tag{B2}$$

The second term of Eq. (B1) is

$$\begin{aligned}
-\frac{1}{2}[k^ak^bl^c\gamma_g{}^f\nabla_c(\nabla_f\phi\nabla_b\nabla_a\phi\nabla_d\phi)](\gamma^g{}_e\nabla^e k^d)&=-\frac{1}{2}(l^c\nabla_c\nabla_{\hat f}\phi)_0(\mathcal{L}_k^2\phi)_1(\nabla_d\phi)(\nabla^{\hat f}k^d)_1\\
&\quad-\frac{1}{2}(D_{\hat f}\phi)_0(k^ak^bl^c\nabla_c\nabla_b\nabla_a\phi)_1(\nabla_d\phi)(\nabla^{\hat f}k^d)_1\\
&\quad-\frac{1}{2}(D_{\hat f}\phi)_0(\mathcal{L}_k^2\phi)_1(l^c\nabla_c\nabla_d\phi)(\nabla^{\hat f}k^d)_1\simeq 0.
\end{aligned}\tag{B3}$$

The third term of Eq. (B1) is

$$\frac{1}{2}[k^a\gamma^{ef}\gamma_g{}^d\nabla_f(\nabla_d\phi\nabla_e\nabla_a\phi\nabla_b\phi)](\gamma^g{}_c\nabla^c k^b) = \frac{1}{2}(D_{\hat{c}}D_{\hat{d}}\phi)_0 D^{\hat{c}}(\mathcal{L}_k\phi)_1(\nabla_b\phi)(\nabla^{\hat{d}}k^b)_1 + \frac{1}{2}(D_{\hat{d}}\phi)_0(k^a\nabla^{\hat{e}}\nabla_{\hat{e}}\nabla_a\phi)_1(\nabla_b\phi)(\nabla^{\hat{d}}k^b)_1 + \frac{1}{2}(D_{\hat{d}}\phi)_0 D^{\hat{e}}(\mathcal{L}_k\phi)_1(\nabla_{\hat{e}}\nabla_b\phi)(\nabla^{\hat{d}}k^b)_1 \simeq 0. \tag{B4}$$

The fourth term of Eq. (B1) is

$$\frac{1}{2}[k^a k^b l^c\gamma_g{}^f\nabla_b(\nabla_a\phi\nabla_c\nabla_f\phi\nabla_d\phi)](\gamma^g{}_e\nabla^e k^d) = \frac{1}{2}(\mathcal{L}_k^2\phi)_1(l^c\nabla_c\nabla_{\hat{e}}\phi)_0(\nabla_d\phi)_0(\nabla^{\hat{e}}k^d)_1 + \frac{1}{2}(\mathcal{L}_k\phi)_1(k^b l^c\nabla_b\nabla_c\nabla_{\hat{e}}\phi)_0(\nabla_d\phi)(\nabla^{\hat{e}}k^d)_1 + \frac{1}{2}(\mathcal{L}_k\phi)_1(l^c\nabla_c\nabla_{\hat{e}}\phi)_0(k^b\nabla_b\nabla_d\phi)_1(\nabla^{\hat{e}}k^d)_1 \simeq 0. \tag{B5}$$

The fifth term of Eq. (B1) is

$$\frac{1}{2}[k^a k^b l^c\gamma_g{}^f\nabla_c(\nabla_a\phi\nabla_b\nabla_f\phi\nabla_d\phi)](\gamma^g{}_e\nabla^e k^d) = \frac{1}{2}(k^a l^c\nabla_c\nabla_a\phi)_0 D_{\hat{e}}(\mathcal{L}_k\phi)_1(\nabla_d\phi)(\nabla^{\hat{e}}k^d)_1 + \frac{1}{2}(\mathcal{L}_k\phi)_1(k^b l^c\nabla_c\nabla_b\nabla_{\hat{e}}\phi)_0(\nabla_d\phi)(\nabla^{\hat{e}}k^d)_1 + \frac{1}{2}(\mathcal{L}_k\phi)_1 D_{\hat{e}}(\mathcal{L}_k\phi)_1(l^c\nabla_c\nabla_d\phi)(\nabla^{\hat{e}}k^d)_1 \simeq 0. \tag{B6}$$

The sixth term of Eq. (B1) is

$$-\frac{1}{2}[k^a\gamma^{ef}\gamma_g{}^d\nabla_f(\nabla_a\phi\nabla_e\nabla_d\phi\nabla_b\phi)](\gamma^g{}_c\nabla^c k^b) = -\frac{1}{2}\nabla_{\hat{e}}(\mathcal{L}_k\phi)_1(D^{\hat{e}}D_{\hat{c}}\phi)_0(\nabla_b\phi)(\nabla^{\hat{c}}k^b)_1 - \frac{1}{2}(\mathcal{L}_k\phi)_1(D^{\hat{d}}D_{\hat{d}}D_{\hat{c}}\phi)_0(\nabla_b\phi)(\nabla^{\hat{c}}k^b)_1 - \frac{1}{2}(\mathcal{L}_k\phi)_1(D_{\hat{e}}D_{\hat{c}}\phi)_0(\nabla^{\hat{e}}\nabla_b\phi)(\nabla^{\hat{c}}k^b)_1 \simeq 0. \tag{B7}$$

The seventh term of Eq. (B1) is

$$\frac{1}{2}[k^a k^b l^c\gamma_g{}^f\nabla_b(\nabla_f\phi\nabla_c\phi\nabla_d\nabla_a\phi)](\gamma^g{}_e\nabla^e k^d) = \frac{1}{2}D_{\hat{e}}(\mathcal{L}_k\phi)_1(l^c\nabla_c\phi)_0(k^a\nabla_d\nabla_a\phi)(\nabla^{\hat{e}}k^d)_1 + \frac{1}{2}(D_{\hat{e}}\phi)_0(k^b l^c\nabla_b\nabla_c\phi)_0 D_{\hat{d}}(\mathcal{L}_k\phi)_1(B^{\hat{e}\hat{d}})_1 - \frac{1}{2}(D_{\hat{e}}\phi)_0(k^b l^c\nabla_b\nabla_c\phi)_0(\mathcal{L}_k^2\phi)_1(l^f\nabla^{\hat{e}}k_f)_1 + \frac{1}{2}(D_{\hat{e}}\phi)_0(l^c\nabla_c\phi)_0(k^a k^b\nabla_b\nabla_{\hat{d}}\nabla_a\phi)_1(B^{\hat{e}\hat{d}})_1 - \frac{1}{2}(D_{\hat{e}}\phi)_0(l^c\nabla_c\phi)_0(\mathcal{L}_k^3\phi)_1(l^f\nabla^{\hat{e}}k_f)_1 \simeq 0. \tag{B8}$$

The eighth term of Eq. (B1) is

$$\begin{aligned}\frac{1}{2}[k^a k^b l^c \gamma_g{}^f \nabla_c(\nabla_f\phi\nabla_b\phi\nabla_d\nabla_a\phi)](\gamma^g{}_e\nabla^e k^d) = &\frac{1}{2}(l^c\nabla_c\nabla_{\hat{e}}\phi)_0(\mathcal{L}_k\phi)_1(k^a\nabla_d\nabla_a\phi)(\nabla^{\hat{e}}k^d)_1\\ &+\frac{1}{2}(D_{\hat{e}}\phi)_0(k^b l^c\nabla_c\nabla_b\phi)_0 D_{\hat{d}}(\mathcal{L}_k\phi)_1(B^{\hat{e}\hat{d}})_1\\ &-\frac{1}{2}(D_{\hat{e}}\phi)_0(k^b l^c\nabla_c\nabla_b\phi)_0(\mathcal{L}_k^2\phi)_1(l^f\nabla^{\hat{e}}k_f)_1\\ &+\frac{1}{2}(D_{\hat{e}}\phi)_0(\mathcal{L}_k\phi)_1(k^a l^c\nabla_c\nabla_d\nabla_a\phi)(\nabla^{\hat{e}}k^d)_1 \simeq 0. \end{aligned} \tag{B9}$$

The ninth term of Eq. (B1) is

$$\begin{aligned}-\frac{1}{2}[k^a\gamma^{ef}\gamma_g{}^d\nabla_f(\nabla_d\phi\nabla_e\phi\nabla_b\nabla_a\phi)](\gamma^g{}_c\nabla^c k^b) = &-\frac{1}{2}(D^{\hat{e}}D^{\hat{d}}\phi)_0(D_{\hat{e}}\phi)_0 D^{\hat{b}}(\mathcal{L}_k\phi)_1(B_{\hat{d}\hat{b}})_1\\ &+\frac{1}{2}(D^{\hat{e}}D^{\hat{d}}\phi)_0(D_{\hat{e}}\phi)_0(\mathcal{L}_k^2\phi)_1(l^f\nabla_{\hat{d}}k_f)_1\\ &-\frac{1}{2}(D^{\hat{d}}\phi)_0(D^{\hat{e}}D_{\hat{e}}\phi)_0 D^{\hat{b}}(\mathcal{L}_k\phi)_1(B_{\hat{d}\hat{b}})_1\\ &+\frac{1}{2}(D^{\hat{d}}\phi)_0(D^{\hat{e}}D_{\hat{e}}\phi)_0(\mathcal{L}_k^2\phi)_1(l^f\nabla_{\hat{d}}k_f)_1\\ &-\frac{1}{2}(D^{\hat{d}}\phi)_0(D^{\hat{e}}\phi)_0(k^a\nabla_{\hat{e}}\nabla^{\hat{b}}\nabla_a\phi)_1(B_{\hat{d}\hat{b}})_1\\ &+\frac{1}{2}(D^{\hat{d}}\phi)_0(D^{\hat{e}}\phi)_0(k^b k^a\nabla_{\hat{e}}\nabla_b\nabla_a\phi)_1(l^f\nabla_{\hat{d}}k_f)_1 \simeq 0. \end{aligned} \tag{B10}$$

The tenth term of Eq. (B1) is

$$\begin{aligned}-\frac{1}{2}[k^a k^b l^c\gamma_g{}^f\nabla_b(\nabla_a\phi\nabla_c\phi\nabla_d\nabla_f\phi)](\gamma^g{}_e\nabla^e k^d) = &-\frac{1}{2}(\mathcal{L}_k^2\phi)_1(l^c\nabla_c\phi)_0(\nabla_d\nabla_{\hat{e}}\phi)(\nabla^{\hat{e}}k^d)_1\\ &-\frac{1}{2}(\mathcal{L}_k\phi)_1(k^b l^c\nabla_b\nabla_c\phi)_0(\nabla_d\nabla_{\hat{e}}\phi)(\nabla^{\hat{e}}k^d)_1\\ &-\frac{1}{2}(\mathcal{L}_k\phi)_1(l^c\nabla_c\phi)_0(k^b\nabla_b\nabla_d\nabla_{\hat{e}}\phi)(\nabla^{\hat{e}}k^d)_1 \simeq 0. \end{aligned} \tag{B11}$$

The eleventh term of Eq. (B1) is

$$\begin{aligned}-\frac{1}{2}[k^a k^b l^c\gamma_g{}^f\nabla_c(\nabla_a\phi\nabla_b\phi\nabla_d\nabla_b\phi)](\gamma^g{}_e\nabla^e k^d) = &-\frac{1}{2}(k^a l^c\nabla_c\nabla_a\phi)_0(\mathcal{L}_k\phi)_1(\nabla_d\nabla_{\hat{e}}\phi)(\nabla^{\hat{e}}k^d)_1\\ &-\frac{1}{2}(\mathcal{L}_k\phi)_1(k^b l^c\nabla_c\nabla_b\phi)_0(\nabla_d\nabla_{\hat{e}}\phi)(\nabla^{\hat{e}}k^d)_1\\ &-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^c\nabla_c\nabla_d\nabla_{\hat{f}}\phi)(\nabla^{\hat{f}}k^d)_1 \simeq 0. \end{aligned} \tag{B12}$$

The twelfth term of Eq. (B1) is

$$\begin{aligned}\frac{1}{2}[k^a\gamma^{ef}\gamma_g{}^d\nabla_f(\nabla_a\phi\nabla_e\phi\nabla_b\nabla_d\phi)](\gamma^g{}_c\nabla^c k^b) = &\frac{1}{2}D_{\hat{e}}(\mathcal{L}_k\phi)_1(D^{\hat{e}}\phi)_0(\nabla_b\nabla_{\hat{c}}\phi)(\nabla^{\hat{c}}k^b)_1\\ &+\frac{1}{2}(\mathcal{L}_k\phi)_1(D_{\hat{e}}D^{\hat{e}}\phi)_0(\nabla_b\nabla_{\hat{c}}\phi)(\nabla^{\hat{c}}k^b)_1\\ &+\frac{1}{2}(\mathcal{L}_k\phi)_1(D_{\hat{e}}\phi)_0(\nabla^{\hat{e}}\nabla_b\nabla_{\hat{c}}\phi)(\nabla^{\hat{c}}k^b)_1 \simeq 0. \end{aligned} \tag{B13}$$

So the first four terms of Eq. (45) under the linear order approximation are expressed as

$$\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(\nabla_c\phi\nabla_b\nabla_a\phi\nabla_d\phi) - \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(\nabla_a\phi\nabla_b\nabla_c\phi\nabla_d\phi) - \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(\nabla_c\phi\nabla_b\phi\nabla_d\nabla_a\phi) + \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(\nabla_a\phi\nabla_b\phi\nabla_d\nabla_c\phi) \simeq 0. \tag{B14}$$

The fifth, sixth, seventh, and eighth terms of Eq. (45) are

$$\begin{aligned} &-\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cd}\nabla_a\phi\nabla_b\phi\nabla_e\nabla^e\phi) + \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ad}\nabla_c\phi\nabla_b\phi\nabla_e\nabla^e\phi) \\ &+\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cb}\nabla_a\phi\nabla_d\phi\nabla_e\nabla^e\phi) - \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ab}\nabla_c\phi\nabla_d\phi\nabla_e\nabla^e\phi) \\ &= \frac{1}{2}[k^a k^b l^c\nabla_b(\nabla_a\phi\nabla_c\phi\nabla_h\nabla^h\phi)](\gamma_{ef}\nabla^e k^f) + \frac{1}{2}[k^a k^b l^c\nabla_c(\nabla_a\phi\nabla_b\phi\nabla_h\nabla^h\phi)](\gamma_{ef}\nabla^e k^f) \\ &-\frac{1}{2}[k^a\gamma^{ef}\nabla_f(\nabla_a\phi\nabla_e\phi\nabla_m\nabla^m\phi)](\gamma_{cd}\nabla^c k^d) + \frac{1}{2}[k^a\gamma_g{}^f\nabla_f(\nabla_a\phi\nabla_b\phi\nabla_m\nabla^m\phi)](\gamma^g{}_c\nabla^c k^b) \\ &+\frac{1}{2}[k^b\gamma_g{}^f\nabla_b(\nabla_f\phi\nabla_d\phi\nabla_h\nabla^h\phi)](\gamma^g{}_e\nabla^e k^d). \end{aligned} \tag{B15}$$

The first term of Eq. (B15) is

$$\begin{aligned} \frac{1}{2}[k^a k^b l^c\nabla_b(\nabla_a\phi\nabla_c\phi\nabla_h\nabla^h\phi)](\gamma_{ef}\nabla^e k^f) &= \frac{1}{2}(\mathcal{L}_k^2\phi)_1(l^c\nabla_c\phi)_0(\nabla_h\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 \\ &+\frac{1}{2}(\mathcal{L}_k\phi)_1(k^b l^c\nabla_b\nabla_c\phi)_0(\nabla_h\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 \\ &+\frac{1}{2}(\mathcal{L}_k\phi)_1(l^c\nabla_c\phi)_0(k^b\nabla_b\nabla_h\nabla^h\phi)_1(B^{\hat{e}}{}_{\hat{e}})_1 \simeq 0. \end{aligned} \tag{B16}$$

The second term of Eq. (B15) is

$$\begin{aligned} \frac{1}{2}[k^a k^b l^c\nabla_c(\nabla_a\phi\nabla_b\phi\nabla_h\nabla^h\phi)](\gamma_{ef}\nabla^e k^f) &= \frac{1}{2}(k^a l^c\nabla_c\nabla_a\phi)_0(\mathcal{L}_k\phi)_1(\nabla_h\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 \\ &+\frac{1}{2}(\mathcal{L}_k\phi)_1(k^a l^c\nabla_c\nabla_a\phi)_0(\nabla_h\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 \\ &+\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^c\nabla_c\nabla_h\nabla^h\phi)(B^{\hat{e}}{}_{\hat{e}})_1 \simeq 0. \end{aligned} \tag{B17}$$

The third term of Eq. (B15) is

$$\begin{aligned} -\frac{1}{2}[k^a\gamma^{ef}\nabla_f(\nabla_a\phi\nabla_e\phi\nabla_m\nabla^m\phi)](\gamma_{cd}\nabla^c k^d) &= -\frac{1}{2}D_{\hat{f}}(\mathcal{L}_k\phi)_1(D^{\hat{f}}\phi)_0(\nabla_m\nabla^m\phi)_0(B^{\hat{c}}{}_{\hat{c}})_1 \\ &-\frac{1}{2}(\mathcal{L}_k\phi)_1(D^{\hat{e}}D_{\hat{e}}\phi)_0(\nabla_m\nabla^m\phi)_0(B^{\hat{c}}{}_{\hat{c}})_1 \\ &-\frac{1}{2}(\mathcal{L}_k\phi)_1(D^{\hat{f}}\phi)_0 D_{\hat{f}}(\nabla_m\nabla^m\phi)_0(B^{\hat{c}}{}_{\hat{c}})_1 \simeq 0. \end{aligned} \tag{B18}$$

The fourth term of Eq. (B15) is

$$\frac{1}{2}[k^a\gamma_g{}^f\nabla_f(\nabla_a\phi\nabla_b\phi\nabla_m\nabla^m\phi)](\gamma^g{}_c\nabla^c k^b) = \frac{1}{2}D_{\hat{c}}(\mathcal{L}_k\phi)_1(\nabla_b\phi\nabla_m\nabla^m\phi)(\nabla^{\hat{c}}k^b)_1 + \frac{1}{2}(\mathcal{L}_k\phi)_1(\nabla_{\hat{c}}\nabla_b\phi\nabla_m\nabla^m\phi)(\nabla^{\hat{c}}k^b)_1 + \frac{1}{2}(\mathcal{L}_k\phi)_1 D_{\hat{c}}(\nabla_m\nabla^m\phi)_0(\nabla_b\phi)(\nabla^{\hat{c}}k^b)_1 \simeq 0. \tag{B19}$$

The fifth term of Eq. (B15) is

$$\frac{1}{2}[k^b\gamma_g{}^f\nabla_b(\nabla_f\phi\nabla_d\phi\nabla_h\nabla^h\phi)](\gamma^g{}_e\nabla^e k^d) = \frac{1}{2}D_{\hat{f}}(\mathcal{L}_k\phi)_1(\nabla_h\nabla^h\phi)_0(\nabla_d\phi)(\nabla^{\hat{f}}k^d)_1 + \frac{1}{2}(D_{\hat{f}}\phi)_0 D_{\hat{d}}(\mathcal{L}_k\phi)_1(\nabla_h\nabla^h\phi)_0(B^{\hat{f}\hat{d}})_1 - \frac{1}{2}(D_{\hat{f}}\phi)_0(\mathcal{L}_k^2\phi)_1(\nabla_h\nabla^h\phi)_0(l^d\nabla^{\hat{f}}k_d)_1 + \frac{1}{2}(D_{\hat{f}}\phi)_0(\nabla_d\phi)[\mathcal{L}_k(\nabla_h\nabla^h\phi)]_1(\nabla^{\hat{f}}k^d)_1 \simeq 0. \tag{B20}$$

So the fifth, sixth, seventh, and eighth terms of Eq. (45) under the linear order approximation are

$$-\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cd}\nabla_a\phi\nabla_b\phi\nabla_e\nabla^e\phi) + \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ad}\nabla_c\phi\nabla_b\phi\nabla_e\nabla^e\phi) + \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cb}\nabla_a\phi\nabla_d\phi\nabla_e\nabla^e\phi) - \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ab}\nabla_c\phi\nabla_d\phi\nabla_e\nabla^e\phi) \simeq 0. \tag{B21}$$

The ninth, tenth, eleventh, and twelfth terms of Eq. (45) are

$$\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cd}\nabla_a\phi\nabla_b\nabla_e\phi\nabla^e\phi) - \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ad}\nabla_c\phi\nabla_b\nabla_e\phi\nabla^e\phi) - \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cb}\nabla_a\phi\nabla_d\nabla_e\phi\nabla^e\phi) + \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ab}\nabla_c\phi\nabla_d\nabla_e\phi\nabla^e\phi)$$
$$= -\frac{1}{2}k^a k^b l^c\nabla_b(\nabla_a\phi\nabla_c\nabla_h\phi\nabla^h\phi)(\gamma_{fe}\nabla^e k^f) + \frac{1}{2}k^b\gamma_g{}^f\nabla_b(\nabla_f\phi\nabla_d\nabla_h\phi\nabla^h\phi)(\gamma^g{}_e\nabla^e k^d) - \frac{1}{2}k^a k^b l^c\nabla_c(\nabla_a\phi\nabla_b\nabla_h\phi\nabla^h\phi)(\gamma^f{}_e\nabla^e k_f) + \frac{1}{2}k^a\gamma^{ef}\nabla_f(\nabla_a\phi\nabla_e\nabla_m\phi\nabla^m\phi)(\gamma^d{}_c\nabla^c k_d) - \frac{1}{2}k^a\gamma_g{}^f\nabla_f(\nabla_a\phi\nabla_b\nabla_m\phi\nabla^m\phi)(\gamma^g{}_c\nabla^c k^b). \tag{B22}$$

The first term of Eq. (B22) is

$$-\frac{1}{2}k^a k^b l^c\nabla_b(\nabla_a\phi\nabla_c\nabla_h\phi\nabla^h\phi)(\gamma_{fe}\nabla^e k^f) = -\frac{1}{2}(\mathcal{L}_k^2\phi)_1(l^c\nabla_c\nabla_h\phi\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 - \frac{1}{2}(\mathcal{L}_k\phi)_1(k^b l^c\nabla_b\nabla_c\nabla_h\phi\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 - \frac{1}{2}(\mathcal{L}_k\phi)_1(k^b l^c\nabla_c\nabla_h\phi\nabla_b\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 \simeq 0. \tag{B23}$$

The second term of Eq. (B22) is

$$\frac{1}{2}k^b\gamma_g{}^f\nabla_b(\nabla_f\phi\nabla_d\nabla_h\phi\nabla^h\phi)(\gamma^g{}_e\nabla^e k^d) = \frac{1}{2}D_{\hat{f}}(\mathcal{L}_k\phi)_1(\nabla_d\nabla_h\phi\nabla^h\phi)(\nabla^{\hat{f}}k^d)_1 + \frac{1}{2}(D_{\hat{f}}\phi)_0(k^b\nabla_b\nabla_{\hat{c}}\nabla_h\phi\nabla^h\phi)_1(B^{\hat{f}\hat{c}})_1 - \frac{1}{2}(D_{\hat{f}}\phi)_0(k^b k^d\nabla_b\nabla_d\nabla_h\phi\nabla^h\phi)_1(l^c\nabla^{\hat{f}}k_c)_1 + \frac{1}{2}(D_{\hat{f}}\phi)_0(k^b\nabla_{\hat{c}}\nabla_h\phi\nabla_b\nabla^h\phi)_1(B^{\hat{f}\hat{c}})_1 - \frac{1}{2}(D_{\hat{f}}\phi)_0(k^b k^d\nabla_d\nabla_h\phi\nabla_b\nabla^h\phi)_1(l^c\nabla^{\hat{f}}k_c)_1 \simeq 0. \tag{B24}$$

The third term of Eq. (B22) is

$$-\frac{1}{2}k^a k^b l^c \nabla_c(\nabla_a\phi\nabla_b\nabla_h\phi\nabla^h\phi)(\gamma^f{}_e\nabla^e k_f) = -\frac{1}{2}(k^a l^c\nabla_c\nabla_a\phi)_0(k^b\nabla_b\nabla_h\phi\nabla^h\phi)_1(B^{\hat{e}}{}_{\hat{e}})_1 \\ -\frac{1}{2}(\mathcal{L}_k\phi)_1(k^b l^c\nabla_c\nabla_b\nabla_h\phi\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 \\ -\frac{1}{2}(\mathcal{L}_k\phi)_1(k^b l^c\nabla_b\nabla_h\phi\nabla_c\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 \simeq 0. \tag{B25}$$

The fourth term of Eq. (B22) is

$$\frac{1}{2}k^a\gamma^{ef}\nabla_f(\nabla_a\phi\nabla_e\nabla_m\phi\nabla^m\phi)(\gamma^d{}_c\nabla^c k_d) = \frac{1}{2}D_{\hat{e}}(\mathcal{L}_k\phi)_1(\nabla^{\hat{e}}\nabla_m\phi\nabla^m\phi)_0(B^{\hat{c}}{}_{\hat{c}})_1 + \frac{1}{2}(\mathcal{L}_k\phi)_1(D^{\hat{e}}D_{\hat{e}}\nabla_m\phi\nabla^m\phi)_0(B^{\hat{c}}{}_{\hat{c}})_1 \\ +\frac{1}{2}(\mathcal{L}_k\phi)_1(\nabla^{\hat{e}}\nabla_m\phi\nabla_{\hat{e}}\nabla^m\phi)_0(B^{\hat{c}}{}_{\hat{c}})_1 \simeq 0. \tag{B26}$$

The fifth term of Eq. (B22) is

$$-\frac{1}{2}k^a\gamma_g{}^f\nabla_f(\nabla_a\phi\nabla_b\nabla_m\phi\nabla^m\phi)(\gamma^g{}_c\nabla^c k^b) = -\frac{1}{2}D_{\hat{f}}(\mathcal{L}_k\phi)_1(\nabla_b\nabla_m\phi\nabla^m\phi)(\nabla^{\hat{f}}k^b)_1 - \frac{1}{2}(\mathcal{L}_k\phi)_1(\nabla_{\hat{f}}\nabla_b\nabla_m\phi\nabla^m\phi)(\nabla^{\hat{f}}k^b)_1 \\ -\frac{1}{2}(\mathcal{L}_k\phi)_1(\nabla_b\nabla_m\phi\nabla_{\hat{f}}\nabla^m\phi)(\nabla^{\hat{f}}k^b)_1 \simeq 0. \tag{B27}$$

Therefore, the ninth, tenth, eleventh, and twelfth terms of Eq. (45) can be expressed as

$$\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cd}\nabla_a\phi\nabla_b\nabla_e\phi\nabla^e\phi) - \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ad}\nabla_c\phi\nabla_b\nabla_e\phi\nabla^e\phi) \\ -\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cb}\nabla_a\phi\nabla_d\nabla_e\phi\nabla^e\phi) + \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ab}\nabla_c\phi\nabla_d\nabla_e\phi\nabla^e\phi) \quad \simeq 0. \tag{B28}$$

The thirteenth, fourteenth, fifteenth, and sixteenth terms of Eq. (45) are

$$-\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cd}\nabla_b\nabla_a\phi\nabla_e\phi\nabla^e\phi) + \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ad}\nabla_b\nabla_c\phi\nabla_e\phi\nabla^e\phi) \\ +\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cb}\nabla_d\nabla_a\phi\nabla_e\phi\nabla^e\phi) - \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ac}\nabla_d\nabla_b\phi\nabla_e\phi\nabla^e\phi) \\ = \frac{1}{2}k^a k^b l^c\nabla_b(\nabla_c\nabla_a\phi\nabla_h\phi\nabla^h\phi)(\gamma_{ef}\nabla^e k^f) - \frac{1}{2}k^b\gamma_g{}^f\nabla_b(\nabla_d\nabla_f\phi\nabla_h\phi\nabla^h\phi)(\gamma^g{}_e\nabla^e k^d) \\ +\frac{1}{2}k^a k^b l^c\nabla_c(\nabla_b\nabla_a\phi\nabla_h\phi\nabla^h\phi)(\gamma^f{}_e\nabla^e k_f) - \frac{1}{2}k^a\gamma^{ef}\nabla_f(\nabla_e\nabla_a\phi\nabla_m\phi\nabla^m\phi)(\gamma^d{}_c\nabla^c k_d) \\ +\frac{1}{2}k^a\gamma_g{}^f\nabla_f(\nabla_b\nabla_a\phi\nabla_m\phi\nabla^m\phi)(\gamma^g{}_c\nabla^c k^b). \tag{B29}$$

The first term of Eq. (B29) is

$$\frac{1}{2}k^a k^b l^c\nabla_b(\nabla_c\nabla_a\phi\nabla_h\phi\nabla^h\phi)(\gamma_{ef}\nabla^e k^f) = \frac{1}{2}(k^a k^b l^c\nabla_b\nabla_c\nabla_a\phi)_1(\nabla_h\phi\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 \\ +\frac{1}{2}(k^a l^c\nabla_c\nabla_a\phi)_0(k^b\nabla_b\nabla_h\phi\nabla^h\phi)_1(B^{\hat{e}}{}_{\hat{e}})_1 \\ +\frac{1}{2}(k^a l^c\nabla_c\nabla_a\phi)_0(k^b\nabla_h\phi\nabla_b\nabla^h\phi)_1(B^{\hat{e}}{}_{\hat{e}})_1 \simeq 0. \tag{B30}$$

The second term of Eq. (B29) is

$$\begin{aligned}-\frac{1}{2}k^b\gamma_g{}^f\nabla_b(\nabla_d\nabla_f\phi\nabla_h\phi\nabla^h\phi)(\gamma^g{}_e\nabla^e k^d) = &-\frac{1}{2}(k^b\nabla_b\nabla_{\hat{c}}\nabla_{\hat{e}}\phi)_1(\nabla_h\phi\nabla^h\phi)_0(B^{\hat{e}\hat{c}})_1\\ &+\frac{1}{2}(k^bk^d\nabla_b\nabla_d\nabla_{\hat{e}}\phi)_1(\nabla_h\phi\nabla^h\phi)_0(l^c\nabla^{\hat{e}}k_c)_1\\ &-\frac{1}{2}(\nabla_d\nabla_{\hat{e}}\phi)(k^b\nabla_b\nabla_h\phi\nabla^h\phi)_1(\nabla^{\hat{e}}k^d)_1\\ &-\frac{1}{2}(\nabla_d\nabla_{\hat{e}}\phi)(k^b\nabla_h\phi\nabla_b\nabla^h\phi)_1(\nabla^{\hat{e}}k^d)_1 \simeq 0.\end{aligned} \tag{B31}$$

The third term of Eq. (B29) is

$$\begin{aligned}\frac{1}{2}k^ak^bl^c\nabla_c(\nabla_b\nabla_a\phi\nabla_h\phi\nabla^h\phi)(\gamma^f{}_e\nabla^e k_f) = &\frac{1}{2}(k^ak^bl^c\nabla_c\nabla_b\nabla_a\phi)_1(\nabla_h\phi\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1\\ &+\frac{1}{2}(\mathcal{L}_k^2\phi)_1(l^c\nabla_c\nabla_h\phi\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1\\ &+\frac{1}{2}(\mathcal{L}_k^2\phi)_1(l^c\nabla_h\phi\nabla_c\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 \simeq 0.\end{aligned} \tag{B32}$$

The fourth term of Eq. (B29) is

$$\begin{aligned}-\frac{1}{2}k^a\gamma^{ef}\nabla_f(\nabla_e\nabla_a\phi\nabla_m\phi\nabla^m\phi)(\gamma^d{}_c\nabla^c k_d) = &-\frac{1}{2}(k^a\nabla^{\hat{e}}\nabla_{\hat{e}}\nabla_a\phi)_1(\nabla_m\phi\nabla^m\phi)_0(B^{\hat{c}}{}_{\hat{c}})_1\\ &-\frac{1}{2}D_{\hat{e}}(\mathcal{L}_k\phi)_1(\nabla^{\hat{e}}\nabla_m\phi\nabla^m\phi)_0(B^{\hat{c}}{}_{\hat{c}})_1\\ &-\frac{1}{2}D_{\hat{e}}(\mathcal{L}_k\phi)_1(\nabla_m\phi\nabla^{\hat{e}}\nabla^m\phi)_0(B^{\hat{c}}{}_{\hat{c}})_1 \simeq 0.\end{aligned} \tag{B33}$$

The fifth term of Eq. (B29) is

$$\begin{aligned}\frac{1}{2}k^a\gamma_g{}^f\nabla_f(\nabla_b\nabla_a\phi\nabla_m\phi\nabla^m\phi)(\gamma^g{}_c\nabla^c k^b) = &\frac{1}{2}(k^a\nabla_{\hat{c}}\nabla_{\hat{b}}\nabla_a\phi)_1(\nabla_m\phi\nabla^m\phi)_0(B^{\hat{c}\hat{b}})_1 - \frac{1}{2}(k^ak^b\nabla_{\hat{c}}\nabla_b\nabla_a\phi)_1(\nabla_m\phi\nabla^m\phi)_0(l^d\nabla^{\hat{c}}k_d)_1\\ &+\frac{1}{2}D_{\hat{b}}(\mathcal{L}_k\phi)_1(\nabla_{\hat{c}}\nabla_m\phi\nabla^m\phi)_0(B^{\hat{c}\hat{b}})_1 - \frac{1}{2}(\mathcal{L}_k^2\phi)_1(\nabla_{\hat{c}}\nabla_m\phi\nabla^m\phi)_0(l^d\nabla^{\hat{c}}k_d)_1\\ &+\frac{1}{2}D_{\hat{b}}(\mathcal{L}_k\phi)_1(\nabla_m\phi\nabla_{\hat{c}}\nabla^m\phi)_0(B^{\hat{c}\hat{b}})_1 - \frac{1}{2}(\mathcal{L}_k^2\phi)_1(\nabla_m\phi\nabla_{\hat{c}}\nabla^m\phi)_0(l^d\nabla^{\hat{c}}k_d)_1 \simeq 0.\end{aligned} \tag{B34}$$

So the thirteenth, fourteenth, fifteenth, and sixteenth terms of Eq. (45) under the linear order approximation are given as

$$\begin{aligned}&-\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cd}\nabla_b\nabla_a\phi\nabla_e\phi\nabla^e\phi) + \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ad}\nabla_b\nabla_c\phi\nabla_e\phi\nabla^e\phi)\\ &+\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cb}\nabla_d\nabla_a\phi\nabla_e\phi\nabla^e\phi) - \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ac}\nabla_d\nabla_b\phi\nabla_e\phi\nabla^e\phi) \simeq 0.\end{aligned} \tag{B35}$$

The seventeenth, eighteenth, nineteenth, and twentieth terms of Eq. (45) are

$$\begin{aligned}&\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cd}\nabla_b\phi\nabla_e\nabla_a\phi\nabla^e\phi)-\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cb}\nabla_d\phi\nabla_e\nabla_a\phi\nabla^e\phi)\\&-\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ad}\nabla_b\phi\nabla_e\nabla_c\phi\nabla^e\phi)+\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cb}\nabla_d\phi\nabla_e\nabla_c\phi\nabla^e\phi)\\&=-\frac{1}{2}k^b\nabla_b(k^a l^c\nabla_c\phi\nabla_h\nabla_a\phi\nabla^h\phi)(\gamma_{ef}\nabla^e k^f)+\frac{1}{2}k^b\gamma_g{}^f\nabla_b(\nabla_d\phi\nabla_h\nabla_f\phi\nabla^h\phi)(\gamma^g{}_e\nabla^e k^d)\\&-\frac{1}{2}k^a k^b l^c\nabla_c(\nabla_b\phi\nabla_h\nabla_a\phi\nabla^h\phi)(\gamma^f{}_e\nabla^e k_f)+\frac{1}{2}k^a\gamma^{ef}\nabla_f(\nabla_e\phi\nabla_m\nabla_a\phi\nabla^m\phi)(\gamma^d{}_c\nabla^c k_d)\\&-\frac{1}{2}k^a\gamma_g{}^f\nabla_f(\nabla_b\phi\nabla_m\nabla_a\phi\nabla^m\phi)(\gamma^g{}_c\nabla^c k^b).\end{aligned}\tag{B36}$$

The first term of Eq. (B36) is

$$\begin{aligned}-\frac{1}{2}k^b\nabla_b(k^a l^c\nabla_c\phi\nabla_h\nabla_a\phi\nabla^h\phi)(\gamma_{ef}\nabla^e k^f)=&-\frac{1}{2}(k^b l^c\nabla_b\nabla_c\phi)_1(k^a\nabla_h\nabla_a\phi\nabla^h\phi)_1(B^{\hat{e}}{}_{\hat{e}})_1\\&-\frac{1}{2}(l^c\nabla_c\phi)_0(k^a k^b\nabla_b\nabla_h\nabla_a\phi\nabla^h\phi)_1(B^{\hat{e}}{}_{\hat{e}})_1\\&-\frac{1}{2}(l^c\nabla_c\phi)_0(k^a k^b\nabla_h\nabla_a\phi\nabla_b\nabla^h\phi)_1(B^{\hat{e}}{}_{\hat{e}})_1\simeq 0.\end{aligned}\tag{B37}$$

The second term of Eq. (B36) is

$$\begin{aligned}\frac{1}{2}k^b\gamma_g{}^f\nabla_b(\nabla_d\phi\nabla_h\nabla_f\phi\nabla^h\phi)(\gamma^g{}_e\nabla^e k^d)=&\frac{1}{2}D_{\hat{c}}(\mathcal{L}_k\phi)_1(\nabla_h\nabla_{\hat{d}}\phi\nabla^h\phi)_0(B^{\hat{d}\hat{c}})_1\\&-\frac{1}{2}(\mathcal{L}_k^2\phi)_1(\nabla_h\nabla_{\hat{e}}\phi\nabla^h\phi)_0(l^c\nabla^{\hat{e}}k_c)_1\\&+\frac{1}{2}(\nabla_d\phi)(k^b\nabla_b\nabla_h\nabla_{\hat{e}}\phi\nabla^h\phi)_1(\nabla^{\hat{e}}k^d)_1\\&+\frac{1}{2}(\nabla_d\phi)(k^b\nabla_h\nabla_{\hat{e}}\phi\nabla_b\nabla^h\phi)_1(\nabla^{\hat{e}}k^d)_1\simeq 0.\end{aligned}\tag{B38}$$

The third term of Eq. (B36) is

$$\begin{aligned}-\frac{1}{2}k^a k^b l^c\nabla_c(\nabla_b\phi\nabla_h\nabla_a\phi\nabla^h\phi)(\gamma^f{}_e\nabla^e k_f)=&-\frac{1}{2}(k^b l^c\nabla_c\nabla_b\phi)_0(k^a\nabla_h\nabla_a\phi\nabla^h\phi)_1(B^{\hat{e}}{}_{\hat{e}})_1\\&-\frac{1}{2}(\mathcal{L}_k\phi)_1(k^a l^c\nabla_c\nabla_h\nabla_a\phi\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1\\&-\frac{1}{2}(\mathcal{L}_k\phi)_1(k^a l^c\nabla_h\nabla_a\phi\nabla_c\nabla^h\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1\simeq 0.\end{aligned}\tag{B39}$$

The fourth term of Eq. (B36) is

$$\begin{aligned}\frac{1}{2}k^a\gamma^{ef}\nabla_f(\nabla_e\phi\nabla_m\nabla_a\phi\nabla^m\phi)(\gamma^d{}_c\nabla^c k_d)=&\frac{1}{2}(D_{\hat{e}}D^{\hat{e}}\phi)_0(k^a\nabla_m\nabla_a\phi\nabla^m\phi)_1(B^{\hat{c}}{}_{\hat{c}})_1\\&+\frac{1}{2}(D_{\hat{e}}\phi)_0(k^a\nabla^{\hat{e}}\nabla_m\nabla_a\phi\nabla^m\phi)_1(B^{\hat{c}}{}_{\hat{c}})_1\\&+\frac{1}{2}(D_{\hat{e}}\phi)_0(k^a\nabla_m\nabla_a\phi\nabla^{\hat{e}}\nabla^m\phi)_1(B^{\hat{c}}{}_{\hat{c}})_1\simeq 0.\end{aligned}\tag{B40}$$

The fifth term of Eq. (B36) is

$$-\frac{1}{2}k^a\gamma_g{}^f\nabla_f(\nabla_b\phi\nabla_m\nabla_a\phi\nabla^m\phi)(\gamma^g{}_c\nabla^c k^b) = -\frac{1}{2}(\nabla_{\hat{c}}\nabla_b\phi)(k^a\nabla_m\nabla_a\phi\nabla^m\phi)_1(\nabla^{\hat{c}}k^b)_1 \\ -\frac{1}{2}(\nabla_b\phi)(k^a\nabla_{\hat{c}}\nabla_m\nabla_a\phi\nabla^m\phi)_1(\nabla^{\hat{c}}k^b)_1 \\ -\frac{1}{2}(\nabla_b\phi)(k^a\nabla_m\nabla_a\phi\nabla_{\hat{c}}\nabla^m\phi)_1(\nabla^{\hat{c}}k^b)_1 \simeq 0. \tag{B41}$$

So the seventeenth, eighteenth, nineteenth, and twentieth terms of Eq. (45) can be expressed as

$$\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cd}\nabla_b\phi\nabla_e\nabla_a\phi\nabla^e\phi) - \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cb}\nabla_d\phi\nabla_e\nabla_a\phi\nabla^e\phi) - \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ad}\nabla_b\phi\nabla_e\nabla_c\phi\nabla^e\phi) \\ + \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{cb}\nabla_d\phi\nabla_e\nabla_c\phi\nabla^e\phi) \simeq 0. \tag{B42}$$

The twenty-first and the twenty-second terms of Eq. (45) are

$$-\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ad}g_{cb}\nabla_e\phi\nabla^e\nabla_f\phi\nabla^f\phi) + \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ab}g_{cd}\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi) \\ = -\frac{1}{2}[\mathcal{L}_k(\nabla_m\phi\nabla^m\phi\nabla_n\nabla^n\phi)]_1(B^{\hat{e}}{}_{\hat{e}})_1 \simeq 0. \tag{B43}$$

The twenty-third and the twenty-fourth terms of Eq. (45) are

$$\frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ad}g_{cb}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi) - \frac{1}{2}(k^a\nabla^c k^b)\nabla^d(g_{ab}g_{cd}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi) \\ = -\frac{1}{2}[\mathcal{L}_k(\nabla^m\phi\nabla_n\nabla_m\phi\nabla^n\phi)]_1(B^{\hat{e}}{}_{\hat{e}})_1 \simeq 0. \tag{B44}$$

Therefore, according to the above results, the integrand $H_{kk}^{(G2)2}$ in the second integral of Eq. (43) is vanishing under the linear order approximation of the perturbation, i.e.,

$$H_{kk}^{(G2)2} \simeq 0. \tag{B45}$$

In the second part of Appendix B, we would like to calculate the integrand $H_{kk}^{(G2)3}$ in the third integral of Eq. (43) to obtain the expression under the first-order approximation. The specific expression of $H_{kk}^{(G2)3}$ is given in Eq. (47). Following the calculation method in Appendix A, the first four terms of Eq. (47) are further calculated as

$$-\frac{1}{2}(k^b\nabla^d l^a)k^c(\nabla_a\phi\nabla_b\nabla_c\phi\nabla_d\phi) + \frac{1}{2}(k^b\nabla^d l^a)k^c(\nabla_c\phi\nabla_b\nabla_a\phi\nabla_d\phi) + \frac{1}{2}(k^b\nabla^d l^a)k^c(\nabla_a\phi\nabla_b\phi\nabla_d\nabla_c\phi) \\ -\frac{1}{2}(k^b\nabla^d l^a)k^c(\nabla_c\phi\nabla_b\phi\nabla_d\nabla_a\phi) \\ = \frac{1}{2}(\mathcal{L}_k^2\phi)_1(D_{\hat{e}}\phi)_0(l^d\nabla_d\phi)_0(k^a\nabla_a l^{\hat{e}})_1 - \frac{1}{2}(\mathcal{L}_k\phi)_1 D_{\hat{e}}(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0(k^a\nabla_a l^{\hat{e}})_1 \\ -\frac{1}{2}(D_{\hat{e}}\phi)_0(\mathcal{L}_k\phi)_1(k^b l^d\nabla_b\nabla_d\phi)_0(k^a\nabla_a l^{\hat{e}})_1 + \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^d\nabla_d\nabla_{\hat{e}}\phi)_0(k^a\nabla_a l^{\hat{e}})_1 - \frac{1}{2}(\mathcal{L}_k^2\phi)_1(D_{\hat{c}}\phi)_0(D_{\hat{d}}\phi)_0(\nabla^{\hat{d}}l^{\hat{c}})_0 \\ + \frac{1}{2}(\mathcal{L}_k\phi)_1 D_{\hat{c}}(\mathcal{L}_k\phi)_1(D_{\hat{d}}\phi)_0(\nabla^{\hat{d}}l^{\hat{c}})_0 + \frac{1}{2}(\mathcal{L}_k\phi)_1 D_{\hat{d}}(\mathcal{L}_k\phi)_1(D_{\hat{c}}\phi)_0(\nabla^{\hat{d}}l^{\hat{c}})_0 - \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(D_{\hat{c}}D_{\hat{d}}\phi)_0(\nabla^{\hat{d}}l^{\hat{c}})_0 \\ -\frac{1}{2}(\mathcal{L}_k^2\phi)_1(l^d\nabla_d\phi)_0(D_{\hat{e}}\phi)_0(l_a\nabla^{\hat{e}}k^a)_1 + \frac{1}{2}(\mathcal{L}_k\phi)_1(k^c l^d\nabla_c\nabla_d\phi)_0(D_{\hat{e}}\phi)_0(l_a\nabla^{\hat{e}}k^a)_1 \\ + \frac{1}{2}(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0 D_{\hat{e}}(\mathcal{L}_k\phi)_1(l_a\nabla^{\hat{e}}k^a)_1 - \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^d\nabla_{\hat{e}}\nabla_d\phi)_0(l_a\nabla^{\hat{e}}k^a)_1 \simeq -\frac{1}{2}(\mathcal{L}_k^2\phi)(D_{\hat{c}}\phi)(D_{\hat{d}}\phi)(\nabla^{\hat{d}}l^{\hat{c}}) \sim \mathcal{O}(\epsilon). \tag{B46}$$

The fifth, sixth, seventh, and eighth terms of Eq. (47) are

$$\begin{aligned}&\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ad}\nabla_c\phi\nabla_b\phi\nabla_e\nabla^e\phi)-\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cd}\nabla_a\phi\nabla_b\phi\nabla_e\nabla^e\phi)\\&\quad-\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ab}\nabla_c\phi\nabla_d\phi\nabla_e\nabla^e\phi)+\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cb}\nabla_a\phi\nabla_d\phi\nabla_e\nabla^e\phi)\\&=-\frac{1}{2}(\mathcal{L}_k\phi)_1(D_{\hat{e}}\phi)_0(\nabla_m\nabla^m\phi)_0(k^a\nabla_a l^{\hat{e}})_1+\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(\nabla_m\nabla^m\phi)_0(\nabla^{\hat{c}}l_{\hat{c}})_0\\&\quad+\frac{1}{2}(\mathcal{L}_k\phi)_1(D_{\hat{e}}\phi)_0(\nabla_m\nabla^m\phi)_0(l_a\nabla^{\hat{e}}k^a)_1\simeq 0.\end{aligned}\tag{B47}$$

The ninth, tenth, eleventh, and twelfth terms of Eq. (47) are

$$\begin{aligned}&-\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ad}\nabla_c\phi\nabla_b\nabla_e\phi\nabla^e\phi)+\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cd}\nabla_a\phi\nabla_b\nabla_e\phi\nabla^e\phi)\\&\quad+\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ab}\nabla_c\phi\nabla_d\nabla_e\phi\nabla^e\phi)+\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cb}\nabla_a\phi\nabla_d\nabla_e\phi\nabla^e\phi)\\&=\frac{1}{2}(D_{\hat{e}}\phi)_0(k^c\nabla_c\nabla_m\phi\nabla^m\phi)_1(k^a\nabla_a l^{\hat{e}})_1-\frac{1}{2}(\mathcal{L}_k\phi)_1(k^b\nabla_b\nabla_m\phi\nabla^m\phi)_1(\nabla^{\hat{c}}l_{\hat{c}})_0\\&\quad-\frac{1}{2}(\mathcal{L}_k\phi)_1(\nabla_{\hat{e}}\nabla_m\phi\nabla^m\phi)_0(l_a\nabla^{\hat{e}}k^a)_1\simeq 0.\end{aligned}\tag{B48}$$

The thirteenth, fourteenth, fifteenth, and sixteenth terms of Eq. (47) are

$$\begin{aligned}&\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ad}\nabla_b\nabla_c\phi\nabla_e\phi\nabla^e\phi)-\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cd}\nabla_b\nabla_a\phi\nabla_e\phi\nabla^e\phi)\\&\quad-\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ab}\nabla_d\nabla_c\phi\nabla_e\phi\nabla^e\phi)+\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cb}\nabla_d\nabla_a\phi\nabla_e\phi\nabla^e\phi)\\&=-\frac{1}{2}D_{\hat{e}}(\mathcal{L}_k\phi)_1(\nabla_m\phi\nabla^m\phi)_0(k^a\nabla_a l^{\hat{e}})_1+\frac{1}{2}(\mathcal{L}_k^2\phi)_1(D_{\hat{e}}\phi\nabla^{\hat{e}}\phi)_0(\nabla^{\hat{c}}l_{\hat{c}})_0-(\mathcal{L}_k^2\phi)_1(\mathcal{L}_k\phi)_1(l^e\nabla_e\phi)_0(\nabla^{\hat{c}}l_{\hat{c}})_0\\&\quad+\frac{1}{2}D_{\hat{e}}(\mathcal{L}_k\phi)_1(\nabla_m\phi\nabla^m\phi)_0(l_a\nabla^{\hat{e}}k^a)_1\simeq\frac{1}{2}(\mathcal{L}_k^2\phi)(D_{\hat{e}}\phi\nabla^{\hat{e}}\phi)(\nabla^{\hat{c}}l_{\hat{c}})\sim\mathcal{O}(\epsilon).\end{aligned}\tag{B49}$$

The seventh, eighteenth, nineteenth, and twentieth terms of Eq. (47) are

$$\begin{aligned}&-\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ad}\nabla_b\phi\nabla_e\nabla_c\phi\nabla^e\phi)+\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ab}\nabla_d\phi\nabla_e\nabla_c\phi\nabla^e\phi)\\&\quad+\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cd}\nabla_b\phi\nabla_e\nabla_a\phi\nabla^e\phi)-\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cb}\nabla_d\phi\nabla_e\nabla_a\phi\nabla^e\phi)\\&=\frac{1}{2}(\mathcal{L}_k\phi)_1(\nabla^m\phi\nabla_m\nabla_{\hat{e}}\phi)_0(k^a\nabla_a l^{\hat{e}})_1+\frac{1}{2}(\mathcal{L}_k\phi)_1(k^a\nabla_m\nabla_a\phi\nabla^m\phi)_1(\nabla^{\hat{c}}l_{\hat{c}})_0\\&\quad-\frac{1}{2}(D_{\hat{e}}\phi)_0(k^b\nabla_m\nabla_b\phi\nabla^m\phi)_1(l_a\nabla^{\hat{e}}k^a)_1\simeq 0.\end{aligned}\tag{B50}$$

The twenty-first and twenty-second terms of Eq. (47) are

$$\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cd}g_{ab}\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi)-\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cb}g_{ad}\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi)=0.\tag{B51}$$

The twenty-third and twenty-fourth terms of Eq. (47) are

$$-\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cd}g_{ab}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi)+\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cb}g_{ad}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi)=0.\tag{B52}$$

So Eq. (47) under the first-order approximation can finally be written as

$$H_{kk}^{(G2)3} \simeq -\frac{1}{2}(\mathcal{L}_k^2\phi)(D_{\hat{c}}\phi)(D_{\hat{d}}\phi)(\nabla^{\hat{d}}l^{\hat{c}}) + \frac{1}{2}(\mathcal{L}_k^2\phi)(D_{\hat{e}}\phi D^{\hat{e}}\phi)(\nabla^{\hat{c}}l_{\hat{c}}). \tag{B53}$$

The third part of this appendix focuses on the simplification of the integrand $H_{kk}^{(G2)4}$ under the first-order approximation of the perturbation. Using the same calculation method in the first two parts of this appendix, the first four terms of Eq. (49) can be further calculated as

$$\begin{aligned}
&-\frac{1}{2}(k^b\nabla^d l^a)k^c(\nabla_a\phi\nabla_b\nabla_c\phi\nabla_d\phi) + \frac{1}{2}(k^b\nabla^d l^a)k^c(\nabla_c\phi\nabla_b\nabla_a\phi\nabla_d\phi)\\
&\quad + \frac{1}{2}(k^b\nabla^d l^a)k^c(\nabla_a\phi\nabla_b\phi\nabla_d\nabla_c\phi) - \frac{1}{2}(k^b\nabla^d l^a)k^c(\nabla_c\phi\nabla_b\phi\nabla_d\nabla_a\phi)\\
&= \frac{1}{2}(k^a l^b\nabla_a\nabla_b\phi)_0(D_{\hat{e}}\phi)_0(D_{\hat{f}}\phi)_0(B^{\hat{e}\hat{f}})_1 - \frac{1}{2}(l^b\nabla_b\phi)_0(k^a\nabla_a\nabla_{\hat{e}}\phi\nabla_{\hat{f}}\phi)_1(B^{\hat{e}\hat{f}})_1\\
&\quad - \frac{1}{2}(\mathcal{L}_k\phi)_1(D_{\hat{e}}\phi)_0(l^b\nabla_{\hat{f}}\nabla_b\phi)_0(B^{\hat{e}\hat{f}})_1 + \frac{1}{2}(\mathcal{L}_k\phi)_1(l^b\nabla_b\phi)_0(D_{\hat{f}}D_{\hat{e}}\phi)_0(B^{\hat{e}\hat{f}})_1\\
&\quad + \frac{1}{2}(l^d\nabla_d\phi)_0(\mathcal{L}_k^2\phi)_1(D_{\hat{e}}\phi)_0(l^c\nabla^{\hat{e}}k_c)_1 - \frac{1}{2}(\mathcal{L}_k\phi)_1(k^b l^d\nabla_b\nabla_d\phi)_0(D_{\hat{e}}\phi)_0(l^c\nabla^{\hat{e}}k_c)_1\\
&\quad - \frac{1}{2}(l^d\nabla_d\phi)_0(\mathcal{L}_k\phi)_1(k^a\nabla_{\hat{e}}\nabla_a\phi)_1(l^c\nabla^{\hat{e}}k_c)_1 + \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^d\nabla_{\hat{e}}\nabla_d\phi)_0(l^c\nabla^{\hat{e}}k_c)_1\\
&\simeq \frac{1}{2}(k^a l^b\nabla_a\nabla_b\phi)(D_{\hat{e}}\phi D_{\hat{f}}\phi)(B^{\hat{e}\hat{f}}) \sim \mathcal{O}(\epsilon).
\end{aligned} \tag{B54}$$

The fifth, sixth, seventh, and eighth terms of Eq. (49) are

$$\begin{aligned}
&\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ad}\nabla_c\phi\nabla_b\phi\nabla_e\nabla^e\phi) - \frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cd}\nabla_a\phi\nabla_b\phi\nabla_e\nabla^e\phi)\\
&\quad - \frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ab}\nabla_c\phi\nabla_d\phi\nabla_e\nabla^e\phi) + \frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cb}\nabla_a\phi\nabla_d\phi\nabla_e\nabla^e\phi)\\
&\quad = -\frac{1}{2}(\mathcal{L}_k\phi)_1(l^b\nabla_b\phi)_0(\nabla_m\nabla^m\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 + \frac{1}{2}(D_{\hat{e}}\phi)_0(D_{\hat{f}}\phi)_0(\nabla_m\nabla^m\phi)_0(B^{\hat{e}\hat{f}})_1\\
&\quad - \frac{1}{2}(\mathcal{L}_k\phi)_1(D_{\hat{e}}\phi)_0(\nabla_b\nabla^b\phi)_0(l^c\nabla^{\hat{e}}k_c)_1 \simeq \frac{1}{2}(D_{\hat{e}}\phi D_{\hat{f}}\phi)(\nabla_m\nabla^m\phi)(B^{\hat{e}\hat{f}}) \sim \mathcal{O}(\epsilon).
\end{aligned} \tag{B55}$$

The ninth, tenth, eleventh, and twelfth terms of Eq. (49) are

$$\begin{aligned}
&-\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ad}\nabla_c\phi\nabla_b\nabla_e\phi\nabla^e\phi) + \frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cd}\nabla_a\phi\nabla_b\nabla_e\phi\nabla^e\phi)\\
&\quad + \frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ab}\nabla_c\phi\nabla_d\nabla_e\phi\nabla^e\phi) - \frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cb}\nabla_a\phi\nabla_d\nabla_e\phi\nabla^e\phi)\\
&= \frac{1}{2}(l^b\nabla_b\phi)_0(k^a\nabla_a\nabla_m\phi\nabla^m\phi)_1(B^{\hat{e}}{}_{\hat{e}})_1 - \frac{1}{2}(D_{\hat{e}}\phi)_0(\nabla_{\hat{f}}\nabla_m\phi\nabla^m\phi)_0(B^{\hat{e}\hat{f}})_1\\
&\quad + \frac{1}{2}(\mathcal{L}_k\phi)_1(\nabla_{\hat{e}}\nabla_m\phi\nabla^m\phi)_0(l^c\nabla^{\hat{e}}k_c)_1 \simeq -\frac{1}{2}(D_{\hat{e}}\phi\nabla_{\hat{f}}\nabla_m\phi\nabla^m\phi)(B^{\hat{e}\hat{f}}) \sim \mathcal{O}(\epsilon).
\end{aligned} \tag{B56}$$

The thirteenth, fourteenth, fifteenth, and sixteenth terms of Eq. (49) are

$$\begin{aligned}
&\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ad}\nabla_b\nabla_c\phi\nabla_e\phi\nabla^e\phi) - \frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cd}\nabla_b\nabla_a\phi\nabla_e\phi\nabla^e\phi)\\
&\quad - \frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ab}\nabla_d\nabla_c\phi\nabla_e\phi\nabla^e\phi) + \frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cb}\nabla_d\nabla_a\phi\nabla_e\phi\nabla^e\phi)\\
&= -\frac{1}{2}(k^a l^b\nabla_a\nabla_b\phi)_0(\nabla_m\phi\nabla^m\phi)_0(B^{\hat{e}}{}_{\hat{e}}) + \frac{1}{2}(D_{\hat{f}}D_{\hat{e}}\phi)_0(\nabla_m\phi\nabla^m\phi)_0(B^{\hat{e}\hat{f}})_1 - \frac{1}{2}D_{\hat{e}}(\mathcal{L}_k\phi)_1(\nabla_m\phi\nabla^m\phi)_0(l^c\nabla^{\hat{e}}k_c)_1\\
&\simeq -\frac{1}{2}(k^a l^b\nabla_a\nabla_b\phi)(\nabla_m\phi\nabla^m\phi)(B^{\hat{e}}{}_{\hat{e}}) + \frac{1}{2}(D_{\hat{f}}D_{\hat{e}}\phi\nabla_m\phi\nabla^m\phi)(B^{\hat{e}\hat{f}}) \sim \mathcal{O}(\epsilon).
\end{aligned} \tag{B57}$$

The seventeenth, eighteenth, nineteenth, and twentieth terms of Eq. (49) are

$$\begin{aligned}
&-\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ad}\nabla_b\phi\nabla_e\nabla_c\phi\nabla^e\phi) + \frac{1}{2}(k^b\nabla^d l^a)k^c(g_{ab}\nabla_d\phi\nabla_e\nabla_c\phi\nabla^e\phi)\\
&\quad + \frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cd}\nabla_b\phi\nabla_e\nabla_a\phi\nabla^e\phi) - \frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cb}\nabla_d\phi\nabla_e\nabla_a\phi\nabla^e\phi)\\
&= \frac{1}{2}(\mathcal{L}_k\phi)_1(l^b\nabla_m\nabla_b\phi\nabla^m\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 - \frac{1}{2}(D_{\hat{f}}\phi)_0(\nabla_m\nabla_{\hat{e}}\phi\nabla^m\phi)_0(B^{\hat{e}\hat{f}})_1\\
&\quad + \frac{1}{2}(k^a\nabla_h\nabla_a\phi\nabla^h\phi)_1(D_{\hat{e}}\phi)_0(l^c\nabla^{\hat{e}}k_c)_1 \simeq -\frac{1}{2}(D_{\hat{f}}\phi\nabla_m\nabla_{\hat{e}}\phi\nabla^m\phi)(B^{\hat{e}\hat{f}}) \sim \mathcal{O}(\epsilon).
\end{aligned} \tag{B58}$$

The twenty-first and twenty-second terms of Eq. (49) are

$$\begin{aligned}
&\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cd}g_{ab}\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi) - \frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cb}g_{ad}\nabla_e\phi\nabla^e\phi\nabla_f\nabla^f\phi)\\
&= -\frac{1}{2}(\nabla^m\phi\nabla_m\phi\nabla_n\nabla^n\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 \sim \mathcal{O}(\epsilon).
\end{aligned} \tag{B59}$$

The twenty-third and twenty-fourth terms of Eq. (49) are

$$\begin{aligned}
&-\frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cd}g_{ab}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi) + \frac{1}{2}(k^b\nabla^d l^a)k^c(g_{cb}g_{ad}\nabla^e\phi\nabla_f\nabla_e\phi\nabla^f\phi)\\
&= \frac{1}{2}(\nabla^m\phi\nabla_n\nabla_m\phi\nabla^n\phi)_0(B^{\hat{e}}{}_{\hat{e}})_1 \sim \mathcal{O}(\epsilon).
\end{aligned} \tag{B60}$$

So the remainder terms of $H_{kk}^{(G2)4}$ under the first-order approximation are

$$\begin{aligned}
H_{kk}^{(G2)4} \simeq{}& \frac{1}{2}(k^a l^b\nabla_a\nabla_b\phi)(D_{\hat{e}}\phi D_{\hat{f}}\phi)(B^{\hat{e}\hat{f}}) + \frac{1}{2}(D_{\hat{e}}\phi D_{\hat{f}}\phi)(\nabla_m\nabla^m\phi)(B^{\hat{e}\hat{f}})\\
&- \frac{1}{2}(D_{\hat{e}}\phi\nabla_{\hat{f}}\nabla_m\phi\nabla^m\phi)(B^{\hat{e}\hat{f}}) - \frac{1}{2}(k^a l^b\nabla_a\nabla_b\phi)(\nabla_m\phi\nabla^m\phi)(B^{\hat{e}}{}_{\hat{e}}) + \frac{1}{2}(D_{\hat{f}}D_{\hat{e}}\phi\nabla_m\phi\nabla^m\phi)(B^{\hat{e}\hat{f}})\\
&- \frac{1}{2}(D_{\hat{f}}\phi\nabla_m\nabla_{\hat{e}}\phi\nabla^m\phi)(B^{\hat{e}\hat{f}}) - \frac{1}{2}(\nabla^m\phi\nabla_m\phi\nabla_n\nabla^n\phi)(B^{\hat{e}}{}_{\hat{e}}) + \frac{1}{2}(\nabla^m\phi\nabla_n\nabla_m\phi\nabla^n\phi)(B^{\hat{e}}{}_{\hat{e}}).
\end{aligned} \tag{B61}$$

APPENDIX C: SIMPLIFICATION OF $H_{kk}^{(\phi 1)}$ UNDER THE LINEAR ORDER APPROXIMATION

In this appendix, we hope to simplify $H_{kk}^{(\phi 1)}$ to obtain its expression under the first-order approximation of the perturbation. The specific expression of $H_{kk}^{(\phi 1)}$ is given in Eq. (37). The first term of Eq. (37) is

$$\frac{1}{4}k^a k^b R\nabla^c\phi\nabla_a\nabla_c\phi\nabla_b\phi = \frac{1}{8}(R)_0[\mathcal{L}_k(\nabla_c\phi\nabla^c\phi)]_1(\mathcal{L}_k\phi)_1 \simeq 0. \tag{C1}$$

The second term of Eq. (37) is

$$
\begin{aligned}
-\frac{1}{2}k^a k^b R_{cd}\nabla^c\phi\nabla_a\nabla^d\phi\nabla_b\phi = &-\frac{1}{2}(k^d R_{df} l^f)_0[(\mathcal{L}_k\phi)_1]^2(k^c l^e\nabla_c\nabla_e\phi)_0 + \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^d R_{d\hat{e}})_0 D^{\hat{e}}(\mathcal{L}_k\phi)_1 \\
&-\frac{1}{2}(\mathcal{L}_k\phi)_1 D^{\hat{c}}(\mathcal{L}_k\phi)_1(D^{\hat{d}}\phi)_0(R_{\hat{c}\hat{d}})_0 + \frac{1}{2}(\mathcal{L}_k\phi)_1 D^{\hat{e}}(\mathcal{L}_k\phi)_1(k^c R_{c\hat{e}})_1(l^d\nabla_d\phi)_0 \\
&-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(\mathcal{L}_k^2\phi)_1(l^e l^f R_{ef})_0 - \frac{1}{2}(\mathcal{L}_k\phi)_1(k^b l^f\nabla_b\nabla_f\phi)_0(l^e\nabla_e\phi)_0(k^c k^d R_{cd})_1 \\
&-\frac{1}{2}(\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)(l^e\nabla_e\phi)_0(k^d l^f R_{df})_0 + \frac{1}{2}(\mathcal{L}_k\phi)_1(k^b l^d\nabla_b\nabla_d\phi)_0(D^{\hat{e}}\phi)_0(k^c R_{c\hat{e}})_1 \\
&+\frac{1}{2}(\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)_1(D^{\hat{e}}\phi)_0(l^d R_{d\hat{e}})_0 \simeq 0.
\end{aligned}
\tag{C2}
$$

The third term of Eq. (37) is

$$
\frac{1}{4}k^a k^b R\nabla^c\phi\nabla_c\nabla_a\phi\nabla_b\phi = \frac{1}{4}(R)_0(\mathcal{L}_k\phi)_1(D^{\hat{c}}\phi)_0 D_{\hat{c}}(\mathcal{L}_k\phi)_1 \simeq 0. \tag{C3}
$$

The fourth term of Eq. (37) is

$$
-\frac{1}{2}k^a k^b R\nabla_a\phi\nabla_c\nabla^c\phi\nabla_b\phi = -\frac{1}{2}(R)_0[(\mathcal{L}_k\phi)_1]^2(\nabla_c\nabla^c\phi)_0 \simeq 0. \tag{C4}
$$

The fifth term of Eq. (37) is

$$
\begin{aligned}
-k^a k^b R_{ad}\nabla^c\phi\nabla_c\nabla^d\phi\nabla_b\phi = &(\mathcal{L}_k\phi)_1 D^{\hat{e}}(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0(k^c R_{c\hat{e}})_1 + (k^c R_{c\hat{e}})_1[(\mathcal{L}_k\phi)_1]^2(l^d\nabla_d\nabla^{\hat{e}}\phi)_0 \\
&-(\mathcal{L}_k\phi)_1(k^b l^f\nabla_b\nabla_f\phi)_0(l^e\nabla_e\phi)_0(R_{kk})_1 - (\mathcal{L}_k\phi)_1(l^e\nabla_e\phi)_0(\mathcal{L}_k^2\phi)_1(k^d l^f R_{df})_0 \\
&-[(\mathcal{L}_k\phi)_1]^2(k^c l^e\nabla_e\nabla_c\phi)_0(k^d l^f R_{df})_0 - (\mathcal{L}_k\phi)_1(k^b R_{b\hat{d}})_1(D_{\hat{e}}\phi)_0(D^{\hat{e}}D^{\hat{d}}\phi) \\
&-[(\mathcal{L}_k\phi)_1]^2(l^e l^f\nabla_e\nabla_f\phi)_0(R_{kk})_1 + (k^c l^d R_{cd})_0(\mathcal{L}_k\phi)_1(D_{\hat{f}}\phi)_0 D^{\hat{f}}(\mathcal{L}_k\phi)_1 \\
&+(R_{kk})_1(\mathcal{L}_k\phi)_1(D_{\hat{f}}\phi)_0(l^d\nabla^{\hat{f}}\nabla_d\phi)_0 \simeq 0.
\end{aligned}
\tag{C5}
$$

The sixth term of Eq. (37) is

$$
\begin{aligned}
k^a k^b R_{ac}\nabla^c\phi\nabla_d\nabla^d\phi\nabla_b\phi = &(k^b R_{b\hat{e}})_1(\mathcal{L}_k\phi)_1(D^{\hat{e}}\phi)_0(\nabla^d\nabla_d\phi)_0 - (k^c l^d R_{cd})_0[(\mathcal{L}_k\phi)_1]^2(\nabla_e\nabla^e\phi)_0 \\
&-(\mathcal{L}_k\phi)_1(R_{kk})_1(l^d\nabla_d\phi)_0(\nabla^e\nabla_e\phi)_0 \simeq 0.
\end{aligned}
\tag{C6}
$$

The seventh term of Eq. (37) is

$$
\begin{aligned}
-\frac{1}{2}k^a k^b R_{cd}\nabla^c\phi\nabla^d\nabla_a\phi\nabla_b\phi = &-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(\mathcal{L}_k^2\phi)_1(l^e l^f R_{ef})_0 - \frac{1}{2}(\mathcal{L}_k\phi)_1(D^{\hat{g}}\phi)_0(R_{\hat{g}\hat{h}})_0[\mathcal{L}_k(D^{\hat{h}}\phi)]_1 \\
&-\frac{1}{2}(\mathcal{L}_k\phi)_1(D^{\hat{g}}\phi)_0(R_{\hat{g}\hat{h}})_0(D_{\hat{b}}\phi)_0(B^{\hat{b}\hat{h}})_1 - \frac{1}{2}(\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)_1(l^e\nabla_e\phi)_0(k^d l^f R_{df})_0 \\
&-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(k^c l^e\nabla_e\nabla_c\phi)_0(k^d l^f R_{df})_0 - \frac{1}{2}(l^d R_{d\hat{f}})_0(\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)_1(D^{\hat{f}}\phi)_0 \\
&+\frac{1}{2}(k^c R_{c\hat{f}})_1(\mathcal{L}_k\phi)_1(l^d k^b\nabla_d\nabla_b\phi)_0(D^{\hat{f}}\phi)_0 - \frac{1}{2}(\mathcal{L}_k\phi)_1(l^e\nabla_e\phi)_0(k^b l^f\nabla_f\nabla_b\phi)_0(R_{kk})_1 \\
&+\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(k^c\nabla^{\hat{e}}\nabla_c\phi)_1(l^d R_{d\hat{e}})_0 + \frac{1}{2}(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0(k^c R_{c\hat{e}})_1 D^{\hat{e}}(\mathcal{L}_k\phi)_1 \simeq 0.
\end{aligned}
\tag{C7}
$$

The eighth term of Eq. (37) is

$$
k^a k^b R_{cd}\nabla_a\phi\nabla^d\nabla^c\phi\nabla_b\phi = [(\mathcal{L}_k\phi)_1]^2(R_{cd}\nabla^d\nabla^c\phi) \simeq 0. \tag{C8}
$$

The ninth term of Eq. (37) is

$$\begin{aligned}-\frac{1}{2}k^ak^bR_{adce}\nabla^c\phi\nabla^e\nabla^d\phi\nabla_b\phi = &-\frac{1}{2}(\mathcal{L}_k\phi)_1D^{\hat m}(\mathcal{L}_k\phi)_1(l^e\nabla_e\phi)_0(k^cl^fk^dR_{cfd\hat m})_1+\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^e\nabla_e\nabla^{\hat m}\phi)_0(k^cl^fk^dR_{cfd\hat m})_1\\&-\frac{1}{2}(\mathcal{L}_k\phi)_1D^{\hat n}(\mathcal{L}_k\phi)_1(D^{\hat m}\phi)_0(k^cl^dR_{c\hat n d\hat m})_0-\frac{1}{2}(\mathcal{L}_k\phi)_1(l^d\nabla_d\nabla^{\hat f}\phi)_0(D^{\hat m}\phi)_0(k^bk^cR_{b\hat m c\hat f})_1\\&-\frac{1}{2}(\mathcal{L}_k\phi)_1(k^bR_{b\hat d\hat c\hat e})_1(D^{\hat c}\phi)_0(D^{\hat e}D^{\hat d}\phi)_0+\frac{1}{2}(\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)_1(l^f\nabla_f\phi)(k^dl^gk^el^hR_{dgeh})_0\\&-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(k^cl^f\nabla_f\nabla_c\phi)_0(k^dl^gk^el^hR_{dgeh})_0+\frac{1}{2}(\mathcal{L}_k\phi)_1(D^{\hat g}\phi)_0D^{\hat h}(\mathcal{L}_k\phi)_1(k^cl^dR_{cd\hat g\hat h})_0\\&+\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(D^{\hat f}D^{\hat e}\phi)_0(k^cl^dR_{c\hat e d\hat f})_0+\frac{1}{2}(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0(D^{\hat f}D^{\hat e}\phi)_0(k^bk^cR_{b\hat e c\hat f})_1\\&+\frac{1}{2}(\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)_1(D^{\hat m}\phi)_0(k^dl^el^fR_{def\hat m})_0+\frac{1}{2}(\mathcal{L}_k\phi)_1(l^ek^b\nabla_e\nabla_b\phi)_0(D^{\hat m}\phi)_0(k^ck^dl^fR_{cfd\hat m})_1\\&-\frac{1}{2}(\mathcal{L}_k\phi)_1(l^e\nabla_e\phi)_0D^{\hat m}(\mathcal{L}_k\phi)_1(k^ck^dl^fR_{cfd\hat m})_1-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2D^{\hat m}(\mathcal{L}_k\phi)_1(k^dl^el^fR_{def\hat m})_0\simeq 0.\end{aligned} \tag{C9}$$

The tenth term of Eq. (37) is

$$\begin{aligned}-\frac{1}{2}k^ak^bR_{aecd}\nabla^c\phi\nabla^e\nabla^d\phi\nabla_b\phi = &-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(k^cl^f\nabla_c\nabla_f\phi)_0(k^dk^el^gl^hR_{dgeh})_0-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2D^{\hat g}(\mathcal{L}_k\phi)_1(k^dl^el^fR_{def\hat g})_1\\&-\frac{1}{2}(\mathcal{L}_k\phi)_1D^{\hat g}(\mathcal{L}_k\phi)_1(l^e\nabla_e\phi)_0(k^ck^dl^fR_{cfd\hat g})_1+\frac{1}{2}(\mathcal{L}_k\phi)_1D^{\hat f}(\mathcal{L}_k\phi)_1(D^{\hat g}\phi)_0(k^cl^dR_{cd\hat g\hat f})_0\\&-\frac{1}{2}(\mathcal{L}_k\phi)_1(k^bR_{b\hat c\hat d\hat e})_1(D^{\hat d}\phi)_0(D^{\hat c}D^{\hat e}\phi)_0+\frac{1}{2}(\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)_1(l^f\nabla_f\phi)_0(k^dk^el^gl^hR_{dgeh})_0\\&-\frac{1}{2}(\mathcal{L}_k\phi)_1(D^{\hat g}\phi)_0D^{\hat f}(\mathcal{L}_k\phi)_1(k^cl^dR_{c\hat f d\hat g})_0-\frac{1}{2}(\mathcal{L}_k\phi)_1(D^{\hat g}\phi)_0(l^d\nabla^{\hat f}\nabla_d\phi)_0(k^bk^cR_{b\hat g c\hat f})_1\\&+\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(D^{\hat f}D^{\hat e}\phi)_0(k^cl^dR_{c\hat f d\hat e})_0+\frac{1}{2}(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0(D^{\hat f}D^{\hat e}\phi)_0(k^bk^cR_{b\hat e c\hat f})_1\\&+\frac{1}{2}(\mathcal{L}_k\phi)_1(k^bl^e\nabla_b\nabla_e\phi)_0(D^{\hat g}\phi)_0(k^cl^fk^dR_{cfd\hat g})_1+\frac{1}{2}(\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)_1(D^{\hat g}\phi)_0(k^dl^el^fR_{de\hat g h})_0\\&-\frac{1}{2}(\mathcal{L}_k\phi)_1(l^e\nabla_e\phi)_0D^{\hat g}(\mathcal{L}_k\phi)_1(k^ck^dl^fR_{cfd\hat g})_1+\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^e\nabla^{\hat g}\nabla_e\phi)_0(k^ck^dl^fR_{cfd\hat g})_1\simeq 0.\end{aligned} \tag{C10}$$

Therefore, based on the above results, under the linear order approximation of the perturbation, the integrand, $H_{kk}^{(\phi 1)}$, can be neglected, i.e.,

$$H_{kk}^{(\phi 1)}\simeq 0. \tag{C11}$$

APPENDIX D: SIMPLIFICATION OF $H_{kk}^{(\phi 2)}$ UNDER THE LINEAR ORDER APPROXIMATION

In the final appendix, we would like to calculate $H_{kk}^{(\phi 2)}$ under the first-order approximation of the perturbation. The concrete expression of $H_{kk}^{(\phi 2)}$ has been given in Eq. (38). The first term of Eq. (38) is

$$\begin{aligned}-\frac{1}{2}k^ak^b\nabla_a\phi\nabla^c(R\nabla_c\phi\nabla_b\phi) = &\frac{1}{2}k^ak^bk^cl^d\nabla_a\phi\nabla_c(R\nabla_b\phi\nabla_d\phi)+\frac{1}{2}k^ak^bk^cl^d\nabla_a\phi\nabla_d(R\nabla_b\phi\nabla_c\phi)\\&-\frac{1}{2}k^ak^b\gamma^{cd}\nabla_a\phi\nabla_d(R\nabla_b\phi\nabla_c\phi).\end{aligned} \tag{D1}$$

The first term of Eq. (D1) can be further calculated as

$$\frac{1}{2}k^a k^b k^c l^d \nabla_a\phi\nabla_c(R\nabla_b\phi\nabla_d\phi) = \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(\mathcal{L}_k R)_1(l^d\nabla_d\phi)_0 + \frac{1}{2}(\mathcal{L}_k\phi)_1(R)_0(\mathcal{L}_k^2\phi)_1(l^d\nabla_d\phi)_0 + \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(R)_0(k^c l^d\nabla_c\nabla_d\phi)_0 \simeq 0. \tag{D2}$$

The second term of Eq. (D1) can be further calculated as

$$\frac{1}{2}k^a k^b k^c l^d \nabla_a\phi\nabla_d(R\nabla_b\phi\nabla_c\phi) = \frac{1}{2}[(\mathcal{L}_k\phi)_1]^3(l^d\nabla_d R)_0 + \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(R)_0(k^b l^d\nabla_d\nabla_b\phi)_0 + \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(k^c l^d\nabla_d\nabla_c\phi)_0(R)_0 \simeq 0. \tag{D3}$$

The third term of Eq. (D1) can be further calculated as

$$-\frac{1}{2}k^a k^b \gamma^{cd}\nabla_a\phi\nabla_d(R\nabla_b\phi\nabla_c\phi) = -\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(D_{\hat{e}}\phi)_0(D^{\hat{e}}R)_0 - \frac{1}{2}(\mathcal{L}_k\phi)_1 D_{\hat{c}}(\mathcal{L}_k\phi)_1(D^{\hat{c}}\phi)_0(R)_0 - \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(D^{\hat{c}}D_{\hat{c}}\phi)_0(R)_0 \simeq 0. \tag{D4}$$

So the first term of Eq. (38) under the first-order approximation can be simplified as

$$-\frac{1}{2}k^a k^b\nabla_a\phi\nabla^c(R\nabla_c\phi\nabla_b\phi) \simeq 0. \tag{D5}$$

The second term of Eq. (38) is

$$\begin{aligned}
k^a k^b\nabla_a\phi\nabla^c(R_{bd}\nabla_c\phi\nabla^d\phi) &= -k^a k^b k^c l^d\nabla_a\phi\nabla_c(R_{de}\nabla_b\phi\nabla^e\phi) - k^a k^b k^c l^d\nabla_a\phi\nabla_d(R_{ce}\nabla_b\phi\nabla^e\phi) \\
&\quad + k^a k^b\gamma^{cd}\nabla_a\phi\nabla_d(\nabla_b\phi R_{ce}\nabla^e\phi) \\
&= -k^a k^b k^c l^d\nabla_a\phi\nabla_c R_{de}\nabla_b\phi\nabla^e\phi - k^a k^b k^c l^d\nabla_a\phi R_{de}\nabla_c\nabla_b\phi\nabla^e\phi \\
&\quad - k^a k^b k^c l^d\nabla_a\phi R_{de}\nabla_b\phi\nabla_c\nabla^e\phi - k^a k^b k^c l^d\nabla_a\phi\nabla_d R_{ce}\nabla_b\phi\nabla^e\phi \\
&\quad - k^a k^b k^c l^d\nabla_a\phi R_{ce}\nabla_d\nabla_b\phi\nabla^e\phi - k^a k^b k^c l^d\nabla_a\phi R_{ce}\nabla_b\phi\nabla_d\nabla^e\phi \\
&\quad + k^a k^b\gamma^{cd}\nabla_a\phi\nabla_d\nabla_b\phi R_{ce}\nabla^e\phi + k^a k^b\gamma^{cd}\nabla_a\phi\nabla_b\phi\nabla_d R_{ce}\nabla^e\phi \\
&\quad + k^a k^b\gamma^{cd}\nabla_a\phi\nabla_b\phi R_{ce}\nabla_d\nabla^e\phi.
\end{aligned} \tag{D6}$$

The first term of Eq. (D6) is

$$-k^a k^b k^c l^d\nabla_a\phi\nabla_c R_{de}\nabla_b\phi\nabla^e\phi = [(\mathcal{L}_k\phi)_1]^3(k^d l^e l^f\nabla_d R_{ef})_0 + [(\mathcal{L}_k\phi)_1]^2(k^c k^d l^f\nabla_d R_{cf})_1(l^e\nabla_e\phi)_0 - [(\mathcal{L}_k\phi)_1]^2(k^c l^d\nabla_c R_{d\hat{e}})_0(D^{\hat{e}}\phi)_0 \simeq 0. \tag{D7}$$

The second term of Eq. (D6) is

$$-k^a k^b k^c l^d\nabla_a\phi R_{de}\nabla_c\nabla_b\phi\nabla^e\phi = [(\mathcal{L}_k\phi)_1]^2(\mathcal{L}_k^2\phi)_1(l^e l^f R_{ef})_0 + (\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)_1(l^e\nabla_e\phi)_0(k^d l^f R_{df})_0 - (\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)_1(D^{\hat{c}}\phi)_0(l^d R_{d\hat{c}})_0 \simeq 0. \tag{D8}$$

The third term of Eq. (D6) is

$$-k^a k^b k^c l^d\nabla_a\phi R_{de}\nabla_b\phi\nabla_c\nabla^e\phi = [(\mathcal{L}_k\phi)_1]^2(k^c l^e\nabla_c\nabla_e\phi)_0(k^d l^f R_{df})_0 - [(\mathcal{L}_k\phi)_1]^2 D^{\hat{e}}(\mathcal{L}_k\phi)_1(l^d R_{d\hat{e}})_0 + [(\mathcal{L}_k\phi)_1]^2(\mathcal{L}_k^2\phi)_1(l^e l^f R_{ef})_0 \simeq 0. \tag{D9}$$

The fourth term of Eq. (D6) is

$$-k^a k^b k^c l^d \nabla_a\phi\nabla_d R_{ce}\nabla_b\phi\nabla^e\phi = -[(\mathcal{L}_k\phi)_1]^2(k^c l^d\nabla_d R_{c\hat{e}})_0(D^{\hat{e}}\phi)_0 + [(\mathcal{L}_k\phi)_1]^2(l^e\nabla_e\phi)_0(k^c k^d l^f\nabla_f R_{cd})_1 + [(\mathcal{L}_k\phi)_1]^3(k^d l^e l^f\nabla_f R_{de})_0 \simeq 0. \tag{D10}$$

The fifth term of Eq. (D6) is

$$-k^a k^b k^c l^d \nabla_a\phi R_{ce}\nabla_d\nabla_b\phi\nabla^e\phi = [(\mathcal{L}_k\phi)_1]^2(k^d l^f R_{df})_0(k^c l^e\nabla_e\nabla_c\phi)_0 - (\mathcal{L}_k\phi)_1(k^b l^d\nabla_d\nabla_b\phi)_0(D^{\hat{e}}\phi)_0(k^c R_{c\hat{e}})_1 + (R_{kk})_1(\mathcal{L}_k\phi)_1(l^e\nabla_e\phi)_0(k^b l^f\nabla_f\nabla_e\phi)_0 \simeq 0. \tag{D11}$$

The sixth term of Eq. (D6) is

$$-k^a k^b k^c l^d \nabla_a\phi R_{ce}\nabla_b\phi\nabla_d\nabla^e\phi = -[(\mathcal{L}_k\phi)_1]^2(l^d\nabla_d\nabla^{\hat{e}}\phi)_0(k^c R_{c\hat{e}})_1 + [(\mathcal{L}_k\phi)_1]^2(l^e l^f\nabla_f\nabla_e\phi)_0(R_{kk})_1 + [(\mathcal{L}_k\phi)_1]^2(k^c l^e\nabla_e\nabla_c\phi)_0(k^d l^f R_{df})_0 \simeq 0. \tag{D12}$$

The seventh term of Eq. (D6) is

$$k^a k^b \gamma^{cd}\nabla_a\phi\nabla_d\nabla_b\phi R_{ce}\nabla^e\phi = (\mathcal{L}_k\phi)_1(D^{\hat{e}}\phi)_0 D^{\hat{d}}(\mathcal{L}_k\phi)_1(R_{\hat{e}\hat{d}})_0 - (\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0 D^{\hat{e}}(\mathcal{L}_k\phi)_1(k^c R_{c\hat{e}})_1 - [(\mathcal{L}_k\phi)_1]^2 D^{\hat{e}}(\mathcal{L}_k\phi)_1(l^d R_{d\hat{e}})_0 \simeq 0. \tag{D13}$$

The eighth term of Eq. (D6) is

$$k^a k^b \gamma^{cd}\nabla_a\phi\nabla_b\phi\nabla_d R_{ce}\nabla^e\phi = [(\mathcal{L}_k\phi)_1]^2(\nabla^{\hat{c}}R_{\hat{c}e}\nabla^e\phi) \simeq 0. \tag{D14}$$

The ninth term of Eq. (D6) is

$$k^a k^b \gamma^{cd}\nabla_a\phi\nabla_b\phi R_{ce}\nabla_d\nabla^e\phi = [(\mathcal{L}_k\phi)_1]^2(R_{\hat{c}e}\nabla^{\hat{c}}\nabla^e\phi) \simeq 0. \tag{D15}$$

So the second term of Eq. (38) can finally be written as

$$k^a k^b\nabla_a\phi\nabla^c(R_{bd}\nabla_c\phi\nabla^d\phi) \simeq 0. \tag{D16}$$

The third term of Eq. (38) is

$$\begin{aligned} k^a k^b\nabla_a\phi\nabla^c(R_{cd}\nabla_b\phi\nabla^d\phi) &= -k^a k^b k^c l^d\nabla_a\phi\nabla_c(R_{be}\nabla_d\phi\nabla^e\phi) - k^a k^b k^c l^d\nabla_a\phi\nabla_d(R_{be}\nabla_c\phi\nabla^e\phi) \\ &\quad + k^a k^b\gamma^{cd}\nabla_a\phi\nabla_d(R_{be}\nabla_c\phi\nabla^e\phi) \\ &= -k^a k^b k^c l^d\nabla_a\phi\nabla_c R_{be}\nabla_d\phi\nabla^e\phi - k^a k^b k^c l^d R_{be}\nabla_a\phi\nabla_c\nabla_d\phi\nabla^e\phi - k^a k^b k^c l^d R_{be}\nabla_a\phi\nabla_d\phi\nabla_c\nabla^e\phi \\ &\quad - k^a k^b k^c l^d\nabla_a\phi\nabla_d R_{be}\nabla_c\phi\nabla^e\phi - k^a k^b k^c l^d\nabla_a\phi R_{be}\nabla_d\nabla_c\phi\nabla^e\phi \\ &\quad - k^a k^b k^c l^d\nabla_a\phi R_{be}\nabla_c\phi\nabla_d\nabla^e\phi + k^a k^b\gamma^{cd}\nabla_a\phi\nabla_d R_{be}\nabla_c\phi\nabla^e\phi \\ &\quad + k^a k^b\gamma^{cd}\nabla_a\phi R_{be}\nabla_d\nabla_c\phi\nabla^e\phi + k^a k^b\gamma^{cd}\nabla_a\phi R_{be}\nabla_c\phi\nabla_d\nabla^e\phi. \end{aligned} \tag{D17}$$

The first term of Eq. (D17) is

$$-k^a k^b k^c l^d\nabla_a\phi\nabla_c R_{be}\nabla_d\phi\nabla^e\phi = [(\mathcal{L}_k\phi)_1]^2(k^d k^c l^f\nabla_d R_{cf})_1(l^e\nabla_e\phi)_0 - (\mathcal{L}_k\phi)_1(k^c k^b\nabla_c R_{b\hat{e}})_1(l^d\nabla_d\phi)_0(D^{\hat{e}}\phi)_0 + (\mathcal{L}_k\phi)_1(k^b k^c k^d\nabla_d R_{bc})_1(l^e\nabla_e\phi)_0(l^f\nabla_f\phi)_0 \simeq 0. \tag{D18}$$

The second term of Eq. (D17) is

$$-k^a k^b k^c l^d R_{be}\nabla_a\phi\nabla_c\nabla_d\phi\nabla^e\phi = (k^d l^f R_{df})_0[(\mathcal{L}_k\phi)_1]^2(k^c l^e\nabla_c\nabla_e\phi)_0 + (R_{kk})_1(\mathcal{L}_k\phi)_1(k^b l^f\nabla_b\nabla_f\phi)_0(l^e\nabla_e\phi)_0 - (k^c R_{c\hat{e}})_1(\mathcal{L}_k\phi)_1(k^b l^d\nabla_b\nabla_d\phi)_0(D^{\hat{e}}\phi)_0 \simeq 0. \tag{D19}$$

The third term of Eq. (D17) is

$$-k^a k^b k^c l^d R_{be} \nabla_a \phi \nabla_d \phi \nabla_c \nabla^e \phi = -(\mathcal{L}_k \phi)_1 D^{\hat{e}} (\mathcal{L}_k \phi)_1 (l^d \nabla_d \phi)_0 (k^c R_{c\hat{e}})_1 + (\mathcal{L}_k \phi)_1 (k^b l^f \nabla_b \nabla_f \phi)_0 (l^e \nabla_e \phi)_0 (R_{kk})_1 + (\mathcal{L}_k \phi)_1 (\mathcal{L}_k^2 \phi)_1 (l^e \nabla_e \phi)_0 (k^d l^f R_{df})_0 \simeq 0. \tag{D20}$$

The fourth term of Eq. (D17) is

$$-k^a k^b k^c l^d \nabla_a \phi \nabla_d R_{be} \nabla_c \phi \nabla^e \phi = -[(\mathcal{L}_k \phi)_1]^2 (k^c l^d \nabla_d R_{c\hat{e}})_0 (D^{\hat{e}} \phi)_0 + [(\mathcal{L}_k \phi)_1]^2 (l^e \nabla_e \phi)_0 (k^c k^d l^f \nabla_f R_{cd})_1 + [(\mathcal{L}_k \phi)_1]^3 (k^d l^e l^f \nabla_f R_{de})_0 \simeq 0. \tag{D21}$$

The fifth term of Eq. (D17) is

$$-k^a k^b k^c l^d \nabla_a \phi R_{be} \nabla_d \nabla_c \phi \nabla^e \phi = [(\mathcal{L}_k \phi)_1]^2 (k^c l^e \nabla_e \nabla_c \phi)_0 (k^d l^f R_{df})_0 - (\mathcal{L}_k \phi)_1 (k^b l^d \nabla_d \nabla_b \phi)_0 (D^{\hat{e}} \phi)_0 (k^c R_{c\hat{e}})_1 + (\mathcal{L}_k \phi)_1 (l^e \nabla_e \phi)_0 (k^b l^f \nabla_f \nabla_b \phi)_0 (R_{kk})_1 \simeq 0. \tag{D22}$$

The sixth term of Eq. (D17) is

$$-k^a k^b k^c l^d \nabla_a \phi R_{be} \nabla_c \phi \nabla_d \nabla^e \phi = -[(\mathcal{L}_k \phi)_1]^2 (l^d \nabla_d \nabla^{\hat{e}} \phi)_0 (k^c R_{c\hat{e}})_1 + [(\mathcal{L}_k \phi)_1]^2 (k^c l^e \nabla_e \nabla_c \phi)_0 (k^d l^f R_{df})_0 + [(\mathcal{L}_k \phi)_1]^2 (l^e l^f \nabla_e \nabla_f \phi)_0 (R_{kk})_1 \simeq 0. \tag{D23}$$

The seventh term of Eq. (D17) is

$$k^a k^b \gamma^{cd} \nabla_a \phi \nabla_d R_{be} \nabla_c \phi \nabla^e \phi = -(\mathcal{L}_k \phi)_1 (l^d \nabla_d \phi)_0 (D^{\hat{e}} \phi)_0 (k^b k^c \nabla_{\hat{e}} R_{bc})_1 + (\mathcal{L}_k \phi)_1 (D^{\hat{e}} \phi)_0 (D^{\hat{f}} \phi)_0 (k^b \nabla_{\hat{f}} R_{b\hat{e}})_1 - [(\mathcal{L}_k \phi)_1]^2 (D^{\hat{e}} \phi)_0 (k^c l^d \nabla_{\hat{e}} R_{cd})_0 \simeq 0. \tag{D24}$$

The eighth term of Eq. (D17) is

$$k^a k^b \gamma^{cd} \nabla_a \phi R_{be} \nabla_d \nabla_c \phi \nabla^e \phi = (\mathcal{L}_k \phi)_1 (D^{\hat{f}} \phi)_0 (D^{\hat{e}} D_{\hat{e}} \phi)_0 (k^b R_{b\hat{f}})_1 - [(\mathcal{L}_k \phi)_1]^2 (D^{\hat{e}} D_{\hat{e}} \phi)_0 (k^c l^d R_{cd})_0 - (\mathcal{L}_k \phi)_1 (l^d \nabla_d \phi)_0 (D^{\hat{e}} D_{\hat{e}} \phi)_0 (R_{kk})_1 \simeq 0. \tag{D25}$$

The ninth term of Eq. (D17) is

$$k^a k^b \gamma^{cd} \nabla_a \phi R_{be} \nabla_c \phi \nabla_d \nabla^e \phi = (\mathcal{L}_k \phi)_1 (D^{\hat{e}} \phi)_0 (D_{\hat{e}} D^{\hat{d}} \phi)_0 (k^b R_{b\hat{d}})_1 - (\mathcal{L}_k \phi)_1 (D^{\hat{e}} \phi)_0 D_{\hat{e}} (\mathcal{L}_k \phi)_1 (k^c l^d R_{cd})_0 - (\mathcal{L}_k \phi)_1 (D^{\hat{e}} \phi)_0 (l^d \nabla_{\hat{e}} \nabla_d \phi)_0 (R_{kk})_1 \simeq 0. \tag{D26}$$

So the third term of Eq. (38) under the linear order approximation can be given as

$$k^a k^b \nabla_a \phi \nabla^c (R_{cd} \nabla_b \phi \nabla^d \phi) \simeq 0. \tag{D27}$$

The fourth term of Eq. (38) is

$$\begin{aligned} -k^a k^b \nabla_a \phi \nabla^c (R_{cb} \nabla_d \phi \nabla^d \phi) &= k^a k^b k^c l^d \nabla_c (R_{bd} \nabla_e \phi \nabla^e \phi) + k^a k^b k^c l^d \nabla_a \phi \nabla_d (R_{bc} \nabla_e \phi \nabla^e \phi) - k^a k^b \gamma^{cd} \nabla_a \phi \nabla_d (R_{bc} \nabla^e \phi \nabla_e \phi) \\ &= k^a k^b k^c l^d \nabla_a \phi \nabla_c R_{bd} \nabla_e \phi \nabla^e \phi + k^a k^b k^c l^d R_{bd} \nabla_a \phi \nabla_c \nabla_e \phi \nabla^e \phi \\ &\quad + k^a k^b k^c l^d R_{bd} \nabla_a \phi \nabla_e \phi \nabla_c \nabla^e \phi + k^a k^b k^c l^d \nabla_a \phi \nabla_d (R_{bc} \nabla_e \phi \nabla^e \phi) \\ &\quad - k^a k^b \gamma^{cd} \nabla_a \phi \nabla_d (R_{bc} \nabla^e \phi \nabla_e \phi). \end{aligned} \tag{D28}$$

The first term of Eq. (D28) is

$$k^a k^b k^c l^d \nabla_a \phi \nabla_c R_{bd} \nabla_e \phi \nabla^e \phi = (\mathcal{L}_k \phi)_1 [\mathcal{L}_k (k^b l^d R_{bd})]_1 (\nabla^e \phi \nabla_e \phi)_0 \simeq 0. \tag{D29}$$

The second term of Eq. (D28) is

$$k^a k^b k^c l^d R_{bd} \nabla_a \phi \nabla_c \nabla_e \phi \nabla^e \phi = -[(\mathcal{L}_k \phi)_1]^2 (k^c l^e \nabla_c \nabla_e \phi)_0 (k^d l^f R_{df})_0 - (\mathcal{L}_k \phi)_1 (\mathcal{L}_k^2 \phi)_1 (l^e \nabla_e \phi)_0 (k^d l^f R_{df})_0 + (\mathcal{L}_k \phi)_1 D_{\hat{e}} (\mathcal{L}_k \phi)_1 (D^{\hat{e}} \phi)_0 (k^c l^d R_{cd})_0 \simeq 0. \tag{D30}$$

The third term of Eq. (D28) is

$$k^a k^b k^c l^d R_{bd} \nabla_a \phi \nabla_e \phi \nabla_c \nabla^e \phi = -[(\mathcal{L}_k \phi)_1]^2 (k^c l^e \nabla_c \nabla_e \phi)_0 (k^d l^f R_{df})_0 - (\mathcal{L}_k \phi)_1 (\mathcal{L}_k^2 \phi)_1 (l^e \nabla_e \phi)_0 (k^d l^f R_{df})_0 + (\mathcal{L}_k \phi)_1 D_{\hat{e}} (\mathcal{L}_k \phi)_1 (D^{\hat{e}} \phi)_0 (k^c l^d R_{cd})_0 \simeq 0. \tag{D31}$$

The fourth term of Eq. (D28) is

$$k^a k^b k^c l^d \nabla_a \phi \nabla_d (R_{bc} \nabla_e \phi \nabla^e \phi) = (\mathcal{L}_k \phi)_1 (k^b k^c l^d \nabla_d R_{bc})_1 (\nabla^e \phi \nabla_e \phi)_0 + (\mathcal{L}_k \phi)_1 (R_{kk})_1 [l^d \nabla_d (\nabla_e \phi \nabla^e \phi)]_0 \simeq 0. \tag{D32}$$

The fifth term of Eq. (D28) is

$$-k^a k^b \gamma^{cd} \nabla_a \phi \nabla_d (R_{bc} \nabla^e \phi \nabla_e \phi) = -(\mathcal{L}_k \phi)_1 (k^b \nabla^{\hat{c}} R_{b\hat{c}})_1 (\nabla_e \phi \nabla^e \phi)_0 - (\mathcal{L}_k \phi)_1 (k^b R_{b\hat{c}})_1 D^{\hat{c}} (\nabla_e \phi \nabla^e \phi)_0 \simeq 0. \tag{D33}$$

So the fourth term of Eq. (38) under the first-order approximation of the perturbation can be written as

$$-k^a k^b \nabla_a \phi \nabla^c (R_{cb} \nabla_d \phi \nabla^d \phi) \simeq 0. \tag{D34}$$

The fifth term of Eq. (38) is

$$\frac{1}{2} k^a k^b \nabla_a \phi \nabla_b (R \nabla_d \phi \nabla^d \phi) = \frac{1}{2} (\mathcal{L}_k \phi)_1 (\mathcal{L}_k R)_1 (\nabla_d \phi \nabla^d \phi)_0 + \frac{1}{2} (R)_0 (\mathcal{L}_k \phi)_1 [\mathcal{L}_k (\nabla_d \phi \nabla^d \phi)]_1 \simeq 0. \tag{D35}$$

The sixth term of Eq. (38) is

$$-k^a k^b \nabla_a \phi \nabla_b (R_{de} \nabla^d \phi \nabla^e \phi) = -k^a k^c \nabla_a \phi \nabla_c R_{ef} \nabla^e \phi \nabla^f \phi - k^a k^c R_{ef} \nabla_a \phi \nabla_c \nabla^e \phi \nabla^f \phi - k^a k^c R_{ef} \nabla_a \phi \nabla^e \phi \nabla_c \nabla^f \phi. \tag{D36}$$

The first term of Eq. (D36) is

$$\begin{aligned} -k^a k^c \nabla_a \phi \nabla_c R_{ef} \nabla^e \phi \nabla^f \phi = & -[(\mathcal{L}_k \phi)_1]^3 (l^f l^g l^h \nabla_f R_{gh})_0 - (\mathcal{L}_k \phi)_1 (k^c \nabla_c R_{\hat{g}\hat{h}})_1 (D^{\hat{g}} \phi)_0 (D^{\hat{h}} \phi)_0 \\ & - 2[(\mathcal{L}_k \phi)_1]^2 (k^e k^f l^h \nabla_f R_{eh})_1 (l^g \nabla_g \phi)_0 + 2[(\mathcal{L}_k \phi)_1]^2 (k^e l^f \nabla_e R_{f\hat{g}})_0 (D^{\hat{g}} \phi)_0 \\ & + 2(\mathcal{L}_k \phi)_1 (k^c k^e \nabla_e R_{c\hat{g}})_1 (l^f \nabla_f \phi)_0 (D^{\hat{g}} \phi)_0 \\ & - (\mathcal{L}_k \phi)_1 (k^c k^e k^f \nabla_f R_{ce})_1 (l^g \nabla_g \phi)_0 (l^h \nabla_h \phi)_0 \simeq 0. \end{aligned} \tag{D37}$$

The second term of Eq. (D36) is

$$\begin{aligned} -k^a k^c R_{ef} \nabla_a \phi \nabla_c \nabla^e \phi \nabla^f \phi = & -[(\mathcal{L}_k \phi)_1]^2 (k^c l^e \nabla_c \nabla_e \phi)_0 (k^d l^f R_{df})_0 + [(\mathcal{L}_k \phi)_1]^2 D^{\hat{e}} (\mathcal{L}_k \phi)_1 (l^d R_{d\hat{e}})_0 \\ & - (\mathcal{L}_k \phi)_1 D^{\hat{d}} (\mathcal{L}_k \phi)_1 (D^{\hat{e}} \phi)_0 (R_{\hat{e}\hat{d}})_0 + (\mathcal{L}_k \phi)_1 D^{\hat{e}} (\mathcal{L}_k \phi)_1 (l^d \nabla_d \phi)_0 (k^c R_{c\hat{e}})_1 \\ & - [(\mathcal{L}_k \phi)_1]^2 (\mathcal{L}_k^2 \phi)_1 (l^e l^f R_{ef})_0 - (\mathcal{L}_k \phi)_1 (k^b l^f \nabla_b \nabla_f \phi)_0 (l^e \nabla_e \phi)_0 (R_{kk})_1 \\ & - (\mathcal{L}_k \phi)_1 (\mathcal{L}_k^2 \phi)_1 (l^e \nabla_e \phi)_0 (k^d l^f R_{df})_0 + (\mathcal{L}_k \phi)_1 (k^b l^d \nabla_b \nabla_d \phi)_0 (D^{\hat{e}} \phi)_0 (k^c R_{c\hat{e}})_1 \\ & + (\mathcal{L}_k \phi)_1 (\mathcal{L}_k^2 \phi)_1 (D^{\hat{e}} \phi)_0 (l^d R_{d\hat{e}})_0 \simeq 0. \end{aligned} \tag{D38}$$

The third term of Eq. (D36) is

$$
\begin{aligned}
-k^a k^c R_{ef} \nabla_a \phi \nabla^e \phi \nabla_c \nabla^f \phi = & -[(\mathcal{L}_k \phi)_1]^2 (k^c l^e \nabla_c \nabla_e \phi)_0 (k^d l^f R_{df})_0 + [(\mathcal{L}_k \phi)_1]^2 D^{\hat{e}} (\mathcal{L}_k \phi)_1 (l^d R_{d\hat{e}})_0 \\
& - (\mathcal{L}_k \phi)_1 D^{\hat{d}} (\mathcal{L}_k \phi)_1 (D^{\hat{e}} \phi)_0 (R_{\hat{e}\hat{d}})_0 + (\mathcal{L}_k \phi)_1 D^{\hat{e}} (\mathcal{L}_k \phi)_1 (l^d \nabla_d \phi)_0 (k^c R_{c\hat{e}})_1 \\
& - [(\mathcal{L}_k \phi)_1]^2 (\mathcal{L}_k^2 \phi)_1 (l^e l^f R_{ef})_0 - (\mathcal{L}_k \phi)_1 (k^b l^f \nabla_b \nabla_f \phi)_0 (l^e \nabla_e \phi)_0 (R_{kk})_1 \\
& - (\mathcal{L}_k \phi)_1 (\mathcal{L}_k^2 \phi)_1 (l^e \nabla_e \phi)_0 (k^d l^f R_{df})_0 + (\mathcal{L}_k \phi)_1 (k^b l^d \nabla_b \nabla_d \phi)_0 (D^{\hat{e}} \phi)_0 (k^c R_{c\hat{e}})_1 \\
& + (\mathcal{L}_k \phi)_1 (\mathcal{L}_k^2 \phi)_1 (D^{\hat{e}} \phi)_0 (l^d R_{d\hat{e}})_0 \simeq 0.
\end{aligned} \tag{D39}
$$

So the sixth term of Eq. (38) under the first-order approximation of the perturbation is

$$
-k^a k^b \nabla_a \phi \nabla_b (R_{de} \nabla^d \phi \nabla^e \phi) \simeq 0. \tag{D40}
$$

The seventh term of Eq. (38) is

$$
\begin{aligned}
-k^a k^b \nabla_a \phi \nabla^c (R_{cdbe} \nabla^d \phi \nabla^e \phi) = & -k^a k^b k^c l^d \nabla_a \phi \nabla_c (R_{bedf} \nabla^e \phi \nabla^f \phi) - k^a k^b k^c l^d \nabla_a \phi \nabla_d (R_{becf} \nabla^e \phi \nabla^f \phi) \\
& + k^a k^b \gamma^{cd} \nabla_a \phi \nabla_d (R_{becf} \nabla^e \phi \nabla^f \phi) \\
= & -k^a k^b k^c l^d \nabla_a \phi \nabla_c R_{bedf} \nabla^e \phi \nabla^f \phi - k^a k^b k^c l^d \nabla_a \phi R_{bedf} \nabla_c \nabla^e \phi \nabla^f \phi \\
& - k^a k^b k^c l^d \nabla_a \phi R_{bedf} \nabla^e \phi \nabla_c \nabla^f \phi - k^a k^b k^c l^d \nabla_a \phi \nabla_d R_{becf} \nabla^e \phi \nabla^f \phi \\
& - k^a k^b k^c l^d \nabla_a \phi R_{becf} \nabla_d \nabla^e \phi \nabla^f \phi - k^a k^b k^c l^d \nabla_a \phi R_{becf} \nabla^e \phi \nabla_d \nabla^f \phi \\
& + k^a k^b \gamma^{cd} \nabla_a \phi \nabla_d R_{becf} \nabla^e \phi \nabla^f \phi + k^a k^b \gamma^{cd} \nabla_a \phi R_{becf} \nabla_d \nabla^e \phi \nabla^f \phi \\
& + k^a k^b \gamma^{cd} \nabla_a \phi R_{becf} \nabla^e \phi \nabla_d \nabla^f \phi.
\end{aligned} \tag{D41}
$$

The first term of Eq. (D41) is

$$
\begin{aligned}
-k^a k^b k^c l^d \nabla_a \phi \nabla_c R_{bedf} \nabla^e \phi \nabla^f \phi = & [(\mathcal{L}_k \phi)_1]^2 (k^c k^d k^e l^g l^h \nabla_e R_{cgdh})_1 (l^f \nabla_f \phi)_0 - (\mathcal{L}_k \phi)_1 (k^b k^c l^d \nabla_c R_{b\hat{m}d\hat{n}})_1 (D^{\hat{m}} \phi)_0 (D^{\hat{n}} \phi)_0 \\
& + [(\mathcal{L}_k \phi)_1]^2 (k^c k^d l^e l^f \nabla_d R_{cef\hat{m}})_0 (D^{\hat{m}} \phi)_0 \\
& - (\mathcal{L}_k \phi)_1 (k^b k^c k^d l^f \nabla_d R_{bfc\hat{m}})_1 (l^e \nabla_e \phi)_0 (D^{\hat{m}} \phi)_0 \simeq 0.
\end{aligned} \tag{D42}
$$

The second term of Eq. (D41) is

$$
\begin{aligned}
-k^a k^b k^c l^d \nabla_a \phi R_{bedf} \nabla_c \nabla^e \phi \nabla^f \phi = & -(\mathcal{L}_k \phi)_1 D^{\hat{g}} (\mathcal{L}_k \phi)_1 (l^e \nabla_e \phi)_0 (k^c k^d l^f R_{cfd\hat{g}})_1 - (\mathcal{L}_k \phi)_1 D^{\hat{f}} (\mathcal{L}_k \phi)_1 (D^{\hat{g}} \phi)_0 (k^c l^d R_{c\hat{f}d\hat{g}})_0 \\
& + (\mathcal{L}_k \phi)_1 (\mathcal{L}_k^2 \phi)_1 (l^f \nabla_f \phi)_0 (k^d k^e l^g l^h R_{dgeh})_0 + (\mathcal{L}_k \phi)_1 (\mathcal{L}_k^2 \phi)_1 (D^{\hat{g}} \phi)_0 (k^d l^e l^f R_{def\hat{g}})_0 \simeq 0.
\end{aligned} \tag{D43}
$$

The third term of Eq. (D41) is

$$
\begin{aligned}
-k^a k^b k^c l^d \nabla_a \phi R_{bedf} \nabla^e \phi \nabla_c \nabla^f \phi = & [(\mathcal{L}_k \phi)_1]^2 (k^c l^f \nabla_c \nabla_f \phi)_0 (k^d k^e l^g l^h R_{dgeh})_0 + [(\mathcal{L}_k \phi)_1]^2 D^{\hat{g}} (\mathcal{L}_k \phi)_1 (k^d l^e l^f R_{def\hat{g}})_0 \\
& - (\mathcal{L}_k \phi)_1 D^{\hat{f}} (\mathcal{L}_k \phi)_1 (D^{\hat{g}} \phi)_0 (k^c l^d R_{c\hat{g}d\hat{f}})_0 \\
& - (\mathcal{L}_k \phi)_1 (k^b l^e \nabla_b \nabla_e \phi)_0 (D^{\hat{g}} \phi)_0 (k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0.
\end{aligned} \tag{D44}
$$

The fourth term of Eq. (D41) is

$$
\begin{aligned}
-k^a k^b k^c l^d \nabla_a \phi \nabla_d R_{becf} \nabla^e \phi \nabla^f \phi = & -(\mathcal{L}_k \phi)_1 (k^b k^c l^d \nabla_d R_{b\hat{g}c\hat{h}})_1 (D^{\hat{g}} \phi)_0 (D^{\hat{h}} \phi)_0 + 2[(\mathcal{L}_k \phi)_1]^2 (k^c k^d l^e l^f \nabla_f R_{ced\hat{g}})_0 (D^{\hat{g}} \phi)_0 \\
& - [(\mathcal{L}_k \phi)_1]^3 (k^d k^e l^f l^g l^h \nabla_h R_{dfeg})_0 \simeq 0.
\end{aligned} \tag{D45}
$$

The fifth term of Eq. (D41) is

$$\begin{aligned}-k^a k^b k^c l^d \nabla_a\phi R_{becf}\nabla_d\nabla^e\phi\nabla^f\phi = {}& [(\mathcal{L}_k\phi)_1]^2 (l^e\nabla_e\nabla^{\hat{g}}\phi)_0 (k^c k^d l^f R_{cfd\hat{g}})_1 - (\mathcal{L}_k\phi)_1 (l^d\nabla_d\nabla^{\hat{f}}\phi)_0 (D^{\hat{g}}\phi)_0 (k^b k^c R_{b\hat{g}c\hat{f}})_1 \\ & - [(\mathcal{L}_k\phi)_1]^2 (k^c l^f\nabla_f\nabla_c\phi)_0 (k^d k^e l^g l^h R_{dgeh})_0 \\ & + (\mathcal{L}_k\phi)_1 (k^b l^e\nabla_e\nabla_b\phi)_0 (D^{\hat{g}}\phi)_0 (k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \end{aligned} \tag{D46}$$

The sixth term of Eq. (D41) is

$$\begin{aligned}-k^a k^b k^c l^d \nabla_a\phi R_{becf}\nabla^e\phi\nabla_d\nabla^f\phi = {}& [(\mathcal{L}_k\phi)_1]^2 (k^c k^d l^f R_{cfd\hat{g}})_1 (l^e\nabla_e\nabla^{\hat{g}}\phi)_0 - (\mathcal{L}_k\phi)_1 (l^d\nabla_d\nabla^{\hat{f}}\phi)_0 (D^{\hat{g}}\phi)_0 (k^b k^c R_{b\hat{g}c\hat{f}})_1 \\ & - [(\mathcal{L}_k\phi)_1]^2 (k^c l^f\nabla_f\nabla_c\phi)_0 (k^d k^e l^g l^h R_{dgeh})_0 \\ & + (\mathcal{L}_k\phi)_1 (k^b l^e\nabla_e\nabla_b\phi)_0 (D^{\hat{g}}\phi)_0 (k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \end{aligned} \tag{D47}$$

The seventh term of Eq. (D41) is

$$\begin{aligned} k^a k^b \gamma^{cd}\nabla_a\phi\nabla_d R_{becf}\nabla^e\phi\nabla^f\phi = {}& -(\mathcal{L}_k\phi)_1 (k^b\nabla^{\hat{g}} R_{b\hat{e}\hat{f}\hat{g}})_1 (D^{\hat{e}}\phi)_0 (D^{\hat{f}}\phi)_0 + (\mathcal{L}_k\phi)_1 (l^d\nabla_d\phi)_0 (D^{\hat{f}}\phi)_0 (k^b k^c\nabla^{\hat{g}} R_{b\hat{f}c\hat{g}})_1 \\ & + [(\mathcal{L}_k\phi)_1]^2 (D^{\hat{f}}\phi)_0 (k^c l^d\nabla^{\hat{g}} R_{cd\hat{f}\hat{g}})_0 - [(\mathcal{L}_k\phi)_1]^2 (l^e\nabla_e\phi)_0 (k^c k^d l^f\nabla^{\hat{g}} R_{cfd\hat{g}})_1 \\ & + [(\mathcal{L}_k\phi)_1]^2 (D^{\hat{f}}\phi)_0 (k^c l^d\nabla^{\hat{g}} R_{c\hat{f}d\hat{g}})_0 - [(\mathcal{L}_k\phi)_1]^3 (k^d l^e l^f\nabla^{\hat{g}} R_{def\hat{g}})_0 \simeq 0. \end{aligned} \tag{D48}$$

The eighth term of Eq. (D41) is

$$\begin{aligned} k^a k^b \gamma^{cd}\nabla_a\phi R_{becf}\nabla_d\nabla^e\phi\nabla^f\phi = {}& -(\mathcal{L}_k\phi)_1 (D^{\hat{f}}\phi)_0 (D^{\hat{e}} D^{\hat{d}}\phi)_0 (k^b R_{b\hat{d}\hat{f}\hat{e}})_1 + (\mathcal{L}_k\phi)_1 (D^{\hat{g}}\phi)_0 D^{\hat{f}} (\mathcal{L}_k\phi)_1 (k^c l^d R_{cd\hat{g}\hat{f}})_0 \\ & + [(\mathcal{L}_k\phi)_1]^2 (D^{\hat{f}} D^{\hat{e}}\phi)_0 (k^c l^d R_{c\hat{e}d\hat{f}})_0 + (\mathcal{L}_k\phi)_1 (l^d\nabla_d\phi)_0 (D^{\hat{f}} D^{\hat{e}}\phi)_0 (k^b k^c R_{b\hat{e}c\hat{f}})_1 \\ & - (\mathcal{L}_k\phi)_1 (l^e\nabla_e\phi)_0 D^{\hat{g}} (\mathcal{L}_k\phi)_1 (k^c k^d l^f R_{cfd\hat{g}})_1 - [(\mathcal{L}_k\phi)_1]^2 D^{\hat{g}} (\mathcal{L}_k\phi)_1 (k^d l^e l^f R_{def\hat{g}})_0 \simeq 0. \end{aligned} \tag{D49}$$

The ninth term of Eq. (D41) is

$$\begin{aligned} k^a k^b \gamma^{cd}\nabla_a\phi R_{becf}\nabla^e\phi\nabla_d\nabla^f\phi = {}& -(\mathcal{L}_k\phi)_1 (D^{\hat{f}}\phi)_0 (D^{\hat{e}} D^{\hat{d}}\phi)_0 (k^b R_{b\hat{f}\hat{d}\hat{e}})_1 + (\mathcal{L}_k\phi)_1 (D^{\hat{g}}\phi)_0 D^{\hat{f}} (\mathcal{L}_k\phi)_1 (k^c l^d R_{c\hat{g}d\hat{f}})_0 \\ & + (\mathcal{L}_k\phi)_1 (D^{\hat{g}}\phi)_0 (l^d\nabla^{\hat{f}}\nabla_d\phi)_0 (k^b k^c R_{b\hat{g}c\hat{f}})_1 + [(\mathcal{L}_k\phi)_1]^2 (D^{\hat{f}} D^{\hat{e}}\phi)_0 (k^c l^d R_{cd\hat{e}\hat{f}})_0 \\ & - [(\mathcal{L}_k\phi)_1]^2 D^{\hat{g}} (\mathcal{L}_k\phi)_1 (k^d l^e l^f R_{def\hat{g}})_0 - [(\mathcal{L}_k\phi)_1]^2 (l^e\nabla^{\hat{g}}\nabla_e\phi)_0 (k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \end{aligned} \tag{D50}$$

So the seventh term of Eq. (38) under the first-order approximation is

$$-k^a k^b\nabla_a\phi\nabla^c (R_{cdbe}\nabla^d\phi\nabla^e\phi) \simeq 0. \tag{D51}$$

The eighth term of Eq. (38) is

$$\begin{aligned}\frac{1}{4}k^a k^b\nabla_c\phi\nabla^c (R\nabla_a\phi\nabla_b\phi) = {}& \frac{1}{4}k^a k^b\gamma_c{}^d\nabla^c\phi\nabla_d (R\nabla_a\phi\nabla_b\phi) - \frac{1}{4}k^a k^b k^c l^d\nabla_a\phi\nabla_d (R\nabla_b\phi\nabla_c\phi) \\ & - \frac{1}{4}k^a k^b k^c l^d\nabla_c (R\nabla_a\phi\nabla_b\phi)\nabla_d\phi. \end{aligned} \tag{D52}$$

The first term of Eq. (D52) is

$$\begin{aligned}\frac{1}{4}k^a k^b\gamma_c{}^d\nabla^c\phi\nabla_d (R\nabla_a\phi\nabla_b\phi) = {}& \frac{1}{4}[(\mathcal{L}_k\phi)_1]^2 (D_{\hat{e}} R)_0 (D^{\hat{e}}\phi)_0 + \frac{1}{4}D^{\hat{e}} (\mathcal{L}_k\phi)_1 (\mathcal{L}_k\phi)_1 (D_{\hat{e}}\phi)_0 (R)_0 \\ & + \frac{1}{4}(\mathcal{L}_k\phi)_1 D_{\hat{e}} (\mathcal{L}_k\phi)_1 (D^{\hat{e}}\phi)_0 (R)_0 \simeq 0. \end{aligned} \tag{D53}$$

The second term of Eq. (D52) is

$$-\frac{1}{4}k^a k^b k^c l^d \nabla_a\phi\nabla_d(R\nabla_b\phi\nabla_c\phi) = -\frac{1}{4}[(\mathcal{L}_k\phi)_1]^3(l^d\nabla_d R)_0 - \frac{1}{4}[(\mathcal{L}_k\phi)_1]^2(R)_0(k^b l^d\nabla_d\nabla_b\phi)_0 - \frac{1}{4}[(\mathcal{L}_k\phi)_1]^2(k^c l^d\nabla_d\nabla_c\phi)_0(R)_0 \simeq 0. \tag{D54}$$

The third term of Eq. (D52) is

$$-\frac{1}{4}k^a k^b k^c l^d \nabla_c(R\nabla_a\phi\nabla_b\phi)\nabla_d\phi = -\frac{1}{4}[(\mathcal{L}_k\phi)_1]^2(l^d\nabla_d\phi)_0(\mathcal{L}_k R)_1 - \frac{1}{4}(R)_0(\mathcal{L}_k^2\phi)_1(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0 - \frac{1}{4}(R)_0(\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)_1(l^d\nabla_d\phi)_0 \simeq 0. \tag{D55}$$

So the eighth term of Eq. (38) under the linear order approximation is

$$\frac{1}{4}k^a k^b \nabla_c\phi\nabla^c(R\nabla_a\phi\nabla_b\phi) \simeq 0. \tag{D56}$$

The ninth term of Eq. (38) is

$$\begin{aligned}
-\frac{1}{2}k^a k^b \nabla_c\phi\nabla^c(R_{bd}\nabla_a\phi\nabla^d\phi) &= -\frac{1}{2}k^a k^b \gamma_c{}^d\nabla^c\phi\nabla_d(R_{be}\nabla_a\phi\nabla^e\phi) + \frac{1}{2}k^a k^b k^c l^d\nabla_a\phi\nabla_d(R_{ce}\nabla_b\phi\nabla^e\phi) \\
&\quad + \frac{1}{2}k^a k^b k^c l^d\nabla_c(R_{be}\nabla_a\phi\nabla^e\phi)\nabla_d\phi \\
&= -\frac{1}{2}k^a k^b \gamma_c{}^d\nabla^c\phi\nabla_d R_{be}\nabla_a\phi\nabla^e\phi - \frac{1}{2}k^a k^b \gamma_c{}^d\nabla^c\phi R_{be}\nabla_d\nabla_a\phi\nabla^e\phi \\
&\quad - \frac{1}{2}k^a k^b \gamma_c{}^d\nabla^c\phi R_{be}\nabla_a\phi\nabla_d\nabla^e\phi + \frac{1}{2}k^a k^b k^c l^d\nabla_a\phi\nabla_d R_{ce}\nabla_b\phi\nabla^e\phi \\
&\quad + \frac{1}{2}k^a k^b k^c l^d\nabla_a\phi R_{ce}\nabla_d\nabla_b\phi\nabla^e\phi + \frac{1}{2}k^a k^b k^c l^d\nabla_a\phi R_{ce}\nabla_b\phi\nabla_d\nabla^e\phi \\
&\quad + \frac{1}{2}k^a k^b k^c l^d\nabla_c R_{be}\nabla_a\phi\nabla^e\phi\nabla_d\phi + \frac{1}{2}k^a k^b k^c l^d R_{be}\nabla_c\nabla_a\phi\nabla^e\phi\nabla_d\phi \\
&\quad + \frac{1}{2}k^a k^b k^c l^d R_{be}\nabla_a\phi\nabla_c\nabla^e\phi\nabla_d\phi.
\end{aligned} \tag{D57}$$

The first term of Eq. (D57) is

$$-\frac{1}{2}k^a k^b \gamma_c{}^d\nabla^c\phi\nabla_d R_{be}\nabla_a\phi\nabla^e\phi = \frac{1}{2}(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0(D^{\hat{e}}\phi)_0(k^b k^c\nabla_{\hat{e}}R_{bc})_1 - \frac{1}{2}(\mathcal{L}_k\phi)_1(k^b\nabla_{\hat{f}}R_{b\hat{e}})_1(D^{\hat{e}}\phi)_0(D^{\hat{f}}\phi)_0 + \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(D^{\hat{f}}\phi)_0(k^c l^d\nabla_{\hat{f}}R_{cd})_0 \simeq 0. \tag{D58}$$

The second term of Eq. (D57) is

$$-\frac{1}{2}k^a k^b \gamma_c{}^d\nabla^c\phi R_{be}\nabla_d\nabla_a\phi\nabla^e\phi = -\frac{1}{2}(k^b R_{b\hat{f}})_1(D^{\hat{e}}\phi)_0(D^{\hat{f}}\phi)_0 D_{\hat{e}}(\mathcal{L}_k\phi)_1 + \frac{1}{2}(R_{kk})_1(l^d\nabla_d\phi)_0(D^{\hat{f}}\phi)_0 D_{\hat{f}}(\mathcal{L}_k\phi)_1 + \frac{1}{2}(k^c l^d R_{cd})_1(\mathcal{L}_k\phi)_1(D^{\hat{f}}\phi)_0 D_{\hat{f}}(\mathcal{L}_k\phi)_1 \simeq 0. \tag{D59}$$

The third term of Eq. (D57) is

$$-\frac{1}{2}k^a k^b \gamma_c{}^d\nabla^c\phi R_{be}\nabla_a\phi\nabla_d\nabla^e\phi = -\frac{1}{2}(\mathcal{L}_k\phi)_1(D_{\hat{e}}\phi)_0(D^{\hat{e}}D^{\hat{d}}\phi)_0(k^b R_{b\hat{d}})_1 + \frac{1}{2}(\mathcal{L}_k\phi)_1(D^{\hat{f}}\phi)_0 D_{\hat{f}}(\mathcal{L}_k\phi)_1(k^c l^d R_{cd})_0 + \frac{1}{2}(R_{kk})_1(D_{\hat{f}}\phi)_0(l^d\nabla^{\hat{f}}\nabla_d\phi)_0(\mathcal{L}_k\phi)_1 \simeq 0. \tag{D60}$$

The fourth term of Eq. (D57) is

$$\frac{1}{2}k^ak^bk^cl^d\nabla_a\phi\nabla_dR_{ce}\nabla_b\phi\nabla^e\phi=\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(k^cl^d\nabla_dR_{c\hat{e}})_0(D^{\hat{e}}\phi)_0-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^e\nabla_e\phi)_0(k^ck^dl^f\nabla_fR_{cd})_1$$
$$-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^3(k^dl^el^f\nabla_fR_{de})_0\simeq 0. \tag{D61}$$

The fifth term of Eq. (D57) is

$$\frac{1}{2}k^ak^bk^cl^d\nabla_a\phi R_{ce}\nabla_d\nabla_b\phi\nabla^e\phi=-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(k^dl^fR_{df})_0(k^cl^e\nabla_e\nabla_c\phi)_0+\frac{1}{2}(\mathcal{L}_k\phi)_1(k^bl^d\nabla_d\nabla_b\phi)_0(D^{\hat{e}}\phi)_0(k^cR_{c\hat{e}})_1$$
$$-\frac{1}{2}(R_{kk})_1(\mathcal{L}_k\phi)_1(l^e\nabla_e\phi)_0(k^bl^f\nabla_f\nabla_b\phi)_0\simeq 0. \tag{D62}$$

The sixth term of Eq. (D57) is

$$\frac{1}{2}k^ak^bk^cl^d\nabla_a\phi R_{ce}\nabla_b\phi\nabla_d\nabla^e\phi=\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^d\nabla_d\nabla^{\hat{e}}\phi)_0(k^cR_{c\hat{e}})_1-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(k^cl^e\nabla_e\nabla_c\phi)_0(k^dl^fR_{df})_0$$
$$-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^el^f\nabla_f\nabla_e\phi)_0(R_{kk})_1\simeq 0. \tag{D63}$$

The seventh term of Eq. (D57) is

$$\frac{1}{2}k^ak^bk^cl^d\nabla_cR_{be}\nabla_a\phi\nabla^e\phi\nabla_d\phi=-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(k^ck^dl^f\nabla_dR_{cf})_1(l^e\nabla_e\phi)_0+\frac{1}{2}(\mathcal{L}_k\phi)_1(k^bk^c\nabla_cR_{b\hat{f}})_1(l^d\nabla_d\phi)_0(D^{\hat{f}}\phi)_0$$
$$-\frac{1}{2}(\mathcal{L}_k\phi)_1(\mathcal{L}_kR_{kk})_1(l^e\nabla_e\phi)_0(l^f\nabla_f\phi)_0\simeq 0. \tag{D64}$$

The eighth term of Eq. (D57) is

$$\frac{1}{2}k^ak^bk^cl^dR_{be}\nabla_c\nabla_a\phi\nabla^e\phi\nabla_d\phi=-\frac{1}{2}(\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)_1(l^e\nabla_e\phi)_0(k^dl^fR_{df})_0+\frac{1}{2}(\mathcal{L}_k^2\phi)_1(l^d\nabla_d\phi)_0(D^{\hat{f}}\phi)_0(k^cR_{c\hat{f}})_1$$
$$-\frac{1}{2}(\mathcal{L}_k^2\phi)(l^e\nabla_e\phi)_0(l^f\nabla_f\phi)_0(R_{kk})_1\simeq 0. \tag{D65}$$

The ninth term of Eq. (D57) is

$$\frac{1}{2}k^ak^bk^cl^dR_{be}\nabla_a\phi\nabla_c\nabla^e\phi\nabla_d\phi=\frac{1}{2}(\mathcal{L}_k\phi)_1D^{\hat{e}}(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0(k^cR_{c\hat{e}})_1-\frac{1}{2}(\mathcal{L}_k\phi)_1(k^bl^f\nabla_b\nabla_f\phi)_0(l^e\nabla_e\phi)_0(R_{kk})_1$$
$$-\frac{1}{2}(\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)_1(l^e\nabla_e\phi)_0(k^dl^fR_{df})_0\simeq 0. \tag{D66}$$

So the ninth term of Eq. (38) under the first-order approximation is

$$-\frac{1}{2}k^ak^b\nabla_c\phi\nabla^c(R_{bd}\nabla_a\phi\nabla^d\phi)\simeq 0. \tag{D67}$$

The tenth term of Eq. (38) is

$$\begin{aligned}-\frac{1}{2}k^ak^b\nabla_c\phi\nabla^c(R_{ad}\nabla_b\phi\nabla^d\phi) &= -\frac{1}{2}k^ak^b\gamma_c{}^d\nabla^c\phi\nabla_d(R_{ae}\nabla_b\phi\nabla^e\phi)+\frac{1}{2}k^ak^bk^cl^d\nabla_a\phi\nabla_d(R_{be}\nabla_c\phi\nabla^e\phi)\\ &\quad+\frac{1}{2}k^ak^bk^cl^d\nabla_c(R_{ae}\nabla_b\phi\nabla^e\phi)\nabla_d\phi\\ &= -\frac{1}{2}k^ak^b\gamma_c{}^d\nabla^c\phi\nabla_dR_{ae}\nabla_b\phi\nabla^e\phi-\frac{1}{2}k^ak^b\gamma_c{}^d\nabla^c\phi R_{ae}\nabla_d\nabla_b\phi\nabla^e\phi\\ &\quad-\frac{1}{2}k^ak^b\gamma_c{}^d\nabla^c\phi R_{ae}\nabla_b\phi\nabla_d\nabla^e\phi+\frac{1}{2}k^ak^bk^cl^d\nabla_a\phi\nabla_dR_{be}\nabla_c\phi\nabla^e\phi\\ &\quad+\frac{1}{2}k^ak^bk^cl^d\nabla_a\phi R_{be}\nabla_d\nabla_c\phi\nabla^e\phi+\frac{1}{2}k^ak^bk^cl^d\nabla_a\phi R_{be}\nabla_c\phi\nabla_d\nabla^e\phi\\ &\quad+\frac{1}{2}k^ak^bk^cl^d\nabla_cR_{ae}\nabla_b\phi\nabla^e\phi\nabla_d\phi+\frac{1}{2}k^ak^bk^cl^dR_{ae}\nabla_c\nabla_b\phi\nabla^e\phi\nabla_d\phi\\ &\quad+\frac{1}{2}k^ak^bk^cl^dR_{ae}\nabla_b\phi\nabla_c\nabla^e\phi\nabla_d\phi. \end{aligned}\tag{D68}$$

The first term of Eq. (D68) is

$$\begin{aligned}-\frac{1}{2}k^ak^b\gamma_c{}^d\nabla^c\phi\nabla_dR_{ae}\nabla_b\phi\nabla^e\phi &= \frac{1}{2}(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0(D^{\hat{f}}\phi)_0(k^bk^c\nabla_{\hat{f}}R_{bc})_1-\frac{1}{2}(\mathcal{L}_k\phi)_1(D^{\hat{e}}\phi)_0(D^{\hat{f}}\phi)_0(k^b\nabla_{\hat{f}}R_{b\hat{e}})_1\\ &\quad+\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(D^{\hat{e}}\phi)_0(k^cl^d\nabla_{\hat{e}}R_{cd})_0\simeq 0. \end{aligned}\tag{D69}$$

The second term of Eq. (D68) is

$$\begin{aligned}-\frac{1}{2}k^ak^b\gamma_c{}^d\nabla^c\phi R_{ae}\nabla_d\nabla_b\phi\nabla^e\phi &= -\frac{1}{2}(k^bR_{b\hat{f}})_1(D^{\hat{e}}\phi)_0(D^{\hat{f}}\phi)_0D_{\hat{e}}(\mathcal{L}_k\phi)_1+\frac{1}{2}D^{\hat{e}}(\mathcal{L}_k\phi)_1(R_{kk})_1(l^d\nabla_d\phi)_0(D_{\hat{e}}\phi)_0\\ &\quad+\frac{1}{2}(k^cl^dR_{cd})_0(\mathcal{L}_k\phi)_1(D^{\hat{e}}\phi)_0D_{\hat{e}}(\mathcal{L}_k\phi)_1\simeq 0. \end{aligned}\tag{D70}$$

The third term of Eq. (D68) is

$$\begin{aligned}-\frac{1}{2}k^ak^b\gamma_c{}^d\nabla^c\phi R_{ae}\nabla_b\phi\nabla_d\nabla_d\nabla^e\phi &= -\frac{1}{2}(\mathcal{L}_k\phi)_1(k^bR_{b\hat{d}})_1(D_{\hat{e}}\phi)_0(D^{\hat{e}}D^{\hat{d}}\phi)_0+\frac{1}{2}(\mathcal{L}_k\phi)_1(D^{\hat{f}}\phi)_0(k^cl^dR_{cd})_0D_{\hat{f}}(\mathcal{L}_k\phi)_1\\ &\quad+\frac{1}{2}(R_{kk})_1(\mathcal{L}_k\phi)_1(D^{\hat{f}}\phi)_0(l^d\nabla_{\hat{f}}\nabla_d\phi)_0\simeq 0. \end{aligned}\tag{D71}$$

The fourth term of Eq. (D68) is

$$\begin{aligned}\frac{1}{2}k^ak^bk^cl^d\nabla_a\phi\nabla_dR_{be}\nabla_c\phi\nabla^e\phi &= \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(k^cl^d\nabla_dR_{c\hat{e}})_0(D^{\hat{e}}\phi)_0-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^e\nabla_e\phi)_0(k^ck^dl^f\nabla_fR_{cd})_1\\ &\quad-\frac{1}{2}[(\mathcal{L}_k\phi)_1]^3(k^dl^el^f\nabla_fR_{de})_0\simeq 0. \end{aligned}\tag{D72}$$

The fifth term of Eq. (D68) is

$$\begin{aligned}\frac{1}{2}k^ak^bk^cl^d\nabla_a\phi R_{be}\nabla_d\nabla_c\phi\nabla^e\phi &= -\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(k^cl^e\nabla_e\nabla_c\phi)_0(k^dl^fR_{df})_0+\frac{1}{2}(\mathcal{L}_k\phi)_1(k^bl^d\nabla_d\nabla_b\phi)_0(D^{\hat{e}}\phi)_0(k^cR_{c\hat{e}})_1\\ &\quad-\frac{1}{2}(\mathcal{L}_k\phi)_1(l^e\nabla_e\phi)_0(k^bl^f\nabla_f\nabla_b\phi)_0(R_{kk})_1\simeq 0. \end{aligned}\tag{D73}$$

The sixth term of Eq. (D68) is

$$\frac{1}{2}k^ak^bk^cl^d\nabla_a\phi R_{be}\nabla_c\phi\nabla_d\nabla^e\phi = \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^d\nabla_d\nabla^{\hat{e}}\phi)_0(k^cR_{c\hat{e}})_1 - \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(k^cl^e\nabla_e\nabla_c\phi)_0(k^dl^fR_{df})_0 - \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(l^el^f\nabla_e\nabla_f\phi)_0(R_{kk})_1 \simeq 0. \quad (D74)$$

The seventh term of Eq. (D68) is

$$\frac{1}{2}k^ak^bk^cl^d\nabla_cR_{ae}\nabla_b\phi\nabla^e\phi\nabla_d\phi = -\frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(k^ck^dl^f\nabla_dR_{cf})_1(l^e\nabla_e\phi)_0 + \frac{1}{2}(\mathcal{L}_k\phi)_1(k^bk^c\nabla_cR_{b\hat{f}})_1(l^d\nabla_d\phi)_0(D^{\hat{f}}\phi)_0 - \frac{1}{2}(\mathcal{L}_k\phi)_1(\mathcal{L}_kR_{kk})_1(l^e\nabla_e\phi)_0(l^f\nabla_f\phi)_0 \simeq 0. \quad (D75)$$

The eighth term of Eq. (D68) is

$$\frac{1}{2}k^ak^bk^cl^dR_{ae}\nabla_c\nabla_b\phi\nabla^e\phi\nabla_d\phi = -\frac{1}{2}(\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)_1(l^e\nabla_e\phi)_0(k^dl^fR_{df})_0 + \frac{1}{2}(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0(D^{\hat{f}}\phi)_0(k^cR_{c\hat{f}})_1 - \frac{1}{2}(\mathcal{L}_k\phi)_1(l^e\nabla_e\phi)_0(l^f\nabla_f\phi)_0(R_{kk})_1 \simeq 0. \quad (D76)$$

The ninth term of Eq. (D68) is

$$\frac{1}{2}k^ak^bk^cl^dR_{ae}\nabla_b\phi\nabla_c\nabla^e\phi\nabla_d\phi = \frac{1}{2}(\mathcal{L}_k\phi)_1D^{\hat{e}}(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0(k^cR_{c\hat{e}})_1 - \frac{1}{2}(\mathcal{L}_k\phi)_1(k^bl^f\nabla_b\nabla_f\phi)_0(l^e\nabla_e\phi)_0(R_{kk})_1 - \frac{1}{2}(\mathcal{L}_k\phi)_1(\mathcal{L}_k^2\phi)_1(l^e\nabla_e\phi)_0(k^dl^fR_{df})_0 \simeq 0. \quad (D77)$$

So the tenth term of Eq. (38) under the first-order approximation is

$$-\frac{1}{2}k^ak^b\nabla_c\phi\nabla^c(R_{ad}\nabla_b\phi\nabla^d\phi) \simeq 0. \quad (D78)$$

The eleventh term of Eq. (38) is

$$\frac{1}{2}k^ak^b\nabla_c\phi\nabla^c(R_{ab}\nabla_d\phi\nabla^d\phi) = \frac{1}{2}k^ak^b\gamma_c{}^d\nabla^c\phi\nabla_d(R_{ab}\nabla_e\phi\nabla^e\phi) - \frac{1}{2}k^ak^bk^cl^d\nabla_a\phi\nabla_d(R_{bc}\nabla_e\phi\nabla^e\phi) - \frac{1}{2}k^ak^bk^cl^d\nabla_c(R_{ab}\nabla_e\phi\nabla^e\phi)\nabla_d\phi. \quad (D79)$$

The first term of Eq. (D79) is

$$\begin{aligned}\frac{1}{2}k^ak^b\gamma_c{}^d\nabla^c\phi\nabla_d(R_{ab}\nabla_e\phi\nabla^e\phi) &= \frac{1}{2}k^ak^b\gamma_c{}^d(\nabla_dR_{ab})\nabla^c\phi\nabla_e\phi\nabla^e\phi + \frac{1}{2}k^ak^bR_{ab}\gamma_c{}^d\nabla^c\phi\nabla_d(\nabla_e\phi\nabla^e\phi)\\ &= \frac{1}{2}D_{\hat{f}}(R_{kk})_1(D^{\hat{f}}\phi)_0(\nabla_e\phi\nabla^e\phi)_0 - \frac{1}{2}(k^aR_{a\hat{b}})_1(B^{\hat{d}\hat{b}})_1(D_{\hat{d}}\phi)_0(\nabla^e\phi\nabla_e\phi)_0\\ &\quad + \frac{1}{2}(R_{kk})_1(l^c\nabla_{\hat{d}}k_c)_1(D^{\hat{d}}\phi)_0(\nabla^e\phi\nabla_e\phi)_0 - \frac{1}{2}(k^bR_{\hat{a}b})_1(B^{\hat{d}\hat{a}})_1(D_{\hat{d}}\phi)_0(\nabla_e\phi\nabla^e\phi)_0\\ &\quad + \frac{1}{2}(R_{kk})_1(l^c\nabla_{\hat{d}}k_c)_1(D^{\hat{d}}\phi)_0(\nabla_e\phi\nabla^e\phi)_0 + \frac{1}{2}(R_{kk})_1(D^{\hat{f}}\phi)_0D_{\hat{f}}(\nabla_e\phi\nabla^e\phi)_0\\ &\simeq \frac{1}{2}D_{\hat{f}}(R_{kk})(D^{\hat{f}}\phi)(\nabla_e\phi\nabla^e\phi) + \frac{1}{2}(R_{kk})(D^{\hat{f}}\phi)D_{\hat{f}}(\nabla_e\phi\nabla^e\phi) \sim \mathcal{O}(\epsilon).\end{aligned} \quad (D80)$$

The second term of Eq. (D79) is

$$-\frac{1}{2}k^a k^b k^c l^d \nabla_a\phi\nabla_d(R_{bc}\nabla_e\phi\nabla^e\phi) = -\frac{1}{2}(\mathcal{L}_k\phi)_1(k^b k^c l^d\nabla_d R_{bc})_1(\nabla^e\phi\nabla_e\phi)_0 - \frac{1}{2}(\mathcal{L}_k\phi)_1(R_{kk})_1[l^d\nabla_d(\nabla_e\phi\nabla^e\phi)]_0 \simeq 0. \tag{D81}$$

The third term of Eq. (D79) is

$$-\frac{1}{2}k^a k^b k^c l^d \nabla_c(R_{ab}\nabla_e\phi\nabla^e\phi)\nabla_d\phi = -\frac{1}{2}(\mathcal{L}_k R_{kk})_1(\nabla_e\phi\nabla^e\phi)_0(l^d\nabla_d\phi)_0 - \frac{1}{2}(R_{kk})_1[\mathcal{L}_k(\nabla_e\phi\nabla^e\phi)]_1(l^d\nabla_d\phi)_0$$
$$\simeq -\frac{1}{2}(\mathcal{L}_k R_{kk})(\nabla_e\phi\nabla^e\phi)(l^d\nabla_d\phi) \sim \mathcal{O}(\epsilon). \tag{D82}$$

So the eleventh term of Eq. (38) under the linear order approximation can be written as

$$\frac{1}{2}k^a k^b\nabla_c\phi\nabla^c(R_{ab}\nabla_d\phi\nabla^d\phi) = \frac{1}{2}D_{\hat{f}}(R_{kk})(D^{\hat{f}}\phi)(\nabla_e\phi\nabla^e\phi) + \frac{1}{2}(R_{kk})(D^{\hat{f}}\phi)D_{\hat{f}}(\nabla_e\phi\nabla^e\phi) - \frac{1}{2}(\mathcal{L}_k R_{kk})(\nabla_e\phi\nabla^e\phi)(l^d\nabla_d\phi). \tag{D83}$$

The twelfth term of Eq. (38) is

$$\begin{aligned}
-\frac{1}{2}k^a k^b\nabla_c\phi\nabla^c(R_{adbe}\nabla^d\phi\nabla^e\phi) &= -\frac{1}{2}k^a k^b\gamma_c{}^d\nabla^c\phi\nabla_d(R_{aebf}\nabla^e\phi\nabla^f\phi) + \frac{1}{2}k^a k^b k^c l^d\nabla_a\phi\nabla_d(R_{becf}\nabla^e\phi\nabla^f\phi)\\
&\quad + \frac{1}{2}k^a k^b k^c l^d\nabla_c(R_{aebf}\nabla^e\phi\nabla^f\phi)\nabla_d\phi\\
&= -\frac{1}{2}k^a k^b\gamma_c{}^d\nabla^c\phi\nabla_d(R_{aebf}\nabla^e\phi\nabla^f\phi) + \frac{1}{2}k^a k^b k^c l^d\nabla_a\phi\nabla_d R_{becf}\nabla^e\phi\nabla^f\phi\\
&\quad + \frac{1}{2}k^a k^b k^c l^d\nabla_a\phi R_{becf}\nabla_d\nabla^e\phi\nabla^f\phi + \frac{1}{2}k^a k^b k^c l^d\nabla_a\phi R_{becf}\nabla^e\phi\nabla_d\nabla^f\phi\\
&\quad + \frac{1}{2}k^a k^b k^c l^d\nabla_c R_{aebf}\nabla^e\phi\nabla^f\phi\nabla_d\phi + \frac{1}{2}k^a k^b k^c l^d R_{aebf}\nabla_c\nabla^e\phi\nabla^f\phi\nabla_d\phi\\
&\quad + \frac{1}{2}k^a k^b k^c l^d R_{aebf}\nabla^e\phi\nabla_c\nabla^f\phi\nabla_d\phi.
\end{aligned} \tag{D84}$$

The first term of Eq. (D84) is

$$\begin{aligned}
-\frac{1}{2}k^a k^b\gamma_c{}^d\nabla^c\phi\nabla_d(R_{aebf}\nabla^e\phi\nabla^f\phi) &= -\frac{1}{2}(D^{\hat{c}}\phi)_0 D_{\hat{c}}(k^a k^b R_{aebf}\nabla^e\phi\nabla^f\phi)_0 + \frac{1}{2}(D^{\hat{c}}\phi)_0(k^a R_{ae\hat{b}f}\nabla^e\phi\nabla^f\phi)_1(B_{\hat{c}}{}^{\hat{b}})_1\\
&\quad - \frac{1}{2}(D^{\hat{c}}\phi)_0(k^a k^b R_{aebf}\nabla^e\phi\nabla^f\phi)_1(l^d\nabla_{\hat{c}}k_d)_1 + \frac{1}{2}(D^{\hat{c}}\phi)_0(k^b R_{\hat{a}ebf}\nabla^e\phi\nabla^f\phi)_1(B_{\hat{c}}{}^{\hat{a}})_1\\
&\quad - \frac{1}{2}(D^{\hat{c}}\phi)_0(k^a k^b R_{aebf}\nabla^e\phi\nabla^f\phi)_1(l^d\nabla_{\hat{c}}k_d)_1\\
&\simeq -\frac{1}{2}(D^{\hat{c}}\phi)D_{\hat{c}}(k^a k^b R_{aebf}\nabla^e\phi\nabla^f\phi) \sim \mathcal{O}(\epsilon).
\end{aligned} \tag{D85}$$

The second term of Eq. (D84) is

$$\begin{aligned}
\frac{1}{2}k^a k^b k^c l^d\nabla_a\phi\nabla_d R_{becf}\nabla^e\phi\nabla^f\phi &= \frac{1}{2}(\mathcal{L}_k\phi)_1(k^b k^c l^d\nabla_d R_{b\hat{g}c\hat{h}})_1(D^{\hat{g}}\phi)_0(D^{\hat{h}}\phi)_0 - [(\mathcal{L}_k\phi)_1]^2(k^c k^d l^e l^f\nabla_f R_{ced\hat{g}})_0(D^{\hat{g}}\phi)_0\\
&\quad + \frac{1}{2}[(\mathcal{L}_k\phi)_1]^3(k^d k^e l^f l^g l^h\nabla_h R_{dfeg})_0 \simeq 0.
\end{aligned} \tag{D86}$$

The third term of Eq. (D84) is

$$\frac{1}{2}k^a k^b k^c l^d \nabla_a \phi R_{becf} \nabla_d \nabla^e \phi \nabla^f \phi = -\frac{1}{2}[(\mathcal{L}_k \phi)_1]^2 (l^e \nabla_e \nabla^{\hat{g}} \phi)_0 (k^c k^d l^f R_{cfd\hat{g}})_1 + \frac{1}{2}(\mathcal{L}_k \phi)_1 (l^d \nabla_d \nabla^{\hat{f}} \phi)_0 (D^{\hat{g}} \phi)_0 (k^b k^c R_{d\hat{g}e\hat{f}})_1$$
$$+ \frac{1}{2}[(\mathcal{L}_k \phi)_1]^2 (k^c l^f \nabla_f \nabla_c \phi)_0 (k^d k^e l^g l^h R_{dgeh})_0$$
$$-\frac{1}{2}(\mathcal{L}_k \phi)_1 (k^b l^e \nabla_e \nabla_b \phi)_0 (D^{\hat{g}} \phi)_0 (k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \tag{D87}$$

The fourth term of Eq. (D84) is

$$\frac{1}{2}k^a k^b k^c l^d \nabla_a \phi R_{becf} \nabla^e \phi \nabla_d \nabla^f \phi = -\frac{1}{2}[(\mathcal{L}_k \phi)_1]^2 (k^c k^d l^f R_{cfd\hat{g}})_1 (l^e \nabla_e \nabla^{\hat{g}} \phi)_0 + \frac{1}{2}(\mathcal{L}_k \phi)_1 (l^d \nabla_d \nabla^{\hat{f}} \phi)_0 (D^{\hat{e}} \phi)_0 (k^b k^c R_{b\hat{e}c\hat{f}})_1$$
$$+ \frac{1}{2}[(\mathcal{L}_k \phi)_1]^2 (k^c l^f \nabla_f \nabla_c \phi)_0 (k^d k^e l^g l^h R_{dgeh})_0$$
$$-\frac{1}{2}(\mathcal{L}_k \phi)_1 (k^b l^e \nabla_e \nabla_b \phi)_0 (D^{\hat{g}} \phi)_0 (k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \tag{D88}$$

The fifth term of Eq. (D84) is

$$\frac{1}{2}k^a k^b k^c l^d \nabla_c R_{aebf} \nabla^e \phi \nabla^f \phi \nabla_d \phi = \frac{1}{2}[(\mathcal{L}_k \phi)_1]^2 (k^c k^d k^e l^g l^h \nabla_e R_{cgdh})_1 (l^f \nabla_f \phi)_0$$
$$+ \frac{1}{2}(k^a k^b k^c \nabla_c R_{a\hat{g}b\hat{h}})_1 (l^d \nabla_d \phi)_0 (D^{\hat{g}} \phi)_0 (D^{\hat{h}} \phi)_0$$
$$- (\mathcal{L}_k \phi)_1 (k^b k^c k^d l^f \nabla_d R_{bfc\hat{g}})_1 (l^e \nabla_e \phi)_0 (D^{\hat{g}} \phi)_0$$
$$\simeq \frac{1}{2}(k^a k^b k^c \nabla_c R_{a\hat{g}b\hat{h}})(l^d \nabla_d \phi)(D^{\hat{g}} \phi)(D^{\hat{h}} \phi) \sim \mathcal{O}(\epsilon). \tag{D89}$$

The sixth term of Eq. (D84) is

$$\frac{1}{2}k^a k^b k^c l^d R_{aebf} \nabla_c \nabla^e \phi \nabla^f \phi \nabla_d \phi = -\frac{1}{2}(\mathcal{L}_k \phi)_1 D^{\hat{g}} (\mathcal{L}_k \phi)_1 (l^e \nabla_e \phi)_0 (k^c k^d l^f R_{cfd\hat{g}})_1$$
$$+ \frac{1}{2} D^{\hat{f}} (\mathcal{L}_k \phi)_1 (l^d \nabla_d \phi)_0 (D^{\hat{g}} \phi)_0 (k^b k^c R_{b\hat{g}c\hat{f}})_1$$
$$+ \frac{1}{2}(\mathcal{L}_k \phi)_1 (\mathcal{L}_k^2 \phi)_1 (l^f \nabla_f \phi)_0 (k^d k^e l^g l^h R_{dgeh})_0$$
$$-\frac{1}{2}(\mathcal{L}_k^2 \phi)_1 (l^e \nabla_e \phi)_0 (D^{\hat{g}} \phi)_0 (k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \tag{D90}$$

The seventh term of Eq. (D84) is

$$\frac{1}{2}k^a k^b k^c l^d R_{aebf} \nabla^e \phi \nabla_c \nabla^f \phi \nabla_d \phi = -\frac{1}{2}(\mathcal{L}_k \phi)_1 D^{\hat{g}} (\mathcal{L}_k \phi)_1 (l^e \nabla_e \phi)_0 (k^c k^d l^f R_{cfd\hat{g}})_1$$
$$+ \frac{1}{2} D^{\hat{f}} (\mathcal{L}_k \phi)_1 (l^d \nabla_d \phi)_0 (D^{\hat{g}} \phi)_0 (k^b k^c R_{b\hat{g}c\hat{f}})_1$$
$$+ \frac{1}{2}(\mathcal{L}_k \phi)_1 (\mathcal{L}_k^2 \phi)_1 (l^f \nabla_f \phi)_0 (k^d k^e l^g l^h R_{dgeh})_0$$
$$-\frac{1}{2}(\mathcal{L}_k^2 \phi)_1 (l^e \nabla_e \phi)_0 (D^{\hat{g}} \phi)_0 (k^c k^d l^f R_{cfd\hat{g}})_1 \simeq 0. \tag{D91}$$

So the twelfth term of Eq. (38) under the first-order approximation is given as

$$-\frac{1}{2}k^a k^b \nabla_c \phi \nabla^c (R_{adbe} \nabla^d \phi \nabla^e \phi) \simeq -\frac{1}{2}(D^{\hat{c}} \phi) D_{\hat{c}} (k^a k^b R_{aebf} \nabla^e \phi \nabla^f \phi) + \frac{1}{2}(k^a k^b k^c \nabla_c R_{a\hat{g}b\hat{h}})(l^d \nabla_d \phi)(D^{\hat{g}} \phi)(D^{\hat{h}} \phi). \tag{D92}$$

The thirteenth term of Eq. (38) is

$$\frac{1}{4}k^a k^b R\nabla_a\phi\nabla_b\phi\nabla_c\nabla^c\phi = -\frac{1}{2}(R)_0[(\mathcal{L}_k\phi)_1]^2(k^a l^d\nabla_a\nabla_d\phi)_0 + \frac{1}{4}(R)_0[(\mathcal{L}_k\phi)_1]^2(D^{\hat{c}}D_{\hat{c}}\phi)_0 \simeq 0. \tag{D93}$$

The fourteenth term of Eq. (38) is

$$\begin{aligned}-\frac{1}{2}k^a k^b R_{bd}\nabla_a\phi\nabla^d\phi\nabla_c\nabla^c\phi = &-(k^d l^f R_{df})_0[(\mathcal{L}_k\phi)_1]^2(k^c l^e\nabla_c\nabla_e\phi)_0 - (\mathcal{L}_k\phi)_1(k^b l^f\nabla_b\nabla_f\phi)_0(l^e\nabla_e\phi)_0(R_{kk})_1\\ &+ (\mathcal{L}_k\phi)_1(k^b l^d\nabla_b\nabla_d\phi)_0(D^{\hat{e}}\phi)_0(k^c R_{c\hat{e}})_1 + \frac{1}{2}[(\mathcal{L}_k\phi)_1](D^{\hat{e}}D_{\hat{e}}\phi)_0(k^c l^d R_{cd})_0\\ &+\frac{1}{2}(\mathcal{L}_k\phi)_1(l^d\nabla_d\phi)_0(D^{\hat{e}}D_{\hat{e}}\phi)_0(R_{kk})_1 - \frac{1}{2}(\mathcal{L}_k\phi)_1(D^{\hat{e}}\phi)_0(D^{\hat{d}}D_{\hat{d}}\phi)_0(k^b R_{b\hat{e}})_1 \simeq 0.\end{aligned} \tag{D94}$$

The fifteenth term of Eq. (38) is

$$\begin{aligned}-\frac{1}{2}k^a k^b R_{ad}\nabla_b\phi\nabla^d\phi\nabla_c\nabla^c\phi = &-[(\mathcal{L}_k\phi)_1]^2(k^c l^e\nabla_c\nabla_e\phi)_0(k^d l^f R_{df})_0 - (\mathcal{L}_k\phi)_1(k^b l^f\nabla_b\nabla_f\phi)_0(l^e\nabla_e\phi)_0(R_{kk})_1\\ &+ (\mathcal{L}_k\phi)_1(k^b l^d\nabla_b\nabla_d\phi)_0(D^{\hat{g}}\phi)_0(k^c R_{c\hat{g}})_1 + \frac{1}{2}[(\mathcal{L}_k\phi)_1]^2(k^c l^d R_{cd})_0(D^{\hat{e}}D_{\hat{e}}\phi)_0\\ &-\frac{1}{2}(\mathcal{L}_k\phi)_1(k^b R_{b\hat{f}})_1(D^{\hat{f}}\phi)_0(D^{\hat{e}}D_{\hat{e}}\phi)_0 + \frac{1}{2}(\mathcal{L}_k\phi)_1(R_{kk})_1(l^d\nabla_d\phi)_0(D^{\hat{e}}D_{\hat{e}}\phi)_0 \simeq 0.\end{aligned} \tag{D95}$$

The sixteenth term of Eq. (38) is

$$\frac{1}{2}k^a k^b R_{ab}\nabla_d\phi\nabla^d\phi\nabla_c\nabla^c\phi = -(R_{kk})_1(\nabla_e\phi\nabla^e\phi)_0(k^a l^d\nabla_a\nabla_d\phi)_0 + \frac{1}{2}(R_{kk})_1(\nabla_e\phi\nabla^e\phi)_0(D^{\hat{c}}D_{\hat{c}}\phi)_0 \sim \mathcal{O}(\epsilon). \tag{D96}$$

The seventeenth term of Eq. (38) is

$$\begin{aligned}-\frac{1}{2}k^a k^b R_{adbe}\nabla^d\phi\nabla^e\phi\nabla_c\nabla^c\phi = &[(\mathcal{L}_k\phi)_1]^2(k^c l^f\nabla_c\nabla_f\phi)_0(k^d k^e l^g l^h R_{dgeh})_0 + (k^a l^d\nabla_a\nabla_d\phi)_0(D^{\hat{g}}\phi)_0(D^{\hat{h}}\phi)_0(k^b k^c R_{b\hat{g}c\hat{h}})_1\\ &-2(\mathcal{L}_k\phi)_1(k^b l^e\nabla_b\nabla_e\phi)_0(D^{\hat{g}}\phi)_0(k^c k^d l^f R_{cfd\hat{g}})_1 - \frac{1}{2}(k^a k^b R_{aebf}\nabla^e\phi\nabla^f\phi)_1(D^{\hat{c}}D_{\hat{c}}\phi)_0\\ \simeq\, &(k^a l^d\nabla_a\nabla_d\phi)(D^{\hat{g}}\phi)(D^{\hat{h}}\phi)(k^b k^c R_{b\hat{g}c\hat{h}}) - \frac{1}{2}(k^a k^b R_{aebf}\nabla^e\phi\nabla^f\phi)(D^{\hat{c}}D_{\hat{c}}\phi) \sim \mathcal{O}(\epsilon).\end{aligned} \tag{D97}$$

Therefore, the expression of $H_{kk}^{(\phi 2)}$ under the linear order approximation of the perturbation can finally be obtained as

$$\begin{aligned}H_{kk}^{(\phi 2)} \simeq\, &\frac{1}{2}(k^a k^b\nabla_{\hat{d}}R_{ab})(D^{\hat{d}}\phi)(\nabla^e\phi\nabla_e\phi) + \frac{1}{2}(R_{kk})(D^{\hat{f}}\phi)D_{\hat{f}}(\nabla_e\phi\nabla^e\phi) + \frac{1}{2}(R_{kk})(\nabla_e\phi\nabla^e\phi)(D^{\hat{c}}D_{\hat{c}}\phi)\\ &-\frac{1}{2}(D^{\hat{c}}\phi)D_{\hat{c}}(k^a k^b R_{aebf}\nabla^e\phi\nabla^f\phi) - \frac{1}{2}(k^a k^b R_{aebf}\nabla^e\phi\nabla^f\phi)(D^{\hat{c}}D_{\hat{c}}\phi) - (R_{kk})(\nabla_e\phi\nabla^e\phi)(k^a l^d\nabla_a\nabla_d\phi)\\ &-\frac{1}{2}(\mathcal{L}_k R_{kk})(\nabla_e\phi\nabla^e\phi)(l^d\nabla_d\phi) + (k^a l^d\nabla_a\nabla_d\phi)(D^{\hat{g}}\phi)(D^{\hat{h}}\phi)(k^b k^c R_{b\hat{g}c\hat{h}})\\ &+\frac{1}{2}(k^a k^b k^c\nabla_c R_{a\hat{g}b\hat{h}})(l^d\nabla_d\phi)(D^{\hat{g}}\phi)(D^{\hat{h}}\phi).\end{aligned} \tag{D98}$$

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