Galactic dark matter effects from purely geometrical aspects of general relativity

Davide Astesiano[*]
Dipartimento di Scienza ed Alta Tecnologia, Università Degli Studi Dell'Insubria, Via Valleggio 11, 22100 Como, Italy and INFN, Via Celoria 16, 20133 Milano, Italy

Matteo Luca Ruggiero[†]
Dipartimento di Matematica "G. Peano", Università Degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy and INFN-LNL, Viale Dell'Università 2, 35020 Legnaro (PD), Italy

(Received 10 May 2022; accepted 12 August 2022; published 26 August 2022)

We study disk galaxies in the framework of general relativity to focus on the possibility that, even in the low energy limit, there are relevant corrections with respect to the purely Newtonian approach. Our analysis encompasses the model by considering both a low energy expansion and exact solutions, making clear the connection between these different approaches. In particular, we focus on two different limits: the well-known gravitomagnetic analogy and a new limit, called "strong gravitomagnetism," which has corrections in c of the same order as the Newtonian terms. We show that these two limits of the general class of solutions can account for the observed flat velocity profile, which is contrary to what happens using Newtonian models, where a dark matter contribution is required. Hence, we suggest a geometrical origin for a certain amount of dark matter effects.

DOI: 10.1103/PhysRevD.106.044061

I. INTRODUCTION

One of the evidences supporting the existence of "dark matter" comes from the observations of the rotation curves of galaxies, which are flat: contrary to what is expected on the basis of Newtonian dynamics [1]. In this context, Newtonian gravity rather than general relativity (GR) is used because far from the galactic center (where the flat behavior is observed), the gravitational field is reasonably supposed to be weak and stars are not moving at relativistic speeds. Nonetheless, it was conjectured that GR may have a role in this context; in particular, the problem of galactic rotation curves was studied by considering both exact solutions of GR equations [2–5] and weak-field approximations [6–8]. Indeed, there are general relativistic effects without a Newtonian analog, such as the gravitomagnetic effects, deriving from mass currents. In the above-cited papers, using different approaches, it was suggested that if these non-Newtonian effects were taken into account, the impact of dark matter in explaining the observations could be different.

The purpose of this paper is to focus on the role that gravitomagnetic and, more in general, post-Newtonian effects might have in galactic dynamics. In order to trace the impact of these effects from a very general viewpoint, we will not resort to a specific galaxy model; but, we will emphasize the modifications introduced by general relativistic effects starting from very few hypotheses, which basically refer to the underlying symmetries. In particular, in Sec. II, we describe how the weak-field approach to the solution of Einstein's equations, which leads to the well-known gravitoelectromagnetic analogy [9,10], can be used to investigate the possible impact of GR effects on galactic rotation curves. Subsequently, in Sec. III, we focus on the exact general relativistic solutions for an axisymmetric stationary system coupled to dust [5], discussing its physical properties and the relevant limits. A new weak-field limit of the general solution that we call strong gravitomagnetism (SGM) is introduced in Sec. IV C; we suggest that it can provide an interesting model for disk galaxies. In this regard, we compare this limit with the Newtonian one (Sec. IV B) and we highlight the differences; unlike the Newtonian model, the SGM limit can naturally provide a flat velocity curve, and the presence of a nondiagonal term in the metric can reduce the amount of energy density needed to sustain the motion of the galaxy. Eventually, in Sec. VI, we consider the rigidly rotating solution; it coincides with the Balasin–Grumiller model [3], which recently gained relevance because, starting from this model, Crosta *et al.* [4] showed a good agreement between the model and the GAIA [11,12] data for the Milky Way. We show that this model, being a rigidly rotating solution, presents some unphysical features that need to be addressed:

[*]davide.astesiano@yahoo.it
[†]matteoluca.ruggiero@unito.it

for instance, the redshift (or blueshift) due to the emission of photons from the galaxy measured from an asymptotic inertial observer is linearly increasing. Before starting with the technical discussion, we point out that the role of gravitomagnetism, and of the other orders in c^{-n}, is usually understood in vacuum, i.e., outside of the source. In this configuration, it makes sense to say that there is a dominant Newtonian contribution plus corrective terms of higher order in c^{-n}. What we do here is substantially different, and the usually adopted expansion in vacuum breaks down. This is due to the fact that we are not analyzing the equations in vacuum but "inside" the matter distribution, i.e., within the galaxy. In Sec. II, we will stress again this important difference.

II. GRAVITOMAGNETIC EFFECTS IN GALAXIES

It possible to write the solution of Einstein's field equations in weak-field and slow-motion approximation by exploiting a well-known analogy with Maxwell equations: this is the so-called "gravitoelectromagnetic" formalism (see, e.g., Ruggiero and Tartaglia [9], and Mashhoon [10]); accordingly, the line element describing this solution is

$$\mathrm{d}s^2 = -c^2\left(1-2\frac{\Phi}{c^2}\right)\mathrm{d}t^2 - \frac{4}{c}(\mathbf{A}\cdot\mathrm{d}\mathbf{x})\mathrm{d}t + \left(1+2\frac{\Phi}{c^2}\right)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{1}$$

In the above equation, the gravitoelectric Φ and gravitomagnetic $\mathbf{A}$ potentials, in stationary conditions, are solutions of the Poisson equations

$$\nabla^2\Phi = -4\pi G\rho, \tag{2}$$

$$\nabla^2\mathbf{A} = -\frac{8\pi G}{c}\mathbf{j}, \tag{3}$$

in terms of the mass density ρ and current $\mathbf{j}$ of the sources.

Notice that, in the gravitoelectromagnetic formalism, in analogy with the electric potential of a point charge, Φ differs by a minus sign from the actual Newtonian potential of point mass M, $U = -\frac{GM}{|\mathbf{x}|}$, which we use in Sec. IV.

Starting from the above potentials, in stationary conditions, we may define the gravitoelectric $\mathbf{E}$ and gravitomagnetic $\mathbf{B}$ fields

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi, \tag{4}$$

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{5}$$

Using these fields, Einstein's equations can be written in analogy to Maxwell's equations. In addition, the spatial component of the geodesic equation (up to linear order in $\frac{|\mathbf{v}|}{c}$) is written in terms of Lorentz-like force acting upon a test mass m:

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -m\mathbf{E} - 2m\frac{\mathbf{v}}{c}\times\mathbf{B}. \tag{6}$$

This formalism is useful because it allows us to express GR effects in terms of known electromagnetic ones; for instance, the Lense–Thirring effect can be explained in analogy with the precession of a magnetic dipole in a magnetic field (see, e.g., Iorio *et al.* [13]). However, this formalism has limitations (for instance, the geodesic equation does not take a Lorentz-like form in nonstationary conditions, as discussed by Ruggiero [14]), and we should not forget that it is just an approximation of the full theory.

In the context of the study of galactic dynamics, Ludwig [7] considered the set of gravitational equations for a fluid of stars modeled as dust; in particular, he solved, in stationary conditions, the momentum equation [Eq. (6)] and the source equations [Eqs. (2) and (3)] to obtain self-consistent solutions for $\mathbf{v}$, $\mathbf{A}$, and Φ; and he showed that the impact of gravitomagnetic effects on the rotation curves is not negligible.

Without using a specific model for the density profile of a galaxy, it is possible to deduce that gravitomagnetic effects may have a relevant impact on the galactic rotation curves, as discussed by Ruggiero *et al.* [8]. To this end, we consider dust particles steadily rotating around a symmetry axis and use cylindrical coordinates $\{r,\varphi,z\}$ such that z is the rotation axis; $\mathbf{u}_r$, $\mathbf{u}_\varphi$, and $\mathbf{u}_z$ are the unit vectors. If $\boldsymbol{\Omega} = \Omega\mathbf{u}_z$ is the rotation rate and $\mathbf{x}$ is the position vector of a dust particle, its velocity turns out to be $\mathbf{v} = \boldsymbol{\Omega}\times\mathbf{x}$; and $\boldsymbol{\Omega}$ can be a function of r and z because axial symmetry is assumed. Accordingly, using a purely Newtonian model in a stationary condition, the Poisson equation can be written as

$$4\pi G\rho = -\boldsymbol{\nabla}\cdot[(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}], \tag{7}$$

where $\mathbf{v}$ is the velocity field of the fluid: taking into account that $\mathbf{v} = \Omega r\mathbf{u}_\varphi = v\mathbf{u}_\varphi$, from Eq. (7), we obtain

$$4\pi G\rho = 2\Omega^2 + 2\Omega\frac{\partial\Omega}{\partial r}r = \frac{2v}{r}\frac{\partial v}{\partial r}. \tag{8}$$

In this equation, the matter density ρ is locally related to the rotation rate Ω and its derivative. If we focus on the regime where the rotation curves are flattened, because it is $v = \Omega r \simeq$ constant, from Eq. (8), we get $\rho = 0$; accordingly, using a Newtonian approach, it is not clear how to link the matter density to the rotation rate in the flat zone.

Things are quite different if we work in a GR context: in weak-field and slow-motion approximation. Indeed, exploiting the above described analogy with electromagnetism, Eq. (8) becomes

$$4\pi G\rho + \frac{2}{c}\mathbf{B}\cdot\boldsymbol{\omega} = -\boldsymbol{\nabla}\cdot[(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}]. \tag{9}$$

In the above equation, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$ is the Newtonian fluid vorticity. Accordingly, the coupling between the gravitomagnetic field and the fluid vorticity modifies the local relation between the density and the velocity of the fluid. If we set $\rho = \rho_N$ in Eq. (8) to denote the density measured in a Newtonian framework and $\rho = \rho_N + \delta\rho$ in Eq. (9), where $\delta\rho$ is the extra density due to the coupling between the gravitomagnetic field and the fluid vorticity, we obtain

$$\delta\rho = -\frac{1}{2\pi G c}\mathbf{B}\cdot\boldsymbol{\omega}. \tag{10}$$

This approach shows that, even in weak-field conditions, neglecting post-Newtonian effects might have an impact on the estimate of the mass density and, in turn, this could contribute to a different evaluation of the dark matter content.

The impact on the evaluation of the matter content can also be deduced by the extension of the virial theorem in the gravitoelectromagnetic case, which was studied by Astesiano [15]:

$$\left\langle \int \rho v^2 d^3x - \frac{1}{2}\int \rho\Phi d^3x - \frac{1}{8\pi G}\int \mathcal{H} d^3x \right\rangle = 0, \tag{11}$$

$$\mathcal{H} \coloneqq (\partial_{\hat{i}} A_{\hat{j}})^2 - (\partial_{\hat{i}} A_{\hat{j}})(\partial_{\hat{j}} A_{\hat{i}}), \tag{12}$$

where $\hat{\imath}, \hat{\jmath} = \{x, y, z\}$. This result can reduce the amount of matter needed to sustain a motion with velocity v as compared to the Newtonian version of the same theorem. Remarkably, the virial theorem can be written in a more suggestive way:

$$\left\langle 2\int \rho v^2 d^3x - \frac{1}{8\pi G}\int (E^2 + B^2) d^3x \right\rangle = 0. \tag{13}$$

Using the analogy with electromagnetism, we see that the second term is the total energy stored in the gravity fields. Therefore, we have the balance equation

$$2 \times \text{energy of free dust (kinetic energy)} = \text{energy of gravity}. \tag{14}$$

There is another important effect of the gravitomagnetic field; in fact, circular orbits in planes orthogonal to the rotation axis are allowed, thanks to the presence of the gravitomagnetic force that balances the Newtonian force in the direction of the rotation axis, which is not possible in purely Newtonian gravity (see, e.g., Bonnor [16]).

The above arguments do not require a specific model for the mass distribution, which is of course important if we want to estimate the order of magnitude of the gravitomagnetic field. The latter was recently estimated in a paper by Toth [17]. In order to evaluate the galactic gravitomagnetic field to estimate its impact on the rotation curves, the author considered the following gravitomagnetic potential:

$$\mathbf{A} = \frac{G}{c}\frac{\mathbf{J}\times\mathbf{x}}{|\mathbf{x}|^3}, \tag{15}$$

which corresponds to the case of a compact source of angular momentum $\mathbf{J}$. From this potential, it is possible to obtain the gravitomagnetic field

$$\mathbf{B} = \frac{G}{c}\left[\frac{3(\mathbf{J}\cdot\mathbf{x})\mathbf{x}}{|\mathbf{x}|^5} - \frac{\mathbf{J}}{|\mathbf{x}|^3}\right] \tag{16}$$

with its dipolelike behavior. We point out that the gravitomagnetic potential [Eq. (15)] is not a solution of the Poisson equation [Eq. (3)] within the mass distribution but in vacuum. Consequently, it is hard to accept that the expression [Eq. (16)] can be used to estimate the galactic gravitomagnetic field. In addition, in doing so, it is assumed that the gravitomagnetic field at a given location is determined only by the internal mass distribution; the underlying idea is that the gravitational field is determined by the internal mass distribution only, in analogy with what happens in Newtonian gravity under suitable symmetry hypotheses. Actually, things are more complicated when we are dealing with gravitomagnetic fields in GR; for instance, the gravitomagnetic field nearby the center of a rotating mass ring (see Ruggiero [18]) is not null, but it is given by

$$\mathbf{B} = \frac{2G}{cR^3}\mathbf{J}, \tag{17}$$

where R is the radius of the ring, and $\mathbf{J}$ is its angular momentum. Or, if we consider a uniformly rotating hollow homogeneous sphere, the gravitomagnetic field (see Ciufolini *et al.* [19]) is

$$\mathbf{B} = \frac{4GM}{3cR}\boldsymbol{\Omega}, \tag{18}$$

where M is the mass of the sphere, R its radius, and $\boldsymbol{\Omega}$ its angular velocity. Notice that in the latter case, the corresponding gravitational field is null; this shows that it is not generally true that gravitomagnetic fields are always smaller than the Newtonian ones.

Accordingly, we suggest that the estimate of the galactic gravitomagnetic field obtained by Toth [17] is based on an oversimplified model, and hence cannot be used as an argument against the impact of GR on galactic rotation curves.

III. THE GENERAL RELATIVISTIC AXISYMMETRIC STATIONARY SYSTEM COUPLED TO DUST

Following Astesiano *et al.* [5], to describe a single disk galaxy, we consider neutral, stationary, and axisymmetric dust coupled to Einstein's equations. Using cylindrical coordinates (ct, r, ϕ, z) with space-time signature $(-1, 1, 1, 1)$,[1] matter is assumed to flow along the Killing vectors ∂_t and ∂_ϕ and, here and henceforth, functional dependence on the coordinates (r, z)—which are not associated to Killing vectors—is allowed only. If ρ denotes the matter density, the energy momentum tensor is given by

$$T^{\mu\nu}(r,z) = \rho(r,z)u^\mu(r,z)u^\nu(r,z),$$
$$u^\mu(r,z) = \frac{1}{\sqrt{-H(r,z)}}(1,0,0,\Omega(r,z)), \tag{19}$$

where $\Omega(r,z) = \frac{d\phi}{dt} = \frac{u^\phi}{u^t}$. As shown by Stephani *et al.* [20], the solution of Einstein's equations is completely determined by the choice of a negative function $H(\eta)$, on which the physical properties depend. Afterward, it is possible to obtain an auxiliary function $\mathcal{F}(\eta)$ using[2]

$$\mathcal{F} = 2\eta + r^2 \int \frac{H'}{H}\frac{d\eta}{\eta} - \int \frac{H'}{H}\eta d\eta. \tag{20}$$

The remaining equations of motion are

$$\mathcal{F}_{,rr} - \frac{1}{r}\mathcal{F}_{,r} + \mathcal{F}_{,zz} = 0, \tag{21}$$

$$\Omega = \frac{1}{2}\int H' \frac{d\eta}{\eta}. \tag{22}$$

After choosing $H(\eta)$ and the solution of Eq. (21), we can calculate the metric components as

$$g_{tt} = \frac{(H - \eta\Omega)^2 - r^2\Omega^2}{H},$$
$$g_{t\phi} = \frac{\eta^2 - r^2}{(-H)}\Omega + \eta,$$
$$g_{\phi\phi} = \frac{r^2 - \eta^2}{(-H)}. \tag{23}$$

In addition, the remaining metric components

$$g_{zz} = g_{rr} =: e^\Psi \tag{24}$$

are determined by the following equations:

$$\Psi_{,r} = \frac{1}{2r}[(g_{tt})_{,r}(g_{\phi\phi})_{,r} - (g_{tt})_{,z}(g_{\phi\phi})_{,z} - ((g_{t\phi})_{,r})^2 + ((g_{t\phi})_{,z})^2], \tag{25}$$

$$\Psi_{,z} = \frac{1}{2r}[(g_{tt})_{,z}(g_{\phi\phi})_{,r} + (g_{tt})_{,r}(g_{\phi\phi})_{,z} - 2(g_{t\phi})_{,r}(g_{t\phi})_{,z}]. \tag{26}$$

Eventually, the matter density is given by

$$8\pi G\rho = \frac{\eta^2 r^{-2}(2-\eta l)^2 - r^2 l^2}{4g_{rr}}\frac{\eta_{,r}^2 + \eta_{,z}^2}{\eta^2}, \tag{27}$$

where $l = \frac{H'}{H}$.

Because it will be useful in what follows, we notice that Eq. (22) corresponds to the two following conditions, with a little abuse of notation:

$$H_{,r} - 2\eta\Omega_{,r} = 0, \qquad H_{,z} - 2\eta\Omega_{,z} = 0. \tag{28}$$

We suppose that a galaxy has a finite extension: as a consequence, flatness at space infinity $r, z \to \infty$ is expected, which means that, in this limit, the metric reduces to the Minkowski one:

$$g_{tt} = -1, \quad g_{t\phi} = 0, \quad g_{\phi\phi} = r^2, \quad g_{rr} = g_{zz} = 1. \tag{29}$$

A. The projection along the word lines of the ZAMO

The space-time metric that describes our model of the galaxy is stationary and axisymmetric; in this case, care must be paid in choosing suitable observers. In fact, it is known that the use of static observers at rest as seen from infinity is not a good choice because these observers are not defined by local properties of space-time; in addition, they cannot exist in some regions (see, e.g., Bardeen [21]). So, rather than referring quantities to a coordinate frame, it is better to use an orthonormal tetrad carried by the so-called "locally nonrotating observers" or "zero angular momentum observers" (ZAMOs) because their angular momentum vanishes. It turns out that these observers are natural candidates to analyze physical processes in the simplest way because their motion compensates, as much as possible, for the dragging effect due to the angular momentum of the source [22,23]. Accordingly, we choose the ZAMO to describe the dust motion. As we said, these observers are nonrotating, in the sense that they are orthogonal to the constant time-spacelike hypersurfaces Σ_t and define a field of one forms

[1]For the sake of simplicity, in this section, we use units such that $c = 1$.

[2]Here and henceforth, we use the following notation: for any function of one argument, like $H(\eta)$, with a prime, we mean the derivative with respect to its argument; in addition, we use a comma to indicate the partial derivative with respect to a given coordinate.

$$n = \frac{r}{\sqrt{g_{\phi\phi}}} dt. \tag{30}$$

The full orthonormal frame they carry with themselves can be constructed choosing

$$e^{(t)} = n, \qquad e^{(\phi)} = \sqrt{g_{\phi\phi}}(d\phi - \chi dt),$$
$$e^{(r)} = e^{\Psi/2} dr, \qquad e^{(z)} = e^{\Psi/2} dz, \tag{31}$$

where, for simplicity, we defined

$$\chi \equiv -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{H\eta}{(r^2 - \eta^2)} + \Omega. \tag{32}$$

Actually, χ is the angular velocity of the ZAMO frame as seen from an asymptotic inertial observer at infinity [22]. Notice that, thanks to the definition in Eq. (32), the ZAMO consistently satisfies the requirement of zero angular momentum: $g_{\phi\phi}\chi + g_{\phi t} = 0$.

The relevant elements of the dual basis are found to be

$$e_{(t)} = \frac{r}{\sqrt{g_{\phi\phi}}} g^{tt} \left(\partial_t + \frac{g^{t\phi}}{g^{tt}} \partial_\phi \right) = \frac{\sqrt{g_{\phi\phi}}}{r} (\partial_t + \chi \partial_\phi), \tag{33}$$

$$e_{(\phi)} = \frac{1}{\sqrt{g_{\phi\phi}}} \partial_\phi. \tag{34}$$

Using these definitions, the four velocity of the dust [Eq. (19)] can be rewritten as

$$u = \frac{r}{\sqrt{g_{\phi\phi}}} \frac{1}{\sqrt{-H}} \left(e_{(t)} + \frac{g_{\phi\phi}}{r} (\Omega - \chi) e_{(\phi)} \right), \tag{35}$$

On the other hand, the three velocity of the dust measured by the ZAMO is

$$v^{(a)} = \frac{u^\mu e_\mu^{(a)}}{u^\mu e_\mu^{(t)}}. \tag{36}$$

Using the expressions [Eq. (35)] of the four velocity and the metric components [Eq. (23)] and taking into account the orthonormal features of the tetrad, we obtain

$$v^{(\phi)} = \frac{u^\mu e_\mu^{(\phi)}}{u^\mu e_\mu^{(t)}} = \frac{\eta}{r} \doteq v. \tag{37}$$

As a consequence, we can use v to give a physical meaning to the mathematical function η: as we are going to show, this function is simply related to the angular momentum per unit mass of a dust element.

In addition, we may define

$$u^\mu e_\mu^{(t)} = \frac{1}{\sqrt{1 - v^2}} \doteq \gamma. \tag{38}$$

Exploiting the above definitions, it is possible to write the four-velocity vector of the dust [Eq. (35)] in the very simple form

$$u = \gamma(e_{(t)} + v e_{(\phi)}). \tag{39}$$

Because $\frac{r}{\sqrt{g_{\phi\phi}}} = \sqrt{-H}\gamma$, we may write the elements of the basis tetrad in the form

$$e^{(t)} = \sqrt{-H}\gamma dt, \qquad e^{(\phi)} = \frac{r}{\sqrt{-H}\gamma}(d\phi - \chi dt), \tag{40}$$

so that the metric turns out to be

$$ds^2 = H\gamma^2 dt^2 - r^2 \frac{1}{H\gamma^2}(d\phi - \chi dt)^2 + e^\Psi (dr^2 + dz^2). \tag{41}$$

Notice that, from Eq. (32), we obtain

$$\chi = \frac{v}{r} H\gamma^2 + \Omega \tag{42}$$

which can be written as

$$r\Omega = r\chi - v\gamma^2 H. \tag{43}$$

This can be seen as a generalization of the usual relation $v = r\Omega$ of Newtonian mechanics, which is restored in the limit where the effects of χ are negligible and $-\gamma^2 H \to 1$. We notice that $r\Omega$ represents the “coordinate velocity,” i.e., the velocity of the dust as measured by inertial observers at rest at infinity.

In addition, we may write the energy momentum tensor [Eq. (19)] in the form

$$T = \rho u \otimes u = \rho\gamma^2 (e_{(t)} + v e_{(\phi)}) \otimes (e_{(t)} + v e_{(\phi)}), \tag{44}$$

from which it is now easy to read the corresponding tetrad components

$$T^{(t)(t)} = \gamma^2 \rho, \qquad T^{(t)(\phi)} = \gamma^2 \rho v, \qquad T^{(\phi)(\phi)} = \gamma^2 \rho v^2. \tag{45}$$

The Killing vectors ∂_t and ∂_ϕ define associated conserved quantities along the flow of the dust:

$$\mathcal{E} \coloneqq -u^\mu (\partial_t)_\mu = \frac{1}{\sqrt{-H}} [\gamma^2 (-H) + r v \chi]$$
$$= \frac{1}{\sqrt{-H}} [\gamma^2 (-H) + \eta\chi] \tag{46}$$

$$\mathcal{M} \coloneqq u^\mu (\partial_\phi)_\mu = \frac{1}{\sqrt{-H}} r v = \frac{1}{\sqrt{-H}} \eta \tag{47}$$

The first one $\mathcal{E}$ is the energy (per unit mass), whereas the second one $\mathcal{M}$ is the angular momentum (per unit mass), which gives a physical interpretation to η.

B. Light frequency shift measured from an inertial asymptotic observer

After having analyzed, using the ZAMO, the features of the metric that constitute our model of a galaxy, we focus on what can be measured by asymptotically inertial observers to investigate galactic dynamics: namely, the frequency shift of a photon emitted by a particle of dust. To this end, the following hypotheses are made: (i) the emitters are supposed to move in (stable) circular geodesics; and (ii) after the emission, photons propagate along null geodesics so that any possible refraction effects are neglected. In our case, the frequency shift is the sum of the gravitational and Doppler contributions: in fact, the photon is emitted from an object moving in a gravitational field. We assume that a photon is emitted with proper frequency ν_e by a dust particle, whereas ν_d is the frequency measured by the detector. The frequency shift is measured by the redshift factor $\tilde{z}$ (negative for a blueshift), which is in general defined by

$$1+\tilde{z} \coloneqq \nu_e/\nu_d. \tag{48}$$

The proper frequencies are (see. e.g., Ruggiero *et al.* [24])

$$\nu_{e,d} \coloneqq -U^{\mu}_{e,d}k_{\mu}, \tag{49}$$

where U^{μ}_d is the four velocity of the detector, U^{μ}_e is the four velocity of the source, and k_μ is the four momentum of the photon at the respective locations.

Under the assumption that gravitational effects could be neglected when measuring the frequency shift of light coming from an external galaxy, the measured redshift $\tilde{z}$ would just be comparable to the pure kinematic Doppler effect in a Minkowski space-time, corresponding to a "special-relativistic" velocity $\mathbf{v}_{\mathrm{SR}}$ of the source, i.e.,

$$1+z=\frac{1+v^{||}_{\mathrm{SR}}}{\sqrt{1-v^2_{\mathrm{SR}}}}. \tag{50}$$

The vector $\mathbf{v}_{\mathrm{SR}}$ is assumed to be directed along ∂_ϕ, and we call its projection along the line of sight $v_{\mathrm{SR}}\sin\theta$.

The general relativistic description leads to a more general and interesting result. Let us now compare the accepted special-relativistic (SR) description, which accounts only for the kinematic Doppler shift, to the general relativistic description, which instead includes the gravitational shift effect. The expressions for the redshift for the model that we are considering were calculated by Astesiano *et al.* [25], and they read

$$1+z=\begin{cases}\mathrm{SR}\ \frac{1+v_{\mathrm{SR}}\sin\theta}{\sqrt{1-v^2_{\mathrm{SR}}}},\\ \mathrm{GR}\ \frac{1}{\sqrt{-H}}\left[1+\frac{r\Omega\sin\theta}{\sqrt{(\gamma^2H)^2-(\gamma^2Hv+r\Omega)^2\cos^2\theta}-(\gamma^2Hv+r\Omega)\sin\theta}\right],\end{cases} \tag{51}$$

where $\frac{\pi}{2}-\theta$ is the angle between ∂_ϕ and the emitted photon. We see that the overall effect depends both on the kinematical effects v and gravitational ones $(\gamma^2(-H),\chi)$. The degree of freedom given by the nondiagonal term χ affects the result, although it is not explicit in the formula. For example, we remember that $r\Omega\neq v$ when χ is not negligible [see Eq. (43)].

As a particular application of this result, if the galaxy is seen edge on ($\theta=\pm\frac{\pi}{2}$), the resulting redshift turns out to be

$$1+\tilde{z}^{||}=\frac{(\gamma\sqrt{-H})}{\gamma^2(-H)\mp r\chi}\frac{(1\pm v)}{\sqrt{1-v^2}}, \tag{52}$$

where the upper and lower signs refer to backward and forward emissions, respectively. If the galaxy is seen face on ($\theta=0$), we obtain the following result:

$$1+\tilde{z}^{\perp}=\frac{1}{\sqrt{-H}}. \tag{53}$$

The same result is obtained if we are observing a disk galaxy tilted with a certain angle with respect to the line of sight and we perform the measurement on the minor axis (see, again, Astesiano *et al.* [25]).

IV. SOME RELEVANT LOW ENERGY LIMITS

Here, we want to further investigate the properties of the general solution studied so far. We expect that a galaxy is a low energy system: as we said, it is reasonable to suppose that far from the galactic center, the gravitational field is weak and stars are not moving at relativistic speeds. Accordingly, we will expand the coefficients of the metric [Eq. (41)] in negative powers[3] of c and make a comparison with known limits of the solutions of Einstein's equations: this will help us to obtain a physical interpretation for the functions $\gamma^2(-H)$ and χ that are, respectively, related to the Newtonian potential and to the gravitomagnetic potential.

The first relevant terms in the low energy expansion of these functions that we consider are given by

$$\gamma^2(r,z)(-H)(r,z)=\mathcal{A}(r,z)_{(0)}c^0+\mathcal{A}(r,z)_{(-2)}c^{-2}+O(c^{-4}), \tag{54}$$

$$\chi(r,z)=\chi(r,z)_{(-1)}c^{-1}+\chi(r,z)_{(-2)}c^{-2}+O(c^{-3}). \tag{55}$$

[3]Hence, physical units are restored throughout this section.

If, in the above expressions, we consider only the $\mathcal{A}_{(0)}$ term, we get the special-relativistic limit, as will be shown in Sec. IV A; on the other, if we also take into account the $\mathcal{A}_{(-2)}$ term, we are led to the usual Newtonian limit, as discussed in Sec. IV B. Further information is obtained by considering the effects of the nondiagonal terms, thus allowing $\chi_{(-1)}$ to be different from zero; in particular, in Sec. IV C, we propose a more general limit where the off diagonal terms are of the same order of the Newtonian effects, which we call the strong gravitomagnetism case: this model provides a simple explanation of the flat velocity profile of disk galaxies. Moreover, in this general relativistic context, it is possible to discuss the mass density needed to produce a flat velocity profile and compare it to what is obtained in Newtonian gravity. In Appendix B, we show that the term $\chi(r,z)_{(-2)}$ gives rise to the usual gravitomagnetism, which was discussed in Sec. II. Eventually, in Sec. VI, we will consider as a particular case the rigidly rotating model $\Omega = \text{constant}$, which was used as a model for our galaxy in previous publications [3,4].

In what follows, we will discuss the impact of these limits on the measured redshift. According to the general expansion given by Eqs. (54) and (55), the form of the redshift [Eq. (51)] at first order in c becomes

$$1+z = \begin{cases} \text{SR} & 1+\frac{v_{\text{SR}}}{c}\sin\theta + O(c^{-2}), \\ \text{GR} & 1+\frac{r\Omega\sin\theta}{c} + O(c^{-2}), \end{cases} \tag{56}$$

where $\Omega = \frac{v}{r} + \chi$ at the leading order. Thanks to the above expressions, it is manifest that, when the off diagonal term χ is negligible, the general relativistic result coincides with the special-relativistic one at the first order, with the identification $r\Omega = v = v_{\text{SR}}$.

A. The special-relativistic limit

Setting $\mathcal{A}(r,z)_{(0)} = 1$ and neglecting the other contributions from the expansions [Eqs. (54) and (55)], we obtain

$$\gamma^2(-H) = 1, \qquad \chi = 0. \tag{57}$$

Hence, taking also into account Eqs. (25) and (26), we get the special-relativistic limit, i.e.,

$$ds^2 = -c^2dt^2 + r^2d\phi^2 + dr^2 + dz^2. \tag{58}$$

Of course, this is a limiting case because there is no matter as the source of the gravitational fields [$\rho = 0$; see Eq. (27)]; and then the four vector of the dust u can be seen as referring to free particles in the metric [Eq. (58)].

In this limit, the parallel and transverse redshifts in Eqs. (52) and (53) give, respectively,

$$1+\tilde{z}^{||} = \frac{(1\pm\frac{v}{c})}{\sqrt{1-\frac{v^2}{c^2}}} = \pm\frac{v}{c} + \frac{v^2}{2c^2} + O(c^{-3}), \tag{59}$$

$$1+\tilde{z}^{\perp} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{v^2}{2c^2} + O(c^{-3}), \tag{60}$$

as expected from the usual relations for the Doppler effect. Therefore, we have a simple relation between the two redshifts:

$$\tilde{z}^{||} = \pm\frac{v}{c} + \tilde{z}^{\perp} + O(c^{-3}). \tag{61}$$

B. The Newtonian limit

In this section, we will show how to obtain the Newtonian limit up to order c^{-2}, therefore neglecting post-Newtonian corrections. Anyhow, the post-Newtonian limit can be obtained by just adding the appropriate terms in the expansion [Eq. (54)]. The Newtonian limit is obtained by taking

$$\gamma^2(-H) = 1 - \frac{2U(r)}{c^2} + O(c^{-4}), \quad \chi = 0, \tag{62}$$

where $U(r)$ is the Newtonian potential; consequently, up to order c^{-2}, we may write

$$-H(r) = 1 - \frac{2U(r)}{c^2} - \frac{v(r)^2}{c^2} + O(c^{-3}). \tag{63}$$

We get a well-known result: the Newtonian limit is cylindrically symmetric. Equation (42) gives

$$\Omega r = v + O(c^{-3}). \tag{64}$$

In this case, the coordinate velocity $r\Omega$ equals the velocity measured by the ZAMO: this is not a surprise because, in this case, the ZAMOs are the same as the observers at rest at infinity. Because we already know the function Ω, we can use it to impose Eq. (22) [or, equivalently, Eq. (28)] to obtain

$$\partial_r U = -\frac{v^2}{r}, \tag{65}$$

from which we directly read the condition for circular orbits.

In this context, we can check what happens to the auxiliary function $\mathcal{F}$, which is defined by Eq. (20). To this end, we calculate the integrals

$$c^2r^2\int\frac{H'}{H}\frac{d\eta}{\eta} = -c^2r^2\int H'\frac{d\eta}{\eta} + O(c^{-2}) \\ = -2r^2\Omega + O(c^{-2}) = -2rv + O(c^{-2}), \tag{66}$$

$$-\int \frac{H'}{H}\eta d\eta = -\int \frac{\partial_r H}{H}\frac{vr}{c}dr = 0 + O(c^{-2}). \quad (67)$$

Therefore, $\mathcal{F} = 0$ up to order c^{-2} and automatically solves Eq. (21). In this case, the gravitomagnetic effects related to the off diagonal terms of the metric $g_{t\phi}$ are suppressed because

$$g_{t\phi} = 0 + O(c^{-3}), \qquad g_{tt} = -c^2\left(1 + \frac{2U}{c^2}\right) + O(c^{-2}),$$
$$g_{\phi\phi} = r^2\left(1 + \frac{2U}{c^2}\right) + O(c^{-4}). \quad (68)$$

Eventually, these results allow us to obtain from Eq. (25) the last unknown element of the metric,

$$e^{\Psi} = e^{2\frac{U}{c^2}+O(c^{-4})} = 1 + \frac{2U}{c^2} + O(c^{-4}). \quad (69)$$

With all the ingredients, we can write down the metric [Eq. (41)]

$$ds^2 = c^2\left(-1 + \frac{2U}{c^2}\right)dt^2 + \left(1 + \frac{2U}{c^2}\right) \times (r^2 d\phi^2 + dr^2 + dz^2) + O(c^{-3}), \quad (70)$$

whereas the density [Eq. (27)] is

$$\frac{8\pi G}{c^2}\rho = 4\frac{v}{c^2 r}\partial_r v + O(c^{-4}), \quad (71)$$

which is in agreement with Eq. (8). The energy and the angular momentum are

$$\frac{\mathcal{E}}{c^2} = 1 + \frac{v^2}{2c^2} - \frac{U}{c^2} + O(c^{-4}), \quad (72)$$

$$\mathcal{M} = vr + O(c^{-3}) = \Omega^2 r + O(c^{-3}). \quad (73)$$

The above results can be used to get further insight into the application of a Newtonian approach to the description of galactic dynamics. In fact, a flat velocity profile in the Newtonian regime would lead to $v =$ constant,[4] as can be seen from Eq. (52). This cannot be achieved because imposing the constraint $v =$ constant implies $\rho = 0$ in Eq. (71).

The parallel and transverse redshifts in Eqs. (52) and (53) are, respectively,

$$1 + \tilde{z}^{||}(\pm\pi/2, r, z) = \frac{(1 \pm \frac{v}{c})}{\sqrt{1 - \frac{v^2}{c^2} - \frac{2U}{c^2}}},$$
$$1 + \tilde{z}^{\perp}(\pm\pi/2, r, z) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2} - \frac{2U}{c^2}}}. \quad (74)$$

Therefore, we have the same relation as in the special-relativistic case [Eq. (61)] between the two redshifts:

$$\tilde{z}^{||} = \pm\frac{v}{c} + \tilde{z}^{\perp} + O(c^{-3}), \quad (75)$$

As a consequence, we may neglect the frequency shift due to the gravitational field in the Newtonian approximation. As it is clear from the above relations, the Doppler effect is of order c^{-1}, whereas the gravitational shift is of order c^{-2}.

C. The strong gravitomagnetic limit and a proposed model for disk galaxies

The presence of off diagonal or gravitomagnetic terms in the metric leads to the "dragging of inertial frames": a gyroscope, which defines the orientation of a local inertial frame, rotates relative to observers at rest at infinity because the gravitational field of the source drags the space-time around it. This effect can be explained as the action of the gravitomagnetic field of the source on the gyroscope spin (see, e.g., Misner *et al.* [26], Ciufolini and Wheeler [27], and Bosi *et al.* [28]). The analogy can be done with a solid sphere, rotating in a viscous fluid: because of its rotation, the fluid is dragged along with the sphere. In order to take into account this effect, in the low energy expansion [Eqs. (54) and (55)], we consider a dragging term in addition to the usual Newtonian potential; in other words, we take the expansion defined by

$$\gamma^2(-H) = 1 - \frac{2U(r,z)}{c^2} + O(c^{-3}), \qquad \chi = \frac{a(r,z)}{r^2} + O(c^{-1}), \quad (76)$$

and call it the strong gravitomagnetic limit. Accordingly, in this case, Eq. (42) gives

$$\Omega(r,z) = \frac{a(r,z)}{r^2} + \frac{v(r,z)}{r}. \quad (77)$$

It is interesting to rephrase Eq. (77) in the form

$$r\Omega = r\chi + v \quad (78)$$

which can be interpreted as a classical velocity-addition relation: the velocity $r\Omega$ measured by inertial observers is the sum of the velocity v measured by the ZAMO and the velocity of the ZAMO with respect to inertial observers $r\chi$.

[4]Remember that, in this case, v equals the coordinate velocity [see, e.g., Eq. (64)].

Once again, knowing Ω, we can use it to impose Eq. (22) and obtain

$$\partial_r U + \frac{v^2}{r} - vr\partial_r\left(\frac{a}{r^2}\right) = 0, \quad (79)$$

$$\partial_z U - \frac{v}{r}\partial_z a = 0. \quad (80)$$

It is relevant to emphasize the role of the function a: we explicitly see from the above equations that when $a = 0$, $\partial_z U = 0$: hence, the Newtonian potential is cylindrical symmetric; this symmetry is broken by the presence of this function.

As for the auxiliary function $\mathcal{F}$ defined in Eq. (20), we calculate the integrals

$$c^2r^2\int\frac{H'}{H}\frac{d\eta}{\eta} = -c^2r^2\int H'\frac{d\eta}{\eta} + O(c^{-4}) = -2r^2\Omega + O(c^{-2}), \quad (81)$$

$$-\int\frac{H'}{H}\eta d\eta = -\int\frac{\partial_r H}{H}vrdr = 0 + O(c^{-2}); \quad (82)$$

therefore,

$$\mathcal{F} = -2\frac{a}{c}. \quad (83)$$

Using these relations, we obtain the following expressions for the metric elements:

$$g_{tt} = -c^2\left(1 - \frac{2U}{c^2} - \frac{a^2}{c^2r^2}\right), \qquad g_{t\phi} = -a,$$
$$g_{\phi\phi} = r^2\left(1 + \frac{2U}{c^2}\right), \quad (84)$$

with a non-negligible off diagonal term depending on a. The equation of motion [Eq. (21)] is written in the simple form:

$$a_{,rr} - \frac{a_{,r}}{r} + a_{,zz} = 0, \quad (85)$$

which is exactly the condition for the integration of Eqs. (25) and (26), which read

$$\Psi_{,r} = \frac{1}{2r}\left[2r\partial_r\left(\frac{2U}{c^2} + \frac{a^2}{c^2r^2}\right) + \frac{a_{,z}^2 - a_{,r}^2}{c^2}\right] + O(c^{-4}), \quad (86)$$

$$\Psi_{,z} = \frac{1}{2r}\left[2r\partial_z\left(\frac{2U}{c^2} + \frac{a^2}{c^2r^2}\right) - \frac{2}{c^2}a_{,r}a_{,z}\right] + O(c^{-4}). \quad (87)$$

They can be directly integrated as

$$\Psi = \frac{2U}{c^2} + \frac{a^2}{c^2r^2} + \frac{1}{2c^2}\int\frac{a_{,z}^2 - a_{,r}^2}{r}dr + O(c^{-4}); \quad (88)$$

therefore, the last element of the metric is

$$g_{rr} = g_{zz} = e^\Psi = \left[1 + \frac{2U}{c^2} + \frac{a^2}{c^2r^2} + \frac{1}{2c^2}\int\frac{a_{,z}^2 - a_{,r}^2}{r}dr\right] + O(c^{-4}). \quad (89)$$

To obtain the energy density [Eq. (27)] for this model, we start evaluating the function ℓ and, taking into account Eq. (63) (which holds true also in the dragging limit thanks to the definitions [Eq. (76)]), we get

$$\ell = \frac{H'}{H} = \frac{(-H)_{,r}}{\eta_{,r}}\frac{1}{(-H)} = \frac{-2\eta\Omega_{,r}}{c^2\eta_{,r}(-H)} = \frac{-2\eta\Omega_{,r}}{c^2\eta_{,r}} + O(c^{-4}). \quad (90)$$

At order c^{-2}, the relevant terms in the energy density are

$$\frac{8\pi G}{c^2}\rho = \left(\frac{v^2}{c^2} - \frac{c^2r^2}{4}\ell^2\right)\frac{\eta_{,r}^2 + \eta_{,z}^2}{\eta^2}. \quad (91)$$

After substituting, we obtain

$$\frac{8\pi G}{c^2}\rho_D = \frac{1}{c^2}\left[4v\frac{v_{,r}}{r} + 2(v - rv_{,r})b_{,r} - r^2b_{,r}^2\right] \times\frac{(v + rv_{,r})^2 + r^2v_{,z}^2}{(v + rv_{,r})^2}, \quad (92)$$

where $b = r^{-2}a$; and we used ρ_D to denote the density obtained in this dragging limit. It is interesting to evaluate the difference between this density ρ_D and the Newtonian density ρ_N given in Eq. (71) for the same value of the velocity v in both models. Because the Newtonian limit is cylindrically symmetric, we focus on the equatorial plane $z = 0$ (where, for symmetry, $\partial_z v = 0$):

$$\frac{8\pi G}{c^2}\delta\rho \equiv \frac{8\pi G}{c^2}\rho_D(z = 0) - \frac{8\pi G}{c^2}\rho_N = \frac{b_{,r}}{c^2}[2(v - rv_{,r}) - r^2b_{,r}]. \quad (93)$$

It is clear that the presence of a nondiagonal dragging term a greatly affects the density required to sustain the motion. As we will show in Appendix B, the first term is also present in the gravitomagnetic limit, whereas the pure negative term $-r^2b_{,r}^2$ is a peculiar feature of the SGM limit: this term can significantly reduce the required mass as compared to the Newtonian case, where these effects are not present.

Hence, the analysis of exact solutions leads to the same conclusion obtained in Sec. II in the weak-field limit: nondiagonal terms in the space-time metric can lead to a re-evaluation of the weight of dark matter in galaxies.

For completeness, we calculate the values of the energy [Eq. (46)] and angular momentum [Eq. (47)] in this limit:

$$\frac{\mathcal{E}}{c^2} = 1 + \frac{v^2}{2c^2} - \frac{U}{c^2} + \frac{av}{rc^2} + O(c^{-4}), \tag{94}$$

$$\mathcal{M} = vr + O(c^{-3}). \tag{95}$$

V. REDSHIFT ANALYSIS OF THE ROTATION CURVES

Here, we will apply the results obtained so far to show that it could be possible to propose a SGM model for a disk galaxy that agrees with the current observations of a flat velocity profile. In doing so, we will point out the effect of the SGM terms that we discussed in Sec. IV C. Accordingly, from Eq. (52) for an edge-on galaxy,[5] we obtain

$$1 + \tilde{z}^{||} = 1 \pm \frac{1}{c}\left(v + \frac{a}{r}\right) + O(c^{-2}). \tag{96}$$

All functions depend on r and z; however, to emphasize the physical content, we will restrict the attention to the galactic plane ($z = 0$) where, due to the symmetry of the system, we have

$$\partial_z v|_{z=0} = 0, \qquad \partial_z a|_{z=0} = 0, \qquad \partial_z U|_{z=0} = 0. \tag{97}$$

From observations based on redshift (or blueshift) measurements, we know that far from the center of the disk galaxy, we observe a flat velocity profile: accordingly, from Eq. (96), and taking into account the relation [Eq. (77)], we get

$$\Omega(r, 0) = \frac{\alpha}{r} + O(c^{-1}), \tag{98}$$

where α is a constant defined by $a(r, 0) = r(\alpha - v(r, 0))$. As already discussed, a flat velocity profile cannot be obtained in the Newtonian limit because, in that case, $a = 0$; on the other hand, in the SGM limit, this observational property can be easily obtained, as we have shown above.

From Eqs. (79) and (80), we get the only condition

$$U(r, 0)_{,r} = -v(r, 0)v(r, 0)_r - \alpha\frac{v(r, 0)}{r}, \tag{99}$$

or their integrated version

$$U(r, 0) = -\frac{v(r, 0)^2}{2} - \alpha\int\frac{v(r, 0)}{r}dr. \tag{100}$$

Then, the energy density in this SGM limit $\rho_D(r, 0)$ for the flat velocity profile is

$$\frac{8\pi G}{c^2}\rho_D(r, 0) = \frac{1}{c^2 r^2}[(v(r, 0) - rv(r, 0)_{,r})^2 - \alpha^2] + O(c^{-4}). \tag{101}$$

Let us now evaluate the transverse redshift given by Eq. (53); we obtain

$$1 + \tilde{z}^{\perp} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2} - \frac{2U}{c^2}}}, \tag{102}$$

exactly as in the Newtonian limit; see Eq. (75). Therefore, the only knowledge of the transverse redshift is not enough to discriminate between the Newtonian and the SGM limits. At order c^{-2}, we can write

$$\tilde{z}^{\perp} = \frac{U}{c^2} + \frac{v^2}{2c^2} + O(c^{-4}), \qquad \tilde{z}^{||} = \pm\frac{1}{c}\left(v + \frac{a}{r}\right) + O(c^{-3}). \tag{103}$$

If we are able to measure both $\tilde{z}^{\perp}$ and $\tilde{z}^{||}$, we can fix both U and a as

$$U = c^2\tilde{z}^{\perp} - \frac{v^2}{2}, \qquad \frac{a}{r} = c\tilde{z}^{||} - \frac{v}{c}, \tag{104}$$

leaving v as the only unknown function that, in turn, can be found using the equations of motion [Eqs. (79) and (80)]

$$\partial_z(\tilde{z}^{\perp}) = \pm\frac{v}{c}\partial_z(\tilde{z}^{||}), \tag{105}$$

$$\partial_r(\tilde{z}^{\perp}) = \pm\frac{v}{c}\partial_r(\tilde{z}^{||}) \mp \frac{v}{cr}\tilde{z}^{||}. \tag{106}$$

VI. THE RIGIDLY ROTATING CASE

We discuss in some detail the case of constant angular velocity $\Omega(r, z) = \Omega_0$, which was considered by Balasin and Grumiller [3] and Crosta *et al.* [4] as a model for our galaxy. The first feature is that in this rigidly rotating solution, the dust fills the entire space-time because the equations of motion for constant angular velocity of the matter Ω are not consistent with $\rho = 0$. This rigidly rotating case can be seen as a very particular case of the SGM limit considered before.

Even though, in this case, the function $H(r, z)$ is fixed to $H = -1$, the presence of the off diagonal term χ allows for a nontrivial profile for v; see Eq. (43):

[5]The extension to the case of generic angle θ is trivial: see Eq. (56).

$$\chi = \Omega_0 - \frac{v}{r}\frac{1}{1-\frac{v^2}{c^2}} = \Omega_0 - \frac{v}{r} + O(c^2). \tag{107}$$

We note that rigid rotation does not imply $r^{-1}v$ constant because v is the velocity measured by the ZAMO and Ω_0 is the angular velocity measured from an asymptotic inertial observer. Using the expressions for H and χ given above, the redshift given by Eq. (52) takes the simple form

$$1 + \tilde{z}(\pm\pi/2, r, z) = 1 \pm r\frac{\Omega_0}{c} + O(c^{-2}); \tag{108}$$

this linear behaviour is expected for the redshift from a rigidly rotating system. Such behavior is not reproduced generally in disk galaxies, except for the inner regions.

For constant angular velocity Ω_0, we can perform a rigid rotation of the coordinates

$$\phi' = \phi - \frac{\Omega_0}{c}ct \tag{109}$$

to rewrite the four velocity of the dust given by Eq. (19) as

$$u = \partial_{t'}, \tag{110}$$

where the rotation is equivalent to impose $\Omega_0 = 0$ everywhere in the coordinates (t, r, z, ϕ). Then, restoring the notation without the $'$, the metric [Eq. (23)] is now given by

$$ds^2 = -\left(cdt - \frac{\eta}{c}d\phi\right)^2 + r^2 d\phi^2 + e^{\Psi}(dr^2 + dz^2), \tag{111}$$

$$\eta_{,rr} + \eta_{,zz} - \frac{\eta_{,r}}{r} = 0, \quad \Psi_{,r} = \frac{(\eta_{,z})^2 - (\eta_{,r})^2}{2r}, \quad \Psi_{,z} = -\frac{\eta_{,r}\eta_{,z}}{r}. \tag{112}$$

Notice that the metric in the form of Eq. (112) is exactly the one used by Cooperstock and Tieu [2], Balasin and Grumiller [3], and subsequently by Crosta *et al.* [4].

The energy density $\rho(r, z)$ given by Eq. (27) boils down to

$$\frac{8\pi G}{c^2}\rho = \frac{e^{-\Psi}}{c^2}\frac{(\eta_{,r})^2 + (\eta_{,z})^2}{r^2}. \tag{113}$$

The rigidly rotating dust metric in Eq. (111) in the ZAMO frame reads

$$ds^2 = -\frac{c^2}{1-\frac{v^2}{c^2}}dt^2 + r^2\left(1-\frac{v^2}{c^2}\right)(d\phi - \chi dt)^2 + (e^2)^2 + (e^3)^2. \tag{114}$$

As shown by Astesiano [15], we can get physical insight into this solution: namely, the gravitational potential U due to the presence of the dust exactly balances the gravitational potential U_C of the noninertial force determined by the rotation of the reference frame.

Using the formalism introduced in Sec. III A, the potential is

$$U = \gamma\sqrt{(-H)} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + O(c^{-3}); \tag{115}$$

therefore, $U = \frac{v^2}{2c^2}$. The balance equation is

$$U + U_C = 0, \tag{116}$$

where $U_C = -\frac{1}{2}\frac{v^2}{c^2}$. This equality is the reason why $g_{tt} = -c^2$ in Eq. (111); as a consequence, if a light signal is emitted from a generic element of the galaxy and it is received from another element, the measured redshift (or blueshift) is zero: the frequency of light does not change [15].

Eventually, the conserved quantities given in Eqs. (46) and (47) read

$$\mathcal{E} = c^2, \qquad \mathcal{M} = rv, \tag{117}$$

Clearly, the reason why the energy $\mathcal{E}$ is equal to c^2 is again the balance [Eq. (116)].

Balasin and Grumiller [3] noted the discrepancy between the density in the rigidly rotating case and in the Newtonian case. This is not due to some mysterious effects but because these two limits (i.e., the rigid one and the Newtonian one) are different limiting cases of the same class of exact solutions and they coincide only in a single point: that is, where $r^{-1}v$ is constant and equal to Ω_0. In fact, in this point, $\chi = O(c^{-2})$ and the densities [Eqs. (113) and (71)] coincide.

VII. FINAL REMARKS AND PERSPECTIVES

Explaining the observed flat velocity profile in disk galaxies is one of the most challenging problems in current astrophysics. Motivated by various suggestions in the literature, which contributed to focus on the role of general relativity in this context, we analyzed the impact of post-Newtonian corrections on the description of rotation curves.

To begin with, under the hypothesis that the gravitational field of a galaxy can be considered sufficiently weak in its outer regions, we started from a well-known low energy limit and used the gravitomagnetic analogy to show that the coupling between the fluid vorticity and the gravitomagnetic field leads to a local relation between density and velocity, which is different from the Newtonian case; this suggests that the post-Newtonian corrections might have an impact on the estimate of the mass density.

Subsequently, we analyzed the problem in the framework of the exact solutions of Einstein's equations. In particular, we considered neutral, stationary, and axisymmetric rotating dust as the model for a disk galaxy; and we described the mathematical properties of the corresponding exact solution. Then, in order to give a physical insight into this solution, we studied it using the so-called zero angular momentum observers, which are suitable to analyze physical processes in presence of the symmetries considered. Using this formalism, we defined some useful observables, such as the dust velocity and the conserved quantities; in addition, we expressed the space-time metric exploiting the corresponding orthonormal tetrad. However, actual measurements on far away galaxies are not performed by the ZAMOs but by asymptotic inertial observers that measure frequency shifts; to this end, we calculated the exact relations for the frequency shift of light, which can be used to explore the physical content of the exact solution.

We obtained further insight into the exact solution by considering some low energy limits, thanks to an expansion in (negative) powers of c. Besides the trivial special-relativistic limit, we investigated the Newtonian limit and what we called the "strong gravitomagnetic limit"; notice that in this limit, which is naturally obtained from the exact solution, dragging effects are of the same order as Newtonian ones. In particular, we showed that in the Newtonian limit, a flat velocity profile can be achieved only on the basis of unphysical constraints on the mass distribution, for which the density should vanish in the flat region. Things are quite different in the SGM limit, where dragging effects have an impact on the density profile required to match the flat velocity profile, which is in agreement with the analysis performed in the weak-field limit using the gravitomagnetic analogy. In addition, using the frequency shift analysis, we showed that a flat velocity profile cannot be obtained in the Newtonian limit, whereas it naturally emerges when dragging effects are taken into account. As a particular case, we discussed the solution proposed by Balasin and Grumiller [3], which corresponds to a rigid rotation of the dust, and we pointed out some unphysical features; for instance, in this solution, the redshift as seen from an inertial observer is always linearly increasing; whereas for an observer corotating with the galaxy, the redshift is zero (see also Astesiano [15]). It is worth remarking that even though the results we discussed refer to galaxies, the same system of equations can provide a good description of other self-gravitating systems in the universe, such as clusters of galaxies.

Our theoretical analysis, which encompasses both the weak-field limit of GR and exact solutions, shows that dragging effects may have a relevant impact in understanding galactic dynamics due to the fact that they introduce an additional degree of freedom with respect to the Newtonian case. In particular, our work provides a theoretical background to the recent publications by Crosta *et al.* [4] and Ludwig [7], where models based on dragging effects were successfully used to fit data coming from galaxies' rotation curves.

Accordingly, we suggest that a better understanding of the mass content can be achieved using this approach, which might shed new light on the role of the dark matter, for which the origin can partly be geometric.

ACKNOWLEDGMENTS

The authors thank Antonello Ortolan, Clive C. Speake, and Federico Re for stimulating discussions on this topic.

APPENDIX A: EXACT SOLUTION FOR ROTATING DUST

Here, we make a comparison with the notation used in the book by Stephani *et al.* [20] (see pages 330–333), where the general class of exact solutions is given in terms of different functions. We give the explicit map. Using their notations, after defining

$$\beta_a \equiv \frac{H_a}{H\eta}, \tag{A1}$$

they write down the following two last equations of motion:

$$W^{-1}\left[(\beta W)_{,a} + \left(\frac{H}{\eta}\right)\left(\frac{\eta^2}{H}\right)_{,a}\right] = \epsilon_{ab}\gamma^{,b} \to \Delta\gamma = 0, \tag{A2}$$

$$D^a W_{,a} = 0, \tag{A3}$$

where

$$\Delta\gamma = \gamma_{rr} + \frac{\gamma_r}{r} + \gamma_{zz}. \tag{A4}$$

For the second equation, they state that it is always possible to choose $W = r$.

They claim that the full solution is given by choosing a function γ and an axisymmetric solution of $\Delta\gamma = 0$. Once $\eta(H)$ and $\gamma(r,z)$ are given, one obtains the function

$$2\eta + r^2\beta - \int \frac{\eta}{H} dH, \tag{A5}$$

from Eq. (A2), and consequently $H(r,z)$; finally,

$$H_a = 2\eta\Omega_{,a}, \tag{A6}$$

which we wrote in Eq. (22). From our perspective, the function [Eq. (A5)] is exactly the function $\mathcal{F}$, which is related to γ through Eq. (A2) or explicitly as

$$\frac{\mathcal{F}_{,r}}{r} = \gamma_{,z}, \qquad \frac{\mathcal{F}_{,z}}{r} = -\gamma_{,r}. \tag{A7}$$

Substituting these relations into $\Delta\gamma = 0$ automatically solves the equations, but the closure of the form gives the consistency equation

$$\gamma_{,zr} - \gamma_{,rz} = 0, \rightarrow \mathcal{F}_{,rr} - \frac{1}{r}\mathcal{F}_{,r} + \mathcal{F}_{,zz} = 0, \quad \text{(A8)}$$

which is exactly the equation of motion [Eq. (21)].

APPENDIX B: MORE DETAILS ON THE SGM LIMIT

The SGM limit is not equivalent to the standard gravitomagnetic approach discussed in Sec. II, but it can be seen as a strong version of it. A simple inspection of the SGM metric

$$ds^2 = -c^2\left(1 - \frac{2U}{c^2} - \frac{a^2}{c^2 r^2}\right)dt^2 - 2a\,dt\,d\phi + r^2\left(1 + \frac{2U}{c^2}\right)d\phi^2 + e^{\Psi}(dr^2 + dz^2) \quad \text{(B1)}$$

and the gravitomagnetic one in the axisymmetric case

$$ds^2 = -c^2\left(1 - 2\frac{\Phi}{c^2}\right)dt^2 - \frac{4}{c}A\,d\phi\,dt + \left(1 + 2\frac{\Phi}{c^2}\right)\delta_{ij}dx^i dx^j, \quad \text{(B2)}$$

shows the substantial difference: the $g_{t\phi}$ term is of order c^0 in the former and c^{-1} in the latter. In fact, assuming a form of H as in the dragging limit, and setting $a \rightarrow 2A/c$ in χ, it is possible to obtain the standard gravitomagnetism at the leading order from the general system of equations. This is the reason why we introduced the strong gravitomagnetism term. In fact, let us check that with the substitution $a \rightarrow 2A/c$, the density is given by Eqs. (93) and (10) is obtained at the leading order. Remembering that $b = r^{-2}a$ and, after the substitution, $a \rightarrow 2A/c$, Eq. (92) on the galactic plane ($z = 0$) becomes

$$\frac{8\pi G}{c^2}\rho(z=0) = \frac{4v}{c^2}\frac{v_{,r}}{r} - \frac{4}{c^3}\left(2\frac{A}{r^3} - \frac{A_{,r}}{r^2}\right)(v - rv_{,r}) + O(c^{-4}). \quad \text{(B3)}$$

The velocity v used in the second part of the work is the velocity with respect to the ZAMO, whereas in the first part, it is the coordinate velocity $v = r\Omega$. Therefore, we must send $v \rightarrow v - \frac{2A}{cr}$ to obtain

$$\frac{8\pi G}{c^2}\rho(z=0) = \frac{4v}{c^2}\frac{v_{,r}}{r} - \frac{4}{c^3}\frac{A_r}{r^2}(v + rv_{,r}) + O(c^{-4}), \quad \text{(B4)}$$

which means

$$\frac{8\pi G}{c^2}\delta\rho = -\frac{4}{c^3}\frac{A_r}{r^2}(v + rv_{,r}) + O(c^{-4}). \quad \text{(B5)}$$

Let us check that this coincides with the result of the application of Eq. (10) in the axisymmetric case. Taking into account Eq. (B2), we make use of the usual cylindrical vector bases $\mathbf{u}_r$, $\mathbf{u}_\varphi$, and $\mathbf{u}_z$ to write

$$\mathbf{A} = \frac{A}{r}\mathbf{u}_\varphi, \qquad \mathbf{v} = v\mathbf{u}_\varphi. \quad \text{(B6)}$$

These fields have the following rotors:

$$\mathbf{B} = \nabla\times\mathbf{A} = \nabla\left(\frac{A}{r}\right)\times\mathbf{u}_\varphi + \frac{A}{r}\nabla\times\mathbf{u}_\varphi = \frac{A_{,r}}{r}\mathbf{u}_z - \frac{A_{,z}}{r}\mathbf{u}_r, \quad \text{(B7)}$$

$$\boldsymbol{\omega} = \nabla\times\mathbf{v} = (rv_{,r} + v)\frac{\mathbf{u}_z}{r} - v_{,z}\mathbf{u}_r, \quad \text{(B8)}$$

where we used the known fact $\nabla\times\mathbf{u}_\varphi = r^{-1}\mathbf{u}_z$. Eventually, we get

$$\delta\rho(z=0) = -\frac{1}{2\pi Gc}\mathbf{B}\cdot\boldsymbol{\omega} = -\frac{1}{2\pi Gc}\frac{A_r}{r^2}(v + rv_{,r}) + O(c^{-2}), \quad \text{(B9)}$$

which coincides with Eq. (B5).

[1] L. E. Strigari, Phys. Rep. **531**, 1 (2013).

[2] F. I. Cooperstock and S. Tieu, Int. J. Mod. Phys. A **22**, 2293 (2007).

[3] H. Balasin and D. Grumiller, Int. J. Mod. Phys. D **17**, 475 (2008).

[4] M. Crosta, M. Giammaria, M. G. Lattanzi, and E. Poggio, Mon. Not. R. Astron. Soc. **496**, 2107 (2020).

[5] D. Astesiano, S. L. Cacciatori, and F. Re, Eur. Phys. J. C **82**, 554 (2022).

[6] J. Ramos-Caro, C. A. Agon, and J. F. Pedraza, Phys. Rev. D **86**, 043008 (2012).
[7] G. Ludwig, Eur. Phys. J. C **81**, 186 (2021).
[8] M. L. Ruggiero, A. Ortolan, and C. C. Speake, arXiv:2112.08290.
[9] M. L. Ruggiero and A. Tartaglia, Nuovo Cimento B **117**, 743 (2002), https://www.sif.it/riviste/sif/ncb/econtents/2002/117/07/article/5.
[10] B. Mashhoon, arXiv:gr-qc/0311030.
[11] GAIA Collaboration, Astron. Astrophys. **595**, A1 (2016).
[12] GAIA Collaboration, Astron. Astrophys. **616**, A1 (2018).
[13] L. Iorio, H. I. M. Lichtenegger, M. L. Ruggiero, and C. Corda, Astrophys. Space Sci. **331**, 351 (2011).
[14] M. L. Ruggiero, Universe **7**, 451 (2021).
[15] D. Astesiano, Gen. Relativ. Gravit. **54**, 63 (2022).
[16] W. Bonnor, J. Phys. A **10**, 1673 (1977).
[17] V. T. Toth, Int. J. Mod. Phys. D **30**, 2150102 (2021).
[18] M. L. Ruggiero, Astrophys. Space Sci. **361**, 140 (2016).
[19] I. Ciufolini, S. Kopeikin, B. Mashhoon, and F. Ricci, Phys. Lett. A **308**, 101 (2003).
[20] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*, 2nd ed., Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2003).
[21] J. M. Bardeen, Astrophys. J. **162**, 71 (1970).
[22] J. M. Bardeen, W. H. Press, and S. A. Teukolsky, Astrophys. J. **178**, 347 (1972).
[23] K. S. Thorne, K. S. Thorne, R. H. Price, and D. A. MacDonald, *Black Holes: The Membrane Paradigm* (Yale University, New Haven, CT, 1986).
[24] M. L. Ruggiero, A. Tartaglia, and L. Iorio, Int. J. Mod. Phys. D **15**, 1183 (2006).
[25] D. Astesiano, S. L. Cacciatori, M. Dotti, F. Haardt, and F. Re, arXiv:2204.05143.
[26] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[27] I. Ciufolini and J. A. Wheeler, *Gravitation and Inertia* (Princeton University, Princeton, NJ, 1995), Vol. 101.
[28] F. Bosi, G. Cella, A. Di Virgilio, A. Ortolan, A. Porzio, S. Solimeno, M. Cerdonio, J. Zendri, M. Allegrini, J. Belfi *et al.*, Phys. Rev. D **84**, 122002 (2011).