

Kinetic axion $F(R)$ gravity inflation

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In this work we investigate the quantitative effects of the misalignment kinetic axion on R^2 inflation. Due to the fact that the kinetic axion possesses a large kinetic energy that dominates its potential energy, during inflation its energy density redshifts as stiff matter fluid and evolves in a constant-roll way, making the second slow-roll index to be nontrivial. At the equations of motion level, the R^2 term dominates the evolution; thus the next possible effect of the axion could be found at the cosmological perturbations level, via the second slow-roll index which is nontrivial. As we show, the latter elegantly cancels from the observational indices; however, the kinetic axion extends the duration of the inflationary era to an extent that it may cause a 15% decrease in the tensor-to-scalar ratio of the vacuum R^2 model. This occurs because as the R^2 model approaches its unstable quasi-de Sitter attractor in the phase space of $F(R)$ gravity due to the $\langle R^2 \rangle$ fluctuations, the kinetic axion dominates over the R^2 inflation and in effect the background equation of state is described by a stiff era, or equivalently a kination era, different from the ordinary radiation domination era. This in turn affects the duration of the inflationary era, increasing the e -foldings number up to $5e$ -foldings in some cases, depending on the reheating temperature, which in turn has a significant quantitative effect on the observational indices of inflation and especially on the tensor-to-scalar ratio.

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I. INTRODUCTION

Dark matter, inflation, dark energy, and the mysterious reheating-early radiation domination era are the mysteries of modern theoretical physics. These problems have been puzzling theoretical physicists for decades, and to date no definite answers are given for the questions imposed by these problems. Of the above evolution eras of our Universe, only the dark energy era has been observationally verified, whereas the rest of the eras remain at the speculation level. However, inflation and the closely related postinflationary reheating era are going to be severely scrutinized in the next 15 years, both by stage-4 cosmic microwave background (CMB) experiments [1,2] and by interferometric and not only gravitational waves experiments such as the LISA, DECIGO, BBO, and Einstein telescope [3–10]; see also [11]. The gravitational wave interferometric experiments will directly probe tensor modes that reentered the Hubble horizon during the mysterious reheating era, thus small wavelength modes that carry information both for the observational indices of inflation and for the reheating era, while the stage-4 CMB experiments will seek the B -mode polarization in the CMB. The B -modes can be generated by two distinct effects: by the E -mode conversion to B -modes via gravitational lensing for small angular scales or large- ℓ CMB modes,

or by tensor modes for large angular scales or small- ℓ CMB modes. Dark matter though seems unreachable to us for the time being. Although there are many proposals for dark matter [12–17], currently it still remains a mystery what dark matter is composed of. The basic known facts about dark matter are that it has a particle nature, based on observations such as the bullet cluster, and the dark matter particle definitely has a small mass. An appealing candidate, as elusive among other particles as dark matter itself, is the axion [18–77]. With the terminology axion, we do not refer to the QCD axion, but to an axionlike particle, in which case the primordial $U(1)$ Peccei-Quinn symmetry of the axion is broken during inflation, and the axion develops a nonzero vacuum expectation value. The axion is a light scalar field, and thus it is highly motivated from a string theory point of view, since scalar fields are the string moduli, which are a basic and profound characteristic of string theory. The axion is an elusive particle with an extremely small mass and is admittedly quite hard to detect; however, there are direct and indirect ways to detect it, for example, from observations of neutron stars [67] or due to black hole superradiance effects [53–56]. However, in the future it might also be detected on ground experiments, utilizing the conversion of the axion to photons in the presence of a strong magnetic field. The most fascinating feature of the axion and of axionlike particles in general is the fact that during inflation, the primordial $U(1)$ symmetry is broken, thus allowing the axion to have a large vacuum

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expectation value, and no cosmic string remnants pollute the postinflationary era. Another fascinating fact about the axion is that when the Hubble rate of the Universe becomes of the same order as the axion mass, the axion commences oscillations around its vacuum expectation value and its energy density redshifts as $\rho_a \sim a^{-3}$, thus as cold dark matter. Hence the axion can be the predominant component of cold dark matter in the Universe.

Modified gravity [78–82] offers an appealing theoretical framework, in the context of which the dark energy era and the inflationary era can be described in a unified way (see the pioneer article [83] for the first attempt toward this direction and also Refs. [76,84–90] for later developments in this research line). In both the inflationary era and the dark energy era general relativistic descriptions, the use of a scalar field, minimally or nonminimally coupled to gravity, is inevitably needed. With regard to the inflationary era, the scalar field description can be somewhat problematic, since the scalar field must inevitably be coupled to the Standard Model particles, and the couplings are arbitrary. Also with regard to the dark energy era, the scalar field description is problematic, since the dark energy equation of state (EoS) parameter is allowed to take values beyond the phantom divide line, so it can be less than -1 . The scalar field description of such an evolution requires tachyon fields, which are not appealing at all in any context. Modified gravity in its various forms offers a consistent framework that can describe both the inflationary and the dark energy eras, without the shortcomings of the scalar field description.

Among all the modified gravities, the $f(R, \phi)$ theories are the most motivated, since in a fundamental primordial scalar field in its vacuum configuration, the first quantum corrections are higher powers of the Ricci scalar (see [91] for more details, and also combinations of the Ricci scalar with Riemann and Ricci tensors, such as the Einstein-Gauss-Bonnet theories). In this article we shall assume that the inflationary era is controlled by an $F(R)$ gravity in the presence of a primordial axion field. The axion field shall be assumed to be the misalignment axion, in which case the primordial $U(1)$ Peccei-Quinn symmetry that the axion possessed is broken during inflation. There are two misalignment axion models in the literature: the canonical misalignment axion [21], and the kinetic misalignment axion models [25–27]. The difference between the two models is that during inflation, the canonical misalignment axion possesses no kinetic energy, and on the contrary in the case of the kinetic misalignment axion case, the axion possesses a large kinetic energy, which dominates the potential energy. Thus the axion oscillations in the latter case commence much more later, compared to the former axion model. In this work we shall investigate the effects of the kinetic misalignment axion model on the inflationary era generated by an $F(R)$ gravity and specifically on the R^2 inflationary era. At the equations of motion level, the effects are absent; however, at the cosmological

perturbations level, the axion may affect directly the evolution via the second slow-roll parameter ϵ_2 . As we show, the kinetic axion obeys a constant-roll evolution, which dominates the evolution at the late stages of the R^2 controlled inflationary era. The kination era caused by the kinetic axion basically dominates over the $\langle R^2 \rangle$ fluctuations that destabilize the inflationary quasi-de Sitter vacuum. This causes the total EoS of the Universe to be described by a short kination era, described by a stiff perfect fluid evolution, which eventually affects the total number of the e -foldings. Remarkably though, the fact that the scalar field obeys a constant-roll evolution during inflation does not affect at all the observational indices of inflation, since the contribution of the slow-roll parameter ϵ_2 is elegantly canceled.

This paper is organized as follows: In Sec. II we present the essential features of the kinetic axion $F(R)$ gravity model. We describe in brief the kinetic axion model, and we also present the way in which the axion postinflationary may mimic cold dark matter. In Sec. III, we investigate in detail the inflationary dynamics of the kinetic axion $F(R)$ gravity model. We show how the constant-roll evolution of the kinetic axion during inflation eventually leaves unaffected the dynamics of inflation at the cosmological perturbations level, and also we show how the kination era at the last stages of the R^2 controlled inflationary era eventually prolongs the inflationary era, increasing the total number of e -foldings. The conclusions of this work follow in the end of the article.

II. ESSENTIAL FEATURES OF THE $F(R)$ GRAVITY-KINETIC AXION MODEL

Before we get to the study of the inflationary dynamics for the $F(R)$ gravity-kinetic axion model, let us first present the theoretical framework of the model in some detail. The $F(R)$ gravity-kinetic axion model is basically an $f(R, \phi)$ gravity theory, in which case the gravitational action has the following form:

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} F(R) - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) + \mathcal{L}_m \right], \quad (1)$$

where $\kappa^2 = \frac{1}{8\pi G} = \frac{1}{M_p^2}$, with G being Newton's gravitational constant and M_p standing for the reduced Planck mass. Also, \mathcal{L}_m denotes the Lagrangian density of the perfect matter fluids that are present, and we will assume that only radiation is present. The dark matter perfect fluid will be composed solely by the axion particles present, with the latter being identified with the scalar field ϕ . Now, with regard to the $F(R)$ gravity model, for phenomenological reasons we will assume that it has the following form:

$$F(R) = R + \frac{1}{M^2} R^2 - \frac{\Lambda \left(\frac{R}{m_s^2}\right)^\delta}{\zeta}, \quad (2)$$

with m_s being defined as $m_s^2 = \frac{\kappa^2 \rho_m^{(0)}}{3}$; also $\rho_m^{(0)}$ is the energy density of cold dark matter at present day, and $0 < \delta < 1$. Finally, ζ and γ are some freely chosen dimensionless constants, which we shall discuss in a later section. The $F(R)$ gravity model is composed by an R^2 model that will control the primordial inflationary era and by a power-law term $\sim R^\delta$, which eventually will control the late-time dynamics. In fact, the model (2) can lead to a viable dark energy era, as was shown in detail in [76,77,91] so we will not address the late-time dynamics issue here.

With regard to the parameter M appearing in the R^2 term, it will be chosen to be $M = 1.5 \times 10^{-5} \left(\frac{N}{50}\right)^{-1} M_p$, a value imposed by inflationary phenomenological reasoning [92], with N being the e -foldings number. Also the parameter Λ in Eq. (2) is assumed to be of the same order as the cosmological constant at present day. In this work we shall consider a flat Friedmann-Robertson-Walker (FRW) background with line element

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2, \quad (3)$$

so the field equations for the $f(R)$ gravity with the axion scalar field in the presence of radiation are

$$3H^2 F_R = \frac{R F_R - F}{2} - 3H \dot{F}_R + \kappa^2 \left(\rho_r + \frac{1}{2} \dot{\phi}^2 + V(\phi) \right),$$

$$-2\dot{H}F = \kappa^2 \dot{\phi}^2 + \dot{F}_R - H \dot{F}_R + \frac{4\kappa^2}{3} \rho_r, \quad (4)$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad (5)$$

where $F_R = \frac{\partial F}{\partial R}$, while the ‘‘dot’’ denotes differentiation with respect to the cosmic time, and the ‘‘prime’’ denotes differentiation with respect to the axion scalar field. With regard to the axion field, we shall consider the misalignment axion [21,76], in which case the axion should not be related to the QCD axion, but it is some axionlike particle, which we call axion. In the literature there are two misalignment axion models, the canonical misalignment model [21] and the kinetic misalignment model [25–27]. In this work we shall consider the effects of the kinetic misalignment axion model on the inflationary dynamics of $F(R)$ gravity, and we shall see in which way it affects eventually the duration of the inflationary era. In the kinetic misalignment axion model, the axion has a primordial Peccei-Quinn $U(1)$ symmetry which is broken during the inflationary era. The fact that the original Peccei-Quinn symmetry is broken is particularly important for the inflationary phenomenology since no remnant cosmic strings remain after inflation ends. This, however, was a problem in standard QCD axion models, which is absent though in all misalignment axion models. After the $U(1)$ symmetry is broken, the axion acquires a large vacuum expectation

value $\langle \phi \rangle = \theta_a f_a$, where θ_a is the initial misalignment angle and f_a is the axion decay constant. The misalignment angle is in reality a dynamical field and can take values in the range $0 < \theta_a < 1$; however, in the way that it enters in the vacuum expectation value of the axion, it is not considered as a dynamical field, but as an average value throughout the whole universe at the time of inflation. With regard to the axion decay constant f_a , this parameter is of fundamental phenomenological importance, and in conjunction with the axion mass, constitutes the two most important phenomenological parameters for the axion dynamics. Regarding the axion having a vacuum expectation value during inflation, this fact does not mean that the axion is actually constant during inflation, but it basically has small deviations about its vacuum expectation value, different from the small oscillations about its vacuum expectation value after the inflationary era ends. Let us describe in brief the kinetic misalignment axion dynamics during inflation. Schematically, this is depicted in Fig. 1. As it can be seen in Fig. 1, the axion during inflation has a small initial displacement from its vacuum expectation value, and more importantly it has a nonzero initial velocity. This initial velocity is what justifies the terminology ‘‘kinetic.’’ The axion rolls down its potential, and due to the initial velocity it ends up uphill again, at a position different from the one corresponding to the canonical misalignment axion model, in which case the axion after it rolls down and reaches the minimum, starts to oscillate around its vacuum expectation value. Thus in the kinetic misalignment case, the axion ends up uphill, and it then rolls downhill until it reaches the minimum and commences to oscillate around its vacuum expectation value when the Hubble rate of the Universe becomes comparable to the axion mass $H \sim m_a$. In the kinetic misalignment axion mechanism the oscillations start at a later time compared to the canonical misalignment mechanism, and thus in the kinetic axion model, the temperature at which oscillations commence is lower compared to the canonical misalignment axion model. Let us briefly quantify the above picture in terms of the potential and the axion mass. Primordially, the axion potential has the following form:

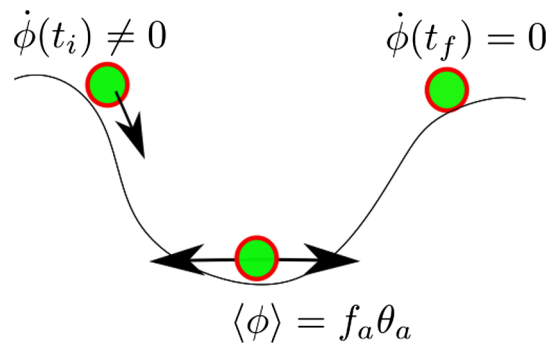


FIG. 1. The kinetic misalignment axion dynamics.

$$V(\phi) = m_a^2 f_a^2 \left(1 - \cos\left(\frac{\phi}{f_a}\right)\right), \quad (6)$$

and when the axion acquires a vacuum expectation value during inflation, for small displacements from its vacuum expectation value, its potential can be approximated as follows:

$$V(\phi) \simeq \frac{1}{2} m_a^2 \phi^2, \quad (7)$$

an approximation that is valid when $\phi \ll f_a$ or similarly $\phi \ll \langle \phi \rangle$. Initially, when the axion is uphill at both ends of the potential, we have $H \gg m_a$; however, when the axion reaches the minimum for the second time, it starts to oscillate around its vacuum expectation value when $H \leq m_a$, and at that point the axion energy density redshifts as dark matter [21,76]. The most important feature of the kinetic misalignment axion model is the fact that initially the axion kinetic energy term $\dot{\phi}^2$ is quite larger than the axion potential energy $\dot{\phi}^2 \gg V$. This continues until some time instance during the reheating, the potential and the kinetic energy become comparable, when the temperature of the Universe is of the order

$$\dot{\phi}(\tilde{T}) = m_a(\tilde{T}), \quad (8)$$

at which temperature the axion does not have a large kinetic energy anymore so it becomes trapped in the potential barrier and the oscillations around the minimum commence. The canonical misalignment temperature when the oscillations start T_* is larger than \tilde{T} . So basically, when the axion mass is larger than the Hubble rate $m_a(\tilde{T}) \geq H(\tilde{T}) = \frac{3}{M_p} \sqrt{\frac{\pi^2}{10} g_*} \tilde{T}^2$, the roll down and up of the axion occurs, and when $m_a(\tilde{T}) \leq H(\tilde{T}) = \frac{3}{M_p} \sqrt{\frac{\pi^2}{10} g_*} \tilde{T}^2$, the axion starts to oscillate with abundance $\rho_a \sim m_a(T=0) \frac{\dot{\phi} f_a^2}{s}$, where s is the entropy density and $m_a(T=0) = m_a$ is the actual axion mass as a dark matter particle. In general, in the kinetic axion misalignment model, the axion dark matter mass is larger than the one compared to the canonical misalignment axion case. A useful relation that connects the axion mass with the axion decay constant is the following:

$$m_a(T) = 6 \text{ meV} \frac{10^9 \text{ GeV}}{f_a}, \quad (9)$$

and in order for the kinetic axion to account for the current dark matter abundance, the axion decay constant must satisfy $f_a \leq 1.5 \times 10^{11} \text{ GeV}$.

Let us further quantify the dynamics of the axion during inflation, since this will be important for the study of the inflationary phenomenology. Since initially, the kinetic

energy of the axion is quite larger than the potential energy, that is, $\dot{\phi}^2 \gg m_a^2 \phi^2$, the field equation for the axion becomes approximately

$$\ddot{\phi} + 3H\dot{\phi} \simeq 0, \quad (10)$$

which can be solved to yield

$$\dot{\phi} \sim a^{-3}. \quad (11)$$

So primordially, the energy density of the axion which is $\rho_a = \frac{\dot{\phi}^2}{2} + V(\phi) \simeq \frac{\dot{\phi}^2}{2}$ becomes $\rho_a \sim a^{-6}$. Thus this is an era of kination for the axion dynamics, with its effective equation of state parameter being $w = 1$, which describes stiff matter fluid. Hence the kinetic axion during inflation behaves as a stiff perfect matter fluid. In the next section we shall consider in a quantitative way the direct effects of the kinetic misalignment axion scalar on the inflationary dynamics of $F(R)$ gravity, and we shall reveal how the stiff axion fluid eventually prolongs the inflationary era.

Before closing this section, let us note that the kinetic misalignment axion mechanism is inherently related to the initial explicit breaking of the Peccei-Quinn symmetry which is broken by a higher dimensional effective operator in the same way as in the Affleck-Dine mechanism.

III. DYNAMICS OF INFLATION FOR THE KINETIC AXION $F(R)$ GRAVITY MODEL

Let us now use the results of the previous section in order to determine the inflationary dynamics and the corresponding phenomenology in terms of the slow-roll indices. Recall that in the previous section we showed that the kinetic axion field redshifts as a perfect matter fluid during inflation with a stiff EoS, since $\rho_a \sim a^{-6}$, so its energy density is smaller compared to the radiation fluid energy density. Assuming a low scale inflationary era, with the Hubble rate during inflation being of the order $H_I = 10^{13} \text{ GeV}$, let us investigate which terms effectively dominate in the field equations of the kinetic axion $F(R)$ gravity. The Ricci scalar takes quite large values for $H_I = 10^{13} \text{ GeV}$, and thus R^2 dominates the evolution roughly speaking. Let us see this in some detail, and recall that $m_s^2 \simeq 1.87101 \times 10^{-67} \text{ eV}^2$ and also the parameter M which appears in the R^2 term in Eq. (2) is $M = 1.5 \times 10^{-5} \left(\frac{N}{50}\right)^{-1} M_p$ [92]; hence by roughly taking for $N \sim 60$, M is of the order $M \simeq 3.04375 \times 10^{22} \text{ eV}$. Also, due to the fact that during inflation the slow-roll conditions are satisfied, we approximately have $R \simeq 12H^2$ and therefore $R \sim 1.2 \times 10^{45} \text{ eV}^2$. Furthermore, $M_p \simeq 2.435 \times 10^{27} \text{ eV}$, and also the parameter Λ is of the order of the cosmological constant at present day, that is, $\Lambda \simeq 11.895 \times 10^{-67} \text{ eV}^2$. Finally, the vacuum expectation value of the axion is roughly of the same order as the axion decay constant;

therefore $\langle \phi \rangle = \phi_i \simeq \mathcal{O}(10^{15})$ GeV and approximately $m_a \simeq \mathcal{O}(10^{-14})$ eV. Thus, the potential term is of the order $\kappa^2 V(\phi_i) \sim \mathcal{O}(8.41897 \times 10^{-36})$ eV², while the two curvature terms R and R^2 are of the order $R \sim 1.2 \times \mathcal{O}(10^{45})$ eV² and $R^2/M^2 \sim \mathcal{O}(1.55 \times 10^{45})$ eV², and the power-law curvature term is of the order $\frac{\Lambda(\frac{R}{m_a})^{0.1}}{0.2} \sim \mathcal{O}(10^{-55})$ eV² for $\delta = 0.1$ and $\zeta = 0.2$, with the latter being phenomenologically acceptable values. Also, during inflation the radiation density term $\kappa^2 \rho_r \sim \kappa^2 e^{-4N}$, a similar relation applies for the kinetic misalignment axion energy density ρ_a , and the corresponding term scales as $\kappa^2 \rho_r \sim \kappa^2 e^{-6N}$. Therefore, at the equations of motion level, the resulting theory is basically effectively described by a vacuum R^2 , in which case

$$F(R) \simeq R + \frac{1}{M^2} R^2. \quad (12)$$

Then the field equations take the form

$$\ddot{H} - \frac{\dot{H}^2}{2H} + \frac{HM^2}{2} = -3H\dot{H}. \quad (13)$$

and due to the slow-roll conditions,

$$-\frac{M^2}{6} = \dot{H}, \quad (14)$$

which has as a solution

$$H(t) = H_I - \frac{M^2}{6} t, \quad (15)$$

which is a quasi-de Sitter solution, with H_I being an arbitrary integration constant, with profound physical significance since this is the scale of inflation.

Now one might consider that effectively the kinetic axion does not affect at all the dynamics of inflation; however, this is not true. It is certain that the axion does not control the Hubble rate at the level of the equations of motion for sure; however, the dynamics of inflation are not affected only by the background evolution. As we will show, the axion may affect inflation in two ways: first, it may directly affect the scalar curvature perturbations, and second, it prolongs the inflationary era, since the axion effective equation of state is $w = 1$ so that inflation is prolonged as we show shortly.

The cosmological scalar curvature perturbations are dynamically quantified by the slow-roll indices, which for the $f(R, \phi)$ theory at hand are defined to be [78,93,94],

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = \frac{\ddot{\phi}}{H\dot{\phi}}, \quad \epsilon_3 = \frac{\dot{F}_R}{2HF_R}, \quad \epsilon_4 = \frac{\dot{E}}{2HE}, \quad (16)$$

where the function E for the $f(R, \phi)$ theory at hand has the following form:

$$E = F_R + \frac{3\dot{F}_R^2}{2\kappa^2\dot{\phi}^2}. \quad (17)$$

Now the most important effect that the kinetic axion theory brings along in the $F(R)$ gravity is contained in the parameter ϵ_2 . Since the axion obeys the stiff scalar differential equation (10), this means that in our case, the slow-roll parameter ϵ_2 takes the value $\epsilon_2 = -3$; therefore, the axion obeys a constant-roll condition in its dynamics. The question is, does ϵ_2 affect the inflationary dynamics? As we now show, at leading order, the contribution of the axion field is elegantly canceled in the observational indices of inflation and specifically with the spectral index of the primordial scalar curvature perturbations. To this end, let us present the details of the calculation of the parameter ϵ_4 in which the dynamics of the axion is found. We have explicitly at leading order during inflation

$$E \simeq \frac{3\dot{F}_R^2}{2\kappa^2\dot{\phi}^2}, \quad (18)$$

so ϵ_4 is approximately equal to

$$\epsilon_4 \simeq \frac{3}{2\kappa^2} \frac{2\dot{F}_R\ddot{F}_R\dot{\phi}^2 - \dot{F}_R^2\dot{\phi}\ddot{\phi}}{\dot{\phi}^4}, \quad (19)$$

which is simplified to

$$\epsilon_4 \simeq \frac{\ddot{F}_{RR}}{H\dot{F}_R} - \frac{\ddot{\phi}}{H\dot{\phi}} = \frac{\ddot{F}_{RR}}{H\dot{F}_R} - \epsilon_2. \quad (20)$$

Let us further elaborate on the parameter ϵ_4 which after some algebra is written as follows:

$$\epsilon_4 \simeq -\frac{24F_{RRR}H^2}{F_{RR}}\epsilon_1 - 3\epsilon_1 + \frac{\dot{\epsilon}_1}{H\epsilon_1} - \epsilon_2. \quad (21)$$

The term $\dot{\epsilon}_1$ can be written as

$$\dot{\epsilon}_1 = -\frac{\ddot{H}H^2 - 2\dot{H}^2H}{H^4} = -\frac{\ddot{H}}{H^2} + \frac{2\dot{H}^2}{H^3} \simeq 2H\epsilon_1^2; \quad (22)$$

hence ϵ_4 becomes

$$\epsilon_4 \simeq -\frac{24F_{RRR}H^2}{F_{RR}}\epsilon_1 - \epsilon_1 - \epsilon_2. \quad (23)$$

Upon introducing x we have

$$x = \frac{48F_{RRR}H^2}{F_{RR}}, \quad (24)$$

and ϵ_4 can be written in terms of it as follows:

$$\epsilon_4 \simeq -\frac{x}{2}\epsilon_1 - \epsilon_1 - \epsilon_2. \quad (25)$$

Now for the $f(R, \phi)$ gravity, the scalar spectral index of the scalar curvature perturbations is [78,93,94]

$$n_S = 1 - 4\epsilon_1 - 2\epsilon_2 + 2\epsilon_3 - 2\epsilon_4. \quad (26)$$

Thus by substituting the expression for ϵ_4 we obtained in Eq. (23), we can see that the contribution of ϵ_2 elegantly cancels, and the spectral index takes the form

$$n_S \simeq 1 - (2 - x)\epsilon_1 + 2\epsilon_3. \quad (27)$$

Also the scalar-to-tensor ratio for the case at hand is equal to [78,93,94]

$$r \simeq 48\epsilon_1^2. \quad (28)$$

Since the dominant part of the $F(R)$ gravity during inflation is an R^2 gravity, the term x is equal to zero; thus, the scalar spectral index is greatly simplified. For the quasi-de Sitter solution at hand, the first slow-roll index is easily calculated to be

$$\epsilon_1 = -\frac{6M^2}{(M^2t - 6H_I)^2}, \quad (29)$$

and by solving the algebraic equation $\epsilon_1(t_f) = 1$, the time instance at which inflation ends is

$$t_f = (6H_I + \sqrt{6M})/M^2. \quad (30)$$

Using the definition of the e -foldings number N ,

$$N = \int_{t_i}^{t_f} H(t)dt, \quad (31)$$

the time instance at which inflation commences is

$$t_i = \frac{2\sqrt{9H_I^2 - 3M^2N} + 6H_I}{M^2}, \quad (32)$$

and so the first slow-roll index at the first horizon crossing is

$$\epsilon_1(t_i) = \frac{1}{1 + 2N}. \quad (33)$$

Hence at leading order in terms of the e -foldings number, the spectral index and the tensor-to-scalar ratio take the forms $n_s \sim 1 - \frac{2}{N}$ and $r \sim \frac{12}{N^2}$. Now for $N \sim 60$ the resulting phenomenology is identical to the Starobinsky model; however, the axion stiff equation of state causes another effect on inflation. Basically it prolongs the inflationary era

to some extent as we now evince. As inflation comes to an end near the time instance t_f , the background total EoS of the Universe is no longer described by a quasi-de Sitter EoS. Hence the stiff EoS of the axion describes the Universe, since the matter perfect fluids become more dominant slowly by slowly. Therefore the total EoS parameter of the background evolution approaches the stiff EoS value $w = 1$. This fact prolongs the inflationary era, causing the e -foldings number to be larger than 60. The physical picture behind the increase of the e -foldings number relies on the combined presence of the R^2 term and the large kinetic term of the kinetic misalignment axion. In standard R^2 gravity, inflation tends to its end when the curvature fluctuations $\langle R^2 \rangle$ become quite strong and make the de Sitter attractor unstable. This phenomenological picture is possible only with the R^2 gravity, and as it was shown in Ref. [95], the Starobinsky model has an unstable de Sitter attractor. Let us show this in brief (see also [95] for more details). By introducing the dimensionless variables,

$$x_1 = -\frac{\dot{F}_R(R)}{F_R(R)H}, \quad x_2 = -\frac{F(R)}{6F(R)H^2}, \quad x_3 = \frac{R}{6H^2}, \quad (34)$$

and by using the e -foldings number as a dynamical variable instead of the cosmic time, the field equations of vacuum $F(R)$ gravity can be written in terms of the following dynamical system:

$$\begin{aligned} \frac{dx_1}{dN} &= -4 - 3x_1 + 2x_3 - x_1x_3 + x_1^2, \\ \frac{dx_2}{dN} &= 8 + m - 4x_3 + x_2x_1 - 2x_2x_3 + 4x_2, \\ \frac{dx_3}{dN} &= -8 - m + 8x_3 - 2x_3^2, \end{aligned} \quad (35)$$

with the dynamical parameter m being equal to

$$m = -\frac{\ddot{H}}{H^3}. \quad (36)$$

The dynamical system (35) is autonomous when the parameter m takes constant values, and for a quasi-de Sitter evolution $a(t) = e^{H_0t - H_I t^2}$ the parameter m is equal to zero. The total EoS of the cosmological system is defined as [78]

$$w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2}, \quad (37)$$

and it can be expressed directly in terms of the variable x_3 in the following way:

$$w_{\text{eff}} = -\frac{1}{3}(2x_3 - 1). \quad (38)$$

Now, by performing a fixed point analysis of the dynamical system for $m = 0$, we easily obtain the fixed points

$$\phi_*^1 = (-1, 0, 2), \quad \phi_*^2 = (0, -1, 2), \quad (39)$$

and the corresponding eigenvalues of the matrix that corresponds to the dynamical system for ϕ_*^1 are $(-1, -1, 0)$, while in the case of the fixed point ϕ_*^2 these are $(1, 0, 0)$. Therefore, the dynamical system possesses two that are nonhyperbolic, but the fixed point ϕ_*^1 is stable and, in contrast, the fixed point ϕ_*^2 is unstable, with the latter being the most interesting fixed point from a phenomenological point of view. It is noticeable that for both the fixed points, we have $x_3 = 2$, and hence, from Eq. (38), we get $w_{\text{eff}} = -1$. This feature basically shows that both fixed points are de Sitter fixed points. As we already mentioned, the second de Sitter fixed point, namely, $\phi_*^2 = (0, -1, 2)$, is the most interesting phenomenologically, since for this equilibrium, the conditions $x_1 \simeq 0$ and $x_2 \simeq -1$ yield

$$-\frac{d^2 F}{dR^2} \frac{\dot{R}}{H} \frac{dF}{dR} \simeq 0, \quad -\frac{F}{H^2} \frac{dF}{dR} \simeq -1. \quad (40)$$

Using the slow-roll approximation during inflation for the Ricci scalar curvature $R \simeq 12H^2$, for the quasi-de Sitter evolution, we can write the second differential equation as follows:

$$F \simeq \frac{dF}{dR} R, \quad (41)$$

which when solved yields

$$F(R) \simeq \alpha R^2, \quad (42)$$

where α is some arbitrary integration constant, which describes the $F(R)$ gravity that generates the quasi-de Sitter evolution. Clearly the R^2 model possesses an unstable de Sitter point. Thus when this unstable de Sitter attractor is reached, the system is repelled from it in the phase space. The time instance for which this happens is determined roughly by the condition $\epsilon_1(t_f) = 1$. Now in the presence of the large axion kinetic term that dominates over its potentials, things are somewhat different when the cosmological system reaches the de Sitter attractor. Particularly, the cosmological system initially is controlled by the R^2 term so it reaches the quasi-de Sitter attractor. However, when it is repelled from the unstable de Sitter attractor point, the cosmological system does not enter directly the reheating era and the $\langle R \rangle$ reheating fluctuations do not commence directly, but the kinetic

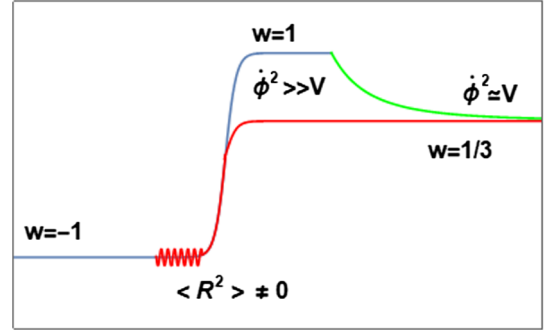


FIG. 2. The kinetic misalignment axion $F(R)$ gravity total EoS dynamical evolution. The cosmological system reaches an unstable de Sitter point in both the vacuum R^2 gravity and the kinetic misalignment axion $F(R)$ gravity. Eventually, the $\langle R^2 \rangle$ fluctuations make the system to be repelled from the de Sitter attractor, and the cosmological system enters the reheating era controlled by the $\langle R^2 \rangle$ fluctuations. In the presence of the kinetic axion, after the $\langle R^2 \rangle$ fluctuations cause the system to be repelled from the de Sitter attractor, the cosmological system does not enter the reheating era directly, but the kinetic term dominates the evolution and the background EoS is not the one corresponding to an ordinary reheating era $w = 1/3$ but it corresponds to a stiff era with $w = 1$. The system stays in this stiff era and the ordinary reheating era commences when the axion oscillations begin.

term, which was subdominant, dominates over the R^2 term and thus controls the dynamics at the end of inflation, after the cosmological system is repelled from the quasi-de Sitter attractor. Thus the end of inflation is somewhat prolonged for the kinetic axion R^2 model. This can be schematically seen in Fig. 2, in which it is shown that in the ordinary R^2 model, the system after it reaches the unstable de Sitter attractor, the $\langle R^2 \rangle$ fluctuations make the system to be repelled from the attractor, and the cosmological system enters the reheating era controlled by the $\langle R \rangle$ fluctuations. In the presence of the kinetic axion, after the $\langle R^2 \rangle$ fluctuations cause the system to be repelled from the de Sitter attractor, the cosmological system does not enter directly the reheating era, but the kinetic term dominates the evolution and the background EoS is not the one corresponding to an ordinary reheating era $w = 1/3$ but it corresponds to a stiff era with $w = 1$. The system stays in this stiff era and the ordinary reheating era commences when the axion oscillations begin, so when $\dot{\phi}^2 \sim V$, at which point the axion redshifts as ordinary dark matter and the radiation fluid controls the evolution thereafter. Thus the number of e -foldings is somewhat extended in the kinetic axion $F(R)$ gravity picture.

Another striking feature of the kinetic axion $F(R)$ gravity model is the fact that the R^2 term actually enhances significantly the kinetic axion physics, delaying further the kinetic axion to start its oscillations. In a future work we shall demonstrate using a dynamical systems approach how this can happen. Now let us quantify the qualitative picture

we described above, and let us see how the stiff era affects the e -foldings number, thus somewhat extending inflation for some e -foldings. As we will show, this feature is strongly affected by the reheating temperature. In a general setting, the e -foldings number for a primordial scalar mode with wave number k , which became a superhorizon at the beginning of inflation, is equal to [96]

$$\frac{a_k H_k}{a_0 H_0} = e^{-N} \frac{H_k a_{\text{end}}}{a_{\text{reh}} H_{\text{reh}}} \frac{H_{\text{reh}} a_{\text{reh}}}{a_{\text{eq}} H_{\text{eq}}} \frac{H_{\text{eq}} a_{\text{eq}}}{a_0 H_0}, \quad (43)$$

with a_k and H_k being the scale factor and the Hubble rate at the time instance where the primordial mode k became a superhorizon at the beginning of inflation (at the first horizon crossing), a_{end} stands for the scale factor at the end of the inflationary era, and finally a_{reh} and H_{reh} denote the scale factor and the Hubble rate when the reheating era ends. Furthermore, a_{eq} and H_{eq} denote the scale factor and the Hubble rate at the time instance that the matter-radiation equality occurs, and moreover a_0 and H_0 denote the present day scale factor and the Hubble rate, respectively. Now, if near the end of inflation, the total EoS parameter is w (different from the value $w = 1/3$), we get

$$\ln\left(\frac{a_{\text{end}} H_{\text{end}}}{a_{\text{reh}} H_{\text{reh}}}\right) = -\frac{1+3w}{6(1+w)} \ln\left(\frac{\rho_{\text{reh}}}{\rho_{\text{end}}}\right), \quad (44)$$

with H_{end} being the Hubble rate when inflation ends, and the energy densities ρ_{end} and ρ_{reh} stand for the total energy density of the Universe when inflation ends and when the reheating era ends. Note that for the derivation of Eq. (44), we assumed that the total EoS parameter at the end of the reheating era and at the end of inflation is constant and equal to w . Then, when the $\langle R^2 \rangle$ commence, causing instability to the de Sitter period, the constant EoS stiff era of the kinetic axion commences, so the e -foldings number of the inflationary era is extended as follows [96]:

$$N = 56.12 - \ln\left(\frac{k}{k_*}\right) + \frac{1}{3(1+w)} \ln\left(\frac{2}{3}\right) + \ln\left(\frac{\rho_k^{1/4}}{\rho_{\text{end}}^{1/4}}\right) + \frac{1-3w}{3(1+w)} \ln\left(\frac{\rho_{\text{reh}}^{1/4}}{\rho_{\text{end}}^{1/4}}\right) + \ln\left(\frac{\rho_k^{1/4}}{10^{16} \text{ GeV}}\right), \quad (45)$$

with ρ_k being the Universe's total energy density at the beginning of the inflationary era, exactly when the mode k became a superhorizon. We shall also assume that the pivot scale k_* is $k_* = 0.05 \text{ Mpc}^{-1}$, and furthermore, we shall assume that the degrees of freedom of particles g_* during the inflationary era, just after this era, are nearly constant. Thus the energy density of the Universe at a temperature T is equal to $\rho = \frac{\pi^2}{30} g_* T^4$. Hence, the expression of Eq. (45) can be rewritten in terms of the temperatures at the various

TABLE I. The e -foldings number for the kinetic axion $F(R)$ gravity model for various reheating temperatures, to be compared with the standard R^2 model results $n_S = 0.966667$ and $r = 0.00333333$ and a standard reheating scenario.

e -foldings number and inflationary indices	$T_R = 10^{12} \text{ GeV}$	$T_R = 10^7 \text{ GeV}$
e -foldings number N	65.3439	61.5063
Spectral index n_S	0.969393	0.967483
Tensor-to-scalar ratio r	0.00281042	0.00317206

epochs and not in terms of the energy densities. In effect, if the total number of e -foldings changes, the parameter M coupled to the R^2 gravity will also be somewhat affected, and this should be taken into account for the inflationary phenomenology of the current model. In Table I we present the phenomenological behavior of the basically prolonged R^2 inflationary model, for three reheating temperatures, namely a large reheating temperature $T_R = 10^{12} \text{ GeV}$, and an intermediate reheating temperature $T_R = 10^7 \text{ GeV}$. The perspective of having low reheating temperatures is already discussed in the literature, even having MeV scale reheating temperatures (see, for example, [97]). As it can be seen, in all cases, the inflationary era is prolonged and the results are different from the standard R^2 model for $N = 60$ with the changes being of the order 15% for the case of the tensor-to-scalar ratio. Also as expected, since the inflationary era generated by the kinetic axion $F(R)$ gravity theory is a deformation of the R^2 model, it produces a viable phenomenology. This can be seen in Fig. 3 where we confront the kinetic axion $F(R)$ gravity model with the Planck likelihood curves for various reheating temperatures in the range 10^7 – 10^{12} GeV . As it can be seen, the model is well fitted in the sweet spot of the Planck data. In the plots, the green dots correspond to the vacuum R^2 model, and the red dots correspond to the kinetic axion R^2 model. As it can be seen, the kinetic axion R^2 model is a measurable deformation of the vacuum R^2 model.

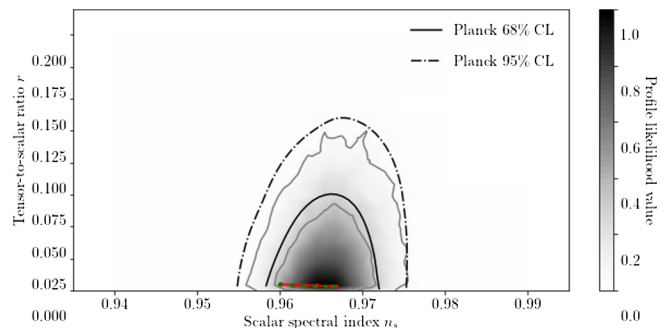


FIG. 3. The Planck likelihood curves and the kinetic axion $F(R)$ gravity model (red dots) and the vacuum R^2 model (green dots). The kinetic R^2 model serves as a viable deformation of the vacuum Starobinsky model.

IV. CONCLUSIONS

In this work we investigated how a kinetic misalignment axion can affect the inflationary era generated by an R^2 model of $F(R)$ gravity. In the context of the kinetic misalignment axion, the primordial $U(1)$ Peccei-Quinn symmetry is broken in the axion sector during inflation, and thus the axion has a nonzero vacuum expectation value; however, it also possesses a large kinetic energy. The kinetic energy term of the axion dominates over its potential; however, during inflation and at the equations of motion level, the vacuum R^2 model dominates the evolution. Thus the axion may affect the dynamics of the inflationary era at the cosmological perturbations level through the second slow-roll index. Due to the dominance of the axion's kinetic energy over its potential, the axion evolves in a constant-roll way, and thus the second slow-roll index is constant and large. We calculated the observational indices including the kinetic axion effects, and as we showed, the contribution of the second slow-roll index elegantly cancels. Thus, at the cosmological perturbations level, the kinetic axion does not affect the R^2 inflationary era. However, the kinetic axion affects the duration of the inflationary era, causing in some cases 15% differences in the tensor-to-scalar ratio compared with the vacuum R^2

model. This change is due to the fact that the kinetic axion has an EoS parameter that corresponds to that of a stiff era. As the R^2 inflationary era reaches its unstable quasi-de Sitter attractor in the phase space, the kinetic axion starts to dominate the evolution over the R^2 term. Thus, the Universe enters a stiff evolution era and an era of kination with background total EoS parameter $w = 1$. This stiff background directly affects the e -foldings number, thus extending the inflationary era up to 5 e -foldings in some cases. Quantitatively in some cases this amounts to a decrease of the tensor-to-scalar ratio of about 15% compared to the vacuum R^2 model. A particularly interesting extension of this work is to further consider in the Lagrangian an Einstein-Gauss-Bonnet term. Due to the fact that the axion is not constant during inflation, but it is fluctuating around its vacuum expectation value, the Einstein-Gauss-Bonnet does not trivially vanish; thus it would be interesting to investigate the consequences of the kinetic axion in this class of theories. In fact, it would furthermore be interesting to investigate the late-time evolution of the unified model, because when the axion starts to oscillate around its vacuum expectation value, it redshifts as dark matter. These issues shall be addressed in a future work.

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