Sound from extra dimensions: Quasinormal modes of a thick brane

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In this work, we investigate the quasinormal modes of a thick brane system. Considering the transverse-traceless tensor perturbation of the brane metric, we obtain the Schrödinger-like equation of the Kaluza-Klein modes of the tensor perturbation. Then we use the Wentzel-Kramers-Brillouin approximation and the asymptotic iteration method to solve this Schrödinger-like equation. We also study the numeric evolution of an initial wave packet against the thick brane. The results show that there is a set of discrete quasinormal modes in the thick brane model. These quasinormal modes appear as the decaying massive gravitons for a brane observer. They are characteristic modes of the thick brane and can reflect the structure of the thick brane.

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I. INTRODUCTION

As the characteristic modes of a dissipative system, quasinormal modes (QNMs) exist in every aspect of our world. These QNMs contain the key features that are characteristics of the physical systems. Studying them would help us to unravel the mysteries of the physical systems. In black hole physics, the QNMs are thought to be able to carry information about black holes and have attracted much attention [1–7], especially after the detection of gravitational waves [8]. In other physical systems such as leaky resonant cavities, QNMs also play an important role [9]. So we are curious about what role QNMs might play in the braneworld model.

The braneworld models were originally introduced as a solution to the hierarchical problem between the weak and Plank scales. Among them, the warped extra dimension models proposed by Randall and Sundrum (RS) have attracted a lot of interest [10,11]. They consist of one brane (RS-II model) or two branes (RS-I model) embedded in a five-dimensional anti–de Sitter spacetime. Since the RS models were proposed, they have been studied in many realms such as particle physics, cosmology, and black hole physics. The applicability has gone far beyond its original scope [12–17]. Combining the RS-II model [11] and the domain wall model [18,19], the thick brane models were developed [20–22]. It is a smooth extension of the RS-II model. The inclusion of brane thickness gives us new possibilities. Usually, most thick branes are generated by one or more scalar fields, but they can also be generated by a vector or spinor field [23–25]. In order to recover the physics in our four-dimensional world, the zero modes of various fields should be confined on the brane. In previous literature, some thick brane models and the localization of various matter fields on the brane were investigated [26–42]. Besides these zero modes, there are massive Kaluza-Klein (KK) modes that might propagate along extra dimensions. These massive KK modes provide the possibility of detecting extra dimensions. In addition, cosmological thick brane solutions were also investigated [43–45].

Quasinormal modes in higher dimensional theories also attract the interest of researchers [46–52]. It is expected that the signatures of extra dimensions can be extracted from QNMs of black holes on the brane. These signals can be used to constrain the extra dimensional models [53–59]. But these researches are mainly focused on the QNMs of black holes on the brane. Does a brane have a characteristic sound? That is, does it have a set of discrete QNMs as characteristic modes of a braneworld model? For the RS-II model, the answer is yes [60,61]. Seahra studied the scattering of KK gravitons in the RS-II model and found that the brane possesses a series of discrete QNMs [60,61].

Reference [62] investigated the evolution of massive modes in the thick brane model and found that the evolution behavior is similar to QNMs. This arouse our interest in QNMs in thick brane models. As far as we know, the QNMs of a thick brane have not been investigated. As the characteristic modes of a brane, it can reflect the structure of the thick brane. On the other hand, since the QNMs dominating the time evolution of some initial

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fluctuations from the physic system's equilibrium state, they can be used to verify the stability of the brane [63]. It is undoubtedly interesting to study the QNMs of a thick brane. We will use semianalytical and numerical methods to study QNMs in a thick brane model.

The organization of the rest of this paper is as follows. In Sec. II, we review a solution of the thick brane and the linear metric tensor perturbation. Based on this solution, we solve for the QNMs of this thick brane. In Sec. III, we compute the quasinormal frequencies of the thick brane by using semianalytical methods. We also compare them with the results of numerical evolution. Finally, Sec. IV gives the conclusions and discussions.

II. BRANEWORLD MODEL IN GENERAL RELATIVITY

In this section, we will briefly review the thick brane solution and its gravitational perturbation. Usually, a thick brane can be generated by a wide variety of matter fields like scalar fields and vector fields. Here we choose a canonical scalar field to generate the brane. The action of this thick brane model is the Einstein-Hilbert action minimally coupled to a canonical scalar field,

$$S = \int d^5x\sqrt{-g}\left(\frac{1}{2\kappa_5^2}R - \frac{1}{2}g^{MN}\partial_M\varphi\partial_N\varphi - V(\varphi)\right), \tag{1}$$

where κ_5 is the five-dimensional gravitational constant. Hereafter, capital latin letters $M, N, \ldots = 0, 1, 2, 3, 5$ label the five-dimensional indices, while greek letters $\mu, \nu \ldots = 0, 1, 2, 3$ and latin letters $i, j \ldots = 1, 2, 3$ label the four-dimensional ones and three-dimensional space ones on the brane, respectively. The dynamical field equations are

$$R_{MN} - \frac{1}{2}Rg_{MN} = -g_{MN}\kappa_5^2\left(\frac{1}{2}\partial^A\varphi\partial_A\varphi + V(\varphi)\right) + \kappa_5^2\partial_M\varphi\partial_N\varphi, \tag{2}$$

$$g^{MN}\nabla_M\nabla_N\varphi = \frac{\partial V(\varphi)}{\partial\varphi}. \tag{3}$$

The five-dimensional metric ensuring the four-dimensional Poincaré symmetry is [22]

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{4}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the four-dimensional Minkowski metric. Now, the specific dynamical equations can be written as

$$6A'^2 + 3A'' = -\frac{\kappa_5^2}{2}\varphi'^2 - \kappa_5^2 V, \tag{5}$$

$$6A'^2 = \frac{\kappa_5^2}{2}\varphi'^2 - \kappa_5^2 V, \tag{6}$$

$$\varphi'' + 4A'\varphi' = \frac{\partial V}{\partial\varphi}, \tag{7}$$

where a prime denotes the derivative with respect to the extra dimensional coordinate y. By using the first-order formalism, the thick brane solution was investigated in Ref. [21],

$$A(y) = -b\ln(\cosh(ky)), \tag{8}$$

$$\varphi(y) = \sqrt{\frac{3b}{\kappa_5^2}}\arcsin(\tanh(ky)), \tag{9}$$

$$V(\varphi) = \frac{3bk^2}{4\kappa_5^2}\left(1 - 4b + (1+4b)\cos\left(\sqrt{\frac{4\kappa_5^2}{3b}}\varphi\right)\right). \tag{10}$$

Here, b is a dimensionless parameter and k is a parameter with mass dimension one. If we choose $\kappa_5 = \sqrt{2}$, the above solutions for the scalar field φ and potential V are the same to Ref. [21] because $\arcsin(\tanh(ky)) = 2\arctan(\tanh(ky/2))$ [64]. Besides, the warp factor (8) differs from the one in Ref. [21] by a factor 2 in front of $\cosh(ky)$. It does not matter, because this factor can be absorbed into a new four-dimensional coordinate. Next, we consider the linear transverse-traceless tensor perturbation of the metric. The perturbed metric is given by

$$g_{MN} = \begin{pmatrix} e^{2A(y)}(\eta_{\mu\nu} + h_{\mu\nu}) & 0 \\ 0 & 1 \end{pmatrix}, \tag{11}$$

where $h_{\mu\nu}$ satisfies the transverse-traceless condition,

$$\partial_\mu h^{\mu\nu} = 0 = \eta^{\mu\nu}h_{\mu\nu}. \tag{12}$$

Substituting the perturbed metric (11) into the field equation (2), we obtain the linear equation of the tensor fluctuation,

$$(e^{-2A}\Box^{(4)}h_{\mu\nu} + h''_{\mu\nu} + 4A'h'_{\mu\nu}) = 0, \tag{13}$$

where $\Box^{(4)} = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$. Introducing the following coordinate transformation $dz = e^{-A}dy$, the metric (4) can be written as

$$ds^2 = e^{2A(z)}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2), \tag{14}$$

and Eq. (13) becomes

$$[\partial_z^2 + 3(\partial_z A)\partial_z + \Box^{(4)}]h_{\mu\nu} = 0. \tag{15}$$

The perturbation $h_{\mu\nu}$ can be written as [61]

$$h_{\mu\nu} = e^{-\frac{3}{2}A(z)}\Phi(t, z)e^{-ia_jx^j}\epsilon_{\mu\nu}, \qquad \epsilon_{\mu\nu} = \text{constant}. \tag{16}$$

Substituting the above decomposition (16) into Eq. (15), we obtain a one-dimensional wave equation of $\Phi(t,z)$,

$$-\partial_t^2\Phi + \partial_z^2\Phi - U(z)\Phi - a^2\Phi = 0, \tag{17}$$

where

$$U(z) = \frac{3}{2}\partial_z^2 A + \frac{9}{4}(\partial_z A)^2 \tag{18}$$

is the effective potential and a is a constant coming from the separation of variables. Furthermore, separability means that the function $\Phi(t,z)$ can be decomposed as

$$\Phi(t,z) = e^{-i\omega t}\phi(z). \tag{19}$$

So we can obtain a Schrödinger-like equation of the extra dimensional part $\phi(z)$,

$$-\partial_z^2\phi(z) + U(z)\phi(z) = m^2\phi(z), \tag{20}$$

where $m^2 = \omega^2 - a^2$ is the mass of the KK modes. Equation (20) supports a bound zero mode $\phi_0(z) \propto e^{\frac{3}{2}A(z)}$ that is localized on the brane for the solution (8) with $b > 0$, and a series of massive KK modes. Usually, the massive KK modes stay on the brane for a finite time and eventually escape to infinity of the extra dimension. Thus, the thick brane is a dissipative system for the massive KK modes. Similar to QNMs in the black hole system, there are also characteristic modes with complex frequencies in the thick brane model. These modes can also reflect the properties of the thick brane model. We will discuss them in the next section.

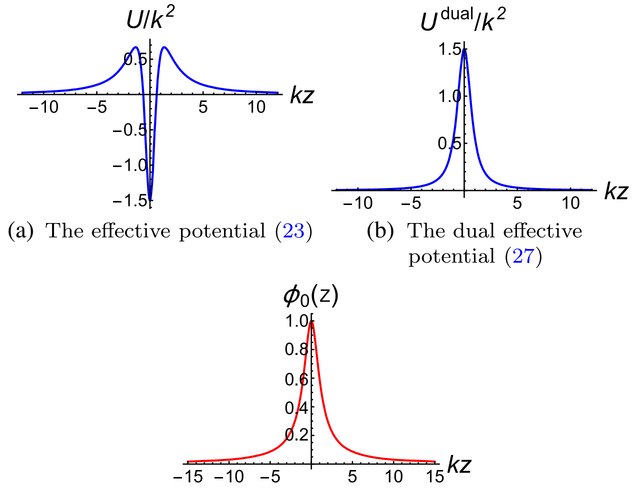
III. QUASINORMAL MODES OF THICK BRANE

In this section, we will use some semianalytical methods to solve the QNMs of the thick brane. As can be seen from the Schrödinger-like equation (20), it is the effective potential $U(z)$ that determines the QNMs. Substituting the thick brane solution (8) into the effective potential (18), we can obtain the specific form of the effective potential. Note that we only consider $b = 1$, because the coordinate transformation relation between y and z is analytical for this case. The specific forms of the warp factor $A(z)$, the effective potential $U(z)$, and the zero mode $\phi_0(z)$ are given by

$$A(z) = -\frac{1}{2}\ln(k^2z^2 + 1), \tag{21}$$

$$U(z) = \frac{3k^2(5k^2z^2 - 2)}{4(k^2z^2 + 1)^2}, \tag{22}$$

$$\phi_0(z) = \frac{1}{(1 + k^2z^2)^{3/4}}. \tag{23}$$



(a) The effective potential (23) (b) The dual effective potential (27)

(c) The zero mode (23)

FIG. 1. The shapes of the effective potential (22), the dual effective potential (27), and the zero mode (23).

We plot the effective potential and the zero mode in Fig. 1. It can be seen that the effective potential is volcanolike and $U(z) \to 0$ when $|z| \to \infty$. This potential is a smooth extension of the effective potential in the RS-II model. In the thin brane scenario, the general solution of the massive KK modes is the Hankel function. So the QNMs can be analytically obtained by imposing the outgoing boundary condition to the Hankel function [60]. But there is not any analytical solution of massive KK modes for this thick brane. So we use some semianalytical method to obtain the QNMs of the thick brane. Unlike the case of a black hole, there is a potential well but not a pure barrier for our brane case. Therefore, some methods of solving the QNMs commonly used in black holes, such as the Wentzel-Kramers-Brillouin (WKB) approximation [65], cannot solve the QNMs of this thick brane directly. But we notice that the Schrödinger-like equation (20) can be factorized as a supersymmetric form,

$$QQ^\dagger\phi(z) = m^2\phi(z), \tag{24}$$

where Q and $Q^\dagger$ are

$$Q = \partial_z + \frac{3}{2}\partial_z A, \qquad Q^\dagger = -\partial_z + \frac{3}{2}\partial_z A. \tag{25}$$

The above Eq. (24) has a corresponding Schrödinger-like equation with the supersymmetric partner potential,

$$Q^\dagger Q\tilde{\phi}(z) = (-\partial_z^2 + U^{\mathrm{dual}}(z))\tilde{\phi}(z) = m^2\tilde{\phi}(z), \tag{26}$$

where

$$U^{\mathrm{dual}}(z) = -\frac{3}{2}\partial_z^2 A + \frac{9}{4}(\partial_z A)^2 = \frac{3k^2(k^2z^2 + 2)}{4(k^2z^2 + 1)^2}. \tag{27}$$

According to the supersymmetric quantum mechanics, the Schrödinger-like equations (20) and (26) have the same spectrum of massive KK modes [66]. So the effective potential and the supersymmetric partner potential have the same spectrum of QNMs [67]. Plot of the supersymmetric partner potential (27) is shown in Fig. 1(b). We can see that the shape of the dual potential is similar to the effective potentials in the case of the Schwarzschild black hole, for which the QNMs can be solved by the asymptotic iteration method [68–70] and the WKB approximation. Therefore, we can obtain the quasinormal frequencies of the thick brane by using the dual potential (27).

A. Solve the QNMs of thick brane by using the asymptotic iteration method

First, we use the asymptotic iteration method to solve the QNMs of the thick brane. Then we compare the results with those obtained by the WKB approximation. At the beginning, we give a brief review on the idea of the asymptotic iteration method. Consider a second-order homogeneous linear differential equation for the function $y(x)$,

$$y''(x) = \lambda_0(x)y'(x) + s_0(x)y(x), \tag{28}$$

where $\lambda_0(x) \neq 0$ and $s_0(x)$ are C^∞ functions. Based on the symmetric structure of the right-hand side of Eq. (28), a general solution can be solved. Indeed, differentiating Eq. (28) with respect to x, we find that

$$y'''(x) = \lambda_1(x)y'(x) + s_1(x)y(x), \tag{29}$$

where

$$\lambda_1(x) = \lambda_0' + s_0 + \lambda_0^2, \tag{30}$$

$$s_1(x) = s_0' + s_0\lambda_0. \tag{31}$$

Iteratively, the $(n-1)$th and nth differentiations of Eq. (28) give

$$y^{(n+1)}(x) = \lambda_{n-1}(x)y'(x) + s_{n-1}(x)y(x), \tag{32}$$

$$y^{(n+2)}(x) = \lambda_n(x)y'(x) + s_n(x)y(x), \tag{33}$$

where

$$\lambda_n(x) = \lambda_{n-1}' + s_{n-1} + \lambda_0\lambda_{n-1}, \tag{34}$$

$$s_n(x) = s_{n-1}' + s_0\lambda_{n-1}. \tag{35}$$

The asymptotic aspect is introduced as follows for sufficiently large n:

$$\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)} = \beta(x). \tag{36}$$

We can obtain the QNMs from the "quantization condition,"

$$s_n(x)\lambda_{n-1}(x) - s_{n-1}(x)\lambda_n(x) = 0. \tag{37}$$

To be more precise, we adopt the improved version of the asymptotic iteration method by Cho *et al.* [70]. The original asymptotic iteration method has the "weakness" that for each iteration one must take the derivative of the $s(x)$ and $\lambda(x)$ terms of the previous iteration. This might bring difficulties for numerical calculations. Cho *et al.* reduced the asymptotic iteration method into a set of recursion relations that no longer require derivative operators. This greatly improves the speed and precision of numerical calculation. In the asymptotic iteration method, when solving Eq. (37), we should take a specific point χ. The two functions λ_n and s_n can be expanded in a Taylor series at the point χ,

$$\lambda_n(x) = \sum_{i=0}^{\infty} c_n^i (x-\chi)^i, \tag{38}$$

$$s_n(x) = \sum_{i=0}^{\infty} d_n^i (x-\chi)^i. \tag{39}$$

Here, c_n^i and d_n^i denote the i-th Taylor coefficients of λ_n and s_n, respectively. Substituting the above expressions into Eqs. (34) and (35), we can obtain a set of recursion relations,

$$c_n^i = (i+1)c_{n-1}^{i+1} + d_{n-1}^i + \sum_{k=0}^{i} c_0^k c_{n-1}^{i-k}, \tag{40}$$

$$d_n^i = (i+1)d_{n-1}^{i+1} + \sum_{k=0}^{i} d_0^k c_{n-1}^{i-k}. \tag{41}$$

Now the "quantization condition" (37) can be rewritten as

$$d_n^0 c_{n-1}^0 - d_{n-1}^0 c_n^0 = 0. \tag{42}$$

In this way, the "quantization condition" (37) reduced to a set of recursion relations which do not require derivative operators.

The Schrödinger-like equation with the dual potential is

$$-\partial_z^2\tilde{\phi}(z) + \left(\frac{3k^2(k^2z^2+2)}{4(k^2z^2+1)^2} - m^2\right)\tilde{\phi}(z) = 0. \tag{43}$$

The boundary conditions are

$$\tilde{\phi}(z) \propto \begin{cases} e^{imz}, & z \to \infty. \\ e^{-imz}, & z \to -\infty. \end{cases} \tag{44}$$

Obviously, there is no first derivative term in the above equation, which means $\lambda_0 = 0$. The asymptotic iteration method cannot be used directly in this situation. We need to transform our coordinates to obtain the equation whose first derivative term is nonvanishing. On the other hand, transforming the infinity to be finite is necessary. So we perform the transformation $u = \frac{\sqrt{4k^2z^2+1}-1}{2kz}$. Then, Eq. (43) becomes

$$\frac{(u^2-1)^3((u^4-1)\tilde{\phi}''(u) + 2u(u^2+3)\tilde{\phi}'(u))}{(u^2+1)^3} + \left(\frac{m^2}{k^2} - \frac{3(u^2-1)^2(2u^4-3u^2+2)}{4(u^4-u^2+1)^2}\right)\tilde{\phi}(u) = 0, \quad (45)$$

where $-1 < u < 1$. The boundary conditions (44) can be rewritten as

$$\tilde{\phi}(u) \propto \begin{cases} e^{-\frac{im/k}{2u-2}}, & u \to 1. \\ e^{\frac{im/k}{2u+2}}, & u \to -1. \end{cases} \quad (46)$$

Thus, $\tilde{\phi}(u)$ can be written in the form,

$$\tilde{\phi}(u) = \psi(u) e^{-\frac{im/k}{2u-2}} e^{\frac{im/k}{2u+2}}. \quad (47)$$

Now the boundary condition becomes that the function $\psi(u)$ is finite at $u \to \pm 1$. Substituting the expression (47) into Eq. (45), we have

$$\psi''(u) = \lambda_0(u)\psi'(u) + s_0(u)\psi(u), \quad (48)$$

where

$$\lambda_0(u) = -\frac{2u(u^4 + 2i(u^2+1)\frac{m}{k} + 2u^2 - 3)}{(u^2-1)^2(u^2+1)}, \quad (49)$$

$$s_0(u) = \frac{1}{4(u^2+1)(u^6-2u^4+2u^2-1)^2} \times \left[-4(u^4-u^2+1)^2(u^2+1)\frac{m^2}{k^2} + 8i(u^2-1)(u^4-u^2+1)^2\frac{m}{k} + 3(2u^4-3u^2+2)(u^2+1)^3\right]. \quad (50)$$

With λ_0 and s_0 obtained, we can solve the quasinormal frequencies of the thick brane using the reduced "quantization condition" (42). Using this method we obtain several QNMs of the thick brane. Plot of the first twenty QNMs obtained by the asymptotic iteration method is shown in Fig. 2. It can be seen that all the QNMs obtained by the asymptotic iteration method have a negative imaginary part. This means that the QNMs will dissipate.

We also compute the quasinormal frequencies through the WKB approximation [71]. In black hole physics, the WKB method was first applied to the scattering problem around black holes by Schutz and Will [72]. The method is based on matching of the asymptotic WKB solutions at the event horizon and spatial infinity with the Taylor expansion near the peak of the potential barrier through the two turning points. Since the shape of the dual potential in the thick brane is similar to the effective potential in the case of the Schwarzschild black hole, the QNMs of the thick brane can be solved by the WKB approximation. Here, we use the sixth order WKB approximation to solve the QNMs of the thick brane. The form of the sixth order WKB formula is

$$i\frac{\omega^2 - U_{\max}}{\sqrt{-2U''_{\max}}} - \sum_{j=2}^{6}\Lambda_j = n + 1/2, \quad n = 1, 2, 3..., \quad (51)$$

where $U_{\max}$ is the maximum value of the dual potential, Λ_j is the correction term of the jth order that depends on the value of the dual potential and its derivatives at the peak value. The explicit form of the correction term Λ_j can be found in Refs. [52,73,74]. We can solve the QNMs of the thick brane using the above expression. The results are listed in Table I. Since the WKB approximation is more applicable to low overtones, i.e., QNMs with a small imaginary part. When the overtone number n is moderately higher, the results of the WKB approximation become discredited [1]. Therefore, we neglect $n \geq 4$ for the results of the WKB approximation. We can see that for the first three QNMs, the results of the asymptotic iteration method are in good agreement with the results of the WKB

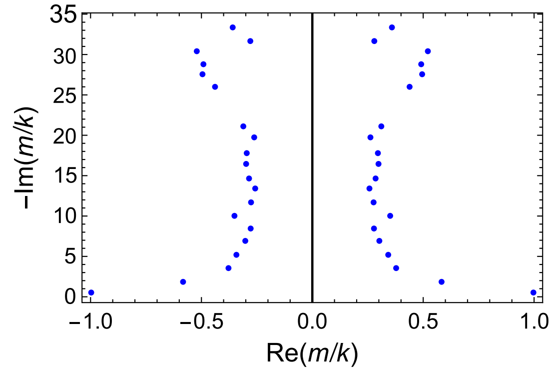


FIG. 2. The first twenty quasinormal frequencies of the thick brane solved by the asymptotic iteration method. The iteration of asymptotic iteration method is 150.

TABLE I. Low overtone modes using the asymptotic iteration method and WKB method.

	Asymptotic iteration method		WKB method	
n	Re(m/k)	Im(m/k)	Re(m/k)	Im(m/k)
1	0.997018	−0.526362	1.04357	−0.459859
2	0.581489	−1.85128	0.536087	−1.71224
3	0.306005	−3.53366	0.279715	−3.70181

approximation. This increases the credibility of our results. For higher overtone modes, we expect to explore new methods to compare with the results of the asymptotic iteration method.

B. Evolution of initial wave packet

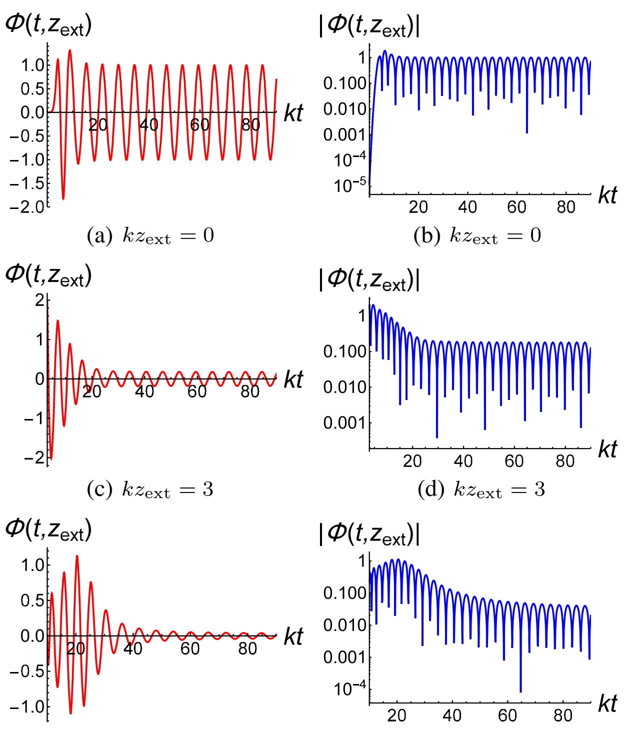
Now we consider the numeric evolution of an initial wave packet against the thick brane. We use the $u-v$ coordinate, where $u = t - z$ and $v = t + z$, to perform the evolution of Eq. (17). Then Eq. (17) can be written as

$$\left(4\frac{\partial^2}{\partial u \partial v} + U + a^2\right)\Phi = 0. \tag{52}$$

The incident wave packet is assumed to be a Gaussian pulse,

$$\Phi(0, v) = e^{\frac{-(v-v_c)^2}{2\sigma^2}}, \qquad \Phi(u, 0) = e^{\frac{-v_c^2}{2\sigma^2}}. \tag{53}$$

Here, we focus on the Gaussian pulse with $kv_c = 5$ and $k\sigma = 1$. The parameter a is set to $a/k = 1$. u and v belong to $(0, 90/k)$. The evolution of the Gauss pulse is shown in Fig. 3. In the early time, the waveform is affected by the initial data. Then the waveform evolves into a plane wave. The frequency and the maximum amplitude of the plane wave do not vary with time. From Figs. 3(a), 3(c), and 3(e),



(a) $kz_{\rm ext} = 0$ (b) $kz_{\rm ext} = 0$ (c) $kz_{\rm ext} = 3$ (d) $kz_{\rm ext} = 3$ (e) $kz_{\rm ext} = 10$ (f) $kz_{\rm ext} = 10$

FIG. 3. Left panel: Time evolution of the Gauss pulse at different locations. The signals are extracted at the points $kz_{\rm ext} = 0$, 3, 10. Right panel: Same as left panel but in a logarithmic scale.

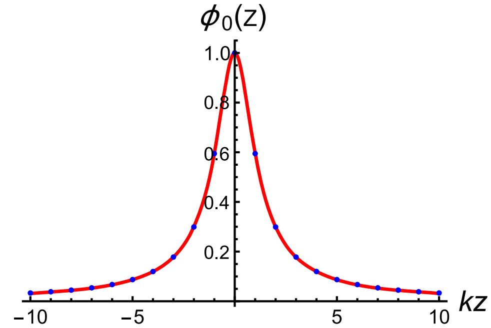


FIG. 4. Comparing the results of the zero mode (the blue dots) exited by the Gauss pulse with the analytical zero mode (23) (the red curve) obtained from the Schrödinger-like equation (20) or equivalently the linear perturbation equation (15).

we can see that the frequencies of the plane waves do not depend on the extracting points. But the maximum amplitudes of the plane waves depend on the extracting points. Observing the maximum amplitude at each extracting point for the same Gauss pulse, we can see that the final maximum amplitude decreases with $kz_{\rm ext}$. That is to say, the further away from the brane, the smaller the amplitude. We compare the maximum amplitudes extracted from different points with the profile of the zero mode (23). The result is shown in Fig. 4, which shows that the maximum amplitude as a function of kz is consistent with the analytical zero mode (23). Thus, after the pulse hits the brane, the incident pulse excites the zero mode localized on the brane. According to the expression (19), we can obtain the function of the plane wave: $\Phi_0(t, z) = e^{-i\omega t}\phi_0(z)$. In addition, from the relation $\omega^2 = m^2 + a^2$, we know that the frequency becomes $\omega = a$ for the zero mode with $m = 0$.

On the other hand, because the potential is symmetric, the wave functions are either even or odd. Specially, the bound zero mode is even. To investigate the character of the odd QNMs, we give an odd initial wave packet,

$$\Phi(0, v) = \sin\left(\frac{kv}{2}\right)e^{\frac{-k^2v^2}{4}}, \tag{54}$$

$$\Phi(u, 0) = \sin\left(\frac{ku}{2}\right)e^{\frac{-k^2u^2}{4}}. \tag{55}$$

Plots of the evolution of the waveform are shown in Fig. 5. To study the effect of the parameter a, we choose $a/k = 0$ and $a/k = 1$. Obviously, there are two stages through the evolution for the case of $a/k = 0$. a) The exponentially decay stage. The frequency and damping time of these oscillations in this stage depend only on the characteristic structure of the thick brane. They are completely independent of the particular initial configuration that causes the excitation of such vibrations. b) The power-law damping stage. This situation is similar to the case of a massless field around a Schwarzschild black hole. Because the first

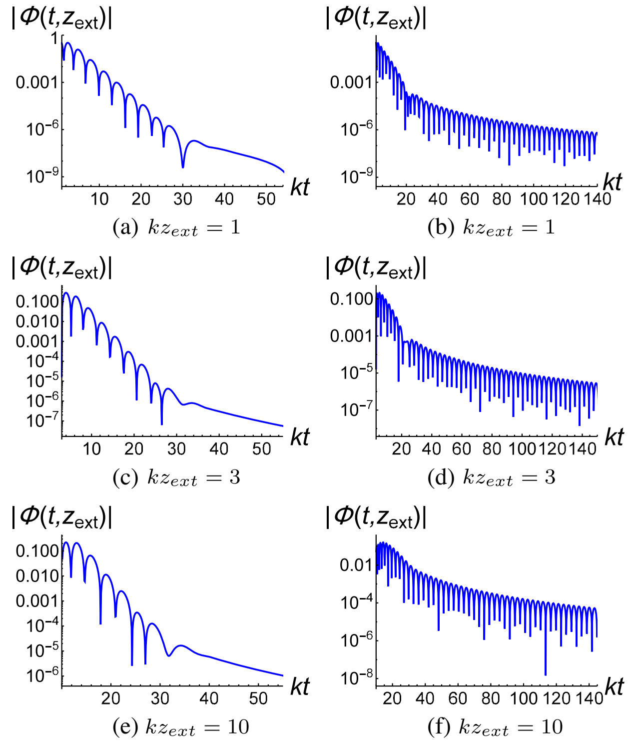


FIG. 5. Left panel: Time evolution of the odd wave packet at different selected extraction points for $a/k = 0$. Right panel: Time evolution of the odd wave packet at different selected extraction points for $a/k = 1$.

QNM dominates the evolution process, we can obtain the frequency of the first QNM by fitting the evolution data. For the case of Fig. 5(a), the frequency is $\omega/k = 1.01079 - 0.501256i$. This result is good agree with the result of the asymptotic iteration method. For the case of the $a/k = 1$, we can see that the quasinormal ringing governs the decay of the perturbation all the time. This is similar to the case of a massive field around a Schwarzschild black hole. It seems that the QNMs in the thick brane model has both two tail characteristics, which is an interesting property. We will investigate the tails of the QNMs for more braneworld models in detail in the future. The above results indicate that there is a normal mode called the zero mode and a series of discrete QNMs in this thick brane model. These modes are the characteristic modes of the brane. The detection of these QNMs can reflect the structure of the brane. From this perspective, these modes are the fingerprints of the brane. This provides a new way for the investigation of the gravitational perturbation in thick brane models.

To more intuitively understand the character of these modes, following the method of Ref. [61], we consider a wave packet on the brane,

$$\delta h_{\mu\nu} \sim \epsilon_{\mu\nu} \int da \left(\alpha(a) \sum_n c_n \exp[i(\omega_n t - ax)] \right). \quad (56)$$

Here, we consider a motion in the x-direction, where $\alpha(a)$ denotes the amplitude of each modes, c_n is the expansion coefficient determined by the initial extra dimensional pulse profile, and n runs over the zero mode and QNMs. Obviously, the zero mode acts like it is traveling in a vacuum with the speed of light since $\omega_0 = a$. Besides the zero mode, since ω_n has a negative imaginary part, the behavior of each massive mode is that it is propagating in an absorptive medium with a speed slower than light. If the amplitude $\alpha(a)$ is peaked sharply around some value $a = a_0$, then the frequency $\omega_n(a)$ can be expanded at that value of a. So, we can define the lifetime τ_n and the group velocity [75],

$$\tau_n = \frac{1}{\mathrm{Im}\omega_n}, \qquad v_n = \mathrm{Re}(\partial_a \omega_n) = \mathrm{Re}\left(\frac{a}{\omega_n}\right). \quad (57)$$

Then we can obtain

$$d_n = v_n \tau_n = \frac{a\mathrm{Re}\omega_n}{\mathrm{Im}\omega_n((\mathrm{Re}\omega_n)^2 + (\mathrm{Im}\omega_n)^2)}, \quad (58)$$

which is the distance that a massive mode propagates on the brane before its amplitude decreases by a factor of e. Since $\omega_n = \sqrt{a^2 + m_n^2}$, so the real part of ω_n increases with a, while the imaginary part $\mathrm{Im}(\omega_n)$ decreases with a. It can also be seen from Fig. 6. This distance is very short for the QNMs with a smaller a. For example, when $k = a = 10^{-3}$ eV, the distance d_n of the first QNM is about 0.2 mm. If the distance is of the Galactic scale, i.e., 10^{21} m, the frequency of the first QNM is of order 10^{39} Hz for $k = 10^{-3}$ eV. Obviously, it is impossible to find these massive modes from laser interferometer gravitational wave detectors currently in use or under construction [76]. These results are consistent with thin brane [61]. Furthermore, Ref. [61] pointed out that, these QNMs might play an important role in the early Universe. We expect that the stochastic gravitational wave background could carry potentially information of massive KK modes. In addition, other thick brane models might support long-lived QNMs. In the future, we will investigate the properties of these long-lived modes.

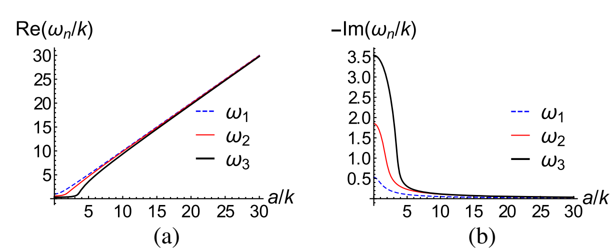


FIG. 6. Left panel: The relations of the real parts of the first three frequencies ω_n and the parameter a. Right panel: The relations of the imaginary parts of the first three frequencies ω_n and the parameter a.

IV. CONCLUSION AND DISCUSSION

In this paper, we investigated the QNMs of the thick brane model by the semianalytical and numerical methods. The results obtained by these methods are in good agreement with each other. It shows that there is a zero mode (normal mode) and a series of discrete QNMs in the thick brane model. This is consistent with the results of the RS-II brane [61]. As the characteristic modes of the thick brane, these QNMs play an indispensable role on understanding the structure of the thick brane. This is a new way for the investigation of the gravitational perturbation in thick brane models. It also may provide new ideas for studying thick brane models.

Starting from the solution and the linear metric tensor perturbation given in Ref. [21], we obtained the wave equation (17) and the Schrödinger-like equation (20). Since the Schrödinger-like equation can be factorized as a supersymmetric form, we can obtain the supersymmetric partner potential which provides the same spectrum of QNMs of the brane. The supersymmetric partner potential is similar to the effective potentials in the case of the Schwarzschild black hole. Some semianalytical methods can be used to solve the QNMs. In this way, the QNMs of the thick brane were obtained indirectly. We used the asymptotic iteration method and the WKB approximation to solve the QNMs. The results of the two methods agree with each other in the low overtone region, which can be seen from Table I. To further confirm the above results, we studied the numerical evolution of the wave equation (17). The results show that a zero mode is excited by the incident Gaussian pulse. And the evolution of the odd wave packet reveals the property of the QNMs, which can be seen from Fig. 5. In addition, the frequency extracted from the data is consistent with the frequency of the first QNM obtained using the asymptotic iteration method and the WKB approximation. This enhances the credibility of our results. Finally, we investigated the propagation distance d_n of the massive mode on the brane. We found that, for the same m_n, the distance d_n increases with the parameter a. If the propagation distance is of the Galactic scale, the frequency of the massive mode is extremely high, far beyond the ability of the current detectors. However, the massive mode might play a key role in the early Universe. It might be detected as a stochastic gravitational wave background.

Our work could be strengthened in a number of ways. First,we need to develop more methods to calculate higher overtone modes and compare with the asymptotic iteration method. Second, some thick brane models might support long-lived QNMs, which deserve further study. Third, the QNMs of other test fields could be investigated in the future.

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