

Multihorizons black hole solutions, photon sphere, and perihelion shift in weak ghost-free Gauss-Bonnet theory of gravity

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Among the modified gravitational theories, the ghost-free Gauss-Bonnet (GFGB) theory of gravity has been considered from the viewpoint of cosmology. The best way to check its applicability could be to elicit observable predicts which give guidelines or limitations on the theory, which could be contrasted with the actual observations. In the present study, we derive consistent field equations for GFGB and by applying the equations to a spherically symmetric space-time, we obtain new spherically symmetric black hole (BH) solutions. We study the physical properties of these BH solutions and show that the obtained space-time possesses multihorizons and the Gauss-Bonnet invariants in the space-time are not trivial. We also investigate the thermodynamical quantities related to these BH solutions and we show that these quantities are consistent with what is known in the previous works. Finally, we study the geodesic equations of these solutions which give the photon spheres and we find the perihelion shift for weak GFGB. In addition, we calculate the first-order GFGB perturbations in the Schwarzschild solution and new BH solutions and show that we improve and extend existing results in the past literature on the spherically symmetric solutions.

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I. INTRODUCTION

Although more than 100 years have passed after the construction of Einstein's theory of general relativity (GR), GR is still the most established macroscopic theory of gravity that is widely accepted. In spite of its vast success in both weak and strong couplings [1,2], however, there is still no harmonic way to link the macroscopic theory of GR to a quantum field theory. Moreover, GR predicts space-time singularity which has mathematical results in its construction. The problem of singularity leads scientists to search for other theories of gravity that could coincide with GR in the scale of daily life and/or the scale of the solar system. It is interesting to note that Lovelock's theory [3] has explained that in four dimensions, Einstein's GR is the unique metric theory of gravity that could yield symmetric, covariant second-order field equations. Therefore, one of the attempts to amend Einstein's GR is to work in space-times with extra dimensions [4]. In these attempts, the most general set of theories could be the Lovelock theories which yield symmetric, covariant second-order field equations regarding the metric tensor in any space-time

dimensions [5]. The Lagrangian of the Lovelock theory is given as follows:

$$\mathcal{L} = \sqrt{-g}(-2\Lambda + R + \alpha\mathcal{G} + \dots), \quad (1)$$

with $\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ being the Gauss-Bonnet (GB) invariant which yields the first-order correction to the action of Einstein's theory with a cosmological constant Λ . Although the GB invariant yields nontrivial effects when the space-time dimensions are larger than four, the invariant is topological in four dimensions [6]. Regardless of being quadratic in curvatures, the GB invariant has theoretical wide advantages from the viewpoints of string theory [7–13].

Many researchers have been tempted by the idea of harmony merging the effect of the GB invariant in a four-dimensional theory of gravity, which could yield equations of motions different from GR, avoiding Lovelock's theorem. Glavan and Lin [14] have investigated the idea to rescale the GB coupling constant γ in N dimensions as $\gamma \rightarrow \gamma/(N-4)$, so that there remains the contribution from the GB invariant in the limit $N \rightarrow 4$. After that, there have appeared works, where spherical black hole solutions [15–18], the construction of cosmological solutions [19,20], the radiation of black holes and the collapse to the

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black hole [21–23], starlike objects [24], the extension to more higher-curvature Lovelock theories [25], the thermodynamical behavior of black hole solutions [26–29], and the physical properties of such objects [25,30–41] have been investigated. In spite of all of these researches, the regularization method used in the four-dimensional Einstein-GB theory [14] has been shown to be inconsistent for many reasons [42–52], which yield to the construction of different models of the regularized (harmonic) four-dimensional Einstein GB theories [20,53–57].

Due to several reasons, some researchers are suspicious about the procedure proposed in [14]. One reason is that the field equations of the Einstein-GB theory defined in higher dimensions can be divided into two various sets. One set yields the field equations which always come from higher dimensional theories and this set makes the specific action in the limit of $N \rightarrow 4$ nontrivial [42–45,49,58]. The tree-level graviton scattering amplitude was also investigated in this frame, apart from the Lagrangian, and it turned out that the dimensional continuation, $N \rightarrow 4$, does not make the GB amplitude create any new four-dimensional GB gravitational amplitude [59]. All of these attempts yield the fact that the existence of the solution in the limit of $N \rightarrow 4$ does not mean that there is a four-dimensional theory as proposed in [14]. In spite of this situation, it could be important to mention that the field equations different from the four-dimensional Einstein GB gravity [20,53–56] support the same static spherically symmetric BH solution as constructed in [14]. Following the $N \rightarrow 4$ regularization of the scalar and vector type gravitational perturbation of the $N > 4$ Einstein-GB BH [60,61], it has been investigated that the asymptotically flat or AdS/dS BHs are unstable for large positive values of the GB coupling parameter [62,63]. The quasinormal modes of the four-dimensional Einstein GB BH in the asymptotically AdS/dS space-time due to scalar, electromagnetic, and Dirac perturbations have been investigated in [40,41]. The quasibound states of massless scalar, electromagnetic, and Dirac fields in the asymptotically flat four-dimensional Einstein GB BH and the associated stability issue have been studied in [64].

Because of the significance of the theories involving the GB scalar, which are encouraged by string theories in many cases, in this study, we shall briefly discuss the drawback of these theories, specifically the existence of ghosts. Generally, higher-derivative gravitational theories involve ghost degrees of freedom due to the Ostrogradsky's instability [65]. It was explained in [66], that ghost degrees of freedom could happen at different levels of the theory, despite the cosmological perturbations level of $f(R, \mathcal{G})$ theories. It is the aim of the present study to derive the spherically symmetric BH solutions in the ghost-free $f(\mathcal{G})$ gravitational theory proposed in [67,68].

This paper is organized as follows: In Sec. II, we present the basic tools for the ghost-free $f(\mathcal{G})$ gravitational theory that is capable to describe the formulation of BH horizons.

In Sec. III, we apply the field equations of ghost-free Gauss-Bonnet (GFGB) to the spherically symmetric space-time and derive BH solutions with multihorizons. In Sec. IV, we study the relevant physics of the BH solutions derived in Sec. III by showing their asymptote at $r \rightarrow \infty$. Moreover, we show that by studying the thermodynamical behavior of these BH solutions by calculating their thermodynamical quantities like Hawking temperature, heat capacity, and the Gibbs's free energy, we show that all these quantities related to the BHs derived in Sec. II are consistent with the results presented in the past literature. In Sec. V, we study the particle motion phenomenology for these BHs and derived their potential for the Schwarzschild background. Moreover, we derive the deviation from Einstein's general relativity of the photon sphere and the perihelion shift. We close our study with the conclusion of the main results in Sec. VI.

Throughout the present study, we assume the relativistic units, i.e., $G = c = 1$.

II. BRIEF SUMMARY OF GHOST-FREE $f(\mathcal{G})$ GRAVITATIONAL THEORY

In the present section, we will present briefly the ghost-free $f(\mathcal{G})$ gravity in the formulation using the Lagrange multipliers. Moreover, we shall investigate how to obtain a ghost-free $f(\mathcal{G})$ gravity, and we shall employ the Lagrange multipliers formalism in order to achieve this. Before going to the details of the formalism, we will start the derivation by showing in detail how ghost modes could exist in $f(\mathcal{G})$ gravity at the field equations level, and then construct the ghost-free model construction of the theory.

A. Ghosts in $f(\mathcal{G})$ gravity

Nojiri *et al.* [67,68] have constructed a ghost-free $f(\mathcal{G})$ gravity theory by using the Lagrange multiplier field. The original $f(\mathcal{G})$, whose action has the following form,

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} R + f(\mathcal{G}) + \mathcal{L}_{\text{matter}} \right), \quad (2)$$

has ghost as we show below. Here $\mathcal{L}_{\text{matter}}$ is the Lagrangian density of the matters. The above action (2) can be rewritten as follows,

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} R + h(\chi)\mathcal{G} - V(\chi) + \mathcal{L}_{\text{matter}} \right), \quad (3)$$

where R is the Ricci scalar, χ is an auxiliary field, \mathcal{G} is the GB invariant, $V(\chi)$ is the potential, and $h(\chi)$ is a function of the auxiliary field. The variation of the action (3) with respect to χ gives

$$0 = h'(\chi)\mathcal{G} - V'(\chi). \quad (4)$$

Equation (4) can be solved with respect to χ as a function of the GB invariant \mathcal{G} , $\chi = \chi(\mathcal{G})$. Then by substituting the obtained expression of $\chi(\mathcal{G})$ into Eq. (4), one can reobtain the action of Eq. (2) where $f(\mathcal{G})$ is defined as

$$f(\mathcal{G}) = h(\chi(\mathcal{G}))\mathcal{G} - V(\chi(\mathcal{G})). \quad (5)$$

Furthermore, the varying of the action (4) with respect to the metric tensor yields

$$0 = \frac{1}{2\kappa^2} \left(-R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R \right) + \frac{1}{2}T_{\text{matter}\mu\nu} - \frac{1}{2}g_{\mu\nu}V(\chi) + D_{\mu\nu}^{\tau\eta}\nabla_\tau\nabla_\eta h(\chi), \quad (6)$$

where the tensor $D_{\mu\nu}^{\tau\eta}$ is defined as

$$D_{\mu\nu}^{\tau\eta} \equiv (\delta_\mu^\tau\delta_\nu^\eta + \delta_\nu^\tau\delta_\mu^\eta - 2g_{\mu\nu}g^{\tau\eta})R + (-4g^{\rho\tau}\delta_\mu^\eta\delta_\nu^\sigma - 4g^{\rho\tau}\delta_\nu^\eta\delta_\mu^\sigma + 4g_{\mu\nu}g^{\rho\tau}g^{\sigma\eta})R_{\rho\sigma} + 4R_{\mu\nu}g^{\tau\eta} - 2R_{\rho\mu\sigma\nu}(g^{\rho\tau}g^{\sigma\eta} + g^{\rho\eta}g^{\sigma\tau}). \quad (7)$$

Since the auxiliary field χ can be rewritten as a function of the GB \mathcal{G} , then Eq. (6) is a fourth-order differential equation for the metric which may contain ghost modes.

In order to eliminate the ghost modes, we may add a canonical kinetic term of χ in the action (3)

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2}R + h(\chi)\mathcal{G} - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V(\chi) + \mathcal{L}_{\text{matter}} \right), \quad (8)$$

where we have chosen the mass dimension of χ to be unity. Then variation of the action (8) with respect to χ and metric gives [67–69]

$$0 = \square\chi + h'(\chi)\mathcal{G} - V'(\chi), \quad (9)$$

$$0 = \frac{1}{2\kappa^2} \left(-R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R \right) + \frac{1}{2}T_{\text{matter}\mu\nu} + \frac{1}{2}\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}g_{\mu\nu} \left(\frac{1}{2}\partial_\rho\chi\partial^\rho\chi + V(\chi) \right) + D_{\mu\nu}^{\tau\eta}\nabla_\tau\nabla_\eta h(\chi). \quad (10)$$

The equations derived in Eq. (9) do not have higher order except the second-order derivatives which mean that we could not have ghosts.

The model (8) has a new dynamical degree of freedom, i.e., χ ; however, if we like to minimize the dynamical degrees of freedom, we can insert a constraint as in the mimetic theory [70–72], by using the Lagrange multiplier field λ , as follows,

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2}R + \lambda \left(\frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{\mu^4}{2} \right) - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + h(\chi)\mathcal{G} - V(\chi) + \mathcal{L}_{\text{matter}} \right), \quad (11)$$

where μ is a constant which has a mass dimension. Thus, by varying action (11) with respect to λ , we obtain

$$0 = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{\mu^4}{2}. \quad (12)$$

Because of the fact that the kinetic term becomes a constant, the kinetic term in Eq. (11) can be absorbed by using the redefinition of potential $V(\chi)$,

$$\tilde{V}(\chi) \equiv \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + V(\chi) = -\frac{\mu^4}{2} + V(\chi). \quad (13)$$

Now, the action of Eq. (11) can be rewritten as follows:

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2}R + \lambda \left(\frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{\mu^4}{2} \right) + h(\chi)\mathcal{G} - \tilde{V}(\chi) + \mathcal{L}_{\text{matter}} \right). \quad (14)$$

The action given in Eq. (14) yields, in addition to Eq. (12), the following two equations of motion:

$$0 = -\frac{1}{\sqrt{-g}}\partial_\mu(\lambda\omega(\chi)g^{\mu\nu}\sqrt{-g}\partial_\nu\chi) + h'(\chi)\mathcal{G} - \tilde{V}'(\chi) + \frac{1}{2}\lambda\omega'(\chi)g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi, \quad (15)$$

$$0 = \frac{1}{2\kappa^2} \left(-R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R \right) + \frac{1}{2}T_{\text{matter}\mu\nu} - \frac{1}{2}\lambda\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}g_{\mu\nu}\tilde{V}(\chi) + D_{\mu\nu}^{\tau\eta}\nabla_\tau\nabla_\eta h(\chi). \quad (16)$$

We should note that the absence of the ghost in the model (16) has been established in [67,68].

It has been shown that the constraint (12), which is related to the mimetic condition, is not consistent with the formation of BH horizons [73]. Therefore, we need to introduce a function ω in the term of the mimetic constraint so that the resulting field equations can describe the construction of the BH horizon [73]. Applying this philosophy, we rewrite the action (14) in the following form:

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} R + \lambda \left(\frac{1}{2} \omega(\chi) \partial_\mu \chi \partial^\mu \chi + \frac{\mu^4}{2} \right) + h(\chi) \mathcal{G} - \tilde{V}(\chi) + \mathcal{L}_{\text{matter}} \right). \quad (17)$$

Variations of the action (17) with respect to the Lagrange multiplier λ , the auxiliary field χ , and the metric give

$$0 = \frac{1}{2} \omega(\chi) \partial_\mu \chi \partial^\mu \chi + \frac{\mu^4}{2}, \quad (18)$$

$$0 = -\frac{1}{\sqrt{-g}} \partial_\mu (\lambda \omega(\chi) g^{\mu\nu} \sqrt{-g} \partial_\nu \chi) + h'(\chi) \mathcal{G} - \tilde{V}'(\chi) + \frac{1}{2} \lambda \omega'(\chi) g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi, \quad (19)$$

$$0 = \frac{1}{2\kappa^2} \left(-R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R \right) + \frac{1}{2} T_{\text{matter}\mu\nu} - \frac{1}{2} \lambda \omega(\chi) \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} g_{\mu\nu} \tilde{V}(\chi) + D_{\mu\nu}{}^{\tau\eta} \nabla_\tau \nabla_\eta h(\chi). \quad (20)$$

In the following, we forget the matter energy-momentum tensor because we are interested in vacuum solution. We are going to apply the field equations (18), (19), and (20) to a spherically symmetric space-time.

III. SPHERICALLY SYMMETRIC BH SOLUTIONS IN GHOST-FREE $f(\mathcal{G})$ GRAVITY

In this section, we will study the spherically symmetric space-time created by solving Eqs. (18), (19), and (20) given by the ghost-free $f(\mathcal{G})$ gravitational theory defined by (17). Specifically, we investigate if it is possible to derive spherically symmetric BH solutions.

A. Schwarzschild-type black hole solutions

Now, we investigate how the field equations for the theory (17) behave in the case of the spherically symmetric metric with the following line element:

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (21)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

For this metric, we have,

$$\begin{aligned} \Gamma_{tt}^r &= f^2 \Gamma_{tr}^t = -f^2 \Gamma_{rr}^r = \frac{1}{2} f f', & \Gamma_{r\theta}^\theta &= \Gamma_{r\phi}^\phi = \frac{1}{r}, \\ \Gamma_{\theta\theta}^r &= \frac{\Gamma_{\phi\phi}^r}{\sin^2 \theta} = -f r, & \Gamma_{\theta\phi}^\phi &= -\frac{\Gamma_{\phi\phi}^\theta}{\sin^2 \theta} = \frac{\cos \theta}{\sin \theta}, \\ \mathcal{G} &= \frac{4(f'^2 + f f'' - f'')}{r^2}, \end{aligned} \quad (22)$$

where $f' \equiv f'(r) \equiv \frac{df(r)}{dr}$. Moreover, we assume that λ , ω , and χ only depend on the radial coordinate r , i.e., $\lambda = \lambda(r)$, $\omega = \omega(r)$, and $\chi = \chi(r)$.

Actually, the (t, t) component, (r, r) component, and $(\theta, \theta) = (\phi, \phi)$ components of the field (20) give

$$0 = \frac{4h'f' - 8f^2h'' + r^2V + rf' + 8fh'' - 1 - 12ff'h' + f}{r^2}, \quad (23)$$

$$0 = \frac{f + 4f'h' + r^2f\lambda\omega\chi'^2 + r^2V + rf' - 1 - 12ff'h'}{r^2}, \quad (24)$$

$$0 = \frac{2rV + 2f' + rf'' - 8h'f'^2 - 8h'ff'' - 8ff'h''}{2r}. \quad (25)$$

On the other hand, Eqs. (18) and (19) yield

$$0 = \mu^4 + \omega f \chi'^2, \quad (26)$$

$$0 = \frac{8h'f''(f-1) - 2r^2f\lambda\omega\chi'\chi'' - r\chi'^2[2r\lambda\omega f' + f\{2r\omega\lambda' + \lambda(4\omega + r\omega')\}] + 8f'^2h' - 2rV'}{2r^2\chi'}. \quad (27)$$

Equations (23)–(27) are five nonlinear differential equations in six unknown functions f , h , V , λ , ω , and χ ; therefore, we are going to fix some of these unknown functions to derive the other ones. First, we solve Eq. (26) and obtain

$$\chi = c_0 r \Rightarrow \omega = -\frac{\mu^4}{c_0^2 f}. \quad (28)$$

Substituting Eq. (28) into Eqs. (23)–(27), we obtain

$$\begin{aligned}
f &= 1 - \frac{2M}{r} + \frac{c_1}{r^2} + \frac{c_2}{r^6}, \\
V &= \frac{1}{(3Mr^{20} - 2r^{19}c_1 - 4r^{15}c_2)} \left\{ 2\Upsilon(r)\Upsilon_1(r) \int \frac{r^6(c_1r^4 + 10c_2)}{\Upsilon_1(r)(2Mr^5 - r^6 - c_1r^4 - c_2)(3Mr^5 - 2c_1r^4 - 4c_2)} dr \right. \\
&\quad \left. - 8c_3\Upsilon(r)\Upsilon_1(r) - r^{11}(r^5Mc_1 + 25rMc_2 - 8c_1c_2) \right\}, \\
h &= c_4 + \int \Upsilon_1(r) \left(\int \frac{r^6(c_1r^4 + 10c_2)}{4\Upsilon_1(r)(r^6 + c_1r^4 + c_2 - 2Mr^5)(3Mr^5 - 2c_1r^4 - 4c_2)} dr - 4c_3 \right) dr, \\
\lambda &= \frac{2(2Mr^5 - c_2 - c_1r^4)}{r^{15}\mu^4(3Mr^5 - 2c_1r^4 - 4c_2)} \left\{ 4\Upsilon_1(r)\Upsilon_2(r) \int \frac{r^6(c_1r^4 + 10c_2)}{\Upsilon_1(r)(2Mr^5 - r^6 - c_1r^4 - c_2)(3Mr^5 - 2c_1r^4 - 4c_2)} dr \right. \\
&\quad \left. - c_3\Upsilon_1(r)\Upsilon_2(r) - (c_1r^4 + 10c_2)r^7 \right\}, \tag{29}
\end{aligned}$$

where

$$\begin{aligned}
\Upsilon(r) &= (2M^2r^{16} - 2M(2c_1 + 3M^2)r^{15} + (15M^2c_1 + c_1^2)r^{14} - 10Mr^{13}c_1^2 + 2c_1^3r^{12} + (79M^2c_2 + 12c_1c_2)r^{10} \\
&\quad - 32Mr^{11}c_2 - 80Mr^9c_1c_2 + 20c_1^2r^8c_2 + 3c_2^2r^6 - 62Mr^5c_2^2 + 30c_1r^4c_2^2 + 12c_2^3), \\
\Upsilon_1(r) &= e^{\int \frac{(12M^2+5c_1)r^{10}-4Mr^{11}-21Mr^9c_1+8c_1^2r^8+27c_2r^6-77Mr^5c_2+48c_1r^4c_2+48c_2^2}{r(2Mr^5-r^6-c_1r^4-c_2)(3Mr^5-2c_1r^4-4c_2)} dr}, \\
\Upsilon_2(r) &= 4Mr^{11} - (12M^2 + 5c_1)r^{10} + 21Mr^9c_1 - 8c_1^2r^8 - 27c_2r^6 + 77Mr^5c_2 - 48c_1r^4c_2 - 48c_2^2. \tag{30}
\end{aligned}$$

The curvature invariants associated with solution (28) take the following form:

$$\begin{aligned}
K &= R_{\alpha\beta\gamma\rho}R^{\alpha\beta\gamma\rho} = \frac{48M^2}{r^6} - \frac{96Mc_1}{r^7} + \frac{56c_1^2}{r^8} - \frac{488Mc_2}{r^{11}} + \frac{608c_1c_2}{r^{12}} + \frac{1912c_2^2}{r^{16}}, \\
R_{\alpha\beta}R^{\alpha\beta} &= \frac{4c_1^2}{r^8} + \frac{80c_1c_2}{r^{12}} + \frac{500c_2^2}{r^{16}}, \\
R &= -\frac{20}{r^8}, \tag{31}
\end{aligned}$$

$$\mathcal{G} = \frac{8(6M^2r^{10} - 12Mc_1r^9 - 56Mc_2r^5 + 5c_1^2r^8 + 36c_1c_2r^4 + 39c_2^2)}{r^{16}}. \tag{32}$$

Equation (31) shows that the BH solution given by Eq. (28) has a hard singularity when $r \rightarrow 0$ compared with the Schwarzschild solution of GR [74] where the Kreschmann scalar K behaves as $K \sim r^{-6}$.

B. More general black hole

Now, let us investigate how the field equations in the theory (17) behave in the case of a spherically symmetric metric with the following line element:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f_1(r)} - r^2d\Omega^2$$

$$\text{where } d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{33}$$

For this metric, we have

$$\begin{aligned}
\Gamma_{tt}^r &= ff_1\Gamma_{tr}^t = -f_1^2\Gamma_{rr}^r = \frac{1}{2}f_1f', \quad \Gamma_{r\theta}^\theta = \Gamma_{r\phi}^\phi = \frac{1}{r}, \\
\Gamma_{\theta\theta}^r &= \frac{\Gamma_{\phi\phi}^r}{\sin^2\theta} = -f_1r, \quad \Gamma_{\theta\phi}^\phi = -\frac{\Gamma_{\phi\phi}^\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin\theta}, \\
\mathcal{G} &= \frac{2(f_1[1-f_1]f'^2 + f[3f_1-1]f'f'_1 - 2ff_1f''[1-f_1])}{r^2f^2}, \tag{34}
\end{aligned}$$

where $f'_1 \equiv f'_1(r) \equiv \frac{df_1(r)}{dr}$. We assume that λ , ω , and χ only depend on the radial coordinate r , i.e., $\lambda = \lambda(r)$, $\omega = \omega(r)$, and $\chi = \chi(r)$, again.

The (t, t) component, (r, r) component, and $(\theta, \theta) = (\phi, \phi)$ components of the field equation (20) have the following forms:

$$0 = \frac{f_1 - 1 + rf_1' + 8f_1 h'' - 8f_1^2 h'' + 4f_1' h' + r^2 V - 12f_1 f_1' h'}{r^2}, \quad (35)$$

$$0 = \frac{ff_1 + 4f_1 f_1' h' + r^2 f f_1 \lambda \omega \chi'^2 + r^2 f V + rf_1 f_1' - f - 12f_1^2 f_1' h'}{fr^2}, \quad (36)$$

$$0 = \frac{2f^2 f_1' + 4rf^2 V + 2ff_1 f_1' - 24ff_1 h' f_1' f_1' - rf_1 f_1'^2 + 2rf f_1 f_1'' + rf f_1' f_1' + 8f_1^2 f_1'^2 h' - 16f_1^2 h' f_1'' - 16ff_1^2 f_1' h''}{4rf^2}. \quad (37)$$

On the other hand, Eqs. (18) and (19) yield

$$0 = \mu^4 + \omega f_1 \chi'^2, \quad (38)$$

$$0 = \frac{1}{2r^2 f^2 \chi'} \{8ff_1 h' f_1'' (f_1 - 1) - 2r^2 f^2 f_1 \lambda \omega \chi' \chi'' - rf \chi'^2 [f_1 r \lambda \omega f_1' + f \{r \lambda \omega f_1' + f_1 [2r \omega \lambda' + \lambda (4\omega + r \omega')]\}] + 4f' h' [f_1' f_1 (1 - f_1) + f f_1' (4f_1 - 1)] - 2r^2 f^2 V'\}. \quad (39)$$

Equations (35)–(39) are five nonlinear differential equations in seven unknown functions f , f_1 , h , V , λ , ω , and χ ; therefore, we are going to fix some of these unknown functions to derive the other ones. By using Eq. (38), we obtain

$$\chi = c_0 r \Rightarrow \omega = -\frac{\mu^4}{c_0^2 f_1}. \quad (40)$$

By substituting Eq. (40) into Eqs. (35)–(39), we obtain

$$\begin{aligned} f &= 1 + \frac{\alpha r^2}{\beta + r^3}, & f_1 &= 1 + \frac{\alpha}{r} + \frac{\beta}{r^3}, \\ V &= \frac{1}{4r^7 (r^3 + \beta)(3\alpha r^5 + 2r^3 \beta + 2\beta^2)} \left\{ \beta \alpha \Upsilon_3(r) \Upsilon_4(r) \int \frac{r^3 (2\beta^2 - 27\alpha r^5 + 4r^3 \beta + 2r^6)}{\Upsilon_3(r)(r^3 + \beta)(3\alpha r^5 + 2r^3 \beta + 2\beta^2)(r^3 + \beta + \alpha r^2)} dr \right. \\ &\quad \left. - 16c_5 \Upsilon_4(r) \alpha \Upsilon_3(r) - 4r^4 \beta^3 \alpha + 70r^7 \beta^2 \alpha + 20r^{10} \beta \alpha + 12r^2 \beta^4 + 24r^5 \beta^3 + 12r^8 \beta^2 + 54\beta r^9 \alpha^2 \right\}, \\ h &= c_6 + \int \left(\beta \int \frac{r^3 (27\alpha r^5 - 2\beta^2 - 4r^3 \beta - 2r^6)}{16\Upsilon_3(r)(r^3 + \beta)(3\alpha r^5 + 2r^3 \beta + 2\beta^2)(r^3 + \beta + \alpha r^2)} dr - c_5 \right) \Upsilon_3(r) dr, \\ \lambda &= \frac{1}{2\mu^4 (r^3 + \beta) r^7 (3\alpha r^5 + 2r^3 \beta + 2\beta^2)} \left\{ \beta \alpha \Upsilon_3(r) \Upsilon_5(r) \int \frac{r^3 (2\beta^2 - 27\alpha r^5 + 4r^3 \beta + 2r^6)}{\Upsilon_3(r)(r^3 + \beta)(3\alpha r^5 + 2r^3 \beta + 2\beta^2)(r^3 + \beta + \alpha r^2)} dr \right. \\ &\quad \left. - 16\alpha c_5 \Upsilon_3(r) \Upsilon_5(r) + 16r^{10} \beta \alpha + 10r^8 \beta^2 + 45\beta r^9 \alpha^2 + 10r^2 \beta^4 + 20r^5 \beta^3 + 10r^4 \beta^3 \alpha + 53r^7 \beta^2 \alpha \right\}, \end{aligned} \quad (41)$$

where

$$\Upsilon_3(r) = e^{\int \frac{12\alpha^2 r^{10} - 15\alpha^2 r^7 \beta + 10\beta^3 \alpha^2 + 19r^5 \alpha \beta^2 + 17r^8 \alpha \beta + 8r^{11} \alpha + 18\beta^4 + 48r^3 \beta^3 + 42r^6 \beta^2 + 12r^9 \beta}{2(3\alpha r^5 + 2r^3 \beta + 2\beta^2)(r^3 + \beta + \alpha r^2)r(r^3 + \beta)}} dr}, \quad (42)$$

$$\Upsilon_4(r) = 12r^3 \beta^3 - 2r^{11} \alpha - 3\alpha^2 r^{10} - 4\beta^3 \alpha r^2 + 44r^5 \alpha \beta^2 - 4\beta^4 + 18r^6 \beta^2 + 19r^8 \alpha \beta + 24\alpha^2 r^7 \beta + 2r^9 \beta, \quad (43)$$

$$\Upsilon_5(r) = 8r^6 \beta^2 + 8r^8 \alpha \beta + 21\alpha^2 r^7 \beta - r^9 \beta + 13r^3 \beta^3 + 4\beta^3 \alpha r^2 + 34r^5 \alpha \beta^2 + 4\beta^4 - 4r^{11} \alpha - 6\alpha^2 r^{10}, \quad (44)$$

and the functions χ and ω have the same form as given by Eq. (29). Calculating the curvature invariants of solution (41) we obtain

$$K = R_{\alpha\beta\gamma\rho}R^{\alpha\beta\gamma\rho} = \frac{3}{4r^{10}(r^3 + \beta)^4} \{16\alpha^2 r^{16} - 40\alpha^2\beta r^{13} + 243\alpha^2\beta^2 r^{10} + 76\alpha^2\beta^3 r^7 + 20\alpha^2\beta^4 r^4 + 80\alpha\beta r^5 + 44\beta^6 + 66\beta^2 r^{12} + 66\beta^3 r^9 + 156\beta^4 r^6 + 80\alpha\beta r^{14} + 80\alpha\beta^2 r^{11} + 160\alpha\beta^3 r^8 + 196\beta^5 r^3 + 80\alpha\beta^5 r^2\}, \quad (45)$$

$$R_{\alpha\beta}R^{\alpha\beta} = \frac{\beta^2}{8r^{10}(r^3 + \beta)^4} \{116\beta^2 r^6 + 40\beta^3 r^3 + 19\beta^4 + 801\alpha^2 r^{10} + 76r^{12} + 93\alpha r^{11} + 76\beta r^9 + 198\beta\alpha r^8 + 117\alpha\beta^3 r^2 + 112\alpha^2\beta r^7 + 88\alpha^2\beta^2 r^4\}, \quad (46)$$

$$R = \frac{\beta(8r^6 + 21\alpha r^5 + 16\beta r^3 - 6\alpha\beta r^2 + 8\beta^2)}{2r^5(r^3 + \beta)^2}, \quad (47)$$

$$\mathcal{G} = \frac{2\alpha(6\alpha r^8 + 10\beta r^6 - 27\beta\alpha r^5 - 31\beta^2 r^3 - 14\beta^3 - 6\alpha\beta^2 r^2)}{2r^5(r^3 + \beta)^2}. \quad (48)$$

The above invariants show that there is no singularity at $r = 0$.

IV. RELEVANT PHYSICS AND THERMODYNAMICS OF THE BHS (28), (29) AND (40), (41)

In this section, we are going to investigate the essential physics of solutions (28), (29) and (40), (41).

A. Relevant physics and thermodynamics of the BH (28), (29)

For the BH (29), we write the line element as

$$ds^2 = -\left[1 - \frac{2M}{r} + \frac{c_1}{r^2} + \frac{c_2}{r^6}\right] dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{c_1}{r^2} + \frac{c_2}{r^6}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (49)$$

The metric of the line element (49) has multihorizons as Fig. 1(a) shows. These multihorizons, three of them, are created due to a specific value of the constant c_2 and other values will create two horizons only. These three multihorizons are created from the constants M , c_1 , and c_2 and the vanishing of the dimensional parameter c_2 reproduces geometry with two horizons. Moreover, when the dimensional parameters c_1 and c_2 vanish, the geometry with one horizon, i.e., the Schwarzschild geometry, is reproduced. As Eq. (32) shows, the BH solution (49) gives a nontrivial form of the GB invariant, whose behavior is shown in Fig. 1(b). The behavior of the physical quantities $h(\chi)$, $V(\chi)$, and the Lagrange multiplier field λ for the BH solution (29) are shown in Figs. 1(c), 1(d), and 1(e). Using Eq. (49), we obtain M as a function of the radial coordinate r ,

$$M = \frac{r}{2} \left(1 + \frac{c_1}{r^2} + \frac{c_2}{r^6}\right). \quad (50)$$

Now we investigate the thermodynamics for the BH (29). The Hawking temperature is defined as [75–78]

$$T_2 = \frac{f'(r_2)}{4\pi}, \quad (51)$$

where r_2 is the event horizon located at $r = r_2$ which is the largest positive root of $f(r_2) = 0$ which satisfies $f'(r_2) \neq 0$. Using Eq. (51), we obtain the Hawking temperature of the BH solution in the form

$$T_2 = \frac{Mr_2^5 - c_1 r_2^4 - 3c_2}{2\pi r_2^7}. \quad (52)$$

The Hawking entropy is defined as [75–80]

$$S(r_2) = \frac{1}{4}A(r_2), \quad (53)$$

where A is the area of the event horizon.

To show the number of horizons of the BH solution Eq. (28), we plot g_{00} in Fig. 2(a). As Fig. 1(a) shows that for the specific value of c_2 , we have three horizons and for other values c_2 or when $c_2 = 0$ we have two horizons as in Fig. 2(a). Also in Fig. 2(a), we show the region where the black hole has no singularity, i.e., the naked singularity.

Using Eq. (53), the entropy of the BH (28) is computed as

$$S_2 = \pi r_2^2. \quad (54)$$

We plot Eq. (54) in Fig. 2(b). As this figure shows, we have always positive entropy. The Hawking temperature associated with the BH solution (28) is plotted in Fig. 2(c).

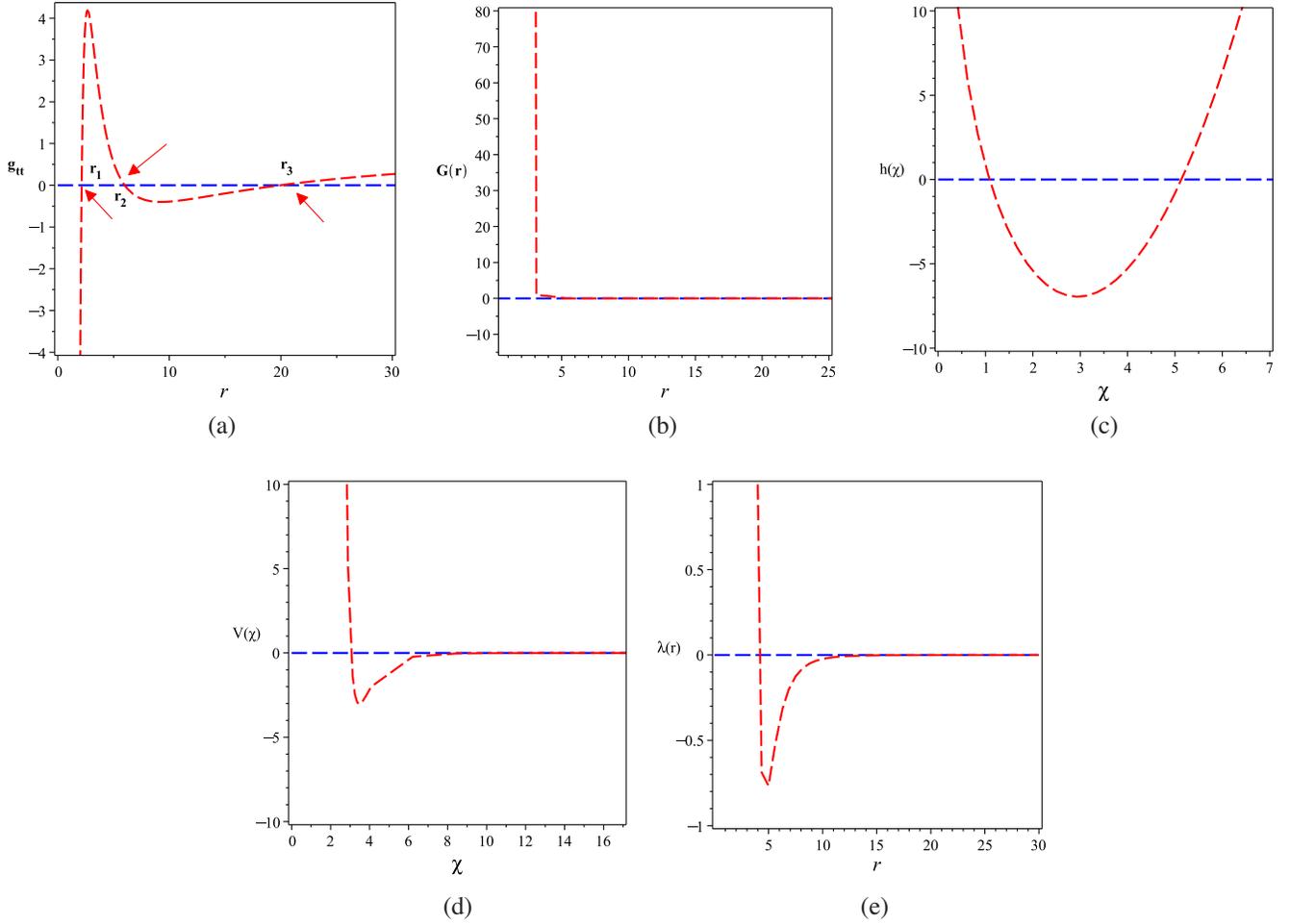


FIG. 1. Schematic plots of the radial coordinate r (a) vs the function f given by Eq. (29); (b) vs the GB invariant given by Eq. (32); (c) the function h vs χ ; (d) the potential V vs χ ; and (e) the Lagrange multiplier $\lambda(r)$ given by Eq. (29) vs r .

From this figure, one can show that we have always a positive temperature for r_2 . To investigate the thermodynamical stability of BHs, the formula of the heat capacity $H(r_2)$ at the event horizon should be derived. The heat capacity is defined as follows [81–83]:

$$H_c \equiv H(r_2) = \frac{\partial M_2}{\partial T_2} = \frac{\partial M_2}{\partial r_2} \left(\frac{\partial T_2}{\partial r_2} \right)^{-1}. \quad (55)$$

The BH will be thermodynamically stable if its heat capacity H_c is positive. On the other hand, it will be unstable if H_c is negative. As is well known, the heat capacity of the Schwarzschild black hole in GR is negative and therefore the solution is unstable, which corresponds to the Hawking evaporation. Substituting (50) and (52) into (55), we obtain the heat capacity as follows:

$$H_c = \frac{\pi r_2^2 (c_1 r_2^4 - r_2^6 + 5c_2)}{2Mr_2^5 - 3c_1 r_2^4 - 21c_2}. \quad (56)$$

The free energy in the grand canonical ensemble, which is called the Gibbs's free energy, can be defined as [80,84]

$$G(r_2) = E(r_2) - T(r_2)S(r_2) \quad (57)$$

where $E(r_2)$, $T(r_2)$, and $S(r_2)$ are the quasilocal energy, the temperature, and entropy at the event horizons, respectively. Substituting Eqs. (50), (52), and (53) into (57), we obtain the Gibbs's free energy of the BH (28) in the following form:

$$G(r_2) = \frac{r_2^6 + 3c_1 r_2^4 + 7c_2}{4r_2^5}. \quad (58)$$

We plot the behavior of the Gibbs's free energy in Fig. 2(e), which shows that the BH solution (28) with r_2 is unstable.

B. Thermodynamics of the BH (40), (41)

In this section, we will study the thermodynamics of the BH solution in (40), (41). For this aim, by assuming r is large, we rewrite the metric as

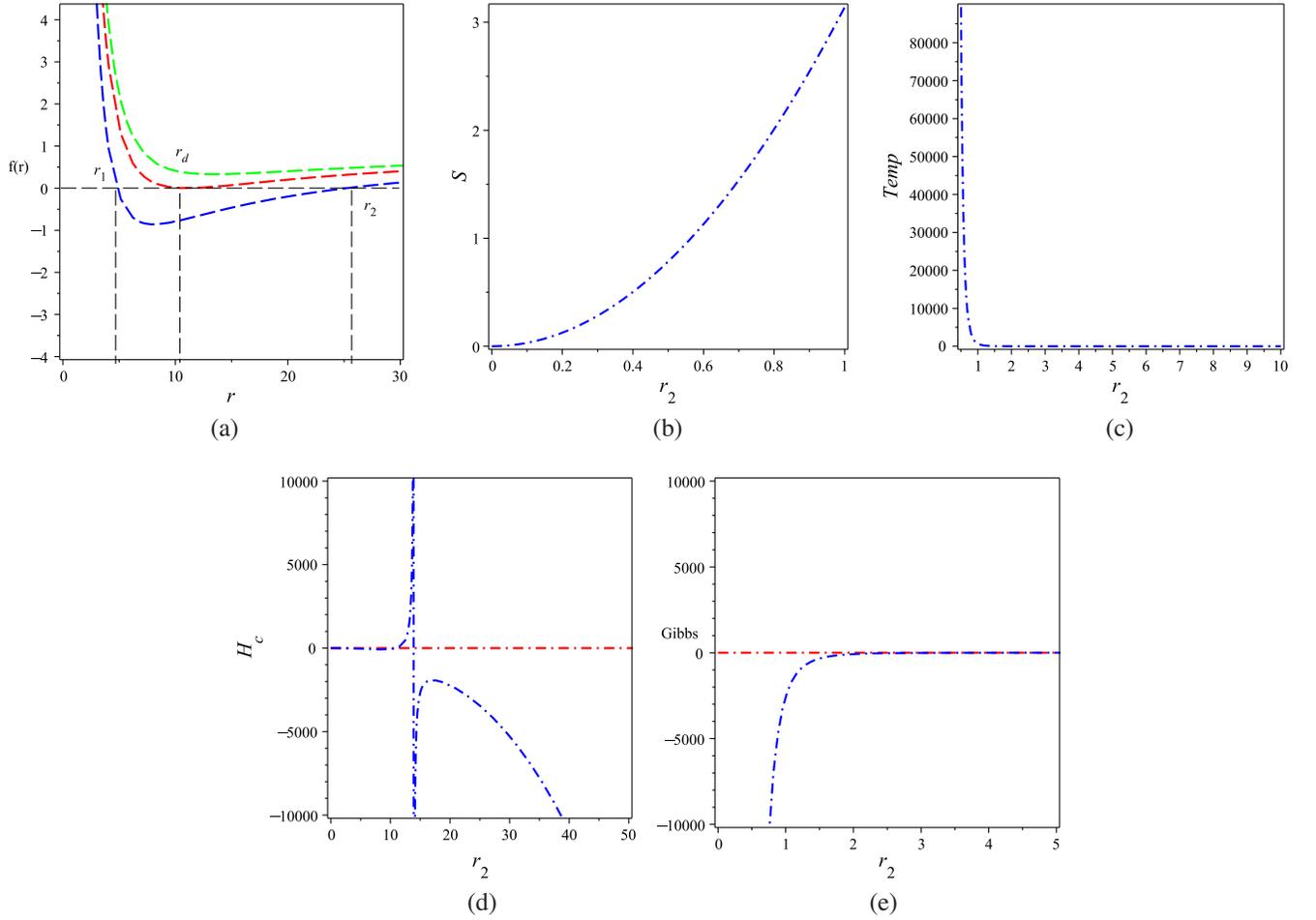


FIG. 2. Schematic plot of (a) the horizons, r_1 and r_2 , of the BH solution (28); (b) entropy of the BH solution (28); (c) Hawking temperature of the BH solution (28); (d) heat capacity of the BH solution (28); finally, (e) Gibbs's free energy of the BH solution (28).

$$f(r) = 1 + \frac{\alpha r^2}{r^3 + \beta} \approx 1 - \frac{2M}{r} + \frac{2M\beta}{r^4} - \frac{2M\beta^2}{r^7}, \quad (59)$$

where $\alpha = -2M$.

By using Eqs. (59) and (40) we obtain

$$ds^2 = - \left[1 - \frac{2M}{r} + \frac{2M\beta}{r^4} - \frac{2M\beta^2}{r^7} \right] dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{\beta}{r^3}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (60)$$

which asymptotically approaches flat space-time but is not equal to the Schwarzschild space-time due to the contribution of the extra term including β . It is easy to check that when the term β vanishes, the geometry reduces to the Schwarzschild space-time. From Eq. (60), we obtain an expression of M in terms of the radial coordinate r as

$$M = \frac{r}{2} \left(1 + \frac{2M\beta}{r^4} - \frac{2M\beta^2}{r^7} \right). \quad (61)$$

The metric of the line element given by (60) has two horizons as shown in Fig. 3(a). These two horizons are created by the constants M and β . When β vanishes, the geometry of the Schwarzschild geometry is reproduced. The behavior of the metric is drawn in Fig. 3(b), which shows clearly that there are two horizons related to the BH solution (40). As Eq. (48) shows, the BH solution (60) has a nontrivial expression of the GB invariant, whose behavior is shown in Fig. 3(c). The behavior of the physical quantities related to the BH solution (41) like $h(\chi)$, $V(\chi)$, and the Lagrange multiplier field λ are shown in Figs. 3(d), 3(e), and 3(f). To show how many horizons appear in the BH solution of Eq. (40), we plot the metric g_{00} in Fig. 4(a). As Fig. 4(b) shows that in the case $\beta = 0.3$,¹ we have two horizons and when $\beta = 1.3$, we have no horizon. Also in Fig. 4(b), we show the region where the black hole has naked singularity, i.e., when $\beta > 1.3$.

By using Eq. (51), we obtain the Hawking temperature of the BH solution (40) in the following form:

¹In this study we use Eq. (59) and put $\alpha = -2$ which yields $M = 1$.

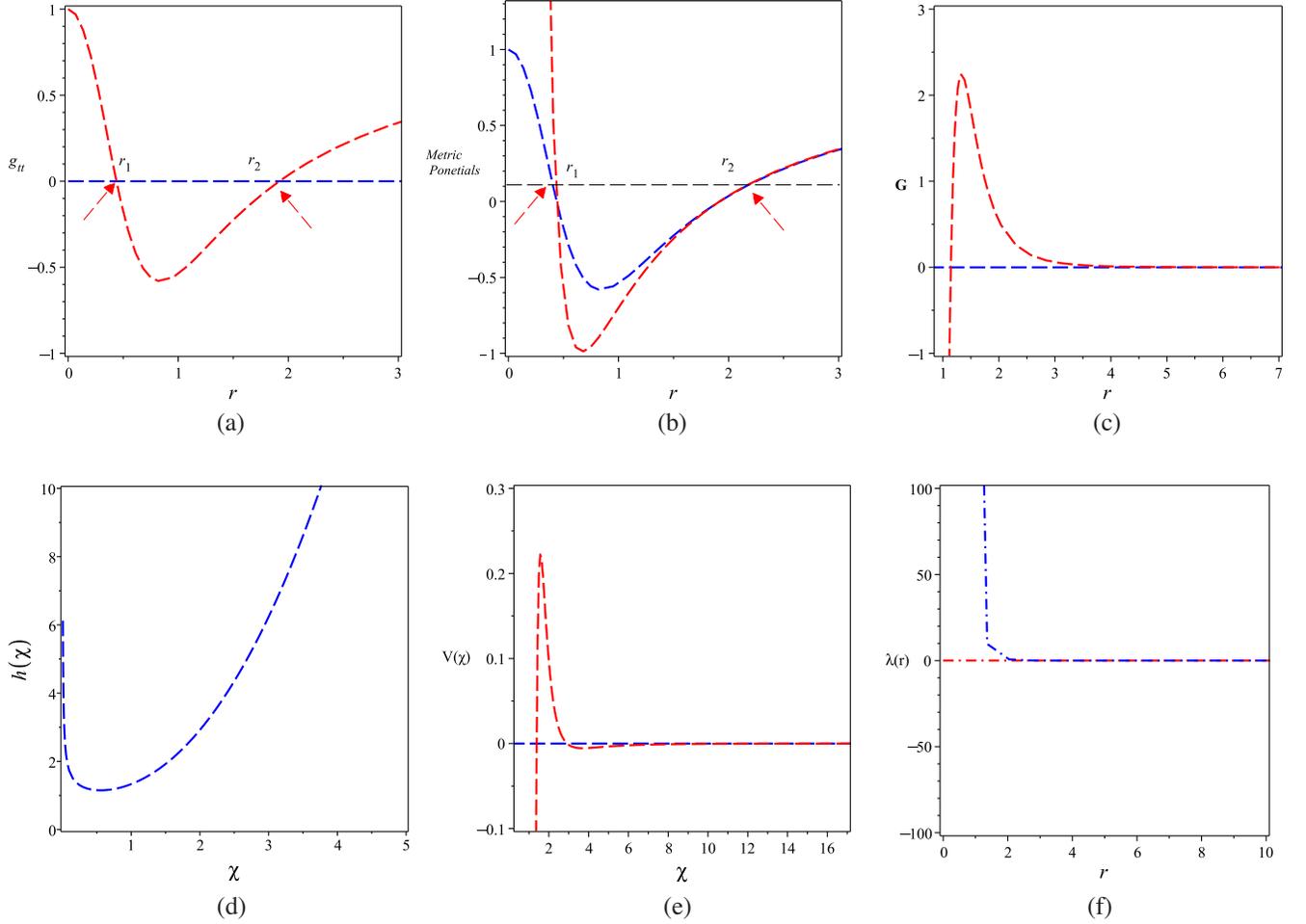


FIG. 3. Schematic plots of the radial coordinate r (a) vs the function f ; (b) vs the metric potentials; (c) vs the GB invariant given by Eq. (48); (d) the function h vs χ ; (e) the potential V vs χ ; and (f) the Lagrange multiplier $\lambda(r)$ vs r .

$$T_2 = \frac{\alpha r_2 (2\beta - r_2^3)}{4\pi (r_2^4 + \beta)^2}. \quad (62)$$

We show the behavior of Eq. (62) in Fig. 4(b). From this figure one can show that we have a positive temperature for $r_2 > r_d$ and negative temperature for $r_2 < r_d$ where r_d is the degenerate horizon as shown in Fig. 4(a). Substituting (61) and (62) into (55), we obtain the heat capacity as follows:

$$H_c = \frac{2\pi(\beta - r_2^3)(r_2^3 + \beta)}{r_2^3 \alpha (r_2^6 - 7\beta r_2^3 + \beta^2)}. \quad (63)$$

We show the behavior of Eq. (63) in Fig. 4(c). By substituting Eqs. (53), (61), and (62) into (58), we obtain the Gibbs's free energy of the BH (40) in the following form:

$$G(r_+) = -\frac{4r_2^9 + 12\beta r_2^6 + 12\beta^2 r_2^3 + 4\beta^3 - \alpha r_2^8 + 2\alpha\beta r_2^5}{4r_2^2 (r_2^3 + \beta)^2}. \quad (64)$$

We plot the behavior of the Gibbs's free energy in Fig. 4(d), which shows that the BH solution (40) with r_2 is unstable.

V. MOTION OF PARTICLE

To show the effect of modified GB theory on observables, we study the motion of a test particle in the background solution given by the metric (28) and (40). We consider the photon sphere around the BH and the perihelion shift of circular orbits. For the time being, the photon sphere becomes of particular interest because it explains the edge of the shadow of a BH while the perihelion shift was already derived in [85,86].

A. Geodesic equation and effective potential

In this subsection, we study the geodesic equation in the space-time given by Eq. (28). For this aim, we define the worldline $q(\tau)$ of a test particle in a curved space-time by the Euler-Lagrange equations which is defined by

$$\frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} \right) - \frac{\partial \mathcal{L}}{\partial q^\mu} = 0, \quad (65)$$

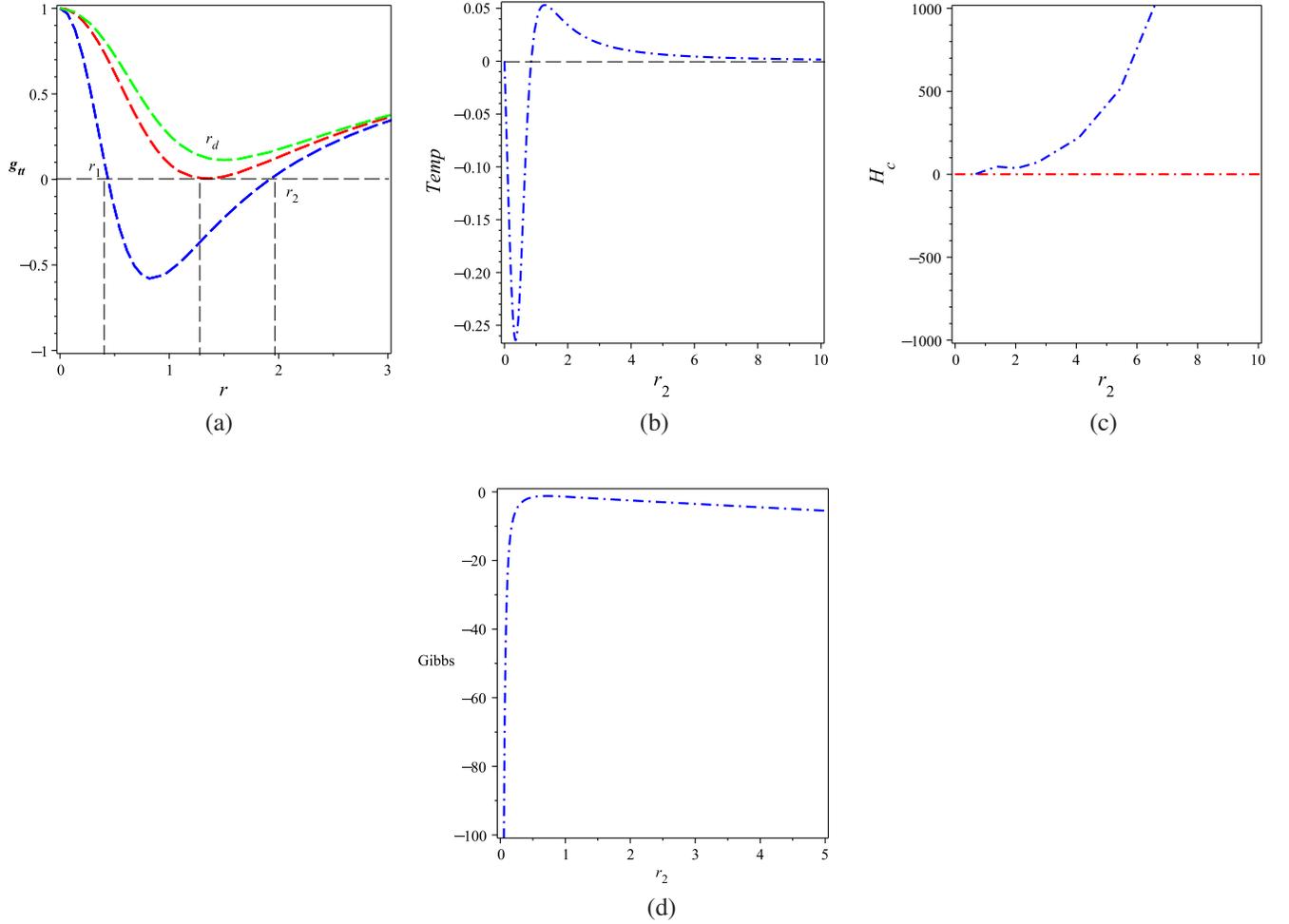


FIG. 4. Schematic plot of (a) horizons r_1 and r_2 of the BH solution (40); (b) Hawking temperature of the BH solution (40); (c) heat capacity of the BH solution (28); and (d) the Gibbs's free energy of the BH solution (40).

for the Lagrangian

$$2\mathcal{L} = g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu = f(r)\dot{t}^2 - \frac{\dot{r}^2}{f(r)} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2, \quad (66)$$

with $q^\mu(\tau) = (t(\tau), r(\tau), \theta(\tau), \phi(\tau))$ and \dot{q}^μ refers to the derivative of q^μ with respect to the affine parameter τ .

We solve the Euler-Lagrange equations (65) in the spherically symmetric space-times and we focus on the motion of the equatorial plane with $\theta = \pi/2$. Under that assumption, we obtain the conserved quantities, i.e., the energy E and angular momentum L , as

$$E = \frac{\partial\mathcal{L}}{\partial\dot{t}} = f(r)\dot{t} = \left(1 - \frac{2M}{r} + \frac{c_1}{r^2} + \frac{c_2}{r^6}\right)\dot{t}, \quad (67)$$

$$L = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = r^2\dot{\phi}. \quad (68)$$

Using the above conserved quantities (67) and (68), we obtain the effective potential in classical mechanics.

Because $2\mathcal{L} = 0$ for the massless particle and $2\mathcal{L} = 1$ for the massive particle, by deleting \dot{t} and $\dot{\phi}$ by using Eqs. (67) and (68), and by putting $\theta = \pi/2$ (constant), we obtain

$$\frac{E^2}{1 - \frac{2M}{r} + \frac{c_1}{r^2} + \frac{c_2}{r^6}} - \frac{\dot{r}^2}{1 - \frac{2M}{r} + \frac{c_1}{r^2} + \frac{c_2}{r^6}} - \frac{L^2}{r^2} = \sigma, \quad (69)$$

where $\sigma = 0$ for massless particles and $\sigma = 1$. We rewrite Eq. (69) as

$$0 = \frac{1}{2}\dot{r}^2 - \frac{1}{2}E^2 + \frac{1}{2}\frac{L^2}{r^2} \left(1 - \frac{2M}{r} + \frac{c_1}{r^2} + \frac{c_2}{r^6}\right) + \frac{1}{2}\sigma \left(1 - \frac{2M}{r} + \frac{c_1}{r^2} + \frac{c_2}{r^6}\right), \quad (70)$$

from which we can read off the effective potential $\mathcal{V}(r)$,

$$\mathcal{V}(r) = \frac{1}{2} \left(1 - \frac{2M}{r} + \frac{c_1}{r^2} + \frac{c_2}{r^6}\right) \left(\frac{L^2}{r^2} + \sigma\right) - \frac{1}{2}E^2, \quad (71)$$

and we rewrite (70) as

$$\frac{1}{2}\dot{r}^2 + \mathcal{V}(r) = 0. \quad (72)$$

For the study of the perihelion shift, we reparametrize $r(\tau)$ as $r(\phi)$, which yields

$$\frac{1}{2}\frac{\dot{r}^2}{\dot{\phi}^2} + \frac{1}{\dot{\phi}^2}\mathcal{V}(r) = \frac{1}{2}\left(\frac{dr}{d\phi}\right)^2 + \frac{r^4}{L^2}\mathcal{V}(r) = 0. \quad (73)$$

B. Photon sphere and perihelion shift of space-time (28)

For a circular orbit where $r = \text{const}$, $\dot{r} = 0$, the effective potential and its derivative have to vanish i.e., we have to solve both equations $\mathcal{V} = 0$ and $\mathcal{V}' = 0$.

When $c_2 = \sigma = 0$, the effective potential $\mathcal{V}(r)$ reduces to

$$\mathcal{V}(r) = \frac{1}{2}\left(1 - \frac{2M}{r} + \frac{c_1}{r^2}\right)\frac{L^2}{r^2} - \frac{1}{2}E^2. \quad (74)$$

When r is large, $\mathcal{V}(r)$ is a monotonically decreasing function of r . On the other hand, when r is small, $\mathcal{V}(r)$ behaves as $\mathcal{V}(r) \sim \frac{c_1 L^2}{2r^4}$ and therefore if $c_1 > 0$, $\mathcal{V}(r)$ goes to positive infinity and if $c_1 < 0$, $\mathcal{V}(r)$ goes to negative infinity.

For circular photon orbits, by solving the equations $\mathcal{V}(r) = \mathcal{V}'(r) = 0$ for the potential $\mathcal{V}(r)$ in Eq. (74), we find

$$r = \frac{3}{2}M \pm \frac{1}{2}\sqrt{9M^2 - 8c_1},$$

$$L_{\pm} = \pm \frac{(3M + \sqrt{9M^2 - 8c_1})^2 E}{2\sqrt{6M^2 + 2M\sqrt{9M^2 - 8c_1} - 4c_1}}, \quad (75)$$

where the value of r given in the first equation of Eq. (75) is used in the second equation of (75). Equation (75) gives the value of the Schwarzschild when $c_1 = 0$, i.e., $r = 3M$ and $L_{\pm} = 3\sqrt{3}ME$.

The expression of r in (75) tells us that when $c_1 < 0$, there is only one extremum $r = \frac{3}{2}M + \frac{1}{2}\sqrt{9M^2 - 8c_1}$. The behavior of the potential tells us that the extremum is a

maximum and therefore the orbit r is unstable. On the other hand, when $\frac{9M^2}{8} > c_1 > 0$, there are two extrema $r = \frac{3}{2}M \pm \frac{1}{2}\sqrt{9M^2 - 8c_1}$. The behavior of the potential shows that the larger extremum $r = \frac{3}{2}M + \frac{1}{2}\sqrt{9M^2 - 8c_1}$ is a local maximum and therefore the orbit corresponding to the extremum is unstable but the smaller extremum $r = \frac{3}{2}M - \frac{1}{2}\sqrt{9M^2 - 8c_1}$ is a local minimum and therefore the orbit corresponding to the extremum is stable.

For circular timelike orbits $\sigma = 1$ for a massive particle, it is also possible to solve the equations $\mathcal{V} = 0$ and $\mathcal{V}' = 0$. The obtained expressions are, however, not so insightful. We consider a perturbation around a circular orbit $r = r_{\text{crc}}$ and by plugging in the ansatz $r(\phi) = r_{\text{crc}} + r_{\phi}(\phi)$ for (73), we obtain

$$\left(\frac{dr_{\phi}}{d\phi}\right)^2 = -2\frac{(r_{\text{crc}} + r_{\phi})^4}{h^2}\mathcal{V}(r_{\text{crc}} + r_{\phi}). \quad (76)$$

Assuming that the ratio r_{ϕ}/r_c is small, the right-hand side can be expanded into powers of this parameter to second order

$$\left(\frac{dr_{\phi}}{d\phi}\right)^2 = -\frac{r_{\text{crc}}^4}{h^2}\mathcal{V}''(r_{\text{crc}})r_{\phi}^2 + \mathcal{O}\left(\frac{r_{\phi}^3}{r_{\text{crc}}^3}\right), \quad (77)$$

where we use the fact that $V(r_{\text{crc}}) = 0$ and $V'(r_{\text{crc}}) = 0$ for circular orbits, as discussed above. The above equation, which represents a simple harmonic oscillation, shows that the solution of r_{ϕ} oscillates with a wave number $K = \sqrt{\frac{r_{\text{crc}}^4}{h^2}\mathcal{V}''(r_{\text{crc}})}$ and thus the perihelion shift is given as

$$\Delta\phi = 2\pi\left(\frac{1}{K} - 1\right) = 2\pi\left(\frac{h}{r_{\text{crc}}^2\sqrt{\mathcal{V}''(r_{\text{crc}})}} - 1\right). \quad (78)$$

Now, we derive the explicit form of the perihelion shift for massive objects where the potential is \mathcal{V} with $\sigma = 1$. We evaluate the equations $\mathcal{V}(r_{\text{crc}}) = 0$ and $\mathcal{V}'(r_{\text{crc}}) = 0$ with $L = L_0 + \epsilon L_1$ and $E = E_0 + \epsilon E_1$. The zeroth order behaviors of these equations determine $L_0(r_{\text{crc}})$ and $E_0(r_{\text{crc}})$ as follows:

$$L_{0\pm} = \pm \frac{\sqrt{Mr_{\text{crc}}^5 - c_1 r_{\text{crc}}^4 - 3c_2 r_{\text{crc}}}}{\sqrt{r_{\text{crc}}^6 - 3Mr_{\text{crc}}^5 + 2c_1 r_{\text{crc}}^4 + 4c_2}},$$

$$E_{0\pm} = \pm \frac{\sqrt{4M^2 r_{\text{crc}}^5 - 2c_1 r_{\text{crc}}^4 M - 4r_{\text{crc}}^6 M + r_{\text{crc}}^7 + c_2 r_{\text{crc}} + c_1 r_{\text{crc}}^5 - 2c_2 M}}{\sqrt{r_{\text{crc}} - 3Mr_{\text{crc}}^3}}. \quad (79)$$

Having obtained the constants of motion for the circular orbit, we derive the perihelion shift by plugging the values into $\mathcal{V}''(r_{\text{crc}}, L_0, E_0)$ to obtain $\mathcal{V}''(r_{\text{crc}})$ alone. Due to the different solutions for the constants of motion, there exist two options to derive the perihelion shift,

$$\Delta\phi(L_{0+}), \quad \Delta\phi(L_{0-}), \quad (80)$$

which are related to each other through

$$\Delta\phi(L_{0-}) = -4\pi - \Delta\phi(L_{0+}). \quad (81)$$

By expanding the perihelion shift into a power series in the variables $q = \frac{M}{r_{\text{cre}}}$, $q_1 = \frac{c_1}{r_{\text{cre}}}$, and $q_2 = \frac{c_2}{r_{\text{cre}}}$, we obtain

$$\begin{aligned} \Delta\phi(L_{0-}) = & 12 \left(\frac{\pi(\sqrt{q - q_1 - 3q_2} + \sqrt{q + 12q_2})}{(q + 12q_2)^{5/2}} - \frac{\pi}{(q + 12q_2)^2} \right) q_1^4 \\ & + \left\{ \frac{18\pi(\sqrt{q - q_1 - 3q_2} + \sqrt{q + 12q_2})(4q_2 - 3q)}{(q + 12q_2)^{5/2}} - \frac{3\pi(72q - 96q_2)}{4(q + 12q_2)^2} \right\} q_1^3 \\ & - \pi \left(\frac{4}{q + 12q_2} + \frac{4(\sqrt{q - q_1 - 3q_2} + \sqrt{q + 12q_2})}{(q + 12q_2)^{3/2}} + \frac{3}{4(q + 12q_2)^2} (8[24q_2^2 - qq_2 + 6q^2] + 3(4q_2 - 3q)^2) \right. \\ & \left. + \frac{3\pi(\sqrt{q - q_1 - 3q_2} + \sqrt{q + 12q_2})(192q_2^2 - 8qq_2 + 48q^2 + (12q_2 - 9q)^2)}{4(q + 12q_2)^{5/2}} \right) q_1^2 \\ & + \left(\frac{\pi(12q_2 - 9q)}{-q - 12q_2} - \frac{3\pi(\sqrt{q - q_1 - 3q_2} + \sqrt{q + 12q_2})(4q_2 - 3q)}{(q + 12q_2)^{3/2}} - \frac{9\pi(24q_2^2 - qq_2 + 6q^2)(4q_2 - 3q)}{2(-q - 12q_2)^2} \right. \\ & \left. + \frac{3\pi(\sqrt{q - q_1 - 3q_2} + \sqrt{q + 12q_2})(24q_2^2 - qq_2 + 6q^2)(12q_2 - 9q)}{2(q + 12q_2)^{5/2}} \right) q_1 \\ & + 2 \frac{2\pi(\sqrt{q - q_1 - 3q_2} + \sqrt{q + 12q_2})}{\sqrt{q + 12q_2}} + \frac{3\pi(\sqrt{q - q_1 - 3q_2} + \sqrt{q + 12q_2})(24q_2^2 - qq_2 + 6q^2)^2}{4(q + 12q_2)^{5/2}} \\ & - \frac{3\pi(24q_2^2 - qq_2 + 6q^2)^2}{4(-q - 12q_2)^2} - \frac{\pi(\sqrt{q - q_1 - 3q_2} + \sqrt{q + 12q_2})(24q_2^2 - qq_2 + 6q^2)}{(q + 12q_2)^{3/2}} \\ & - \frac{\pi(24q_2^2 - qq_2 + 6q^2)}{q + 12q_2} + \mathcal{O}((qq_1q_2)^3). \end{aligned} \quad (82)$$

Equation (82) when $q_1 = q_2 = 0$, i.e., $c_1 = c_2 = 0$, yields

$$\Delta\phi(L_{0-}) = 6\pi q + 27\pi q^2 + \mathcal{O}(q^3), \quad (83)$$

which coincides with the perihelion of the Schwarzschild solution.

The qualitative behavior of the perihelion shift is always the same, only the numerical values differ. Equation (82) shows that $q > q_1 + 3q_2$, and the higher q_1 and q_2 , the smaller the influence of the perturbation and corrections to the perihelion shift appear only in higher orders in q .

Now we repeat the above perihelion of the BH solution (28) to the BH (40).

C. Photon sphere and perihelion shift in space-time (40)

For the Lagrangian

$$2\mathcal{L} = g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu = f(r) \dot{t}^2 - \frac{\dot{r}^2}{f_1(r)} - r^2 \dot{\theta}^2 - r^2 \sin^2 \theta \dot{\phi}^2, \quad (84)$$

with $q^\mu(\tau) = (t(\tau), r(\tau), \theta(\tau), \phi(\tau))$, and \dot{q}^μ refers to the derivative of q^μ with respect to the affine parameter τ .

To solve the Euler-Lagrange equations, we apply the same procedure used above for the BH (28). For the BH

solution (40), we obtain the energy E and angular momentum L as follows:

$$E = \frac{\partial \mathcal{L}}{\partial \dot{t}} = f(r) \dot{t} = \left(1 + \frac{\alpha r^2}{r^3 + \beta} \right) \dot{t}, \quad (85)$$

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2 \dot{\phi}. \quad (86)$$

Using the above expressions, we obtain the effective potential by rewriting the Lagrangian (84),

$$\frac{E^2}{1 + \frac{\alpha r^2}{r^3 + \beta}} - \frac{\dot{r}^2}{1 + \frac{\alpha}{r} + \frac{\beta}{r^3}} - \frac{L^2}{r^2} = \sigma. \quad (87)$$

The corresponding effective potential of the BH solution (40) takes the form

$$\begin{aligned} \mathcal{V}(r) = & \frac{L^2}{2r^2} \left(1 + \frac{\alpha}{r} + \frac{\beta}{r^3} \right) + \frac{\sigma}{2} \left(1 + \frac{\alpha}{r} + \frac{\beta}{r^3} \right) \\ & - \frac{E^2 \left(1 + \frac{\alpha}{r} + \frac{\beta}{r^3} \right)}{2 \left(1 + \frac{\alpha r^2}{r^3 + \beta} \right)}. \end{aligned} \quad (88)$$

For circular photon orbits, $\sigma = 0$, solving the zeroth order equations yields

$$0 = -2r_0^3 L_0^2 + 3r_0^2 \beta E_0^2 + 6r_0^2 M L_0^2 - 5\beta L_0^2,$$

$$L_{0\pm} = \pm E_0 r_0 \sqrt{\frac{r_0^3 + \beta}{r_0^3 + \beta - 2M r_0^2}}. \quad (89)$$

Equation (89) shows that when the dimensional constant β vanishes, we obtain the zeroth order terms of r_0 and L_0 of the Schwarzschild space-time when $\sigma = 0$, i.e., $r_0 = 3M$ and $L_{0\pm} = \pm 3\sqrt{3}ME_0$. The above equation has three roots for r_0 ; one of them has a real value and takes the form

$$r_0 = \frac{\sqrt[3]{\beta^3 E_0^6 - 10\beta L_0^6 + 6\beta^2 E_0^4 L_0^2 M + 12\beta E_0^2 L_0^4 M^2 - 8L_0^6 M^3 - 16SL_0^3}}{2L_0^2} + \frac{(\beta E_0^2 + 2L_0^2 M)^2}{2L_0^2 \sqrt[3]{\beta^3 E_0^6 - 10\beta L_0^6 + 6\beta^2 E_0^4 L_0^2 M + 12\beta E_0^2 L_0^4 M^2 - 8L_0^6 M^3 - 16SL_0^3}} + \frac{\beta E_0^2 + 2ML_0^2}{2L_0^2}, \quad (90)$$

where $S = \sqrt{5\beta(L_0^6 M^3 + 5\beta L_0^6 - \beta^3 E_0^6 - 6\beta^2 E_0^4 L_0^2 M + 12\beta E_0^2 L_0^4 M^2)}$. The above equation gives the value of the Schwarzschild space-time when $\beta = 0$.

Now we derive the explicit form of the perihelion shift for massive particles for the BH (40) by using the potential \mathcal{V} (88) with $\sigma = 1$. We evaluate the equations $\mathcal{V}(r_{\text{crc}}) = 0$ and $\mathcal{V}'(r_{\text{crc}}) = 0$ by considering the perturbation with $L = L_0 + \epsilon h_1$ and $k = k_0 + \epsilon k_1$. The zeroth order terms of these equations determine $L_0(r_{\text{crc}})$ and $E_0(r_{\text{crc}})$ as

$$E_{0\pm} = \pm \frac{\sqrt{2}(r_{\text{crc}}^3 - 2Mr_{\text{crc}}^3 + \beta)}{\sqrt{2r_{\text{crc}}^6 - 6Mr_{\text{crc}}^5 + 4\beta r_{\text{crc}}^3 + 2\beta^2}},$$

$$L_{0\pm} = \pm r_{\text{crc}}^2 \sqrt{\frac{2Mr_{\text{crc}}^3 + 4M\beta}{2r_{\text{crc}}^6 - 6Mr_{\text{crc}}^5 + 4\beta r_{\text{crc}}^3 + 2\beta^2}}. \quad (91)$$

By using the obtained constants of motion for the circular orbit, we derive the perihelion shift by plugging in the expressions of $E_0 = E_{0\pm}$ and $L_0 = L_{0\pm}$ into $V''(r_{\text{crc}}, E_0, L_0)$. Corresponding to the signatures \pm in the expressions of $E_0 = E_{0\pm}$ and $L_0 = L_{0\pm}$, there exist two options to derive the perihelion shift,

$$\Delta\phi(L_{0+}), \quad \Delta\phi(L_{0-}), \quad (92)$$

which are related to each other through

$$\Delta\phi(L_{0-}) = -4\pi - \Delta\phi(L_{0+}). \quad (93)$$

By expanding the perihelion shift into a power series of $q = \frac{M}{r_{\text{crc}}}$ and $q_1 = \frac{\beta}{r_{\text{crc}}^3}$, we obtain

$$\Delta\phi(L_{0-}) = 2 \frac{\sqrt{1-2q_1} - \sqrt{1-6q+11q_1-8q_1^2}}{\sqrt{1-6q+11q_1-8q_1^2}}. \quad (94)$$

Equation (94) when $q_1 = 0$ yields

$$\Delta\phi(L_{0-}) \approx 6\pi q + 27\pi q^2 - 13\pi q_1 - 105\pi q q_1 + \frac{435\pi}{4} q_1^2 + \mathcal{O}((q q_1)^3), \quad (95)$$

which coincides with the perihelion shift of the Schwarzschild solution when $q_1 = 0$.

The qualitative behavior of the perihelion shift is not so changed, only the numerical values differ. As for the photon sphere, the higher q_1 , the higher the influence of the perturbation and corrections to the perihelion shift appear.

VI. CONCLUSIONS

In this study, we constructed a consistent ghost-free modified GB gravitational theory capable of describing a BH with horizons. The field equations of this theory are applied to a spherically symmetric space-time and we succeeded to derive BH solutions with multihorizons. We showed that for the Schwarzschild BH type metric (21), we obtained a BH solution with three horizons and the curvature invariants of this BH show a true singularity at $r = 0$. Moreover, we calculated the thermodynamical quantities associated with this solution and showed that all the thermodynamical quantities and the heat capacity and Gibbs's free energy tell us that this solution is not stable.

We repeated our calculations for a more general case whose metric is given by (33) and showed that the solution has two horizons in spite of the fact that the field equations do not include a cosmological constant nor is there any source of charge to reproduce such two horizons. Moreover, we also showed that such BH yields a true singularity at $r = 0$. We also calculated the thermodynamical quantities and showed that the Gibbs's free energy is negative. Furthermore, for both BH solutions in (21) and (33), we calculated all of the physical quantities which appear in the

GFGB theory, that is, the potential, the Lagrange multiplier, and the function f and showed their behaviors in Figs. 1 and 3.

We should note that the present study is a first trial in the direction of a full phenomenological classification of observables, which is derived in the weak GFGB gravity, to

compare them with observations. The future work in this direction could be to study axially symmetric perturbations around rotating space-time, to obtain the shift in the photon regions, that will give an important imprint on the predictions of the shape of the BH shadow. This case will be studied elsewhere.

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