


# Traversing through a black hole singularity

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We show that the Kantowski-Sachs model of a Schwarzschild black hole interior can be slightly generalized in order to accommodate spatial metrics of different orientations, and in this formulation the equations of motion admit a variable redefinition that makes the system regular at the singularity. This system will then traverse the singularity in a deterministic way (information will be conserved through it), and evolve into a time-reversed and orientation-flipped Schwarzschild white hole interior.

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## I. INTRODUCTION

Singularities are a striking prediction of General Relativity (GR), and seem to imply that classical determinism breaks down whenever one is formed, in the sense that the equations of motion are unable to predict the evolution of the physical degrees of freedom past the singularity [1]. The issue of the fate of determinism in gravitational singularities has traditionally been discussed in terms of the (weak and strong) cosmic censorship conjectures, conceived by Penrose [2]. These conjectures imply that, even if determinism fails at singularities, General Relativity is nevertheless able to uniquely predict the entire evolution of the Universe, with the exception of some finite regions of space hidden inside event horizons. Counterexamples have been found for both the weak and strong versions of the conjecture [3,4], and there is some degree of disagreement on the precise formulation of the strong version.

In a recent series of papers [5–7],<sup>1</sup> we were able to bypass the cosmic censorship discussion entirely, by showing that there is a sense in which determinism is preserved, at singularities, by the classical dynamics of General Relativity. The version of determinism that is respected does not make reference to the continuability of geodesics,<sup>2</sup>

<sup>1</sup>See also [8–10] for a closely-related approach, and [11] for a possible quantum origin for the quiescence mechanism.

<sup>2</sup>Geodesics, as it turns out, might not admit at all a unique continuation through the singularity in our model, without ruining the predictability of the theory. In fact, geodesics are an abstraction that describes well the motion of the center of mass of a physical observer only when tidal forces are small on the length scales of the observer. Close to a singularity these tidal forces will become arbitrarily large, making the notion of a pointlike observer meaningless. Still, if all the physical (i.e., gauge-invariant) degrees of freedom of classical GR coupled to Standard Model matter fields admit a unique continuation through the singularity, classical determinism is safe. Our physical observers might get irreparably scrambled by the tidal forces at the singularity, but they would do so in a unique and predictable fashion.

or to the fate of observers (which is what the strong cosmic censorship conjecture is concerned with). Our result concerns the physical degrees of freedom of relativistic field theories (gravity included). These are the gauge-invariant degrees of freedom that are necessary to specify on a spacelike hypersurface in order to uniquely fix a solution of the GR + matter fields system. This is the basic ontology of classical field theory, and all the physical predictions of the theory can, in principle, be expressed in terms of those degrees of freedom. In [5–7], we considered the special cases of a homogeneous cosmology with compact spatial slices (Bianchi IX and Friedmann-Lemaître-Robertson-Walker models) with, at most, an arbitrary number of scalar fields. By extending the configuration space of the theory to include the information about the orientation of spatial slices, this result allowed us to prove that to each and every collapsing solution ending up in a singularity, there corresponds one and only one expanding solution that evolves away from the singularity with opposite orientation.

Our result does not involve a spacetime extension beyond the big bang singularity; the 4-dimensional picture is that of a 4-metric with a singular spacelike cross section. This singularity is a degenerate hypersurface which cannot support a nonzero volume because it is effectively one- or two-dimensional. At this hypersurface, the spatial orientation flips. What we proved is that the dynamics of the homogeneous spatial hypersurfaces of the model has, at the big bang, a singularity that can be regularized, much like, for example, the singularities of the  $N$ -body problem [12]. We hope that this result represents a first step towards proving a general conjecture: namely, that a large class of gravitational singularities preserve classical determinism, so that one can predict unambiguously the spacetime geometry that can be found beyond said singularities. There are good reasons to believe that our result can be extended beyond the confines of homogeneous cosmologies: for example, if the Belinski-Khalatnikov-Lifshitz

(BKL) conjecture [13] were proven, it would imply that the result can be extended immediately to all inhomogeneous cosmological solutions with the global topology of the Bianchi IX model (i.e.,  $S^3 \times \mathbb{R}$ ). There is a significant amount of numerical support for the validity of this conjecture [14,15], and there is an actual proof in the case of analytic solutions [16].

In the present paper, we will extend our result to Schwarzschild singularities. Thanks to the fact that empty Kantowski-Sachs homogeneous cosmological models are isometric to the interior of the Schwarzschild spacetime [17], our insights on homogeneous cosmologies can be directly applied to black hole singularities described as singularities of Kantowski–Sachs spacetimes. The result will be an extension of Schwarzschild spacetimes (with, possibly, a spherically symmetric scalar field) into a unique spacetime with a low degree of regularity. We do not evade no-go theorems on continuous extensions of spacetime through the Schwarzschild singularity, e.g., [18–21]. What we do have, is a preservation of determinism, like in the cosmological case [5–7]. Also in this case, a proof of the BKL conjecture would allow to generalize our result to general inhomogeneous solutions of Einstein’s equations with a Schwarzschild-type singularity.

## II. THE EMPTY KANTOWSKI-SACHS MODEL

The interior of the Schwarzschild metric can be understood as a special case of the Kantowski-Sachs class of spacetimes [17]. These are homogeneous cosmological models with an  $S^2 \times \mathbb{R}$  spatial topology, and a spacetime metric of the form,

$$d\tau^2 = -N(\sigma)^2 d\sigma^2 + A(\sigma)^2 d\rho^2 + B(\sigma)^2 d\Omega^2. \quad (1)$$

The ordinary Schwarzschild metric is found as the particular case

$$N = \left(\frac{2M}{\sigma} - 1\right)^{-1/2}, \quad A = \left(\frac{2M}{\sigma} - 1\right)^{1/2}, \quad B = \sigma, \quad (2)$$

if we call  $\sigma = r$  and  $\rho = t$ . This, of course, is only valid when  $r < 2M$ , i.e., the region inside the event horizon, where the  $r$  coordinate is timelike and the  $t$  coordinate is spacelike. The manipulation we described allows us to understand the Schwarzschild singularity in a similar manner to the big bang, and to translate progress in the understanding of homogeneous cosmological singularities into advancement in the physics of black holes.

After imposing the ansatz (1), the Einstein-Hilbert Lagrangian reads

$$L = \frac{1}{2\kappa} \int d^3x \sqrt{-g} R = \frac{4\pi N \lambda}{\kappa} \left( A - \frac{A(\dot{B})^2 + 2B\dot{A}\dot{B}}{N^2} \right) + \dot{K}, \quad (3)$$

where  $\kappa = 8\pi G c^{-4}$  and  $\lambda = \int_{r_1}^{r_2} d\rho$  is the width of a fiducial interval of radii over which we integrate (by homogeneity, the metric outside this interval will be identical to the one inside). Our notation is  $\dot{f} = \frac{df}{ds}$ , where we call our independent timelike variable  $s$  (rather than  $t$ , to avoid confusion between Schwarzschild time and radius).  $K = \frac{4\pi\lambda}{\kappa N} (\dot{A}B^2 + 2\dot{B}AB)$  appears as a total derivative, and is therefore a boundary term that can be removed (it is minus the Gibbons-Hawking-York term [22,23]).

In terms of the canonical momenta  $P_A = \partial L / \partial \dot{A}$ ,  $P_B = \partial L / \partial \dot{B}$ , we can write the total Hamiltonian  $H = P_A \dot{A} + P_B \dot{B} - L$  as

$$H = \frac{N}{\nu^2} \left( \frac{P_A^2 A - 2P_A P_B B}{4B^2} - \nu^4 A \right), \quad (4)$$

where  $\nu = \sqrt{\frac{4\pi\lambda}{\kappa}}$ . With the following canonical transformation,

$$A = \frac{e^{-\frac{x}{\sqrt{2}}}}{\nu}, \quad B = \frac{e^{\frac{x+y}{\sqrt{2}}}}{\nu},$$

$$P_A = -\sqrt{2}\nu e^{\frac{x}{\sqrt{2}}}(p_x - p_y), \quad P_B = \sqrt{2}\nu e^{\frac{x+y}{\sqrt{2}}} p_y, \quad (5)$$

the Hamiltonian takes the simple form,

$$H = \frac{N\nu}{2} e^{\frac{x+2y}{\sqrt{2}}} (p_x^2 - p_y^2 - 2e^{\sqrt{2}y}). \quad (6)$$

We are free to choose the lapse function  $N$ , and the obvious choice is  $N = \frac{1}{\nu} e^{\frac{x+2y}{\sqrt{2}}}$ , which simplifies the prefactor and gives us the elementary Hamiltonian

$$H = \frac{1}{2} (p_x^2 - p_y^2) - e^{\sqrt{2}y}. \quad (7)$$

This Hamiltonian makes  $p_x$  a conserved quantity, and the Hamiltonian constraint  $H \approx 0$  imposes that  $p_x^2 = p_y^2 + 2e^{\sqrt{2}y}$ . This is the Hamiltonian of a one-dimension nonrelativistic point particle with potential  $2e^{\sqrt{2}y}$  and energy  $p_x^2$ . The general solution to Hamilton’s equations is

$$x = p_x(s - s_1), \quad p_y = k \tanh\left(\frac{k(s - s_2)}{\sqrt{2}}\right),$$

$$y = -\sqrt{2} \log \left[ \frac{\sqrt{2}}{|k|} \cosh\left(\frac{k(s - s_2)}{\sqrt{2}}\right) \right], \quad (8)$$

where  $p_x$  now is a constant of motion, and  $k$ ,  $s_1$ , and  $s_2$  are constants of integration. Moreover, the Hamiltonian constraint imposes that  $p_x^2 = k^2$ . The asymptotic components of the velocity are  $\dot{x} = p_x$  and  $\dot{y} \xrightarrow{[s \rightarrow \pm\infty]} \mp |k| = \mp |p_x|$ . In the  $x$ - $y$  plane, all solutions look like a ball bouncing off an

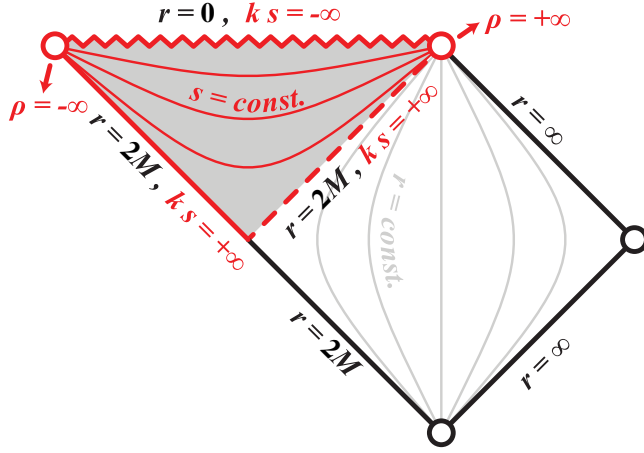


FIG. 1. The shaded region corresponds to the patch of Schwarzschild spacetime that is covered by the  $s, \rho$  coordinates as it appears in the Penrose-Carter diagram. The borders of this coordinate patch are represented by the two red dots, the singularity and the horizon.

exponential slope and rolling inertially to infinity at a  $45^\circ$  angle. In Fig. 1, we show the Penrose diagram of Schwarzschild spacetime, the patch that our coordinates cover, the hypersurfaces of simultaneity that correspond to the  $s = \text{const.}$  condition, and the location of the singularity, which can be seen as an asymptotic condition for the “time” variable  $s$ .

From Fig. 2, we can see that the volume  $v \propto AB^2$  of our fiducial region is concave (as a function of  $s$ ), going to zero as  $s \rightarrow \pm\infty$ , and reaching a unique maximum in between.

We can calculate the Ricci tensor on the solution (8), and only one component turns out to be nonzero

$$R_{\mu\nu} = (k^2 - p_x^2)\delta^0_\mu\delta^0_\nu, \quad (9)$$

and if we impose the Hamiltonian constraint  $p_x^2 = k^2$ , the spacetime we get is Ricci-flat. To highlight the location of

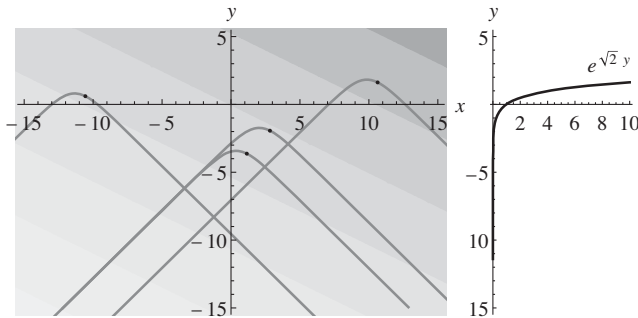


FIG. 2. Examples of solutions of the Kantowski-Sachs system in the  $x$ - $y$  plane, with a plot of the potential on the side. The curves asymptotes to the singularity on the left, and the horizon on the right. The gradient on the background represents the values of the volume degree of freedom  $v$  (darker = larger), and the black dots represent the points of maximum volume.

the singularity, we can calculate the Kretschmann scalar (when  $p_x^2 = k^2$ )

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{3\nu^4(e^{\sqrt{2}k(s_2-s)} + 1)^6}{k^4 e^{-\sqrt{2}k(2s_1+4s_2)}}, \quad (10)$$

and see that it diverges when  $\text{sign}(k)s \rightarrow -\infty$ , while it asymptotes to a constant as  $\text{sign}(k)s \rightarrow +\infty$ . The first limit corresponds to the singularity, while the second is the horizon.

### III. OBTAINING THE SCHWARZSCHILD METRIC

The Schwarzschild solution can be obtained by setting<sup>3</sup>

$$k(s_1 - s_2) = \sqrt{2} \log \nu, \quad (11)$$

then one can see that  $A$  and  $B$ , expressed in terms of the solution  $x, y$  of (8) through the relations (5), satisfy the equation

$$A^2 = \frac{2M}{B} - 1, \quad (12)$$

where, as it turns out,  $2M = \frac{\sqrt{2}|k|}{\nu^2}$ . We can recover the full Schwarzschild metric by making a time reparametrization  $s \rightarrow r$  that transforms  $B[s(r)] = r$  (which is legitimate because, as is easy to check, on shell  $\dot{B}$  is definite, and  $B$  is therefore monotonic), which gives

$$s = s_2 - \frac{1}{\sqrt{2}k} \log \left( \frac{2M}{r} - 1 \right), \quad (13)$$

and transforms the lapse into

$$N[s(r)] \frac{ds}{dr} = \left( \frac{\sqrt{2}|k|}{\nu^2 r} - 1 \right)^{-1/2} = \left( \frac{2M}{r} - 1 \right)^{-1/2}. \quad (14)$$

Notice that all solutions (8) represent a Schwarzschild spacetime. Those whose integration constants fail to satisfy (11) are just associated to a rescaled metric,

$$N = \alpha \left( \frac{2M}{r} - 1 \right)^{-1/2}, \quad A = \alpha \left( \frac{2M}{r} - 1 \right)^{1/2}, \quad B = \alpha r, \quad (15)$$

where  $\alpha = \nu e^{\frac{k(s_1-s_2)-x_0}{\sqrt{2}}}$ , and therefore a redefinition of units can reabsorb this.

The time redefinition (13) gives us another way to identify the values of  $s$  corresponding to the singularity (which is at  $r \rightarrow 0$ ). The reparametrization monotonically

<sup>3</sup>In this section, we assume that  $p_x = k$ , the other case  $p_x = -k$  can be straightforwardly worked out analogously.

maps  $r \in (0, 2M)$  into  $\text{sign}(k)s \in (-\infty, \infty)$ . The singularity  $r \rightarrow 0^+$  coincides with  $\text{sign}(k)s \rightarrow -\infty$ , i.e., when  $x \rightarrow -\infty$ . The other limit  $x \rightarrow +\infty$  coincides with the horizon  $r \rightarrow (2M)^-$ .

#### IV. SHAPE SPACE AND ORIENTATION

One linear combination of the  $x$  and  $y$  variables corresponds to the scale degree of freedom, while the other is conformally invariant and determines the shape of our spatial hypersurface (in particular, it determines the ratio between the radial extension of our coordinate patch and its areal radius). To disentangle scale and shape, consider the determinant of the spatial metric,  $\det g = A^2 B^4 = \nu^{-6} e^{\sqrt{2}(x+2y)}$ , which is a pure scale degree of freedom. Therefore  $x + 2y$  determines the scale, while the orthogonal direction in the  $(x, y)$  plane determines the shape. The following linear canonical transformation separates between scale  $z$  and shape  $w$ ,

$$\begin{aligned} x &= \frac{1}{\sqrt{3}}(2w - z), & y &= \frac{1}{\sqrt{3}}(2z - w), \\ p_x &= \frac{p_z + 2p_w}{\sqrt{3}}, & p_y &= \frac{2p_z + p_w}{\sqrt{3}}, \end{aligned} \quad (16)$$

so that now  $\det g = \nu^{-6} e^{\sqrt{6}z}$  depends on  $z$  alone. In the new variables, the Hamiltonian constraint takes a simple form,

$$H = \frac{1}{2}(p_w^2 - p_z^2) - e^{\sqrt{\frac{2}{3}}(2z-w)}, \quad (17)$$

notice that, as usual in a constant-mean-extrinsic-curvature foliation, the scale degree of freedom gives a negative contribution to the kinetic term [24].

Notice now that the coordinate change from the  $(w, z)$  variables to the original  $(A, B)$  variables,

$$A = \frac{e^{\frac{1}{\sqrt{6}}(z-2w)}}{\nu}, \quad B = \frac{e^{\frac{1}{\sqrt{6}}(w+z)}}{\nu}, \quad (18)$$

is not surjective; it only maps  $\mathbb{R}^2$  to the first quadrant ( $A > 0, B > 0$ ) of  $\mathbb{R}^2$ . Normally this would not be a problem, because the metric (1) depends only on the square of  $A$  and  $B$ , and the configuration space of Kantowski-Sachs metrics is more appropriately defined as the quotient of the  $(A, B)$  plane by reflections of  $A$  and  $B$ . However, there is a bit of information that is erased by this quotienting procedure, which we might want to keep track of instead. This is the orientation of our spatial manifold, which is encoded, for example, in the triad formulation of the metric [25]

$$g_{ij} = \delta_{ab} e^a_i e^b_j, \quad (19)$$

the associated volume form  $e^1 \wedge e^2 \wedge e^3$  defines an orientation on our manifold. In this formulation, under the

Kantowski-Sachs ansatz the frame field components are linear in  $A$  and  $B$ , and the volume form reads  $e^1 \wedge e^2 \wedge e^3 = AB^2$ . Therefore, the sign of  $A$  determines the orientation of our spatial hypersurface.

The variable  $z$  parametrizes the scale degree of freedom, while  $w$  determines the shape of our spatial manifold, and it makes sense to include the information regarding the orientation into the ‘‘shape space’’ of our model [5,24]. We can then extend the shape space, by defining two coordinate patches,  $w_+ \in \mathbb{R}$  and  $w_- \in \mathbb{R}$ , which are mapped to the two possible signs of  $A$ ,

$$A = \begin{cases} \frac{e^{\frac{1}{\sqrt{6}}(z-2w_+)}}{\nu}, & \text{if } A > 0, \\ -\frac{e^{\frac{1}{\sqrt{6}}(z-2w_-)}}{\nu}, & \text{if } A < 0. \end{cases} \quad (20)$$

The above map sends two copies of  $\mathbb{R}$  onto the two halves of the real line.

The two possible signs of  $A$  correspond to the choice between left- and right-handed triads compatible with the metric,  $e_{\pm}$ . Taking the Schwarzschild solution as our guide we expect that, as we approach the singularity  $|A| \rightarrow \infty$ . The singularity is potentially a point of transition between  $e_+$  and  $e_-$ , hence a point at which the orientation of our space may change. By extending our description to the coordinate patches  $w_{\pm}$  we allow for our dynamics to distinguish between orientations.

At this point we could propose a continuation theorem along the lines of what was done in Bianchi IX [5–7], however such a theorem would be, in the present case, trivial. This is because the theorem of [5–7] depends on the presence of more than one shape degree of freedom, and it becomes trivial in the case of a one-dimensional shape space. In fact, at the core of the continuation result, is the fact that one can decouple the scale degree of freedom (which is singular at the singularity) from the shape ones, and express the dynamics as a differential system in which the change of one shape degree of freedom is expressed in terms of the change in the others. This is the fundamental idea behind the ‘‘shape dynamics’’ formulation of General Relativity [24], and the papers [5–7] show how this intrinsic dynamics of pure shapes is regular at the singularity and can be continued deterministically through it. However, when we have only one shape degree of freedom, its change cannot be expressed in terms of other shape degrees of freedom. The intrinsic shape dynamics reduces to the prediction of an unparametrized curve on a one-dimensional manifold (a circle), and there is only one such curve. The fact that this curve continues through the singularity (which is located at a particular point on the circle) is a trivial statement.

For this reason, we are compelled to add some more shape degrees of freedom, in order to have a shape space of dimension at least two, where the fact that the intrinsic shape dynamics continues uniquely through the singularity



is a nontrivial statement. The simplest way to do this is to add a homogeneous scalar field, which contributes with one shape degree of freedom. Notice that in [5,7] too we were forced to add (at least) one scalar field, but for a different reason. In fact, in these papers we were interested in the Bianchi IX cosmological model, which already comes equipped with a two-dimensional shape space. However, unless a stiff matter source is added, this model has an essential singularity at the big bang, which makes continuation impossible. The the simplest form of stiff matter is a scalar field without mass or potential, the addition of which causes the system to transition to a state that is known as “quiescence”, after which the dynamics ceases to be chaotic and admits a deterministic continuation through the singularity. In the present case, we add the scalar field just because we need additional scale degrees of freedom and that is the simplest option. The dynamics of the Kantowski-Sachs model can be continued through the singularity independently of the presence of scalar fields or stiff matter sources, because it is not chaotic like Bianchi IX.

## V. HOMOGENEOUS SCALAR FIELD

To include a homogeneous scalar field to the Einstein-Hilbert Lagrangian (3) we need to add the following term,

$$\begin{aligned} L_\varphi &= - \int d^3x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi) \right] \\ &= 4\pi\lambda AB^2 \left[ \frac{1}{2} N^{-1} (\dot{\varphi})^2 - NV(\varphi) \right]. \end{aligned} \quad (21)$$

Notice that the homogeneous ansatz for the scalar field corresponds, in the limit  $ks \rightarrow \infty$  ( $r \rightarrow (2M)^-$ ), to a field that is constant on the horizon. This could be taken as the  $s$ -wave contribution in an expansion in spherical harmonics around a Schwarzschild background. We can now show how the Hamiltonian (6) generalizes in presence of a minimally-coupled homogeneous scalar field  $\varphi$  (with the convenient choice of lapse  $N = \frac{1}{v} e^{\sqrt{\frac{2}{3}}z}$ ),

$$H = \frac{1}{2} \left( p_w^2 - p_z^2 + \frac{1}{\kappa} \pi_\varphi^2 \right) + U(z, w, \varphi), \quad (22)$$

where  $\pi_\varphi$  is the momentum canonically conjugate to  $\varphi$ , and

$$U(z, w, \varphi) = -e^{\sqrt{\frac{2}{3}}(2z-w)} + \frac{\kappa}{v^2} e^{\sqrt{6}z} V(\varphi), \quad (23)$$

is the sum of the geometric and the scalar field potentials.

The first thing to notice is that the scalar field potential term is coupled to the scale degree of freedom  $z$  only, and in such a way that large negative values of  $z$  make its contribution to the potential negligible. Moreover, if  $V(\varphi)$  is positive, this potential has the opposite sign of

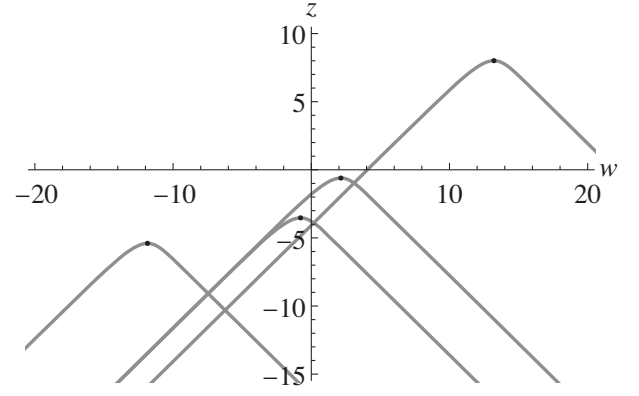


FIG. 3. The same solutions of Fig. 2, this time plotted in the  $w$ - $z$  plane. The singularity is reached asymptotically as  $w \rightarrow -\infty$ , while the horizon is at  $w \rightarrow +\infty$ .

the geometric potential, and as  $z$  grows it quickly dominates it. So it has the ability to reverse the collapse of the solution towards the singularity. Given a solution evolving from the horizon (the region  $w \gg 1$  and  $z \ll -1$ ), it is not guaranteed that it will reach a maximum value of  $z$  and bounce off the potential like in the plots of Fig. 3. If the initial conditions are right, it might get captured by the  $\frac{\kappa}{v^2} e^{\sqrt{6}z} V(\varphi)$  potential, and run off towards the positive- $z$  direction. As is well known, the spherically-symmetric Einstein-scalar system has a nontrivial dynamics, and the general conditions under which gravitational collapse leads to a singularity are not simple to express [26,27].

At any rate, we are interested only in those solutions which do reach the singularity, and depending on the form of  $V(\varphi)$ ,<sup>4</sup> there will be large classes of solutions which run off in the negative- $z$  and negative- $w$  directions, so that the  $e^{\sqrt{2}(x+2y)} V(\varphi)$  term asymptotes to zero, and  $z$  and  $w$  asymptote to the straight-line motion that ends in the singularity at  $w \rightarrow -\infty$ . Calculating the Kretschmann scalar (10) in this case, where  $p_x^2 \leq k^2$ , reveals that such runoff solutions end up in a curvature singularity just like in the matter-free case.

The solutions we are interested in asymptote to a free dynamics for all the configurational variables,  $z$ ,  $u$ , and  $w$ . The whole potential term  $U(z, w, \varphi)$  becomes negligible near the singularity, and the solutions are identical to Eq. (3), with the addition of  $\varphi = p_u(s - s_3)$ ,  $p_u = \text{const}$ . What changes is the form of the Hamiltonian constraint. In the variables  $x$ ,  $y$  used in Eq. (3) it takes the form,

$$p_x^2 + \frac{1}{\kappa} \pi_\varphi^2 - k^2 = 0, \quad (24)$$

<sup>4</sup>The ability of scalar field potentials to change the behavior of solutions near a singularity has been discussed also in the context of Bianchi IX cosmological models in [7].

which implies that the asymptotic motion in the  $x$ - $y$  plane is not at  $45^\circ$  like in Eq. (3), but at a steeper angle. Equation (24) has another consequence; the Ricci tensor vanishes only when  $\pi_\varphi = 0$ . It is only in absence of the scalar field that spacetime is Ricci-flat, and isometric to the Schwarzschild metric (a well-known fact).

## VI. COMPACTIFICATION OF SHAPE SPACE WITH ORIENTATION AND CONTINUATION THROUGH THE SINGULARITY

First, it is convenient to change the scalar field variable  $\varphi$  to a dimensionless one, by means of the following canonical transformation,

$$u = \sqrt{\kappa}\varphi, \quad p_u = \pi_\varphi/\sqrt{\kappa}. \quad (25)$$

Then, we can repeat the transformation (16) in order to separate scale and shape degrees of freedom. In the new variables, the Hamiltonian constraint takes this form,

$$H = \frac{1}{2}(p_w^2 + p_u^2 - p_z^2) + U(w, u, z). \quad (26)$$

The map (20) still applies in presence of a scalar field, however now the two fixed-orientation shape spaces are two-dimensional planes, coordinatized by  $(w_-, u_-) \in \mathbb{R}^2$  and  $(w_+, u_+) \in \mathbb{R}^2$ . This also extends to any number of additional fields; the shape space consists of two  $N$ -dimensional hyperplanes, one for each orientation.

We can now discuss one of the crucial steps allowing us to establish a continuation result; as we did in [5–7], we impose a particular topology on shape-space-with-orientation, which joins the borders of its two fixed-orientation connected components, making the overall space connected. This is done by compactifying shape space through the gnomonic projection; each of the two fixed-orientation planes is mapped onto one of the hemispheres of a 2-sphere, with the origins mapped to the two poles, and the asymptotic borders mapped to the equator (see Fig. 4). The gnomonic projection maps the coordinates  $(w_\pm, u_\pm)$  into the spherical coordinates  $\beta \in [0, \pi]$  and  $\alpha \in [0, 2\pi]$  as follows:

$$|\tan \beta|(\cos \alpha, \sin \alpha) = \begin{cases} (w_+, u_+), & \text{if } \beta < \pi/2, \\ (w_-, u_-), & \text{if } \beta > \pi/2. \end{cases} \quad (27)$$

The equations of motion for the new angular variables are

$$\dot{\alpha} = \cot^2 \beta \lambda \quad \dot{\beta} = \cos^2 \beta \cot \beta \delta, \quad (28)$$

where

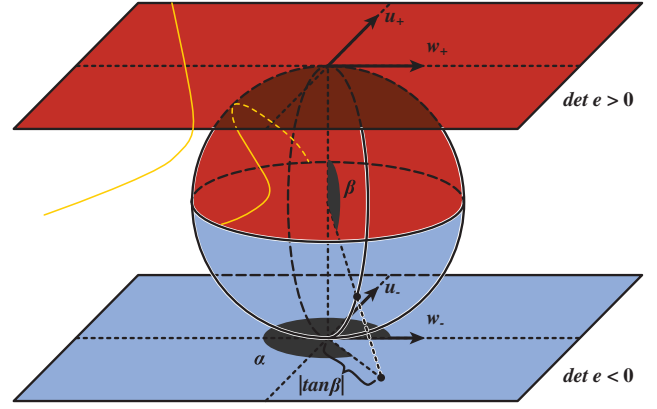


FIG. 4. Shape space with orientation; each hemisphere represents an orientation, and each point on the sphere represents different values of the shape degrees of freedom  $(w, u)$ . The poles coincide with the value  $u = w = 0$ , while the equator corresponds to the border of the  $(w, u)$  plane at infinity. A solution curve is shown on the top plane, together with its projection on the northern hemisphere.

$$\lambda = wp_u - up_w, \quad \delta = wp_w + up_u. \quad (29)$$

The quantity  $\lambda$  is the angular momentum on the  $w$ - $u$  plane, and it is asymptotically conserved (because inertial motion conserves it). The quantity  $\delta$  is neither conserved nor finite at the singularity, and the same holds for the remaining variable  $z$ . We then need to find alternative variables which take finite values at the singularity. In complete analogy with the cosmological models discussed in [5–7], we introduce the following variables,

$$\rho = \cot \beta \delta, \quad \eta = \text{sign}(\tan \beta)z + \frac{|\tan \beta|}{\rho} p_z, \quad (30)$$

finally, we need to consider the fact that the gnomonic projection we introduced glues the negative- $w$  region of one plane with the negative- $w$  region of the other. So, solutions that asymptote to the singularity (as opposed to the horizon), will be glued to solutions that come out of the singularity (with opposite orientation). The volume degree of freedom will have to go to zero on one side, and come out of zero on the other side. Then  $z$  will have to be decreasing (that is,  $p_z > 0$ ) on one side, and then increasing  $p_z < 0$  on the other. The variable  $p_z$  cannot be continuous at the singularity, so we need to replace it with

$$\kappa = -\text{sign}(\tan \beta) p_z. \quad (31)$$

With the variables just defined, one can show that the following equations of motion [when we write  $\frac{\partial U}{\partial z}$  we mean that one first takes the  $z$ -derivative of  $U(w, u, z)$ , and then converts the variables into  $\alpha, \beta, \eta$ ]

$$\begin{aligned} \dot{\alpha} &= \cot^2 \beta \lambda, & \dot{\beta} &= \cos^2 \beta \rho, & \dot{\lambda} &= -\frac{\partial U}{\partial \alpha}, \\ \dot{\eta} &= \frac{1}{\rho^2} \left[ \cot^2 \beta \kappa \lambda^2 - |\tan \beta| \rho \frac{\partial U}{\partial z} - \sin \beta \cos \beta \kappa \frac{\partial U}{\partial \beta} \right], & \dot{\rho} &= \cot^3 \beta \lambda^2 - \cos^2 \beta \frac{\partial U}{\partial \beta}, & \dot{\kappa} &= \text{sign}(\tan \beta) \frac{\partial U}{\partial z}, \end{aligned} \quad (32)$$

are equivalent to the original ones everywhere on the sphere (except at the equator, where the original equations of motion are singular).

The final step is to consider that the parameter time  $s$  diverges at the singularity, so it needs to be compactified. A physically sensible choice is to use a quantity on the shape sphere of the system, to be used as independent variable. The arc of length on the shape sphere is particularly well suited, because it is monotonic everywhere on the solution curves,

$$d\ell = \sqrt{\sin^2 \beta d\alpha^2 + d\beta^2} \Rightarrow \dot{\ell} = \frac{\cos^2 \beta}{\sin \beta} \sqrt{\lambda^2 + \rho^2 \sin^2 \beta}. \quad (33)$$

With respect to this “internal time variable”, the equations of motion take the form

$$\begin{aligned} \frac{d\alpha}{d\ell} &= \frac{\lambda}{\sin \beta \sqrt{\lambda^2 + \rho^2 \sin^2 \beta}}, \\ \frac{d\beta}{d\ell} &= \frac{\rho \sin \beta}{\sqrt{\lambda^2 + \rho^2 \sin^2 \beta}}, \\ \frac{d\eta}{d\ell} &= \frac{1}{\rho^2 \sqrt{\lambda^2 + \rho^2 \sin^2 \beta}} \left[ \frac{\kappa \lambda^2}{\sin \beta} - \frac{\sin^2 \beta}{|\cos^3 \beta|} \rho \frac{\partial u}{\partial z} - \kappa \sin \beta \tan \beta \frac{\partial U}{\partial \beta} \right], \\ \frac{d\lambda}{d\ell} &= -\frac{\sin \beta}{\cos^2 \beta \sqrt{\lambda^2 + \rho^2 \sin^2 \beta}} \frac{\partial U}{\partial \alpha}, \\ \frac{d\rho}{d\ell} &= \frac{1}{\sqrt{\lambda^2 + \rho^2 \sin^2 \beta}} \left( \frac{\cos \beta}{\sin^2 \beta} \lambda^2 - \sin \beta \frac{\partial U}{\partial \beta} \right), \\ \frac{d\kappa}{d\ell} &= \frac{\sin \beta \text{sign}(\tan \beta)}{\cos^2 \beta \sqrt{\lambda^2 + \rho^2 \sin^2 \beta}} \frac{\partial U}{\partial z}. \end{aligned} \quad (34)$$

The Hamiltonian constraint, in these variables, takes the form,

$$\mathcal{H} = \frac{1}{2} (-\kappa^2 + \rho^2 + \cot^2 \beta \lambda^2) + U(\alpha, \beta, \eta) \approx 0, \quad (35)$$

which is regular at the equator, where  $U(\alpha, \beta, \eta) \rightarrow 0$ .

Just as in our previous results [5–7], Eq. (34) satisfy the assumptions of the existence and uniqueness theorem (the Picard-Lindelöf theorem) for solutions of ordinary differential equations, and therefore, to each solution reaching the singularity from one hemisphere we can associate one and only one solution reaching the same point on the equator from the other hemisphere.

The Schwarzschild solution is a special case of the above system, in which there is no matter potential ( $U = 0$ ) and no scalar field momentum. In such a case it can be verified

that  $\alpha = p_\alpha = 0$  is a solution to the equations of motion, which is represented by a great circle through the poles on the shape sphere. At the equator the solution continues along the great circle and crosses from one hemisphere to the other. On each hemisphere of shape space, the solution describes a black hole interior with either a left- or right-handed triad. The Picard-Lindelöf theorem shows then that there is a unique continuation of the Schwarzschild interior beyond the singularity—it is an orientation-flipped interior of an otherwise identical black hole.

To make the result clearer, we can study the system in the vicinity of the singularity. As we approach the singularity  $\beta \rightarrow (\frac{\pi}{2})^\pm$ , all the terms containing the potential  $U$  or its derivatives are exponentially suppressed (they are all multiplied to exponentials of negative constants times  $\tan \beta$ ). The equations of motion tend to the free equations,

$$\begin{aligned}
\frac{d\alpha}{d\ell} &= \frac{\lambda}{\sin\beta\sqrt{\lambda^2 + \rho^2\sin^2\beta}}, \\
\frac{d\beta}{d\ell} &= \frac{\rho\sin\beta}{\sqrt{\lambda^2 + \rho^2\sin^2\beta}}, \\
\frac{d\eta}{d\ell} &= \frac{\kappa\lambda^2}{\rho^2\sin\beta\sqrt{\lambda^2 + \rho^2\sin^2\beta}}, \\
\frac{d\rho}{d\ell} &= \frac{\lambda^2\cos\beta}{\sin^2\beta\sqrt{\lambda^2 + \rho^2\sin^2\beta}}, \\
\frac{d\lambda}{d\ell} &= \frac{d\kappa}{d\ell} = 0.
\end{aligned} \tag{36}$$

Consider now a small region around the equator,  $\beta \sim \pi/2 + \delta\beta$ ,  $\cos\beta \sim -\delta\beta$ ,  $\sin\beta \sim 1 + \mathcal{O}(\delta\beta^2)$ . The equations of motion approximate to

$$\begin{aligned}
\frac{d\alpha}{d\ell} &\simeq \frac{\lambda}{\sqrt{\lambda^2 + \rho^2}}, & \frac{d\delta\beta}{d\ell} &\simeq \frac{\rho}{\sqrt{\lambda^2 + \rho^2}}, \\
\frac{d\eta}{d\ell} &\simeq \frac{1}{\rho^2\sqrt{\lambda^2 + \rho^2}}\kappa\lambda^2, & \frac{d\rho}{d\ell} &\simeq \frac{\delta\beta}{\sqrt{\lambda^2 + \rho^2}}\lambda^2, \\
\frac{d\lambda}{d\ell} &\simeq 0, & \frac{d\kappa}{d\ell} &\simeq 0,
\end{aligned} \tag{37}$$

and the Hamiltonian constraint

$$\mathcal{H} \simeq \frac{1}{2}(-\kappa^2 + \rho^2 + \delta\beta^2\lambda^2) \approx 0, \tag{38}$$

and we can plug the solution to the Hamiltonian constraint into the right-hand side of the equations of motion as follows:

$$\begin{aligned}
\frac{1}{\sqrt{\lambda^2 + \rho^2}} &\simeq \frac{1}{\sqrt{\lambda^2 + \kappa^2 - \delta\beta^2\lambda^2}} \\
&\simeq \frac{1}{\sqrt{\lambda^2 + \kappa^2}} \left( 1 + \frac{\lambda^2}{2(\lambda^2 + \kappa^2)}\delta\beta^2 \right),
\end{aligned} \tag{39}$$

as well as

$$\begin{aligned}
\frac{1}{\rho^2\sqrt{\lambda^2 + \rho^2}} &\simeq \frac{1}{\sqrt{\lambda^2 + \kappa^2 - \delta\beta^2\lambda^2}} \\
&\simeq \frac{1}{\kappa^2\sqrt{\kappa^2 + \lambda^2}} + \frac{\delta\beta^2\lambda^2(3\kappa^2 + 2\lambda^2)}{2\kappa^4(\kappa^2 + \lambda^2)^{3/2}},
\end{aligned} \tag{40}$$

$$\frac{\rho}{\sqrt{\lambda^2 + \rho^2}} \simeq \pm \frac{|\kappa|}{\sqrt{\lambda^2 + \kappa^2}} - \frac{\delta\beta^2\lambda^4}{2(\sqrt{\kappa^2}(\kappa^2 + \lambda^2)^{3/2})}, \tag{41}$$

where we have to choose the sign plus because  $\rho$  needs to be positive. So, at first order in  $\delta\beta$ ,

$$\begin{aligned}
\frac{d\alpha}{d\ell} &\simeq \frac{\lambda}{\sqrt{\lambda^2 + \kappa^2}}, & \frac{d\delta\beta}{d\ell} &\simeq \frac{\kappa}{\sqrt{\lambda^2 + \kappa^2}}, & \frac{d\eta}{d\ell} &\simeq \frac{\lambda^2}{\kappa\sqrt{\lambda^2 + \kappa^2}}, \\
\frac{d\lambda}{d\ell} &\simeq 0, & \frac{d\kappa}{d\ell} &\simeq 0.
\end{aligned} \tag{42}$$

with the approximate solution to the Hamiltonian constraint  $\rho \simeq \kappa$ . The solutions of Eq. (42) are

$$\begin{aligned}
\alpha &\simeq \alpha_0 + \frac{\lambda_0\sqrt{\lambda_0^2 + \kappa_0^2}}{\kappa_0}(\ell - \ell_0), & \beta &\simeq \frac{\pi}{2} + \frac{\kappa_0}{\sqrt{\lambda_0^2 + \kappa_0^2}}(\ell - \ell_0), \\
\eta &\simeq \eta_0 + \frac{\lambda_0^2}{\kappa_0\sqrt{\lambda_0^2 + \kappa_0^2}}(\ell - \ell_0), & \rho &\simeq \kappa_0, & \lambda &\simeq \lambda_0, & \kappa &\simeq \kappa_0,
\end{aligned} \tag{43}$$

where we chose the integration constants so that when  $\ell = \ell_0$ , we are at the equator  $\beta = \pi/2$ . The solution above, written in terms of the original variables  $w, u, z, p_w, p_u, p_z$  is

$$\begin{aligned}
x^0 &= \text{sign}(\tan\beta) \left[ \eta + \frac{\kappa}{\rho}\tan\beta \right], & x^1 &= |\tan\beta|\cos\alpha, & x^2 &= |\tan\beta|\sin\alpha, \\
p_0 &= -\text{sign}(\tan\beta)\kappa, & p_1 &= -\frac{\sin\alpha\lambda}{|\tan\beta|} + \text{sign}(\tan\beta)\cos\alpha\rho, & p_2 &= \frac{\cos\alpha\lambda}{|\tan\beta|} + \text{sign}(\tan\beta)\sin\alpha\rho,
\end{aligned} \tag{44}$$

$$\begin{aligned}
z &\simeq \frac{\sqrt{\lambda_0^2 + \kappa_0^2}}{\kappa_0}|\ell - \ell_0|^{-1} + \text{sign}(\ell_0 - \ell)\eta_0 + \frac{\lambda_0^2/\kappa_0 + \kappa_0/3}{\sqrt{\lambda_0^2 + \kappa_0^2}}|\ell - \ell_0| + \mathcal{O}(\ell - \ell_0)^3, \\
(w, u) &\simeq (\cos\alpha_0, \sin\alpha_0) \frac{\sqrt{\lambda_0^2 + \kappa_0^2}}{\kappa_0}|\ell - \ell_0|^{-1} + (-\sin\alpha_0, \cos\alpha_0) \frac{\lambda_0(\lambda_0^2 + \kappa_0^2)}{\kappa_0^2}\text{sign}(\ell - \ell_0) + \mathcal{O}(\ell - \ell_0), \\
p_z &\simeq \kappa_0\text{sign}(\ell - \ell_0), \\
(p_w, p_u) &\simeq \text{sign}(\ell - \ell_0)\kappa_0(\cos\alpha_0, \sin\alpha_0) + \mathcal{O}(\ell - \ell_0).
\end{aligned} \tag{45}$$



The above solution has the following characteristics:

- (a) the momenta  $p_w, p_u, p_z$  all change sign upon crossing the equator,
- (b) the coordinates  $w, u, z$  all diverge with the same sign, as we approach the equator from the two hemispheres, so it represents two (asymptotic) solutions of the original system with opposite orientations, joined at the singularity.

## VII. DISCUSSION

Our generalized dynamical system allows to continue singular solutions through the Schwarzschild singularity uniquely. As can be deduced by looking at the shape sphere in Fig. 4, a great circle that crosses the equator would not be invariant under reflections with respect to the equator's plane (unless we are in the special case of a vertical, "meridian" circle). Then the solution continues to one that is objectively different; it is not simply the time-reversed repetition of the initial solution. After crossing the singularity, the shape degrees of freedom  $w$  and  $u$  will have a different evolution and will go through different pairs of values.

A legitimate question, at this point, is: what is the structure of the spacetime that corresponds to these continued solutions? Thanks to the work of Christodoulou and Sbierski [18,21], a rather strong formulation of the strong cosmic censorship conjecture for the spherically symmetric Einstein-scalar field system is proven, so we know that there can be no continuous spacetime extending the Schwarzschild singularity. Our picture is that of two spacetimes that are smooth everywhere except for a singularity, which are glued in a particular way at said singularity. Or, alternatively, we can talk about a single nonorientable spacetime with a singular hypersurface. The causal structure of such spacetime is entirely codified in the evolution of the shape variable  $w$ . This reflects the fact that within our ontology, spacetime is not fundamental—the shape degrees of freedom are, and these determine all observations. Spacetime is a useful descriptive tool, and the singularity represents a failure of this tool, not a breakdown in the determinism of the fundamental physics. We know the causal structure associated to any half of each solution that is confined to one hemisphere; it is that of the region of Schwarzschild's spacetime that is inside the horizon (the shaded region in Fig. 1). A full solution can then be associated to two such causal patches, and it is tempting to glue them at the singularity in the manner of Fig. 5; one has two regions with opposite spatial orientations, looking like a black hole interior glued to a white hole interior. Extending these spacetimes beyond the horizons, one finds two asymptotically flat regions of opposite orientations, one in the causal past and one in the future.

This picture, however, is tentative and does not necessarily reflect actual physics. A Penrose diagram makes sense as an effective description of the causal relations

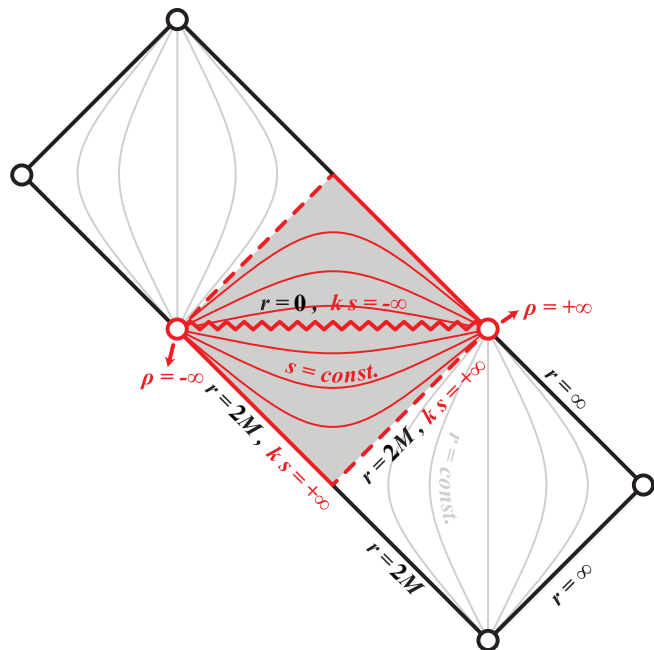


FIG. 5. The continuation of the Schwarzschild solution. At the singularity, the shape system remains well defined, and connects two Schwarzschild interiors described by right- and left-handed triads.

between test particles propagating in a background spacetime, in a regime in which the backreaction of the particles on the geometry can be neglected. This is a reasonable assumption around most points in the Penrose diagram 5, but not in the vicinity of the singularity. We cannot say, at the moment, what a test particle would experience upon crossing the singularity: that would need a dedicated analysis. Until that is done, we cannot be sure that timelike worldlines would behave smoothly at the singularity in the Penrose diagram 5, and therefore the physical meaning of that diagram remains unclear.

This paper has shown how spacelike singularities at the center of black holes do not represent the end of the determinism of the solution. Together with [5–7], this hints that the resolution of spacelike singularities may be a generic feature of the relational approach. However, this is far from the end of the problem of singularities. The Hawking-Penrose theorems still hold, and as yet it is not known how to extend geodesics beyond the singularity itself. Recent work [28], see also [29,30], has shown that despite these problems, given some extensions of spacetime beyond a singularity certain quantum matter degrees of freedom can be deterministically evolved beyond these points. A tantalizing prospect is that relational descriptions may resolve the issues of singularities entirely classically. The ramifications for quantum gravity searches, many of which have their sights set on resolution of singularities, would be profound.

Another issue that should be investigated before proposing causal structures for our singularity-crossing solutions (and, in particular, before extending these structures outside of the horizons, is the fact that the Schwarzschild spacetime represents an eternal black hole, while realistic black holes are created through the collapse of matter. This is better discussed within a matter collapse model that creates the black hole metric in its wake (e.g., a thin-shell [31,32] or a Lemaître-Tolman-Bondi model). Then, the study of the behavior of the collapsing matter upon crossing the singularity should

reveal the nature of the region beyond the singularity. A compelling possibility is that the singularity turns the collapse of the matter into an expansion, and the expanding matter leaves behind a pocket of spacetime with a white-hole metric.

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