Can late-time extensions solve the H_0 and σ_8 tensions?

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We analyze the properties that any late-time modification of the ΛCDM expansion history must have in order to consistently solve both the H_0 and the σ_8 tensions. Taking a model-independent approach, we obtain a set of necessary conditions that can be applied to any late-time extension whose main effect is a deviation from the ΛCDM background. Our results are fully analytical and merely based on the assumptions that the deviations from the ΛCDM background remain small. For the concrete case of a dark energy fluid with equation of state $w(z)$, we derive the following general requirements: (i) Solving the H_0 tension demands $w(z) < -1$ at some z (ii) Solving both the H_0 and σ_8 tensions requires $w(z)$ to cross the phantom divide. Finally, we also allow for small deviations on the effective gravitational constant. In this case, our method is still able to constrain the functional form of these deviations.

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I. INTRODUCTION

Historically, the quest for a satisfactory description of our universe has always been guided by the latest observational data. It is therefore not surprising that the dramatic increase of the quantity and quality of cosmological observations over the last 25 years has allowed for a revolution on the theoretical side as well. The ΛCDM model has emerged as the leading theoretical description of cosmic evolution, explaining key features, such as the distribution of the cosmic microwave background (CMB) anisotropies, with only a few free parameters. However, despite its success, several observations have been, and still are, hard to account for within this paradigm. In particular, the well-known H_0 tension, that is the discrepancy of the Hubble constant inferred from the CMB within ΛCDM [1] compared to the results of local measurements [2–5], has become increasingly worrying in recent years and it is hard to disregard it as a simple statistical fluke. At this point, either there is something wrong with different, independent observations or we must change the theoretical framework to interpret them.

Another mayor concern in the community is the σ_8 tension which, just as the H_0 tension, arises when comparing the CMB-inferred value of the clustering amplitude to alternative observations, in this case large scale structure (LSS) surveys [6–9]. Focusing on these two parameters and disregarding correlations with other parameters, we may say that CMB data favors a lower value of H_0 while at the same time preferring a higher σ_8 value compared to late-time measurements. See [10] for a compilation of recent measurements. Finally note that this tension is commonly referred to as S_8 tension as well, where $S_8 = \sigma_8\sqrt{\Omega_m/0.3}$, since this combination is better constrained by weak lensing surveys.

In recent years, great efforts have been made toward solving the H_0 tension, see e.g., [11–24], as well as the σ_8 tension, see e.g., [10,25–34] (see also [35–40] for an overview of observations and models). While dark energy models have received special attention, the proposals range over ideas of introducing primordial magnetic fields modifying the recombination history [41], over departures from isotropy or homogeneity [39], or considering spacial curvature [42], all the way to introducing new interactions in the dark sector [27,43]. Yet, typically the main effect on H_0 boil down to modifications of the CMB angular scale through departures from the ΛCDM predictions for either the comoving sound horizon or the conformal distance to decoupling, which in many models are sourced by modifications of the expansion history. In this work we choose to focus on the latter and therefore restrict our analysis to modifications of the late-time Hubble parameter with a main application to late-time dark energy. Note, however, that our method is not *a priori* tied to dark energy models and in principle applies to any late time modification of the ΛCDM expansion history, as long as other effects remain negligible. As a particular but relevant example of an additional effect we will in a second step also consider possible changes in the effective gravitational constant at the level of perturbations.

While there exist an abounding amount of proposed solutions to the Hubble tension as mentioned above, it is

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known that most of the attempts run into problems when trying to be consistent with complementary observations. In particular, typical late-time solutions to the H_0 tension, as for example models based on scalar or vector Galileons [44–48], usually lead to an even larger value of σ_8 than within ΛCDM and therefore potentially increase the σ_8 tension. One should however keep in mind that the σ_8 tension is only properly addressed as a tension in the σ_8-Ω_m plane, or in the general multidimensional posterior. While these models predict a slightly larger σ_8 they also prefer smaller values for Ω_m, a result in the line of weak-lensing surveys. However, even if one could argue that strictly speaking such late-time solutions to the Hubble tension are still statistically compatible with current σ_8 measurements, they clearly show the wrong trend, such that they will most likely be difficult to reconcile with a low σ_8 value if it is confirmed by the more precise measurements of the next generation of LSS surveys.

This generic trend, that late-time dark energy models easing the Hubble tension predominantly increase σ_8 as well, can be understood as follows. A realistic dark energy model typically affects σ_8 in two ways: (1) Through its effects on the expansion history, i.e., modifications of the background equation of state. (2) Through its clustering properties, i.e., clustering dark energy that can modify the effective Newton constant G_{eff} which governs the evolution of the matter growth function. In the models mentioned above with a phantom equation of state both effects contribute to an increase in σ_8: (1) A phantom-like evolution of dark energy extends the matter-dominated phase, boosting the matter growth. (2) Dark energy clusters at late times, increasing G_{eff} and further boosting the amplitude of perturbations. The key point is that the same phantom-like equation of state that is crucial to solve the H_0 tension, can only worsen the σ_8 tension.

In face of these problems, one may wonder if it is even possible to solve both tensions modifying only the late-time dark energy behavior. Of course, in a consistent dark energy model one should not only study the background evolution, but the perturbations as well. And while the background evolution is governed by the dark energy equation of state $w(z)$, the perturbations are also affected by the dark energy sound speed $c_{\mathrm{s}}(z)$. Hence, at first sight one could conclude that with two arbitrary functions at hand it should not be difficult to find a dark energy model that solves both tensions at once. However, this is not the case since in realistic scenarios such as vector Galileons both functions are not independent and their observational impact is very different. In fact, while $c_{\mathrm{s}}(z)$ is relevant for observables like the ISW effect, the modifications in $w(z)$ are the main force driving the values of σ_8 and H_0. In the light of these considerations we will therefore narrow down the scope of this work by mostly neglecting the effects of dark energy perturbations and address the following main question:

Can the H_0 and σ_8 tensions be simultaneously relieved modifying only the dark energy equation of state $w(z)$ at late times?

We will show that the answer is no if the dark energy equation of state does not meet some very definite criteria. These conclusions apply to any dark energy model in which the perturbations do not play a leading role in the determination of σ_8. After this, we will generalize the results to the case where dark energy also affects the growth of structure through a change in the effective gravitational constant. Thus, our results provide valuable insights into the behavior of the dark sector and can be seen as hints toward building successful models beyond ΛCDM.

The main steps of the computation can be succinctly summarized as follows:

(i) We will start off with a late-time ΛCDM cosmology, that can effectively be described by two free parameters (h, ω_m), i.e., the Hubble constant and the matter abundance. In a second step, we consider an alternative cosmology with slightly different parameters and with a different expansion history $(h+\delta h,\ \omega_m+\delta\omega_m,\ \delta H(z))$. Here, $\delta H(z)$ is an *arbitrary* function that produces a small deformation of the ΛCDM expansion history, for fixed h and ω_m. Restricting ourselves to late time modifications translates into the assumption that $\delta H(z) = 0$ for roughly $z > 300$. With all the deformations considered to be small, the Hubble parameter in the alternative cosmology can be written as

$$H = H_{\Lambda\mathrm{CDM}} + \Delta H(\delta h, \delta\omega_m, \delta H(z)). \qquad (1)$$

(ii) Working to first order, we compute the variations induced by the modified Hubble parameter in different cosmological observables.

(iii) Because of the deformation $\delta H(z)$, the observationally preferred values for h and ω_m in the new cosmology will be different compared to the initial ΛCDM model. The variations δh and $\delta\omega_m$ can be related to $\delta H(z)$ by choosing two very well measured observables whose value should not change in the new cosmology, i.e., we impose their variation to be zero in order to be compatible with observations. This allows us to compute the response functions

$$\frac{\delta h}{h} = \int \mathcal{R}_h(z)\frac{\delta H(z)}{H(z)}\frac{\mathrm{d}z}{1+z}, \qquad (2a)$$

$$\frac{\delta\omega_m}{\omega_m} = \int \mathcal{R}_{\omega_m}(z)\frac{\delta H(z)}{H(z)}\frac{\mathrm{d}z}{1+z}. \qquad (2b)$$

In this work we will choose the variations of the CMB distance priors [49] to vanish. The response functions $\mathcal{R}_h$ and $\mathcal{R}_{\omega_m}$ are fully analytical and are defined in (22).

(iv) Based on the results above we can then compute the response function of any other quantity, and crucially in our case

$$\frac{\Delta\sigma_8}{\sigma_8} = \int \mathcal{R}_{\sigma_8}(z)\frac{\delta H(z)}{H(z)}\frac{\mathrm{d}z}{1+z}. \tag{3}$$

Depending on the shape of the response functions, this allows us to derive general requirements on the functional form of $\delta H(z)$ in order to achieve the desired variations in h and σ_8. Again, $\mathcal{R}_{\sigma_8}$ can be computed analytically and it will be derived in Sec. IV.

(v) In Sec. IV C, we will generalize these results and include a second free function, $\delta G_{\rm eff}(z)$, that affects the evolution of σ_8, computing its associated response function along the same lines.

This paper is organized as follows. In Sec. II we introduce the deformations of the background and most of the notation. Section III will cover the choice of observational data (CMB priors) used to compute the response functions. Section IV deals with the variations of the growth factor and σ_8. It also includes a generalization for models that modify the effective Newton constant. In Sec. V we address the resolution of the H_0 tension in more detail, analyzing the differences that arise when formulated in terms of the supernova absolute magnitude M. Section VI summarizes the results of this work. Appendix A collects all the analytical formulas used to derive the results in the main text. Finally, in Appendix B we present some tests performed to check the accuracy of the first-order, analytical results against the full numerical computation with a Boltzmann code for a particular dark energy model.

II. DEFORMATIONS OF THE EXPANSION HISTORY

The Hubble parameter in a flat ΛCDM model can be written as

$$\begin{aligned} H^2_{\Lambda\rm CDM} &= H_0^2(\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda) \\ &= C_H^2(\omega_m(1+z)^3 + \omega_r(1+z)^4 + \omega_\Lambda), \end{aligned} \tag{4}$$

where $C_H \equiv 100\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ and

$$\omega_\Lambda = h^2 - \omega_m - \omega_r. \tag{5}$$

Let us consider now a generic extension that slightly modifies the expansion history, for fixed values of all the cosmological parameters, so the new Hubble parameter is

$$H(h,\omega_m) = H_{\Lambda\rm CDM}(h,\omega_m) + \delta H. \tag{6}$$

This deformation of the expansion history will also shift the preferred values for the ΛCDM parameters, by a small amount. The observationally preferred background in ΛCDM and in the generic extension can then be related as

$$H(h+\delta h, \omega_m + \delta\omega_m) = H_{\Lambda\rm CDM}(h,\omega_m) + \Delta H, \tag{7}$$

where we are also assuming that δH only produces late-time changes so the cosmology can be effectively described by h and ω_m. Assuming that all these variations are small and working to first order we have

$$\begin{aligned} \frac{\Delta H}{H} &= \frac{H_0^2}{H^2}\frac{\delta h}{h} + m(z)\frac{\delta\omega_m}{\omega_m} + \frac{\delta H}{H}, \\ m(z) &\equiv \frac{\omega_m C_H^2}{2H^2}((1+z)^3 - 1). \end{aligned} \tag{8}$$

Starting with the general variation (8), we can propagate its effect to any cosmological observable. In general, for every cosmological quantity $g(z)$ we will express its variation as

$$\begin{aligned} \frac{\Delta g(z)}{g(z)} = {}& I_g(z)\frac{\delta h}{h} + J_g(z)\frac{\delta\omega_m}{\omega_m} \\ &+ \int_0^\infty \frac{\mathrm{d}x_z}{1+x_z} R_g(x_z,z)\frac{\delta H(x_z)}{H(x_z)}. \end{aligned} \tag{9}$$

For instance, from the definition of the conformal, luminosity and angular diameter distances,

$$\chi(z) = \int_0^z \frac{\mathrm{d}z}{H(z)}, \tag{10a}$$

$$d_L(z) = (1+z)\chi(z), \tag{10b}$$

$$d_A(z) = \frac{1}{1+z}\chi(z), \tag{10c}$$

we can easily compute

$$\begin{cases} I_\chi(z) = I_{d_L}(z) = I_{d_A}(z) = -\frac{1}{\chi(z)}\int_0^z \mathrm{d}x_z \frac{H_0^2}{H^3}, \\ J_\chi(z) = J_{d_L}(z) = J_{d_A}(z) = -\frac{1}{\chi(z)}\int_0^z \mathrm{d}x_z \frac{H_0^2}{H^3} m(x_z), \\ R_\chi(x_z,z) = R_{d_L}(x_z,z) = R_{d_A}(x_z,z) = -(1+x_z)\frac{\theta(z-x_z)}{\chi(z)H(x_z)}. \end{cases} \tag{11}$$

Notice that, since we are working to first order, every function like $\chi(z)$ and $H(z)$ can be computed in the base ΛCDM cosmology. The full analytical expressions for all the (I,J,R) functions that are used in this work are collected in Appendix A.

Finally, notice that we can also express the previous results in terms of a variation on the energy content

$$H^2(h,\omega_m) = H^2_{\Lambda\rm CDM}(h,\omega_m) + H_0^2\delta\Omega. \tag{12}$$

with $\delta\Omega = 0$ at $z=0$. Working again to first order we have

$$\frac{\delta H}{H} = \frac{H_0^2}{2H^2}\delta\Omega. \tag{13}$$

Note that we did not, and will not, specify a functional form for either δH or $\delta\Omega$. However, in particular models some additional properties may be desirable. For instance, if $\delta\Omega$ arises from a dark energy model, we may want to require that the dark energy density is positive, leading to

$$\Omega_{\rm DE}(z) \equiv \Omega_\Lambda + \delta\Omega(z) > 0 \quad \text{(DE model)} \tag{14}$$

In this case, we can also relate the variation to the equation of state of dark energy $w(z)$

$$\delta\Omega(z) = \Omega_\Lambda\left\{\exp\left(3\int_0^z (1+w(z))\frac{\mathrm{d}z}{1+z}\right) - 1\right\}. \tag{15}$$

Following the same reasoning, in a model with dark matter and dark energy interactions we have

$$\Omega_{\rm DE-DM}(z) \equiv \Omega_\Lambda + \Omega_{\rm cdm}(1+z)^3 + \delta\Omega(z) > 0 \quad \text{(Interacting DM-DE model)} \tag{16}$$

III. CMB PRIORS AND THE H_0 TENSION

In the previous section we considered a generic background modification over a ΛCDM cosmology. However, we know that the extremely precise observations of the CMB severely restrict such modifications. Two combinations of parameters are particularly well measured,

$$\theta_* \equiv \frac{r_{\rm s}(z_*)}{(1+z_*)d_A(z_*)}, \tag{17a}$$

$$R_* \equiv (1+z_*)d_A(z_*)\sqrt{\Omega_m H_0^2}. \tag{17b}$$

where $r_s(z)$ is the comoving sound horizon

$$r_{\rm s}(z) = \int_z^\infty \frac{\mathrm{d}z}{H}c_{\rm s}, \qquad c_{\rm s} = \frac{1}{\sqrt{3(1+R)}}, \qquad R = \frac{3\Omega_b}{4\Omega_\gamma(1+z)}, \tag{18}$$

and $z_* \simeq 1090$ is the redshift at decoupling, see [49] for a more accurate interpolation formula. These are commonly referred to as the CMB distance priors: the acoustic scale (θ_*) and the shift parameter (R_*), that govern the angular position and the height of the peaks in the CMB spectrum, respectively. Their latest values using the Planck 2018 release, for ΛCDM and some extensions, can be found in [49]. We can compute their variation following the steps of the previous section

$$\frac{\Delta\theta_*}{\theta_*} = \frac{\Delta r_{\rm s}^*}{r_{\rm s}^*} - \frac{\Delta d_A^*}{d_A^*}, \tag{19a}$$

$$\frac{\Delta R_*}{R_*} = \frac{\Delta d_A^*}{d_A^*} + \frac{\delta\omega_m}{2\omega_m}. \tag{19b}$$

Here we are using the shorthand notation $d_A^* \equiv d_A(z_*)$. The variation in these two parameters is only a small fraction of all the possible changes that any modified cosmology can produce in the CMB. If we want to compute all these changes and definitively establish the level of agreement of a given model with the CMB, we must resort to a Boltzmann code and perform the numerical computation. However, we can argue that in order not to be directly excluded, any reasonable ΛCDM extension must keep θ_* and R_* approximately fixed. Then, imposing $\Delta\theta_*, \Delta R_* \simeq 0$, we obtain the following system

$$(I_{d_A}^* - I_{r_s}^*)\frac{\delta h}{h} + (J_{d_A}^* - J_{r_s})\frac{\delta\omega_m}{\omega_m} = \int \frac{\mathrm{d}x_z}{1+x_z}(R_{r_s}^* - R_{d_A}^*)\frac{\delta H}{H}, \tag{20a}$$

$$I_{d_A}^*\frac{\delta h}{h} + \left(J_{d_A}^* + \frac{1}{2}\right)\frac{\delta\omega_m}{\omega_m} = -\int_0^\infty \frac{\mathrm{d}x_z}{1+x_z}R_{d_A}^*\frac{\delta H}{H}. \tag{20b}$$

Solving the system we get the response functions for h and ω_m

$$\frac{\delta h}{h} = \int_0^\infty \frac{\mathrm{d}x_z}{1+x_z}\mathcal{R}_h(x_z)\frac{\delta H(x_z)}{H(x_z)}, \tag{21a}$$

$$\frac{\delta\omega_m}{\omega_m} = \int_0^\infty \frac{\mathrm{d}x_z}{1+x_z}\mathcal{R}_{\omega_m}(x_z)\frac{\delta H(x_z)}{H(x_z)}. \tag{21b}$$

where

$$\mathcal{R}_h = \frac{1}{D_*}\left\{\left(J_{d_A}^* + \frac{1}{2}\right)(R_{r_s}(x,z_*) - R_{d_A}(x,z_*)) - (I_{r_s}^* - I_{d_A}^*)R_{d_A}(x,z_*)\right\}, \tag{22a}$$

$$\mathcal{R}_{\omega_m} = \frac{1}{D_*}\{I_{r_s}^* R_{d_A}(x,z_*) - I_{d_A}^* R_{r_s}(x,z_*)\}, \tag{22b}$$

$$D^* = I_{d_A}^*(J_{r_s}^* - J_{d_A}^*) - (I_{r_s}^* - I_{d_A}^*)\left(J_{d_A}^* + \frac{1}{2}\right). \tag{22c}$$

The response functions allow us to connect the changes produced in the expansion history by a generic model with changes of the observationally preferred parameters in the new cosmology. The observations considered in this case are the CMB priors, which ensure that all the modified cosmologies considered are roughly compatible with the

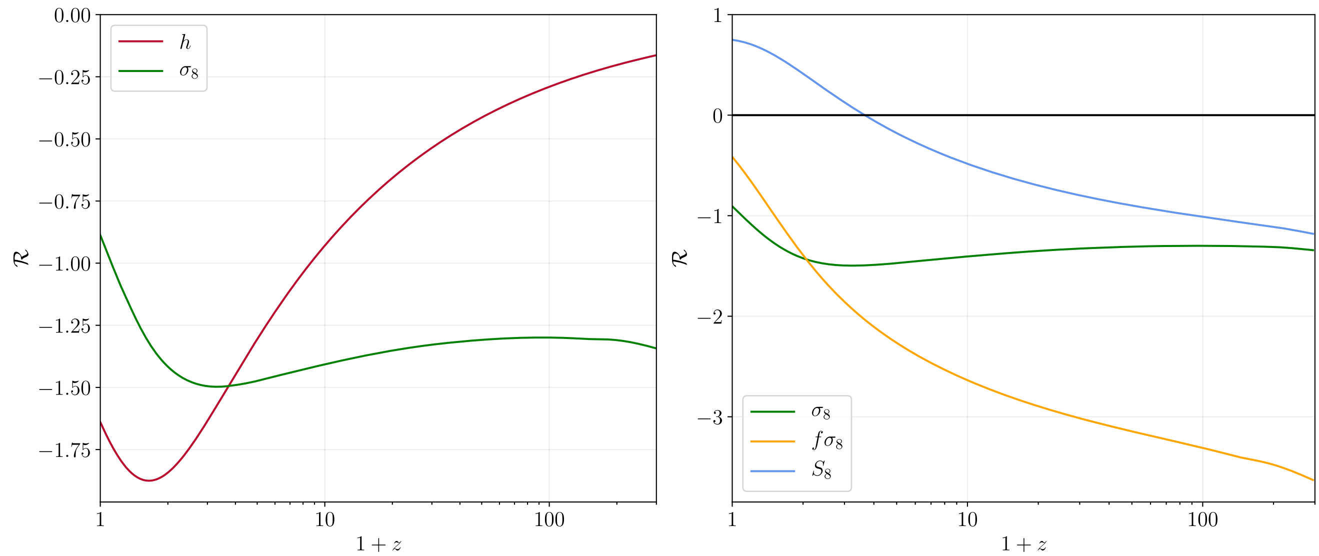

FIG. 1. Left: response functions for h and σ_8. Notice that both have the same sign so, unless δH changes sign, both variations follow the same trend, i.e., if we increase h we also increase σ_8. Right: response functions for other clustering-related quantities. The curves have been computed using the Planck 2018 [1] best-fit values.

CMB. While the expressions derived so far are general, we will restrict ourselves to late-time modifications, so that $\delta H(z) = 0$ for $z > 300$.

We can now use the previous results to compute the response function of any other cosmological quantity. After plugging (22) in the expression for a general variation, (9), we obtain

$$\frac{\Delta g(z)}{g(z)} = \int_0^\infty \frac{\mathrm{d}x_z}{1+x_z} \mathcal{R}_g(x_z, z) \frac{\delta H(x_z)}{H(x_z)}, \tag{23}$$

where the response function $\mathcal{R}_g$ can be expressed as

$$\mathcal{R}_g(x_z,z) \equiv I_g(z)\mathcal{R}_h(x_z) + J_g(z)\mathcal{R}_{\omega_m}(x_z) + R_g(x_z,z). \tag{24}$$

These results allow us to answer one of the main questions of this work. The response function $\mathcal{R}_h$, depicted in Fig. 1, is strictly negative, so to increase the value of h and thus solve the Hubble tension we need $\delta H(z) < 0$ for some z. In the context of dark energy models, according to (15), this means that the equation of state must be phantomlike, i.e., $w(z) < -1$ for some z. To reach this conclusion we only used the fact that $\mathcal{R}_h$ is strictly negative. Its shape will be important in the next section, where we will try to simultaneously solve the H_0 and the σ_8 tensions.

After evaluating (22), one can see that the response function of ω_m is very close to zero in the whole range $0 < z < 300$. Even though we will present the analytical results with full generality, for late-time modifications it is completely justified to keep ω_m fixed, i.e., $\mathcal{R}_{\omega_m} \to 0$. The variation of the Hubble parameter can then be obtained using only the first CMB prior in (19)

$$\frac{\delta h}{h} \simeq -\frac{1}{I^*_{d_A}} \int_0^\infty \frac{\mathrm{d}x_z}{1+x_z} R^*_{d_A} \frac{\delta H}{H}. \tag{25}$$

Finally, notice that in this work we are neglecting the changes in the ΛCDM parameters that could be induced by the modified ISW effect. Late-time changes of the equation of state, and especially modifications to the perturbations (e.g., clustering dark energy), lead to a modification of the ISW effect that might affect the determination of the ΛCDM parameters τ_{reio} and A_s, i.e., the optical depth to reionization and the amplitude of the spectrum of scalar perturbations. However, in particular dark energy models, e.g., [48], the modification on A_s has been shown to be very small and the effects considered in this work (modifications of $w(z)$ and G_{eff}) were the ones driving σ_8 above the ΛCDM value. Hence, we will neglect this contribution here, leaving a detailed analysis of the ISW effect for future work.

IV. GROWTH FACTOR AND THE σ_8 TENSION

A. Growth factor and σ_8 in ΛCDM

After decoupling, the time evolution of matter perturbations can be encapsulated in the growth factor. The growth factor in ΛCDM obeys

$$\frac{\mathrm{d}^2 D}{\mathrm{d}a^2} + \frac{\mathrm{d}\log(a^3 H)}{\mathrm{d}a}\frac{\mathrm{d}D}{\mathrm{d}a} - F(a)D = 0,$$
$$F(a) \equiv \frac{3\Omega_m H_0^2}{2a^5 H^2}. \tag{26}$$

This equation remains valid even if the expansion history $H(a)$ is different from ΛCDM, as long as the equations describing the perturbations are not modified. For a late-time ΛCDM universe, where matter and Λ are the dominant components, the two independent solutions of (26) can be expressed analytically

$$D_+(a) = \frac{5\Omega_m}{2}\frac{H(a)}{H_0} I(a),$$
$$I(a) \equiv \int_0^a \mathrm{d}x_a \frac{H_0^3}{(x_a H(x_a))^3}, \tag{27a}$$
$$D_-(a) \propto H(a), \tag{27b}$$

where D_+ and D_- are the growing and decaying mode, respectively. It is also common to define the linear growth rate f, that in ΛCDM can be approximated as

$$f \equiv \frac{\mathrm{d}\log D_+}{\mathrm{d}\log a} \simeq \left(\frac{\Omega_m H_0^2 a^{-3}}{H^2}\right)^{0.55}. \tag{28}$$

We start with the definition, e.g., see [50],

$$\sigma_R^2 \equiv \langle \delta_{m,R}^2(\mathbf{x})\rangle, \tag{29}$$

where

$$\delta_{m,R}(\boldsymbol{x}) \equiv \int \mathrm{d}^3x'\, \delta_m(\boldsymbol{x}') W_R(|\boldsymbol{x}-\boldsymbol{x}'|),$$
$$W_R(r) = \begin{cases} \frac{3}{4\pi R^3}, & x < R \\ 0, & x > R \end{cases} \tag{30}$$

It is common practice to evaluate this averaged clustering amplitude in spheres of radius $R = 8h^{-1}$ Mpc and denote it as σ_8. It can be equivalently expressed in Fourier space and in terms of the matter power spectrum as

$$\sigma_R^2 = \int \frac{\mathrm{d}k}{k}\mathcal{P}_m(k) W^2(kR), \qquad W(x) \equiv \frac{3j_1(x)}{x}, \tag{31}$$

where j_1 is a spherical Bessel function. The approximate form of the linear matter power spectrum in terms of the matter growth factor and the transfer function is [50]

$$\mathcal{P}_m(k) \equiv \frac{k^3}{2\pi^2}P_m(k) = \frac{4}{25}\frac{k^4}{\Omega_m^2 H_0^4}T^2(k)D_+^2(a)\mathcal{P}_\mathcal{R}(k), \tag{32}$$

where the primordial power spectrum of curvature perturbations is

$$\mathcal{P}_\mathcal{R}(k) = A_s\left(\frac{k}{k_p}\right)^{n_s-1}, \qquad k_p = 0.05\ \mathrm{Mpc}^{-1}. \tag{33}$$

For the transfer function we will adopt the Eisenstein-Hu fitting formula [51] that takes into account the baryonic suppression at small scales and proves important for an accurate computation of σ_8. Following the notation of the original work [51]

$$q = \frac{k}{\mathrm{Mpc}^{-1}}\frac{\Theta_{2.7}^2}{\Gamma_\mathrm{eff}}, \qquad \Gamma_\mathrm{eff} = \omega_m\left(\alpha_\Gamma + \frac{1-\alpha_\Gamma}{1+(0.43ks)^4}\right),$$
$$T_0(q) = \frac{L_0}{L_0 + C_0 q^2}, \qquad \alpha_\Gamma = 1 - 0.328\log(431\omega_m)\frac{\omega_b}{\omega_m} + 0.38\log(22.3\omega_m)\left(\frac{\omega_b}{\omega_m}\right)^2,$$
$$L_0(q) = \log(2\mathrm{e} + 1.8q), \qquad s = \frac{44.5\log(9.83/\omega_m)}{\sqrt{1+10(\omega_b)^{3/4}}}\ \mathrm{Mpc},$$
$$C_0(q) = 14.2 + \frac{731}{1+62.5q}, \tag{34}$$

where $\Theta_{2.7}$ is the temperature of the CMB in 2.7 K units. So finally, the transfer function that we will use is $T_\mathrm{EH}(k) = T_0(q(k))$ Also notice that we will always assume that k in the integral is measured in Mpc^{-1} and not in $\mathrm{Mpc}^{-1}h$ units. After rewriting (31), we can write the σ_8 as

$$\sigma_8^2 = \frac{4}{25\omega_m}D_+^2(a)\mathcal{I}_k, \tag{35}$$

where

$$\mathcal{I}_k = \int_0^\infty \frac{\mathrm{d}k}{k}\left(\frac{k}{C_H}\right)^4 T^2(k)W^2(kR)\mathcal{P}_\mathcal{R}(k), \tag{36}$$

and again $R = 8h^{-1}$ Mpc. Closely related, S_8 is defined as

$$S_8 \equiv \sigma_8\sqrt{\frac{\Omega_m}{0.3}}. \tag{37}$$

This quantity is closer to what is actually measured in weak-lensing surveys and is commonly used to reformulate the σ_8 tension as a S_8 tension. Spectroscopic surveys on the other hand usually target the combination $f\sigma_8$, that can be precisely measured with redshift-space distortions.

B. The σ_8 tension

The evolution of the variation of the growth factor is described by

$$\frac{\mathrm{d}^2}{\mathrm{d}a^2}\Delta D + \frac{\mathrm{d}\log(a^3 H)}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}a}\Delta D - F(a)\Delta D = g(a), \quad (38)$$

where

$$g(a) \equiv -\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\Delta H}{H}\right)\frac{\mathrm{d}D}{\mathrm{d}a} + FD\left(\frac{\delta\omega_m}{\omega_m} - 2\frac{\Delta H}{H}\right). \quad (39)$$

Using the Wronskian method, we can construct the particular solution to the inhomogeneous equation (38) and express the variations of the growth factor and the linear growth rate as

$$\Delta D = \frac{H(a)}{H_0}\int_0^a \mathrm{d}x_a \frac{x_a^3 H^2(x_a)}{H_0^2}(I(a) - I(x_a))g(x_a), \quad (40)$$

$$\Delta f = \frac{\mathrm{d}}{\mathrm{d}\log a}\frac{\Delta D}{D}. \quad (41)$$

The full analytical expressions for the (I, J, R) pieces of the variations can be found in Appendix A. We are now in position to compute the variation in the σ_8 clustering amplitude. This variation can be written as the combination

$$\frac{\Delta\sigma_8}{\sigma_8} = \frac{\Delta D}{D} - \frac{\delta\omega_m}{\omega_m} + \frac{1}{2}\frac{\Delta\mathcal{I}_k}{\mathcal{I}_k}. \quad (42)$$

We still need to compute the variations on the integral $\mathcal{I}_k$

$$\Delta\mathcal{I}_k = 2\int_0^\infty \frac{\mathrm{d}k}{k}T^2(k)\left(\frac{k}{C_H}\right)^4 W(kR)\Delta W(kR)\mathcal{P}_\mathcal{R}(k) + 2\int_0^\infty \frac{\mathrm{d}k}{k}T(k)\Delta T(k)\left(\frac{k}{C_H}\right)^4 W^2(kR)\mathcal{P}_\mathcal{R}(k). \quad (43)$$

Similarly, the variation of S_8 is

$$\frac{\Delta S_8}{S_8} = \frac{\Delta\sigma_8}{\sigma_8} - \frac{\delta h}{h} + \frac{1}{2}\frac{\delta\omega_m}{\omega_m}. \quad (44)$$

The response functions for σ_8, $f\sigma_8$ and S_8 are represented on the right hand side of Fig. 1. Both $\mathcal{R}_{\sigma_8}$ and $\mathcal{R}_{f\sigma_8}$ are strictly negative, which means that in order to reduce them we need $\delta H(z) > 0$ at some z. This can be compared with the result of the previous section, which showed that in order to increase H_0 we need $\delta H(z) < 0$. The bottom line of this analysis is that both conditions must be fulfilled to solve the two cosmological tensions, otherwise we improve one at the cost of worsening the other. In particular, for a dark energy model (15) a change of sign in $\delta H(z)$ implies that the equation of state $w(z)$ must cross the value $w = -1$. However, the results for S_8 are slightly different, since at very late-times the response function changes its sign. This feature could be very positive if, with the results of upcoming LSS surveys, we find ourselves in a situation where the clustering amplitude tension is clearly more severe in S_8 or in $f\sigma_8$. The different behavior of their response functions might be then a clear explanation and could give us hints about the shape of $\delta H(z)$.

C. Deformations beyond the background: G_{eff}

We define G_{eff} as a modification in the sub-Hubble regime that leads to the modified evolution for the growth factor

$$\frac{\mathrm{d}^2 D}{\mathrm{d}a^2} + \frac{\mathrm{d}\log(a^3 H)}{\mathrm{d}a}\frac{\mathrm{d}D}{\mathrm{d}a} - \frac{G_{\mathrm{eff}}}{G}F(a)D = 0. \quad (45)$$

Many realistic scenarios actually produce this kind of modification, e.g., see [48]. Following the same steps as in previous sections, if we assume that the effective gravitational coupling is close to the ΛCDM case, $G_{\mathrm{eff}} = G + \delta G(z)$, we get

$$\frac{\mathrm{d}^2}{\mathrm{d}a^2}\Delta D + \frac{\mathrm{d}\log(a^3 H)}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}a}\Delta D - F(a)\Delta D = F(a)D(a)\frac{\delta G(a)}{G}, \quad (46)$$

where in this case ΔD stands for a variation keeping fixed all the cosmological parameters and $H(z)$. The particular solution is

$$\frac{\Delta D}{D} = \frac{H(a)}{D(a)H_0}\int_0^a \mathrm{d}x_a \frac{x_a^3 H^2(x_a)}{H_0^2}(I(a) - I(x_a))F(x_a)D(x_a)\frac{\delta G(x_a)}{G} = \int_0^\infty \frac{\mathrm{d}x_z}{1+x_z}\mathcal{G}(x_z, z)\frac{\delta G(x_z)}{G}, \quad (47)$$

where the response function for δG is

$$\mathcal{G}(x_z, z) = \frac{H^3(z)D(x_z)}{H_0^3 D(z)}(I(z) - I(x_z))F(x_z)\frac{\theta(x_z - z)}{(1+x_z)^4}. \quad (48)$$

Since we are working to first order, the complete variation, modifying the background as well, is just a linear combination with the results of the previous section, i.e.,

$$\left.\frac{\Delta D}{D}\right|_{\mathrm{full}} = \int_0^\infty \frac{\mathrm{d}x_z}{1+x_z}\left(\mathcal{R}_D(x_z, z)\frac{\delta H(x_z)}{H(x_z)} + \mathcal{G}(x_z, z)\frac{\delta G(x_z)}{G}\right). \quad (49)$$

Including two free functions $\delta H(z)$ and $\delta G(z)$ make the results more general but unfortunately prevent us from making strong statements about the behavior of any of them. In order to proceed further, we restrict ourselves to

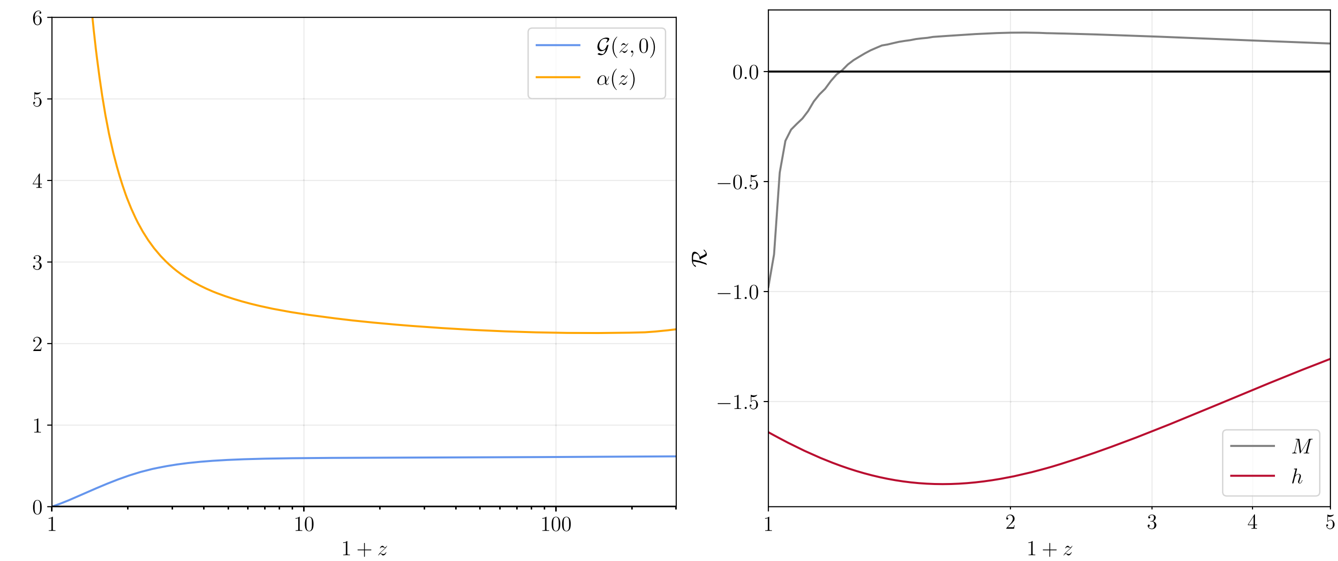

FIG. 2. Left: response of σ_8 to small changes in the effective gravitational constant, $\mathcal{G}$ in (48). The second curve $\alpha(z)$ depicts the bound defined in (50). Right: response function of h and the supernova absolute magnitude M. Notice that the latter becomes negative at very low redshift, so a very late-time modification that tries to increase the value of h would also increase M and thus fail at addressing the H_0 tension between the CMB and supernovae-based direct measurements. The curves have been computed using the Planck 2018 [1] best-fit values.

the case in which $\delta H(z)$ does not change sign. We know that this scenario is realized in many physically relevant models, for instance when we have a dark energy fluid that does not cross the phantom divide.

We already discussed that in this case in order to solve the H_0 tension we require $\delta H < 0$. If we do not modify the evolution of the perturbations, i.e., $G_{\rm eff} = G$, this leads to an increase in σ_8, since $\mathcal{R}_{\sigma_8} < 0$. However, including $\delta G(z)$ we have enough freedom to increase H_0 while reducing σ_8. Intuitively, it seems evident that we can achieve this goal just reducing the effective strength of gravity enough, i.e., $\delta G(z) < 0$. Defining $\alpha(x_z) \equiv -\mathcal{R}_{\sigma_8}(x_z,0)/\mathcal{G}(x_z,0)$ and using the previous results we can derive the stronger condition

$$\frac{\delta G(x_z)}{G} < \alpha(z)\frac{\delta H(x_z)}{H(x_z)} < 0 \quad \text{for some } x_z > 0, \tag{50}$$

that a model must satisfy if we want to reduce the value of σ_8, while increasing H_0. The response function for the gravitational coupling constant and α_z are shown in Fig. 2.

V. SUPERNOVA ABSOLUTE MAGNITUDE

Most discussions on the Hubble tension are formulated in terms of the H_0, or h, parameter. However, the parameter that is closer to what is actually measured by collaborations like SH0ES [2], is the absolute magnitude M used to calibrate the observed apparent magnitudes of SNe. This is the actual source of the Hubble tension, as has been stressed by [52], which also presented models where H_0 is raised without affecting M. In this section we will compute the response function for M, paying special attention to its differences with respect to the response function for h. The apparent magnitude and distance modulus are defined as [53]

$$m \equiv 5\log_{10}\left(\frac{d_L}{\rm Mpc}\right) + 25 + M, \tag{51}$$

$$\mu \equiv m - M, \tag{52}$$

where M is the absolute magnitude, that must be calibrated to infer the distance from the observed apparent magnitude. The χ^2 can then be constructed as

$$\chi^2_{\rm SNe} = (m^i_{\rm obs} - m^i_{\rm th})(C^{-1})_{ij}(m^j_{\rm obs} - m^j_{\rm th}), \tag{53}$$

and it can be analytically minimized for M

$$\frac{\partial\chi^2_{\rm SNe}}{\partial M} = 0 \to M_{\text{best fit}} = \frac{\sum_{ij}(C^{-1})_{ij}(m^j_{\rm obs} - \mu^j_{\rm th})}{\sum_{ij}(C^{-1})_{ij}}. \tag{54}$$

For given values of h and ω_m, this is the absolute magnitude that provides the best fit to SNe data. Its variation is

$$\Delta M = -5\frac{\delta h}{h} - \frac{5}{\sum_{ij}(C^{-1})_{ij}}\sum_{ij}\frac{(C^{-1})_{ij}\Delta d_L(z_j)}{d_L(z_j)}, \tag{55}$$

so we have

$$\mathcal{R}_M(x_z) = -5\mathcal{R}_h(x_z) - \frac{5}{\sum_{ij}(C^{-1})_{ij}}\sum_{ij}(C^{-1})_{ij}\mathcal{R}_{d_L}(x_z, z_j). \tag{56}$$

In this work, we will use the Pantheon sample [53] for the computation of (56). In Fig. 2, we can see that, in contrast with h, the response function for M changes sign at low redshift. This means that models that rely on modifications at very low redshift may increase the Hubble constant without actually decreasing M, a result in line with the conclusions of [52].

VI. SUMMARY AND CONCLUSIONS

In this paper we have addressed the question of why typical late-time dark energy models only solve the H_0 tension at the cost of predicting a large clustering amplitude σ_8 and whether it is therefore actually possible to relieve both tensions simultaneously by perturbatively modifying the expansion history, and maybe the gravitational constant, at late times. Using a model independent approach we derived a set of necessary conditions on the functional form of $\delta H(z)$ which have to be satisfied in order to tackle both the H_0 and σ_8 tensions. For the particularly interesting case in which the deformation is due to dark energy with equation of state $w(z)$ our results can be summarized schematically as follows

(i) Solving the H_0 tension $\Rightarrow$ $\delta H(z) < 0$ for some $z \Rightarrow w(z) < -1$ for some z.

(ii) If the perturbations are not modified ($G_{\mathrm{eff}} = G$) then: Solving the H_0 and σ_8 tensions $\Rightarrow$ $\delta H(z)$ changes sign $\Rightarrow$ $w(z)$ crosses the phantom divide.

(iii) If $G_{\mathrm{eff}} = G + \delta G(z)$ and $\delta H(z)$ does not change sign then: Solving the H_0 and σ_8 tensions $\Rightarrow \frac{\delta G(z)}{G} < \alpha(z)\frac{\delta H(z)}{H(z)} < 0$ for some z. where $\delta H(z) < 0$ and $\alpha(z) > 0$.

(iv) Solutions that rely on significant modifications at low redshift ($z < 1$) can increase H_0 without decreasing the supernova absolute magnitude M, thus failing to address the Hubble tension.

Note that while we chose here to present the implications of our results for the specific case of a dark energy model, the conditions on the form of $\delta H(z)$ are much more general and can be applied to any theory. If the theory in question introduces changes beyond a modification of the background, we can use the previous conditions to gain insight on how these additional changes must behave. For instance, if we know from the previous analysis that the background effects worsen the σ_8 tension, at least some of the changes beyond the background must work toward improving it, otherwise the theory is doomed.

Providing a full catalog of models ruled out by these necessary criteria will be left for future work, as well as the further study of theories meeting them. It would for example be interesting to include low redshift constraints from baryon acoustic oscillation (BAO) data and address concerns along the lines of [54].

Another interesting avenue would be to extend this results to early dark energy models. In this case, the CMB would be modified in a different way and different observational anchors should be used. The variation of ω_m, negligible for this work, would have to be taken into account and would potentially play an important role.

Finally, while the computations in the present paper, in particular the solution (40), explicitly assume a ΛCDM background with only matter and curvature contributions on top of the cosmological constant, it is possible to generalize our analysis to arbitrary backgrounds. This will allow the application of the method to deviations from general DE models or theories beyond Einstein gravity and will be useful especially in the context of effective field theory of dark energy to consider observables at the perturbation level beyond G_{eff} presented here.

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APPENDIX A: FULL ANALYTICAL RESULTS

This appendix contains the full analytical expressions used in this work. Unless otherwise stated, every function inside the integrals depends on the integration variable x_z. Comoving, luminosity and angular diameter distance:

$$\begin{cases} I_\chi(z) = I_{d_L}(z) = I_{d_A}(z) = -\frac{1}{\chi(z)}\int_0^z \mathrm{d}x_z \frac{H_0^2}{H^3} \\ J_\chi(z) = J_{d_L}(z) = J_{d_A}(z) = -\frac{1}{\chi(z)}\int_0^z \mathrm{d}x_z \frac{H_0^2}{H^3} m(x_z) \\ R_\chi(x_z, z) = R_{d_A}(x_z, z) = R_{d_A}(x_z, z) = -(1+x_z)\frac{\theta(z-x_z)}{\chi(z)H(x_z)} \end{cases} \tag{A1}$$

Comoving sound horizon:

$$\begin{cases} I_{r_s}(z) = -\frac{1}{r_s(z)}\int_z^\infty \mathrm{d}x_z \frac{H_0^2}{H^3} c_s(x_z) \\ J_{r_s}(z) = -\frac{1}{r_s(z)}\int_z^\infty \mathrm{d}x_z \frac{H_0^2}{H^3} m(x_z) c_s(x_z) \\ R_{r_s}(x_z, z) = -\frac{(1+x_z)c_s(x_z)}{r_s(z)}\frac{\theta(x_z-z)}{H(x_z)} \end{cases} \tag{A2}$$

Integral defined in (36):

$$\begin{cases} I_{\mathcal{I}_k} = -\frac{2}{\mathcal{I}_k}\int_0^\infty \frac{\mathrm{d}k}{k} T^2(k)\mathcal{P}_{\mathcal{R}}(k)\left(\frac{k}{C_H}\right)^4 kRW(kR)W'(kR) \\ J_{\mathcal{I}_k} = \frac{2\omega_m}{\mathcal{I}_k}\int_0^\infty \frac{\mathrm{d}k}{k} T(k)\frac{\partial T(k)}{\partial\omega_m}\mathcal{P}_{\mathcal{R}}(k)\left(\frac{k}{C_H}\right)^4 W^2(kR) \\ R_{\mathcal{I}_k} = 0 \end{cases} \tag{A3}$$

Growth factor:

$$\begin{cases} I_D(z) = -\frac{H(z)}{H_0 D(z)}\int_z^\infty \frac{\mathrm{d}x_z}{1+x_z}\frac{H_0^2 D}{H^2}\left\{\frac{H_0}{H}f + \frac{N(z,x_z)F}{1+x_z}\left(1 + \frac{(1+x_z)^3}{F}\frac{\mathrm{d}\log H}{\mathrm{d}x_z}f\right)\right\} \\ J_D(z) = \frac{1}{5} + \frac{H(z)}{H_0 D(z)}\int_z^\infty \frac{\mathrm{d}x_z}{(1+x_z)^2}N(z,x_z)FD \\ \qquad - \frac{H(z)}{H_0 D(z)}\int_z^\infty \frac{\mathrm{d}x_z}{1+x_z} mD\left\{\frac{H_0}{H}f + \frac{N(z,x_z)F}{1+x_z}\left(1 + \frac{(1+x_z)^3}{F}\frac{\mathrm{d}\log H}{\mathrm{d}x_z}f\right)\right\} \\ R_D(x_z,z) = -\frac{H(z)D(x_z)}{H_0 D(z)}\left\{\frac{H_0}{H(x_z)}f(x_z) + \frac{N(z,x_z)F(x_z)}{1+x_z}\left(1 + \frac{(1+x_z)^3}{F(x_z)}\frac{\mathrm{d}\log H}{\mathrm{d}x_z}f(x_z)\right)\right\}\theta(x_z - z) \end{cases} \tag{A4}$$

Linear growth rate:

$$\begin{cases} I_f(z) = -I_D(z)\left(1 + \frac{1+z}{f}\frac{\mathrm{d}\log H}{\mathrm{d}z}\right) - \frac{H_0^2}{H^2(z)} \\ \qquad - \frac{(1+z)^2 H_0^2}{f(z)H^2(z)D(z)}\int_z^\infty \frac{\mathrm{d}x_z}{(1+x_z)^5}FD\left(1 + \frac{(1+x_z)^3}{F}\frac{\mathrm{d}\log H}{\mathrm{d}x_z}f\right) \\ J_f(z) = -\left(J_D(z) - \frac{1}{5}\right)\left(1 + \frac{1+z}{f}\frac{\mathrm{d}\log H}{\mathrm{d}z}\right) - m(z) \\ \qquad + \frac{(1+z)^2}{f(z)H^2(z)D(z)}\int_z^\infty \frac{\mathrm{d}x_z}{(1+x_z)^5}H^2 FD\left\{1 - m\left(1 + \frac{(1+x_z)^3}{F}\frac{\mathrm{d}\log H}{\mathrm{d}x_z}f\right)\right\} \\ R_f(x_z,z) = -R_D(x_z,z)\left(1 + \frac{1+z}{f}\frac{\mathrm{d}\log H}{\mathrm{d}z}\right) - \delta(x_z - z) \\ \qquad - \frac{1}{(1+z)f(z)}\frac{(1+z)^3 H^2(x_z)D(x_z)}{(1+x_z)^3 H^2(z)D(z)}\frac{F(x_z)}{1+x_z}\left(1 + \frac{(1+x_z)^3}{F(x_z)}\frac{\mathrm{d}\log H}{\mathrm{d}x_z}f(x_z)\right)\theta(x_z - z) \end{cases} \tag{A5}$$

Supernova absolute magnitude:

$$\begin{cases} I_M = -\frac{5}{M} - \frac{5}{M\sum_{ij}(C^{-1})_{ij}}\sum_{ij}(C^{-1})_{ij}I_\chi(z_j) \\ J_M = -\frac{5}{M\sum_{ij}(C^{-1})_{ij}}\sum_{ij}(C^{-1})_{ij}J_\chi(z_j) \\ R_M(x_z) = -\frac{5}{M\sum_{ij}(C^{-1})_{ij}}\sum_{ij}(C^{-1})_{ij}R_\chi(x_z,z_j) \end{cases} \tag{A6}$$

APPENDIX B: NUMERICAL BENCHMARKS

The analytical expressions of the previous section have been tested for two particular dark energy models, $w = \text{const}$ and the CPL [55,56] parametrization $w = w_0 + w_a(1 - a)$. In the latter, we choose fix parameter $w_0 = -1.05$ to obtain a deformation $\delta H(z)$ that changes sign at late times. We compare the analytical results with the numerical ones obtained using CLASS [57], keeping fixed the acoustic scale θ_* and ω_m. The analytic results show a very satisfactory performance as can be seen in Tables I and II.

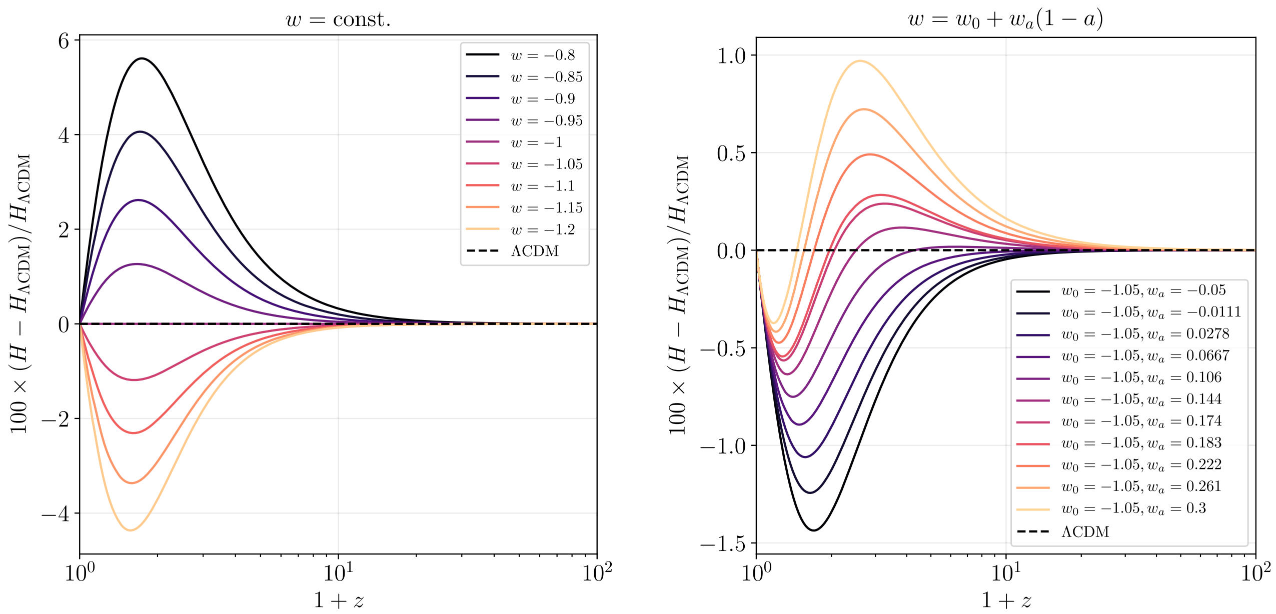

FIG. 3. $\delta H(z)$ for two common dark energy parametrizations.

TABLE I. Comparison of our analytical results with the full computation in CLASS, keeping fixed θ_* and ω_m, for a dark energy model with a constant equation of state w.

	$100 \times \delta h/h$		$100 \times \Delta\sigma_8/\sigma_8$	
w	CLASS	Analytical	CLASS	Analytical
−0.80	−8.57	−10.97	−8.98	−7.13
−0.85	−6.47	−7.76	−6.29	−5.32
−0.90	−4.35	−4.89	−3.93	−3.53
−0.95	−2.19	−2.32	−1.85	−1.76
−1.05	2.22	2.10	1.65	1.75
−1.10	4.47	4.01	3.12	3.48
−1.15	6.75	5.74	4.45	5.21
−1.20	9.07	7.33	5.66	6.93

TABLE II. Comparison of our analytical results with the full computation in CLASS, keeping fixed θ_* and ω_m, for a dark energy model with an equation of state $w(a) = -1.05 + w_a(1-a)$. Notice that in this case $\delta H(z)$ changes sign, as shown in the right panel in Fig. 3, and for the case $w_a = 0.174$ the variations have opposite signs. Even though they are too small to relieve the tensions, this example shows that when $\delta H(z)$ changes sign it is possible to increase h while reducing σ_8.

	$100 \times \delta h/h$		$100 \times \Delta\sigma_8/\sigma_8$	
w_a	CLASS	Analytical	CLASS	Analytical
−0.05	2.81	2.61	2.24	2.08
−0.01	2.35	2.22	1.86	1.75
0.03	1.89	1.80	1.47	1.39
0.07	1.42	1.37	1.07	1.03
0.11	0.946	0.93	0.66	0.64
0.14	0.465	0.46	0.24	0.23
0.174	0.095	0.093	−0.089	−0.092
0.18	−0.022	−0.025	−0.19	−0.20
0.22	−0.52	−0.53	−0.64	−0.65
0.26	−1.02	−1.07	−1.09	−1.13
0.3	−1.54	−1.62	−1.56	−1.64

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