

Linear mode analysis from spin transport equation

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We provide a linear analysis on normal modes of the spin Boltzmann equation proposed by Weickgenannt *et al.* [*Phys. Rev. D* **104**, 016022 (2021)], where the nondiagonal or polarized part of the transition rate is neglected to ensure the Hermitian property of a linearized collision operator. As an instrumental element of spin kinetic theory, the conservation of total angular momentum is explicitly considered; thus, our analysis is relevant to the recent investigation on the issue of local spin polarization. By treating the linearized collision operator as an evolution operator, solving the normal modes turns out to be a degenerate perturbation problem in quantum mechanics. The dispersion relations of spinless modes are in accordance with well-known calculations, while the frequencies of spin modes are also determined up to second-order in wave vectors; the second-order expressions are only formal solutions to be further determined. Moreover, the relaxation of spin density is related to our linear mode analysis, which shall play a big role in investigating the issues of the local spin polarization in the relativistic heavy-ion collisions.

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I. INTRODUCTION

The community that studies heavy-ion collisions shows great interest in the research on spin polarization inspired by the recent measurements of the spin-related observables of Λ hyperons [1,2]. Although satisfying agreements with experiment data in global polarization are reported [3–11], the “spin sign puzzle” has long bothered the scientists, where the dependence of Λ polarization on the azimuthal angle and transverse momentum [12,13] cannot be well reproduced and even the opposite dependence is predicted [14,15]. It is widely believed that the origin of the spin sign problem may be the inappropriate application of the equilibrium picture of spin extensively adopted in those numerical calculations. To overcome it, the framework needs to take nonequilibrium effects into consideration and spin hydrodynamic theory is thought to be a promising one. Compared to ordinary hydrodynamics, spin hydrodynamics includes the conservation law of total angular momentum and focus on the dissipation of spin density. On the other hand, these direct experimental measurements of quantum effects in relativistic heavy-ion collisions provide the opportunity to study the evolution of spinful fluids and stimulate related research on spin kinetic theory.

Many efforts are along the line of spin hydrodynamics trying to get insight into the spin sign problem [16–26]. Though with great progress in investigating spinful fluids, it is noted that the ultimate goal for us is to construct a causal and numerically stable theory of spin hydrodynamics, which allows the numerical implementation and simulation of the evolution of the fluid system and eventually provides us with quantitative explanations for the spin-related experimental phenomena. To that end, the hydrodynamic equations and relevant transport coefficients must be both obtained from the microscopic spin kinetic theory, constructing a self-consistent theory. Therefore, it is necessary to derive the quantum transport equations with proper collision terms, which must incorporate spin as the independent variable and account for the coupling between spin and orbit. To note, the related developments in spin transport can be found in [27–31]. Accounting for the complexity and nonlinearity, these transport equations are hard to solve and the linearization of related transport equations are usually used as a procedure to get some analytical results or spin hydrodynamics, from which the low-energy effective theory of spinful fluids can be constructed on solid ground. Following this manner, a second-order spin hydrodynamic theory [32] has been constructed from the spin Boltzmann equation [27,28] recently, while the causality and stability of resulting theory need further investigation.

In this work, we want to give a detailed linear analysis on normal modes appearing in spin hydrodynamic theory. To achieve it, the framework constructed in [27] is adopted as the starting point of our derivation. Although we are not directly constructing the hydrodynamic theory throughout

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the paper, the theory of hydrodynamics can be completely constructed from these normal modes. After the linearization of the collision term, solving the linear integral equation can be ascribed to the solution to the spectrum of the linear collision operator, while the nonequilibrium distribution can be expanded with corresponding eigenfunctions. However, the full spectrum is not yet solved even for the local Boltzmann collision term. As will be shown around Eq. (60), most modes decay rapidly except for the zero modes protected by all conservation laws respected by the collision term. Considering that the hydrodynamics actually originates from the conservation laws, the normal modes to be solved are equivalent to the description of spin hydrodynamics in this way. This paper is organized as follows. In Sec. II, we review the spin Boltzmann equation [27]. In Sec. III, we linearize the nonlocal collision kernel and neglect the polarized transition rate to get the Hermitian collision operator. In Sec. IV, the dispersion relations of normal modes are determined following the method of degenerate perturbation theory [33,34]. In Sec. V, we relate our results obtained in Sec. IV to the relaxation of spin, which is relevant in the investigation of local spin polarization and thus draws much attention. Discussion and outlook are presented in Sec. VI. We use natural units $k_B = c = \hbar = 1$. The metric tensor is defined as $g^{\mu\nu} \equiv \text{diag}(1, -1, -1, -1)$, while the projection tensor orthogonal to fluid velocity u^μ is given by $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$.

In the following, the shorthand notations are used:

$$A^{(\mu\nu)} \equiv (A^{\mu\nu} + A^{\nu\mu})/2, \quad (1)$$

$$A^{[\mu\nu]} \equiv (A^{\mu\nu} - A^{\nu\mu})/2, \quad (2)$$

$$A^{\langle\mu\nu\rangle} \equiv \left(\frac{\Delta_\alpha^\mu \Delta_\beta^\nu + \Delta_\alpha^\nu \Delta_\beta^\mu}{2} - \frac{\Delta^{\mu\nu} \Delta_{\alpha\beta}}{3} \right) A^{\alpha\beta}, \quad (3)$$

and we decompose the derivative ∂ as

$$\partial^\mu = u^\mu D + \nabla^\mu, \quad D \equiv u^\mu \partial_\mu, \quad \nabla^\mu \equiv \Delta^{\mu\nu} \partial_\nu. \quad (4)$$

II. REVIEW OF THE SPIN BOLTZMANN EQUATION

In this section, we review the spin Boltzmann equation derived in [27], which provides a description of the evolution of the system composed of massive fermions,

$$p \cdot \partial f(x, p, s) = C[f] + C_s[f], \quad (5)$$

where $C[f]$ is the collision kernel in which particles involved change their momentum and spin, while in $C_s[f]$ particles only exchange spin without momentum transfer,

$$C[f] \equiv \int d\Gamma_1 d\Gamma_2 d\Gamma' \mathcal{W}[f(x + \Delta_1, p_1, s_1) f(x + \Delta_2, p_2, s_2) - f(x + \Delta, p, s) f(x + \Delta', p', s')], \quad (6)$$

$$C_s[f] \equiv \int d\Gamma_2 dS_1(p) \mathcal{D} f(x + \Delta_1, p, s_1) f(x + \Delta_2, p_2, s_2), \quad (7)$$

with

$$d\Gamma \equiv d^4 p \delta(p^2 - m^2) dS(p), \quad (8)$$

$$\int dS(p) \equiv \sqrt{\frac{p^2}{3\pi^2}} \int d^4 s \delta(s \cdot s + 3) \delta(p \cdot s). \quad (9)$$

Note that the phase space spanned by particle position x and momentum p is now extended to include the variable s as a classical description of spin [19,28,35–38]. Here the spatial shift Δ is defined as

$$\Delta^\mu \equiv -\frac{1}{2m(p \cdot \hat{t} + m)} \epsilon^{\mu\nu\alpha\beta} p_\nu \hat{t}_\alpha s_\beta. \quad (10)$$

Additionally, the transition rates are shown by

$$\begin{aligned} \mathcal{W} &\equiv \delta^{(4)}(p + p' - p_1 - p_2) \frac{1}{8} \sum_{s,r,s',r',s_1,s_2,r_1,r_2} h_{sr}(p, s) h_{s'r'}(p', s') h_{s_1 r_1}(p_1, s_1) h_{s_2 r_2}(p_2, s_2) \\ &\quad \times \langle p, p'; r, r' | t | p_1, p_2; s_1, s_2 \rangle \langle p_1, p_2; r_1, r_2 | t^\dagger | p, p'; s, s' \rangle, \\ \mathcal{D} &\equiv \frac{\pi \hbar}{4m} \sum_{s_1, s_2, r, r_2} \epsilon_{\mu\nu\alpha\beta} s^\mu s_1^\nu p^\alpha n_{s_1 r}^\beta(p) h_{s_2 r_2}(p_2, s_2) \langle p, p_2; r, r_2 | t + t^\dagger | p, p_2; s_1, s_2 \rangle, \end{aligned} \quad (11)$$

with

$$n_{sr}^\alpha(p, s) \equiv \frac{1}{2m} \bar{u}_s(p) \gamma^5 \gamma^\alpha u_r(p), \quad (12)$$

$$h_{sr}(p, s) \equiv \delta_{sr} + \frac{1}{2m} \bar{u}_s(p) \gamma^5 s \cdot \gamma u_r(p), \quad (13)$$

where γ is the Dirac matrix, $u_s(p)$ is the spinor, r and s are spin indices, and the matrix element of t is defined as the conventional scattering amplitude.

In the following derivation, we choose to neglect $C_s[f]$. The reason for neglecting the spin-exchange term can be argued via an estimation of magnitude. When comparing $C_s[f]$ with $C[f]$, we find that the integration over s_1 in $C_s[f]$ is not zero unless $f(x + \Delta_1, p, s_1)$ can provide another s_1 because of $\int dS(p)s^\mu = 0$. The sources of s_1 coming from $f(x + \Delta_1, p, s_1)$ consist of two contributions. One is Δ_1 accompanied with a derivative ∂ ; the other is $\Sigma_s^{\mu\nu}$, which we expect to appear with $\Omega_{\mu\nu}$. In the following sections, our discussion is limited to small Ω ; thus $C_s[f]$ is estimated as $O(\partial)$ or $O(\Omega)$. While for $C[f]$, there is no such a suppression factor.

Hereafter, we split \mathcal{W} into unpolarized and polarized parts. Note that, if we neglect the term linearized to spin s in Eq. (13), the transition rate turns out to be the familiar unpolarized form,

$$\bar{\mathcal{W}} \equiv \delta^{(4)}(p + p' - p_1 - p_2) \frac{1}{8} \sum_{r, r', r_1, r_2} \langle p, p'; r, r' | t | p_1, p_2; r_1, r_2 \rangle \times \langle p_1, p_2; r_1, r_2 | t^\dagger | p, p'; r, r' \rangle. \quad (14)$$

Equation (14) is nothing but our widely used local collision term of two-body scattering.

As a reminder, we comment that the spatial shift Δ captures the crucial nonlocality of collisions, where $\hat{\gamma}^\mu$ is the timelike unit vector which is (1,0) in the frame where p^μ is measured. To note, such an appealing structure of the collision kernel originates from the nontrivial tensor structure of particle fields or, equivalently, the nontrivial dynamics introduced by spin degrees of freedom compared to related discussions about the scalar field in [39].

It is widely known that it is hard to find a physical decomposition for the total angular momentum tensor valid for all possible systems and we can only choose one specific pseudogauge appropriate for the physical system of our interest [40]. When talking about dynamic spin polarization, it is the mechanism of spin-orbit angular momentum conversion that works. Consequently, we expect the spin angular momentum itself (as internal degrees of freedom) is conserved when interactions are turned off, whereas it is not when interactions are turned on. The pseudogauge proposed by Hilgevoord and Wouthuysen (HW) exactly satisfies the requirement [41]. Thus, it is reasonable and natural to take the HW gauge for our investigation of spin polarization in the present work. With our choice, the relevant particle current, energy-momentum tensor, and spin tensor are defined as

$$N^\mu \equiv \int d\Gamma p^\mu f(x, p, s), \quad (15)$$

$$T_{\text{HW}}^{\mu\nu} \equiv \int d\Gamma p^\mu p^\nu f(x, p, s), \quad (16)$$

$$S_{\text{HW}}^{\lambda, \mu\nu} \equiv \int d\Gamma p^\lambda \left(\frac{1}{2} \Sigma_s^{\mu\nu} - \frac{1}{2m^2} p^{[\mu} \partial^{\nu]} \right) f(x, p, s), \quad (17)$$

where the antisymmetric part of Eq. (16) is omitted because it is at the second order of gradients $O(\partial^2)$ [41] and outside the range of our following linear-order analysis. In the following sections, when nothing confusing occurs, the subscript HW will be omitted.

The above tensors can be conveniently decomposed as

$$N^\mu = nu^\mu + V^\mu, \quad (18)$$

$$T^{\mu\nu} = eu^\mu u^\nu - P\Delta^{\mu\nu} + \pi^{\mu\nu} + \Pi\Delta^{\mu\nu}, \quad (19)$$

$$S^{\lambda, \mu\nu} = u^\lambda S^{\mu\nu} + \delta S^{\lambda, \mu\nu}, \quad (20)$$

where n is the particle number density, e is the energy density, P is the static pressure, and $S^{\mu\nu}$ is the spin density. The dissipative quantities V^μ , $\pi^{\mu\nu}$, and Π are the diffusion current, shear stress tensor, and bulk viscous pressure, respectively, and $V^\mu u_\mu = \pi^{\mu\nu} u_\mu = 0$. Note that we have chosen the Landau choice of fluid velocity and imposed Landau matching conditions by requiring

$$T^{\mu\nu} u_\nu = eu^\mu, \quad u_\mu N^\mu = u_\mu N_{\text{eq}}^\mu, \quad u_\mu T^{\mu\nu} u_\nu = u_\mu T_{\text{eq}}^{\mu\nu} u_\nu. \quad (21)$$

Also another Landau matching condition for unambiguously defining $S^{\mu\nu}$ is taken,

$$u_\lambda J^{\lambda, \mu\nu} = u_\lambda J_{\text{eq}}^{\lambda, \mu\nu}, \quad (22)$$

where the conserved tensor $J^{\lambda, \mu\nu} \equiv S^{\lambda, \mu\nu} + x^\mu T^{\lambda\nu} - x^\nu T^{\mu\lambda}$ is the total angular momentum tensor. In the present case, the HW energy-momentum tensor takes only a symmetric form, and the matching condition (22) degenerates into

$$u_\lambda S^{\lambda, \mu\nu} = u_\lambda S_{\text{eq}}^{\lambda, \mu\nu}. \quad (23)$$

III. LINEAR COLLISION OPERATOR

It is shown in [25,27,28] that the collision term (6) is consistent with the local equilibrium distribution function [16,42]

$$f_{\text{leq}}(x, p, s) = \frac{1}{(2\pi)^3} \exp \left[\xi - \beta \cdot p + \frac{\Omega_{\mu\nu} \Sigma_s^{\mu\nu}}{4} \right], \quad (24)$$

where $\Omega_{\mu\nu}$ denotes spin potential, the dipole moment $\Sigma_s^{\mu\nu}$ is defined as $\Sigma_s^{\mu\nu} \equiv -\frac{1}{m} \epsilon^{\mu\nu\alpha\beta} p_\alpha s_\beta$, and $\beta^\mu \equiv \frac{u^\mu}{T}$, $\xi \equiv \frac{\mu}{T}$ with the temperature T , and the chemical potential μ . Without details, we summarize the conditions for global equilibrium distribution as

$$\beta^\mu = a^\mu + \Omega^{\mu\nu} x_\nu, \quad a^\mu = \text{const.} \quad (25)$$

Here we want to emphasize that such a conclusion relies on the assumption of conservative total angular momentum $J^{\mu\nu}$,

$$J^{\mu\nu} = 2\Delta^{[\mu} p^{\nu]} + \frac{1}{2}\Sigma_s^{\mu\nu}, \quad (26)$$

because the collision term itself does not conserve total angular momentum for lack of a similar delta function like $\delta^{(4)}(p + p' - p_1 - p_2)$. From now on, we follow this necessary assumption, and some modification needs to be built into Eqs. (11) and (14) based on phenomenological consideration. Then Eq. (14) is substituted by

$$\bar{\mathcal{W}} \times F[p, p', p_1, p_2; s, s', s_1, s_2], \quad (27)$$

where the dimensionless function F is inserted respecting the conservation for the total angular momentum and we leave the detailed form of F unspecified. Aside from this constraint on F , we require that F respect the symmetry of interchanging the variables of initial states with that of final states, which is not a severe constraint. One can naively treat F as a function proportional to $\delta^{(6)}(J^{\mu\nu} + J'^{\mu\nu} - J_1^{\mu\nu} - J_2^{\mu\nu})$ with the superscript reminding us of six degrees of freedom of $J^{\mu\nu}$. In addition, such an insertion needs also to be made into Eq. (11). In Appendix D, we prove that the linearized form of the collision kernel is Hermitian and non-negative with the unpolarized transition rate like Eq. (27). There are two points

that deserve special attention. One is that the unpolarized transition rate respects the principle of detailed balance. The other is that the non-negative property is important because it selects kinetic modes protected by conservation laws, which is reported in detail in Sec. V. Therefore, we concentrate only on the non-negative Hermitian linearized collision operator afterward.

After discussing the equilibrium state, we can linearize the transport equation (5) around the global equilibrium state f_G . Before doing that, Eq. (5) can be explicitly decomposed into

$$p \cdot u Df(x, p, s) + p^\nu \nabla_\nu f(x, p, s) = C[f], \quad (28)$$

and the distribution function is

$$f \equiv f_G(1 + \chi(x, p, s)), \quad (29)$$

where f_G is exactly the global equilibrium distribution, equivalently, Eq. (24) satisfying Eq. (25). Insert Eq. (29) into Eq. (28), and we get

$$p^\mu u_\mu D\chi(x, p, s) + p^\mu \nabla_\mu \chi(x, p, s) = -\mathcal{L}[\chi], \quad (30)$$

with the linearized collision operator defined as

$$\begin{aligned} \mathcal{L}[\phi] \equiv & \frac{1}{(2\pi)^3} \left(1 + \frac{\Omega_{\mu\nu} \Sigma_s^{\mu\nu}}{4}\right)^{-1} \int d\Gamma' d\Gamma_1 d\Gamma_2 \bar{\mathcal{W}} F[p, p', p_1, p_2; s, s', s_1, s_2] \exp(\xi - a \cdot p') \\ & \times \left[\frac{1}{2} \Omega_{\mu\nu} (J^{\mu\nu} + J'^{\mu\nu}) (\phi(x, p, s) + \phi(x, p', s') - \phi(x, p_1, s_1) - \phi(x, p_2, s_2)) \right. \\ & \left. + (\phi(x + \Delta, p, s) + \phi(x + \Delta', p', s') - \phi(x + \Delta_1, p_1, s_1) - \phi(x + \Delta_2, p_2, s_2)) \right]. \end{aligned} \quad (31)$$

For simplicity, the background profile is taken to be $u^\mu = (1, 0, 0, 0)$, $\Omega^{\mu\nu} = 0$. Here we note as an aside that the background profile can absolutely be chosen to be various. For example, a finite vorticity can survive in global equilibrium, then the β field can possess coordination dependence. In the meantime, thermodynamic integrals determining the dispersion relations shown in Appendix A are totally different and more complicated because $\hat{\tau}$ does not equal u anymore. On the other hand, only the spin-exchange collision term should be taken into account when Ω is finite. In order to compare with similar analysis on hydrodynamic modes in [20], the calibration should be made ensuring the global equilibrium configurations are the same, i.e., $u^\mu = (1, 0, 0, 0)$, $\Omega^{\mu\nu} = 0$. Thus, $a^\mu = \beta^\mu = \beta u^\mu$ with $\beta \equiv \frac{1}{T}$ and the linearized collision operator is greatly simplified.

Considering that $\phi(x + \Delta)$ is actually the linear approximation to $\phi(x) + \Delta \cdot \partial\phi$ [27], we return to its initial form and naturally split the linearized collision term into a local one,

$$\begin{aligned} \mathcal{L}_1[\phi] \equiv & \frac{1}{(2\pi)^3} \int d\Gamma' d\Gamma_1 d\Gamma_2 \bar{\mathcal{W}} F[p, p', p_1, p_2; s, s', s_1, s_2] \exp(\xi - \beta \cdot p') \\ & \times [\phi(x, p, s) + \phi(x, p', s') - \phi(x, p_1, s_1) - \phi(x, p_2, s_2)], \end{aligned} \quad (32)$$

and a nonlocal one featured by the nonlocal shift Δ ,

$$\begin{aligned} \mathcal{L}_2[\phi] \equiv & \frac{1}{(2\pi)^3} \int d\Gamma' d\Gamma_1 d\Gamma_2 \bar{\mathcal{W}} F[p, p', p_1, p_2; s, s', s_1, s_2] \exp(\xi - \beta \cdot p') \\ & \times [\Delta \cdot \partial_x \phi(x, p, s) + \Delta' \cdot \partial_x \phi(x, p', s') - \Delta_1 \cdot \partial_x \phi(x, p_1, s_1) - \Delta_2 \cdot \partial_x \phi(x, p_2, s_2)]. \end{aligned} \quad (33)$$

As is exhibited in Appendix D, \mathcal{L}_1 is a Hermitian operator, while \mathcal{L}_2 is not for the noncommutative property of the derivative. With our proposed division, Eq. (30) is transformed into

$$p^\mu u_\mu D\chi(x, p, s) + p^\mu \nabla_\mu \chi(x, p, s) + \mathcal{L}_2[\chi] = -\mathcal{L}_1[\chi]. \quad (34)$$

It is convenient to introduce the Fourier transformation of $\chi(x, p, s)$,

$$\tilde{\chi}(k, p, s) = \int d^4x \exp(ik \cdot x) \chi(x, p, s), \quad (35)$$

and cast Eq. (30) into

$$\tau \omega \tilde{\chi} + \hat{p}^\mu \kappa_\mu \tilde{\chi} + L_2[\tilde{\chi}] = -iL_1[\tilde{\chi}], \quad (36)$$

where the following notations are utilized:

$$\begin{aligned} \tau &\equiv \frac{p \cdot u}{T}, & \omega &\equiv \frac{u \cdot k}{n\sigma(T)}, & \hat{p} &\equiv \frac{p}{T}, & \kappa^\alpha &\equiv \frac{\Delta^{\alpha\beta} k_\beta}{n\sigma(T)}, \\ \kappa &\equiv \sqrt{-\kappa \cdot \kappa}, & l^\alpha &\equiv \frac{\kappa^\alpha}{\kappa}, \end{aligned} \quad (37)$$

and $\sigma(T)$ is an arbitrary constant with the dimension of total cross sections. The dimensionless collision operator is

$$\begin{aligned} L_1[\phi] &\equiv \frac{\exp(\xi)}{(2\pi)^3 n\sigma(T) T} \int d\Gamma' d\Gamma_1 d\Gamma_2 \bar{\mathcal{W}}F[p, p', p_1, p_2; s, s', s_1, s_2] \exp(-\beta \cdot p') \\ &\times [\phi(k, p, s) + \phi(k, p', s') - \phi(k, p_1, s_1) - \phi(k, p_2, s_2)], \end{aligned} \quad (38)$$

and meanwhile,

$$\begin{aligned} L_2[\phi] &\equiv \frac{\exp(\xi)}{(2\pi)^3 T} \int d\Gamma' d\Gamma_1 d\Gamma_2 \bar{\mathcal{W}}F[p, p', p_1, p_2; s, s', s_1, s_2] \exp(-\beta \cdot p') \\ &\times [\Delta \cdot \kappa \phi(k, p, s) + \Delta' \cdot \kappa \phi(k, p', s') - \Delta_1 \cdot \kappa \phi(k, p_1, s_1) - \Delta_2 \cdot \kappa \phi(k, p_2, s_2)]. \end{aligned} \quad (39)$$

Here the inner product is the global equilibrium state given by

$$(B, C) = \frac{1}{(2\pi)^3} \int d\Gamma \exp(\xi - \beta \cdot p) B(p, s) C(p, s). \quad (40)$$

To proceed, the adjoint operator of L_2 is denoted by L_2^\dagger and defined such that

$$(L_2^\dagger[B], C) = (B, L_2[C]). \quad (41)$$

with the same argument as in Appendix D, L_2 is also Hermitian. Although \mathcal{L}_2 is not Hermitian, L_2 turns out to be a Hermitian operator for the replacement of the derivative in Eq. (33) by κ in Eq. (39).

IV. DEGENERATE PERTURBATION THEORY AND LINEAR MODE ANALYSIS

As was stressed previously, solving normal modes can be treated in the fashion as used in quantum mechanics. Here we regard the $p \cdot \kappa$ and L_2 terms in Eq. (36) as a perturbation with respect to the linearized collision operator $-iL_1$; equivalently,

By utilizing the Hermitian property of L_1 , one can show that

$$\begin{aligned} (B, L_2[C]) &= n\sigma(T) (B, L_1[\Delta \cdot \kappa C]) = n\sigma(T) (L_1[B], \Delta \cdot \kappa C) \\ &= n\sigma(T) (\Delta \cdot \kappa L_1[B], C). \end{aligned} \quad (42)$$

Therefore, the adjoint operator L_2^\dagger is identified with $n\sigma(T) \Delta \cdot \kappa L_1$ and explicitly takes the following form:

$$\begin{aligned} L_2^\dagger[\phi] &\equiv \frac{\exp(\xi)}{(2\pi)^3 T} \int d\Gamma' d\Gamma_1 d\Gamma_2 \bar{\mathcal{W}}F[p, p', p_1, p_2; s, s', s_1, s_2] \exp(-\beta \cdot p') \\ &\times \Delta \cdot \kappa [\phi(k, p, s) + \phi(k, p', s') - \phi(k, p_1, s_1) - \phi(k, p_2, s_2)], \end{aligned} \quad (43)$$

we take the nonuniformity κ as a small quantity. Then the unperturbed equation reduces to an eigenvalue problem,

$$-iL_1 \tilde{\chi}^{(0)} = \tau \omega^{(0)} \tilde{\chi}^{(0)}. \quad (44)$$

It is easy to show that 11 zero modes (or collision invariants) $1, p^\mu$, and $J^{\mu\nu}$ trivially solve the zeroth-order equation. Compared to ordinary hydrodynamic theory, the six new modes are related to the conservation of total angular momentum in nonlocal collisions between particles. Noting that we have exhausted the global symmetries relevant

for present research, there should be no more conserved currents and no more zero modes.

When it comes to the first-order perturbation of $p \cdot \kappa$ and L_2 , we denote the n th-order contribution with a superscript

n and only focus on 11 zero modes, which are in tight connection with fluid-dynamic variables. The eigenfunctions set for 11-fold degenerate zeros can be readily chosen as

$$\begin{aligned} \tilde{\psi}_1 &= \frac{1}{\sqrt{V_{1,1}}}, & \tilde{\psi}_2 &= \beta \frac{u \cdot p - \frac{\epsilon}{n}}{\sqrt{V_{2,2}}}, & \tilde{\psi}_3 &= \frac{\beta l \cdot p}{\sqrt{V_{3,3}}}, & \tilde{\psi}_4 &= \frac{\beta j \cdot p}{\sqrt{V_{3,3}}}, & \tilde{\psi}_5 &= \frac{\beta v \cdot p}{\sqrt{V_{3,3}}}, \\ \tilde{\psi}_6 &= \frac{u_\mu J^{\mu\nu} l_\nu}{\sqrt{V_{6,6}}}, & \tilde{\psi}_7 &= \frac{u_\mu J^{\mu\nu} j_\nu}{\sqrt{V_{6,6}}}, & \tilde{\psi}_8 &= \frac{u_\mu J^{\mu\nu} v_\nu}{\sqrt{V_{6,6}}}, & \tilde{\psi}_9 &= \frac{l_\mu J^{\mu\nu} j_\nu}{\sqrt{V_{9,9}}}, \\ \tilde{\psi}_{10} &= \frac{l_\mu J^{\mu\nu} v_\nu}{\sqrt{V_{9,9}}}, & \tilde{\psi}_{11} &= \frac{j_\mu J^{\mu\nu} v_\nu}{\sqrt{V_{9,9}}}, \end{aligned} \quad (45)$$

where Schmidt orthogonalization is used and these eigenfunctions satisfy the orthonormal condition

$$(\tilde{\psi}_\alpha, \tau \tilde{\psi}_\beta) = \delta_{\alpha\beta}. \quad (46)$$

The definitions of two auxiliary vectors j and v and the normalized factor $V_{i,j}$ are all defined in Appendix B.

Taking into account the first-order perturbation, we obtain the inhomogeneous integral equation for $\tilde{\chi}^{(1)}$,

$$-iL_1 \tilde{\chi}_\alpha^{(1)} = \tau \omega_\alpha^{(1)} \tilde{\chi}_\alpha^{(0)} + \hat{p}^\mu \kappa_\mu \tilde{\chi}_\alpha^{(0)}, \quad (47)$$

where the property of $L_2 \tilde{\chi}_\alpha^{(0)} = 0$ is used. According to degenerate perturbation theory, the solubility condition is

$$(\tilde{\psi}_\alpha, \tau \omega^{(1)} \tilde{\chi}^{(0)} + p^\mu \kappa_\mu \tilde{\chi}^{(0)}) = 0, \quad (48)$$

where $\tilde{\chi}^{(0)}$ should be understood as the linear combination of the eigenfunctions $\tilde{\psi}_\beta$ in Eq. (45). Therefore, the frequency ω obeys the dispersion relation, i.e., the secular equation,

$$\begin{vmatrix} \omega^{(1)} & 0 & H_{1,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega^{(1)} & H_{2,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ H_{1,3} & H_{2,3} & \omega^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^{(1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega^{(1)} & 0 & H_{7,9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^{(1)} & 0 & H_{7,9} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & H_{7,9} & 0 & \omega^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_{7,9} & 0 & \omega^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^{(1)} \end{vmatrix} = 0, \quad (49)$$

where matrix elements $H_{i,j}$ are given in Appendix C.

The solution to this equation is made up of the eigenvalues

$$\begin{aligned} \omega_1^{(1)} &= -\omega_2^{(1)} = \sqrt{H_{2,3}^2 + H_{1,3}^2}, & \omega_3^{(1)} &= \omega_4^{(1)} = \omega_5^{(1)} = 0, \\ \omega_6^{(1)} &= \omega_{11}^{(1)} = 0, & \omega_7^{(1)} &= \omega_8^{(1)} = H_{7,9}, & \omega_9^{(1)} &= \omega_{10}^{(1)} = -H_{7,9} \end{aligned} \quad (50)$$

and the eigenfunctions

$$\begin{aligned}
 \tilde{\chi}_1^{(0)} &= \frac{1}{\sqrt{2}} \left(-\frac{H_{1,3}}{\sqrt{H_{1,3}^2 + H_{2,3}^2}} \tilde{\psi}_1 - \frac{H_{2,3}}{\sqrt{H_{1,3}^2 + H_{2,3}^2}} \tilde{\psi}_2 + \tilde{\psi}_3 \right), \\
 \tilde{\chi}_2^{(0)} &= \frac{1}{\sqrt{2}} \left(\frac{H_{1,3}}{\sqrt{H_{1,3}^2 + H_{2,3}^2}} \tilde{\psi}_1 + \frac{H_{2,3}}{\sqrt{H_{1,3}^2 + H_{2,3}^2}} \tilde{\psi}_2 + \tilde{\psi}_3 \right), \\
 \tilde{\chi}_3^{(0)} &= \frac{1}{\sqrt{\frac{H_{2,3}^2}{H_{1,3}^2} + 1}} \left(-\frac{H_{2,3}}{H_{1,3}} \tilde{\psi}_1 + \tilde{\psi}_2 \right), \quad \tilde{\chi}_4^{(0)} = \tilde{\psi}_4, \quad \tilde{\chi}_5^{(0)} = \tilde{\psi}_5, \quad \tilde{\chi}_6^{(0)} = \tilde{\psi}_6, \\
 \tilde{\chi}_7^{(0)} &= \frac{1}{\sqrt{2}} (\tilde{\psi}_7 - \tilde{\psi}_9), \quad \tilde{\chi}_8^{(0)} = \frac{1}{\sqrt{2}} (\tilde{\psi}_8 - \tilde{\psi}_{10}), \quad \tilde{\chi}_9^{(0)} = \frac{1}{\sqrt{2}} (\tilde{\psi}_7 + \tilde{\psi}_9), \quad \tilde{\chi}_{10}^{(0)} = \frac{1}{\sqrt{2}} (\tilde{\psi}_8 + \tilde{\psi}_{10}), \quad \tilde{\chi}_{11}^{(0)} = \tilde{\psi}_{11}. \quad (51)
 \end{aligned}$$

It is not hard to verify that the results of the first five modes are exactly those in [43]. As a consistent check, we find the eigenfunctions $\tilde{\chi}_\alpha^{(0)}$, $\alpha = 6, \dots, 11$ indeed satisfy the solubility condition (48) with $\omega_\alpha^{(1)} = 0$.

Likewise, we obtain again the solubility condition for the second-order perturbation equation,

$$(\tilde{\chi}_\alpha^{(0)}, \tau \omega_\beta^{(1)} \tilde{\chi}_\beta^{(1)} + \hat{p}^\mu \kappa_\mu \tilde{\chi}_\beta^{(1)} + \tau \omega_\beta^{(2)} \tilde{\chi}_\beta^{(0)}) = 0, \quad (52)$$

where we have used Eq. (41) to vanish the inner product of $\tilde{\chi}_\alpha^{(0)}$ and $L_2 \tilde{\chi}_\beta^{(1)}$. In order to form a comparison with related results in [43], the second-order frequencies can be written with the assistance of the bracket notation,

$$\omega_\alpha^{(2)} = i[\tilde{\chi}_\alpha^{(1)}, \tilde{\chi}_\alpha^{(1)}], \quad (53)$$

where $\tilde{\chi}^{(1)}$ should be obtained by solving the integral equation (47) and the bracket notation is given by $[B, C] \equiv (L_1[B], C)$. The second-order frequencies in the form of the bracket notation is in accordance with [43].

As seen from the secular equation (49), the spin modes and spinless modes decouple from each other; thus, the dispersion relations for the spinless modes receive no corrections from the spin effects and retain the same form as that in the linear analysis of the ordinary hydrodynamics, which is also confirmed in [20]. Therefore, the frequencies of the spinless modes up to the second order in κ are already available as follows [43]:

$$\begin{aligned}
 \omega_1 &= \sqrt{\frac{\gamma}{\hat{h}}} \kappa - i \frac{n\sigma}{2h} \left(\frac{4}{3} \eta + \zeta + \frac{((\gamma - 1)\hat{h} - \gamma)^2}{\gamma \hat{h}} \lambda \right) \kappa^2, \\
 \omega_2 &= -\sqrt{\frac{\gamma}{\hat{h}}} \kappa - i \frac{n\sigma}{2h} \left(\frac{4}{3} \eta + \zeta + \frac{((\gamma - 1)\hat{h} - \gamma)^2}{\gamma \hat{h}} \lambda \right) \kappa^2, \\
 \omega_3 &= -i \frac{\gamma - 1}{\gamma} \lambda \sigma \kappa^2, \quad \omega_4 = \omega_5 = -i \frac{n\sigma}{h} \eta \kappa^2, \quad (54)
 \end{aligned}$$

with the transport coefficients shear viscosity, bulk viscosity, and diffusion coefficient denoted by η , ζ , and λ ,

respectively, \hat{h} is the reduced enthalpy density $\hat{h} \equiv \frac{e+P}{T}$, and γ denotes the ratio of the heat capacities at constant pressure $c_p \equiv \left(\frac{\partial(h/n)}{\partial T} \right)_p$ and at constant volume $c_v \equiv \left(\frac{\partial(e/n)}{\partial T} \right)_v$. As a low-energy effective theory, the kinetic theory should match the coarse-grained hydrodynamics in the limit of long wavelengths. The one-to-one correspondence between hydrodynamic modes and kinetic modes has a great practical importance as the basis for a theory of transport coefficients, which should be thought of from a more “philosophical” view. A fluid system is a collection of particles moving in a quite disordered manner; however, in the limit of long wavelengths, the only possible modes of motion of the fluid are ordered modes, such as a sound-wave propagation, which originate from the dominant effect of the collisions [33,34].

In a short summary, among the five spinless hydrodynamic modes there are two sound modes traveling with opposite sound speed and the same damping rate, one purely decaying heat mode, and two degenerate shear modes. On the other hand, the frequencies for the other six spin-related modes are

$$\begin{aligned}
 \omega_6 &= i[\tilde{\chi}_6^{(1)}, \tilde{\chi}_6^{(1)}], \quad \omega_{11} = i[\tilde{\chi}_{11}^{(1)}, \tilde{\chi}_{11}^{(1)}], \\
 \omega_8 &= \omega_7 = H_{7,9} + i[\tilde{\chi}_7^{(1)}, \tilde{\chi}_7^{(1)}], \\
 \omega_{10} &= \omega_9 = -H_{7,9} + i[\tilde{\chi}_9^{(1)}, \tilde{\chi}_9^{(1)}]. \quad (55)
 \end{aligned}$$

By a careful comparison, we find that these spin-related dispersion relations cannot match with those in [20]. The reason is that in that work the authors concentrate on the nonconservative spin density, while our spin modes are inherently protected by conservation laws. There are four propagating transverse spin modes: two degenerate modes with $H_{7,9}/\kappa$ as the propagating speed damp at the rate $-[\tilde{\chi}_7^{(1)}, \tilde{\chi}_7^{(1)}]$, while the other two propagate in the opposite direction with a damping rate of $-[\tilde{\chi}_9^{(1)}, \tilde{\chi}_9^{(1)}]$. On the other hand, there are no propagating spin modes in [20]. In addition, the longitudinal modes are purely decaying at their respective damping rates. However, the bracket

notation is rather abstract and not directly related to our familiar physical quantities or formulalike spinless modes. To put it less abstractly, first we notice that

$$\omega_\alpha^{(2)} = -i(\tilde{\chi}_\alpha^{(0)}(\hat{p}^\mu \kappa_\mu + \tau \omega_\alpha^{(1)}), L_1^{-1}[(\hat{p}^\mu \kappa_\mu + \tau \omega_\alpha^{(1)})\tilde{\chi}_\alpha^{(0)}]). \quad (56)$$

If neglecting the nonuniformity (this term is an order of magnitude smaller), Eq. (34) takes the form of the time evolution

$$\beta\tau D\chi(t, p, s) = -L_1[\chi]. \quad (57)$$

When choosing the rest configuration $D = \partial t$ without losing the generality, the solution to the equation of evolution is shown as

$$\chi(t, p, s) = \exp\left(-\frac{L_1}{\beta\tau}t\right)\chi(p, s), \quad (58)$$

with $\chi(t=0, p, s) = \chi(p, s)$. Therefore, Eq. (56) can be cast into

$$\begin{aligned} \omega_\alpha^{(2)} &= -i(\tilde{\chi}_\alpha^{(0)}(\hat{p}^\mu \kappa_\mu + \tau \omega_\alpha^{(1)}), L_1^{-1}[(\hat{p}^\mu \kappa_\mu + \tau \omega_\alpha^{(1)})\tilde{\chi}_\alpha^{(0)}]) \\ &= -i\kappa^2 \left(\tilde{\chi}_\alpha^{(0)} \left(\hat{p}^\mu l_\mu + \tau \frac{\omega_\alpha^{(1)}}{\kappa} \right), L_1^{-1} \left[\left(\hat{p}^\mu l_\mu + \tau \frac{\omega_\alpha^{(1)}}{\kappa} \right) \tilde{\chi}_\alpha^{(0)} \right] \right) \\ &= i \frac{1}{\beta(2\pi)^3} \int d\Gamma dt \exp(\xi - \beta \cdot p) \frac{1}{\tau} g(0, p, s) \exp\left(-\frac{L_1}{\beta\tau}t\right) g(0, p, s) \\ &= i \frac{1}{\beta(2\pi)^3} \int d\Gamma dt \exp(\xi - \beta \cdot p) \frac{1}{\tau} g(0, p, s) g(t, p, s) = i \frac{1}{\beta} \int_0^\infty dt \langle g(0, p, s) g(t, p, s) \rangle_0, \end{aligned} \quad (59)$$

with the definition $g(t=0, p, s) = (\hat{p}^\mu l_\mu + \tau \frac{\omega_\alpha^{(1)}}{\kappa})\tilde{\chi}_\alpha^{(0)}$ and the expectation value in equilibrium identified as $\langle A \rangle_0 \equiv \frac{1}{(2\pi)^3} \int d\Gamma \exp(\xi - \beta \cdot p) \frac{1}{\tau} A$.

From the Green-Kubo-like formula derived here, we can find that the second-order frequencies can all be related to the time correlation functions. It is widely believed that the transport coefficients can be expressed by the Green-Kubo formula, based on which the second-order dispersion relations of spin modes are supposed to be used as definitions for new transport coefficients or their combination. Since there is no well-calibrated spin hydrodynamic theory relevant to the spin kinetic theory adopted here and no corresponding definitions for our proposed novel transport coefficients elsewhere, we have to content ourselves with ending with Eq. (59).

V. THE RELAXATION OF SPIN

In this section, we show how to relate our linear analysis with the relaxation of spin toward the equilibrium state. To that end, first we note that the imaginary parts of spinless hydrodynamic modes encode the information of the relaxation of conserved currents, e.g., the energy-momentum tensor and the particle number current marked by the hydrodynamic transport coefficients. Similarly, there is no doubt that this conclusion can be generalized to new spin modes, encoding related information of the relaxation of total angular momentum. In the community of heavy-ion collisions, the relaxation of spin density toward equilibrium

is important and interesting, stimulated by the experimental research on local spin polarization. Whether the spin density relaxes to their equilibrium value earlier than other hydrodynamic variables or not is under debate. With the spin equilibrated picture still undetermined, further numerical simulations or related modelings lack the stable basis. Inspired by the enlightening discussions about the ordinary hydrodynamic modes analysis, we start with Eq. (58), then a general fluctuation i.e., the deviation function from the equilibrium distribution, can be expanded with the orthogonal normalized sets of the eigenfunctions of linearized collision (evolution) operator,

$$\chi(t, p, s) = \sum_{\alpha=1}^{\infty} \psi_\alpha(\psi_\alpha, \chi) \approx \sum_{\alpha=1}^{11} \psi_\alpha(\psi_\alpha, \chi), \quad (60)$$

where ψ_α is the inverse Fourier transformation of $\tilde{\psi}_\alpha$ [they are actually the same because the thermodynamic parameters appearing in Eq. (45) are chosen to be independent of the coordination x in global equilibrium] and the inner product (ψ_α, χ) represents the corresponding fluctuation amplitude (for instance, if α is chosen to be 1, this inner product denotes the fluctuation amplitude of the particle number density). Here we truncate the summation to the first 11 terms, for the first 11 zero modes are exactly the slowest modes, which are protected by the conservation laws. Because of the non-negative property of the linearized collision operator, other modes with positive eigenvalues

are damped by the exponential factor as time evolves in view of Eq. (58).

After singling out relevant zero modes from infinitely many positive modes, it is our task to see how these zero modes respond to the perturbation of nonuniformity, which will give us dispersion laws as is exhibited in the previous section. To that end, we move back to momentum space, and the eigenfunctions used to expand the fluctuation function $\tilde{\chi}$ turn into $\tilde{\chi}_\alpha$ in Eq. (51),

$$\tilde{\chi}(k, p, s) \approx \sum_{\alpha=1}^{11} \tilde{\chi}_\alpha(\tilde{\chi}_\alpha, \tilde{\chi}). \quad (61)$$

Analogous to the definition of the spin tensor (17), we define a new fluctuation for spin density,

$$\delta S^{\mu\nu}(x, p, s) = u_\lambda \int \frac{d\Gamma}{(2\pi)^3} e^{\xi - \beta \cdot p} p^\lambda \frac{1}{2} \Sigma_s^{\mu\nu} \chi(x, p, s), \quad (62)$$

where we have invoked that $u_\lambda \int d\Gamma p^\lambda \frac{1}{2} \Sigma_s^{\mu\nu} f_{\text{eq}}$ is the spin density in equilibrium compatible with the related definition in [20] and we neglect the derivative term belonging to the smaller order in gradients. By analogy with $\delta T^{\mu\nu}$ or δN^μ , the natural interpretation for $(\frac{u \cdot p}{2} \Sigma_s^{\mu\nu}, \chi)$ is the fluctuation amplitude for spin density.

Considering the overlap of $\Sigma_s^{\mu\nu}$ with the spinless eigenmodes ($\alpha = 1, 2, 3, 4, 5$) vanishes for $\int dS(p) s^\mu = 0$, only six spin-related fluctuation amplitudes ($\chi_\alpha, \tilde{\chi}$) are responsible for the relaxation of spin density,

$$\left(\frac{u \cdot p}{2} \Sigma_s^{\mu\nu}, \tilde{\chi} \right) = \sum_{\alpha=6}^{11} \left(\frac{u \cdot p}{2} \Sigma_s^{\mu\nu}, \tilde{\chi}_\alpha \right) (\tilde{\chi}_\alpha, \tilde{\chi}), \quad (63)$$

and these independent six modes decay at different relaxation times. Therefore, the relaxation time for spin density fluctuation should be identified as the largest one, namely,

$$\tau_s = \max \left\{ \frac{1}{|\omega_\alpha^{(2)}|}, \alpha = 6, \dots, 11 \right\}, \quad (64)$$

where $|A|$ denotes the amplitude of a complex A . Given specific interaction and proper form for the constraint function F in Eq. (31), solve the complicated integral equation (47) and the relaxation times for ordinary hydrodynamic dissipative quantities and spin density can be determined, constructing the related hierarchy for the different relaxation times and clarifying the relevant equilibrium picture.

Before ending this section, some comments about any possible pseudogauge dependencies are presented as follows. As is put forward in previous sections, the energy-momentum tensor in distinct gauges take the same symmetric form as far as only first-order gradients are in consideration. The dispersion relations of spinless modes do not change when taking distinct pseudogauges.

However, the variation of gauge will lead to the variation of definition of spin density and alter the spin density fluctuation (62), too. The variation should be of the form $\sim \int dS(p) s \chi(x, p, s)$ with only the spin-related structure displayed, which indicates that involved modes responsible for spin relaxation are still unchanged and Eq. (64) is formally identical. On the other hand, the precise relaxation rates of these spin modes are also unchanged given a fixed background profile on top of which the linear analysis is carried out. In a summary, our results presented before are independent of the choice of pseudogauges.

Note that our formalism is based on neglecting the antisymmetric part of the HW energy momentum, which manifests that spin angular momentum and orbit angular momentum are separately conserved. Though the present framework is rather restrictive, it is reasonable as $\partial_\mu T^{[\mu\nu]}$ is counted as $O(\partial^3)$ in the motion equation and thus is negligible. Moreover, the spin Boltzmann equation derived from Wigner formalism is limited to first order in semiclassical expansion (or gradients expansion) and does not encode the information of the antisymmetric part of the HW energy-momentum tensor by construction [44]. Therefore, the antisymmetric part of the HW energy-momentum tensor cannot be expressed by distribution function f or collision kernel $C[f]$ presented here and the discussion on it is beyond the application range of the present framework. We comment that it is a more straightforward way to formulate the retarded correlators from underlying field theoretical calculation and then systematically look for nonhydrodynamic modes related to nonconserved spin density.

VI. SUMMARY AND OUTLOOK

We provide a discussion about normal modes of a linearized collision operator based on the spin kinetic theory for massive fermions [27] in this paper. By insisting on Hermitian and non-negative properties of the linearized operator, we neglect the nondiagonal part of the transition rate and solve the dispersion relations for normal modes according to the degenerate perturbation theory we frequently see in quantum mechanics. With the conservation of total angular momentum in a collision event phenomenologically considered, we find 11 zero modes that are protected by conserved laws and thus are in relation to spin hydrodynamic theory.

Following similar procedures that are used in quantum mechanics, we obtain the dispersion relations of these modes, among which the results of five spinless modes are consistent with well-known conclusions. On the other hand, the frequencies of the remaining spin-related modes are also solved up to second order in wave vectors, though the second-order corrections are only formal solutions, which are proved to be in relation to the time correlation functions. In other words, one can directly calculate the

time correlation function without recourse to solving the complicated integral equation. We also show that our framework can be applied to investigate the relaxation of spin density. Our findings manifest that the relaxation of spin density has nothing to do with the spinless modes and the relaxation time is identified as the largest one of the reciprocals of damping rates for those spin-related modes. Therefore, given specific interactions and reasonable parametrized form for the constraint function in Eq. (38), the relaxation for both spin density and other dissipative hydrodynamic quantities can be determined, which at least provides a comparison for these typical timescales and decides which one is the slow process. The clarification of the hierarchy for relaxation times based on reliable quantum kinetic theory is highly nontrivial in resolving the problem of discovering the local spin polarization in the experiments of relativistic heavy-ion collisions. Note that the spin Boltzmann equation adopted here does not encode the antisymmetric part of the HW energy-momentum tensor; therefore, spin is approximately conserved and spin-orbit conversion happens at the second order in gradients expansion.

There is also one thing that deserves attention. Because our discussion is limited to ignoring the polarized part in the transition rate, the linearized collision operator is Hermitian. Without this approximation, the unperturbed operator is not Hermitian and the eigenvalues are not necessarily real, while the eigenfunctions are not necessarily orthogonal, which means that we cannot solve the problem following the same fashion as used in quantum mechanics. For the impact of the polarized transition rate on the relaxation of spin density or other spin-related variables, we leave the research work to the future.

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APPENDIX A: THERMODYNAMIC INTEGRAL

In this appendix, we calculate thermodynamic integrals used in the main text. The first one is

$$\begin{aligned} I^{\alpha_1 \cdots \alpha_n} &\equiv 2 \int \frac{dP}{(2\pi)^3} p^{\alpha_1} p^{\alpha_2} \cdots p^{\alpha_n} e^{\xi - \beta \cdot p} \\ &= I_{n0} u^{\alpha_1} \cdots u^{\alpha_n} \\ &\quad + I_{n1} (\Delta^{\alpha_1 \alpha_2} u^{\alpha_3 \cdots \alpha_n} + \text{permutations}) + \cdots, \end{aligned} \quad (\text{A1})$$

where in the second equality we have employed the analysis of the Lorentz covariance. By projecting u^α and $\Delta^{\alpha\beta}$ onto Eq. (A1), the scalar coefficients are defined as

$$I_{nq} \equiv \frac{2}{(2q+1)!!} \int \frac{dP}{(2\pi)^3} (u \cdot p)^{n-2q} (\Delta_{\alpha\beta} p^\alpha p^\beta)^q e^{\xi - \beta \cdot p}, \quad (\text{A2})$$

with $K_n(z)$ representing the modified Bessel functions of the second kind

$$K_n(z) \equiv \int_0^\infty dx \cosh(nx) e^{-z \cosh x}. \quad (\text{A3})$$

Especially, we have $I_{10} = n$, $I_{20} = e$, $I_{31} = -h$, and

$$I_{30}(z) = \frac{T^5 z^5}{32\pi^2} (K_5(z) + K_3(z) - 2K_1(z)), \quad (\text{A4})$$

with $z \equiv \frac{m}{T}$.

Additionally, the following similar formulas are also of great use:

$$\begin{aligned} L^{\alpha_1 \cdots \alpha_n} &\equiv 2 \int \frac{dP}{(2\pi)^3 (p^0 + m)} p^{\alpha_1} p^{\alpha_2} \cdots p^{\alpha_n} e^{\xi - \beta \cdot p} \\ &= L_{n0} u^{\alpha_1} \cdots u^{\alpha_n} + L_{n1} (\Delta^{\alpha_1 \alpha_2} u^{\alpha_3 \cdots \alpha_n} + \text{permutations}) + \cdots, \\ N^{\alpha_1 \cdots \alpha_n} &\equiv 2 \int \frac{dP}{(2\pi)^3 (p^0 + m)^2} p^{\alpha_1} p^{\alpha_2} \cdots p^{\alpha_n} e^{\xi - \beta \cdot p} \\ &= N_{n0} u^{\alpha_1} \cdots u^{\alpha_n} + N_{n1} (\Delta^{\alpha_1 \alpha_2} u^{\alpha_3 \cdots \alpha_n} + \text{permutations}) + \cdots. \end{aligned} \quad (\text{A5})$$

Similarly, L_{nq} and N_{nq} are also obtained as in Eq. (A2). Generally speaking, these scalar integrals may not be analytically worked by expressing them with the modified Bessel functions of the second kind $K_n(z)$, due to an extra factor appearing in the integrations, and we need to turn to numerical integration.

APPENDIX B: NORMALIZED FACTORS

In the main text, we utilize two auxiliary unit vectors j^μ and v^μ , which form a triad with the vectors u and l ,

$$\begin{aligned} u \cdot l = u \cdot j = u \cdot v = l \cdot j = l \cdot v = j \cdot v = 0, \\ l^2 = j^2 = v^2 = -1. \end{aligned} \quad (\text{B1})$$

Therefore, p^μ and $J^{\mu\nu}$ can be readily expanded as

$$\begin{aligned} p^\mu &= u \cdot p u^\mu + l \cdot p l^\mu + j \cdot p j^\mu + v \cdot p v^\mu, \\ J^{\mu\nu} &= u_\mu J^{\mu\nu} l_\nu - l_\mu J^{\mu\nu} u_\nu + u_\mu J^{\mu\nu} j_\nu - j_\mu J^{\mu\nu} u_\nu + u_\mu J^{\mu\nu} v_\nu - v_\mu J^{\mu\nu} u_\nu \\ &\quad + l_\mu J^{\mu\nu} j_\nu - j_\mu J^{\mu\nu} l_\nu + l_\mu J^{\mu\nu} v_\nu - v_\mu J^{\mu\nu} l_\nu + j_\mu J^{\mu\nu} v_\nu - v_\mu J^{\mu\nu} j_\nu. \end{aligned} \quad (\text{B2})$$

As the total angular momentum $J^{\mu\nu}$ is antisymmetric, the effective basis set is $(1, u \cdot p, l \cdot p, j \cdot p, v \cdot p, u_\mu J^{\mu\nu} l_\nu, u_\mu J^{\mu\nu} j_\nu, u_\mu J^{\mu\nu} v_\nu, l_\mu J^{\mu\nu} j_\nu, l_\mu J^{\mu\nu} v_\nu, j_\mu J^{\mu\nu} v_\nu)$. Throughout the paper, they are ordered this way and are labeled by the i th basis, respectively ($i = 1, 2, \dots, 11$).

Then the normalized factors appearing in Eq. (45) are

$$\begin{aligned} V_{1,1} &= \exp(\xi) \int \frac{d\Gamma}{(2\pi)^3} \frac{u \cdot p}{T} \exp(-\beta \cdot p) = \frac{n}{T}, \\ V_{2,2} &= \exp(\xi) \int \frac{d\Gamma}{(2\pi)^3} \frac{(u \cdot p - \frac{e}{n})^2 (u \cdot p)}{T^3} \exp(-\beta \cdot p) = \frac{I_{30} - \frac{e^2}{n}}{T^3}, \\ V_{3,3} &= V_{4,4} = V_{5,5} = \exp(\xi) \int \frac{d\Gamma}{(2\pi)^3} \frac{(u \cdot p)(l \cdot p)^2}{T^3} \exp(-\beta \cdot p) = \frac{h}{T^2}. \end{aligned} \quad (\text{B3})$$

When calculating the remaining normalized factors, recall the calibration settings mentioned after Eq. (31) $\hat{i} = u = (1, 0, 0, 0)$, and we obtain

$$\begin{aligned} V_{6,6} &= V_{7,7} = V_{8,8} = \exp(\xi) \int \frac{d\Gamma}{(2\pi)^3} \frac{u \cdot p}{T} u_\mu J^{\mu\nu} l_\nu u_\rho J^{\rho\sigma} l_\sigma \exp(-\beta \cdot p) \\ &= \frac{1}{2m^2 T} (-I_{31} + 2L_{41} - N_{51}), \\ V_{9,9} &= V_{10,10} = V_{11,11} = \exp(\xi) \int \frac{d\Gamma}{(2\pi)^3} l_\mu J^{\mu\nu} j_\nu (p \cdot u) l_\rho J^{\rho\sigma} j_\sigma \exp(-\beta \cdot p) \\ &= \frac{I_{30} + I_{31} + 4L_{41} + 10N_{52}}{4m^2 T}, \end{aligned} \quad (\text{B4})$$

where L_{nq} and N_{nq} are defined in Appendix A and the factors $V_{2,2}$, $V_{6,6}$, and $V_{9,9}$ are positive.

APPENDIX C: PERTURBATION MATRIX ELEMENTS

We compute the perturbation matrix elements used in Eq. (49). The matrix elements sandwiching the $p \cdot \kappa$ term are written as

$$\begin{aligned} H_{1,3} &= \frac{\exp(\xi)}{\sqrt{V_{1,1} V_{3,3} T^2}} \int \frac{d\Gamma}{(2\pi)^3} \kappa \cdot p l \cdot p \exp(-\beta \cdot p) = \frac{P\kappa}{\sqrt{V_{1,1} V_{3,3} T^2}}, \\ H_{2,3} &= \frac{\exp(\xi)}{\sqrt{V_{2,2} V_{3,3} T^3}} \int \frac{d\Gamma}{(2\pi)^3} \left(u \cdot p - \frac{e}{n} \right) p \cdot \kappa (l \cdot p) \exp(-\beta \cdot p) = \frac{P\kappa}{\sqrt{V_{2,2} V_{3,3} T^2}}, \\ H_{7,9} &= H_{8,10} = \frac{\exp(\xi)}{\sqrt{V_{6,6} V_{9,9}}} \int \frac{d\Gamma}{(2\pi)^3} \frac{p \cdot \kappa}{T} u_\mu J^{\mu\nu} j_\nu l_\rho J^{\rho\sigma} j_\sigma \exp(-\beta \cdot p) = \frac{(-I_{31} + L_{41} - 5L_{42} + 5N_{52})\kappa}{4m^2 T \sqrt{V_{6,6} V_{9,9}}}. \end{aligned} \quad (\text{C1})$$

Notice that the perturbation matrix owns a symmetry of transposition $H_{i,j} = H_{j,i}$. Except for the above matrix elements and their transpositions, the others are all zero.

APPENDIX D: PROOF OF HERMITICITY OF LINEARIZED COLLISION OPERATOR

First, we note the linearized collision operator to be proved is

$$\begin{aligned} \mathcal{L}_1[\phi] \equiv & \frac{1}{(2\pi)^3} \int d\Gamma' d\Gamma_1 d\Gamma_2 \bar{\mathcal{W}}F[p, p', p_1, p_2; s, s', s_1, s_2] \exp(\xi - u \cdot p') \\ & \times [\phi(x, p, s) + \phi(x, p', s') - \phi(x, p_1, s_1) - \phi(x, p_2, s_2)]. \end{aligned} \quad (\text{D1})$$

In the sense of the definition of the inner product, Eq. (40), we obtain

$$\begin{aligned} (\mathcal{L}_1[\phi], \psi) &= \frac{1}{(2\pi)^6} \int d\Gamma d\Gamma' d\Gamma_1 d\Gamma_2 \bar{\mathcal{W}}F[p, p', p_1, p_2; s, s', s_1, s_2] \exp(2\xi - u \cdot (p + p')) \\ & \quad \times [\phi(x, p, s) + \phi(x, p', s') - \phi(x, p_1, s_1) - \phi(x, p_2, s_2)] \psi(x, p, s) \\ &= \frac{1}{2(2\pi)^6} \int d\Gamma d\Gamma' d\Gamma_1 d\Gamma_2 \bar{\mathcal{W}}F[p, p', p_1, p_2; s, s', s_1, s_2] \exp(2\xi - u \cdot (p + p')) \\ & \quad \times [\phi(x, p, s) + \phi(x, p', s') - \phi(x, p_1, s_1) - \phi(x, p_2, s_2)] (\psi(x, p, s) + \psi(x, p', s')) \\ &= \frac{1}{4(2\pi)^6} \int d\Gamma d\Gamma' d\Gamma_1 d\Gamma_2 \bar{\mathcal{W}}F[p, p', p_1, p_2; s, s', s_1, s_2] \exp(2\xi - u \cdot (p + p')) \\ & \quad \times [(\phi(x, p, s) + \phi(x, p', s'))(\psi(x, p, s) + \psi(x, p', s')) - 2(\phi(x, p_1, s_1) + \phi(x, p_2, s_2))(\psi(x, p, s) + \psi(x, p', s')) \\ & \quad + (\phi(x, p_1, s_1) + \phi(x, p_2, s_2))(\psi(x, p_1, s_1) + \psi(x, p_2, s_2))]. \end{aligned} \quad (\text{D2})$$

Only by neglecting the polarized part in the collision rate can the third step be implemented. To see this, when interchanging (p, s) with (p_1, s_1) [for simplicity, we omit (p', s') and (p_2, s_2) because of duplication and other trivial factors] and taking the full h_{sr} into account,

$$\mathcal{W} = \sum_{s, r, s_1, r_1} h_{sr}(p, s) h_{s_1 r_1}(p_1, s_1) \langle p, r | t | p_1, s_1 \rangle \langle p_1, r_1 | t^\dagger | p, s \rangle, \quad (\text{D3})$$

$$\begin{aligned} \mathcal{W}[(p, s) \leftrightarrow (p_1, s_1)] &= \sum_{s, r, s_1, r_1} h_{sr}(p_1, s_1) h_{s_1 r_1}(p, s) \langle p_1, r | t | p, s_1 \rangle \langle p, r_1 | t^\dagger | p_1, s \rangle \\ &= \sum_{s, r, s_1, r_1} h_{sr}(p, s) h_{s_1 r_1}(p_1, s_1) \langle p_1, r_1 | t | p, s \rangle \langle p, r | t^\dagger | p_1, s_1 \rangle. \end{aligned} \quad (\text{D4})$$

In general, Eq. (D3) is not equivalent to Eq. (D4) unless t is Hermitian. From Eq. (D2), we conclude that $(\mathcal{L}_1[\phi], \phi) \geq 0$ if the transition rate respects this exchanging symmetry. From the above argument, we find the non-negative property of the linearized collision operator lies in the negligence of the polarized part of collision rate \mathcal{W} . When neglecting the polarized vector n_{sr} , the collision rate can be cast into the traditional form of cross section and thus is symmetric for exchanging the momentums of initial and final states, which is called the principle of detailed balance.

Continuing the discussion following Eq. (D2) with the collision rate replaced by the reduced (unpolarized) one, we find with assistance of the detailed balance,

$$(\mathcal{L}_1[\phi], \psi) = (\mathcal{L}_1[\psi], \phi) = (\phi, \mathcal{L}_1[\psi]). \quad (\text{D5})$$

To prove this, interchanging $(p, s), (p', s')$ with $(p_1, s_1), (p_2, s_2)$,

$$(\phi(x, p_1, s_1) + \phi(x, p_2, s_2))(\psi(x, p, s) + \psi(x, p', s')) \rightarrow (\psi(x, p_1, s_1) + \psi(x, p_2, s_2))(\phi(x, p, s) + \phi(x, p', s')), \quad (\text{D6})$$

the integral in Eq. (D2) over this term should be invariant. Such property is employed for the second term within the bracket of Eq. (D2), invoking the symmetry property of the transition rate (the other terms have this exchange symmetry of ϕ and ψ already). Without detailed balance,

Eq. (D5) cannot hold, thus the polarized part of the collision rate is also responsible for non-Hermiticity of the linearized collision term. To conclude, we derive the non-negative self-adjoint property for the unpolarized linearized collision operator.

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