

Baryonic form factors of the pion and kaon in a chiral quark model

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The baryonic form factor of charged pions is studied in detail in the Nambu–Jona-Lasinio model with constituent quarks, where the spontaneously broken chiral symmetry is the key dynamical ingredient guaranteeing the would-be Goldstone boson nature of the pseudoscalar mesons octet. In general, this form factor arises when the isospin symmetry is broken, which is the case if the u and d quark masses are split, as in the real world, or if the electromagnetic effects were taken into account. We obtain estimates for this basic property of the pion resulting from the quark mass splitting for a range of model parameters, and importantly, for different pion masses, going up to the values used in lattice QCD. We find very stable model results, with the mean square radius of π^+ in the range $(0.05\text{--}0.07\text{ fm})^2$. From charge conjugation, the baryonic form factor of π^+ and π^- are equal and opposite. We also obtain the transverse-coordinate, relativistically invariant baryonic density of the charged pion. In π^+ , the inner region carries a negative, and the outside—a positive baryon number density, both cancelling to zero, as obviously the pion carries no net baryon charge. We also carry out an analogous analysis for the kaon, where the effect is much larger due to the sizable s and u/d quark mass splitting. We discuss the prospects of lattice QCD measurements of the baryonic form factors of charged pions and kaons.

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I. INTRODUCTION

As it is well known, by construction mesons carry no net baryon charge. However, in a recent paper [1] we brought up the largely overlooked fact that the *baryonic* form factor of charged pions does not vanish when the isospin symmetry is broken, as is the case when the u and d quark masses are not equal, or when the electromagnetic (EM) effects are taken into account. We carried out several estimates of the effect based on very different approaches, ranging from simple quark models to an extraction from the available $e^+e^- \rightarrow \pi^+\pi^-$ data (BABAR [2] and KLOE [3–6]), made with the help of the vector meson dominance (VMD) involving the $\rho - \omega$ mixing. Our data analysis [1] yielded the following estimate for the baryonic mean squared radius (msr) of π^+ :

$$\langle r^2 \rangle_B^{\pi^+} = (0.041(1)\text{ fm})^2 = 0.0017(1)\text{ fm}^2. \quad (1)$$

As expected from the weak effect of isospin breaking, this is small compared to the central value of the accurately known EM radius $\langle r^2 \rangle_Q^{\pi^+} = (0.659(4)\text{ fm})^2 = 0.434(5)\text{ fm}^2$ [7], and at the level of about one third of the quoted error.

The fact that the charged pion has yet another, hitherto unexplored form factor corresponding to a conserved current is fundamental and definitely worth further dedicated studies. It has a very intuitive physical interpretation in the coordinate space (in the Breit frame), or in the transverse-coordinate space [8–10], where (1) implies that in π^+ the outer region has a net baryon density from the excess of the lighter u quark, whereas the inner region has a net antibaryon density from the heavier \bar{d} antiquark. Of course, over the whole space the baryon and antibaryon densities compensate each other such that the total baryon number of the pion is zero. For π^- , the described geometric picture is opposite, with more antibaryon in the outer region.

The presence of the baryonic form factor of charged pions allows one for a natural interpretation of the VMD modeling with $\rho - \omega$ mixing, where ω couples to the baryon current [1]. The extraction from the data in the

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timelike region, where the mixing becomes most visible, requires conventional model assumptions regarding the shape of the higher resonance profiles. An extrapolation to the spacelike region proceeds via a dispersion relation [1].

In this paper we focus entirely on the estimates of the nonvanishing baryonic form factor of pseudoscalar mesons (pions and kaons) based on a chiral quark model, namely the Nambu–Jona-Lasinio (NJL) model with the Pauli-Villars (PV) regularization (see [11] and references therein). The model ensures that both pions and kaons emerge as would-be pseudoscalar Goldstone bosons, thanks to the spontaneous breakdown of the chiral symmetry. We note that this model has been used to obtain successful phenomenology for a great variety of soft matrix elements involving pseudo-Goldstone bosons, hence one may hope it produces credible results for the baryonic form factor as well. Moreover, in the model we can easily change parameters. In particular, one can increase the pion mass up to larger values, such as those employed in some lattice QCD simulations, which of course could not be made in an extraction from the experimental data. One can also increase the u and d mass splitting, which augments the effect up to the point where it can be easily observed on the lattice.

The baryonic form factor of charged pions and kaons is proportional to the splitting of the masses of the constituents that build up the meson. Compared to [1], in the present work we analyze this splitting more thoroughly, which augments the model estimate for the baryonic msr of the pion. We derive analytic expressions, allowing for a better comprehension of the dependence of the effect on the kinematic and model parameters. The full details of our calculations are given in the appendixes.

We also analyze the case of the neutral kaons, where (for structureless quarks as in the NJL model at the leading- N_c level) the baryonic form factor is, up to an overall sign, equal to the EM form factor, for which experimental and lattice data do exist. For the case of the pion, where $m_d - m_u$ is tiny compared to other scales in the model, the baryonic form factor can be evaluated to first order in $m_d - m_u$. For the kaon, however, all orders in $m_s - m_{u/d}$, which are substantial, should be kept. Hence we carry out the calculation exactly at the one-quark-loop level, which within the NJL model corresponds to the leading- N_c approximation.

II. SYMMETRIES AND BARYONIC FORM FACTOR OF THE PION

We begin by reviewing some well-known basic facts, for completeness and to establish our notation. We then proceed to show that the symmetries do not preclude a nonzero baryonic form factor of charged pions. We recall that in QCD the vector currents corresponding to any flavor $f = u, d, s, c, b, t$, defined as

$$J_f^\mu(x) = \bar{q}_f(x)\gamma^\mu q_f(x), \quad (2)$$

are conserved:

$$\partial_\mu J_f^\mu(x) = 0, \quad (3)$$

with $q_f(x)$ denoting a quark field with $N_c = 3$ colors (the summation over color is understood). One introduces the baryon current and the third isospin component of the isovector current as

$$J_B^\mu = \frac{1}{N_c} \sum_f J_f^\mu, \quad J_3^\mu = \frac{1}{2}(J_u^\mu - J_d^\mu). \quad (4)$$

For the considered case of the pion, one can ignore the strangeness and heavier flavor contributions to matrix elements of J_B^μ , as they are strongly suppressed by the Okubo-Zweig-Iizuka (OZI) rule and subleading in the large- N_c limit. Hence we take $J_B^\mu = (J_u^\mu + J_d^\mu)/N_c$. The EM current follows from the Gell-Mann–Nishijima formula,

$$J_Q^\mu = J_3^\mu + \frac{1}{2}J_B^\mu. \quad (5)$$

The baryon, the (third component of) isospin, and the electromagnetic form factors are defined via matrix elements of the corresponding currents in on-shell pion states of isospin $a = 0, +, -$, namely

$$\langle \pi^a(p+q) | J_{B,3,Q}^\mu(0) | \pi^a(p) \rangle = (2p^\mu + q^\mu) F_{B,3,Q}^a(t), \quad (6)$$

with $F_Q^a(t) = F_3^a(t) + \frac{1}{2}F_B^a(t)$, in line with Eq. (5). Here and in the following, p is the momentum of the initial pion, $p+q$ of the final pion, and $q^2 \equiv t$. We note the following scaling with number of QCD colors: $J_3 \sim 1$ and $J_B \sim 1/N_c$.

Now we pass to symmetries. The currents $J_{B,3}^\mu$ are odd under the charge conjugation C , whereas the neutral pion is an eigenstate, $C|\pi^0\rangle = |\pi^0\rangle$, which immediately implies

$$F_{B,3}^{\pi^0}(t) = 0 \quad (7)$$

identically (at any t). On the other hand, since $C|\pi^\pm\rangle = |\pi^\mp\rangle$, for the charged pions the C conjugation only yields the condition

$$F_{B,3,Q}^{\pi^+}(t) = -F_{B,3,Q}^{\pi^-}(t), \quad (8)$$

and, of course, no vanishing follows.

In the case of an exact isospin symmetry, which holds when the light current quark masses are equal, $m_u = m_d$, and when the small EM effects are ignored, G -parity is also a good symmetry. It is defined as the charge conjugation followed with the rotation by π about the 2-axis in isospin $G = e^{i\pi I_2} C$. All the pions are eigenstates of G , namely

$G|\pi^a\rangle = -|\pi^a\rangle$. Since G involves a flip between u and d , J_3 becomes even under G (yielding no constraints), whereas the baryon current remains odd, implying for any a

$$F_B^{\pi^a}(t) = 0 \quad (\text{for an exact isospin symmetry}). \quad (9)$$

The key point now is that the isospin is only an approximate symmetry of Nature, broken with $m_d \neq m_u$ and with EM interactions. Consequently, G -parity is no longer a good symmetry and the condition (9) need not, and as we explicitly show in [1] and in the present paper, does not hold for charged pions in a field theoretical model with conserved baryon current.

Additional constraints for the form factors at vanishing t follow from the additivity of charges. This feature holds in any local field theory for charges associated with conserved local currents. In particular, $F_3^{\pi^+}(t) = 1$, which is a sum of isospin charges of u and \bar{d} , analogously $F_3^{\pi^-}(t) = -1$, and

$$F_B^{\pi^\pm}(0) = 0, \quad (10)$$

which sums to zero the opposite baryon charges of a quark and antiquark.

To summarize, Eqs. (7), (8) hold as long as C is a good symmetry (strong and EM interactions), Eq. (10) is always true on general field-theoretic grounds, whereas Eq. (9) does not in general hold in the real world, where $m_d > m_u$ and EM interactions are involved.

Although it may seem unusual at first glance, the fact that neutral particles have a corresponding nonzero charge form factor is not uncommon. The neutron carries no electric charge, but has a nonzero electric form factor, with msr equal to $\langle r^2 \rangle_Q^n = -0.1161(22) \text{ fm}^2$. As already mentioned at the end of Sec. I, the same is true for neutral kaons. Even more unexpectedly, the nucleon possesses a nonvanishing strangeness form factor, despite being strangeless [12,13].

In this work, similarly as in [1], we only explore the *charge symmetry breaking* (nomenclature borrowed from nuclear physics, where $m_n > m_p$) in quark models, i.e., the effects of $m_d > m_u$. An analysis of EM effects would be much more involved and extends beyond the scope of the present work.

III. CHIRAL QUARK MODELS AT THE ONE-QUARK-LOOP LEVEL

We carry out our calculations within the NJL model, where the (point-like) four-quark interactions lead to dynamical chiral symmetry breaking, amending quarks with large constituent masses (see [11] for a review, and references therein). At the leading- N_c level, various observable quantities are evaluated using one-quark loop. We stress that in this treatment the pion is described in a covariant, fully relativistic manner in terms of the corresponding Bethe-Salpeter equation. The model is designed for soft physics, where the virtualities of quarks in the loop are small and Euclidean,¹ hence a regularization has to be used to cut off the hard momenta. To preserve the Lorentz, gauge, and chiral symmetries, care is needed here. A regularization scheme that works, applied in this work, is based on the PV regularization with two subtractions [11,15] (see Appendix A for details).

The approach has been used successfully for a great variety of soft matrix elements, shedding light also onto such quantities as the parton distribution functions [16], parton distribution amplitudes [17], generalized parton distribution functions [18], or the double parton distributions of the pion [19,20] (see, e.g., [21] for a review). As the NJL model with unequal quark masses is not so frequently used (see, however, [22,23] for the model in the proper-time regularization), in Appendix B we provide a glossary of the NJL model formulas used in our calculations. We also discuss there the standard strategy of fixing the model parameters such as the average current quark mass and the PV cut-off.

In the following, we use the short-hand notation for current quark mass average and for the splitting,

$$m = \frac{m_u + m_d}{2}, \quad \delta = m_d - m_u, \quad (11)$$

and for the constituent quarks, correspondingly,

$$M = \frac{M_u + M_d}{2}, \quad \Delta = M_d - M_u. \quad (12)$$

The one-loop Feynman diagrams used to evaluate the form factors for currents J_u and J_d are presented in Fig. 1, where we have chosen π^+ for definiteness. The coupling of the pion to the quarks carries the coupling constant $g_{\pi^+u\bar{d}}$ discussed in Appendix B. The expression for the matrix elements corresponding to the two diagrams are

$$\begin{aligned} \langle \pi^+(p) | J_u^\mu(0) | \pi^+(p+q) \rangle &= -(\sqrt{2}g_{\pi^+u\bar{d}})^2 \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[\gamma^\mu \frac{i}{\not{k} - M_u} \gamma_5 \frac{i}{\not{k} - \not{p} - M_d} \gamma_5 \frac{i}{\not{k} + \not{q} - M_u} \right], \\ \langle \pi^+(p) | J_d^\mu(0) | \pi^+(p+q) \rangle &= -(\sqrt{2}g_{\pi^+u\bar{d}})^2 \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[\gamma^\mu \frac{i}{\not{k} - \not{q} - M_d} \gamma_5 \frac{i}{\not{k} - \not{p} - M_u} \gamma_5 \frac{i}{\not{k} - M_d} \right], \end{aligned} \quad (13)$$

¹Note a recent work [14] based on the Bethe-Salpeter equations in the Dyson-Schwinger formalism, where the pion charge form factor is accessible also at physical momenta.

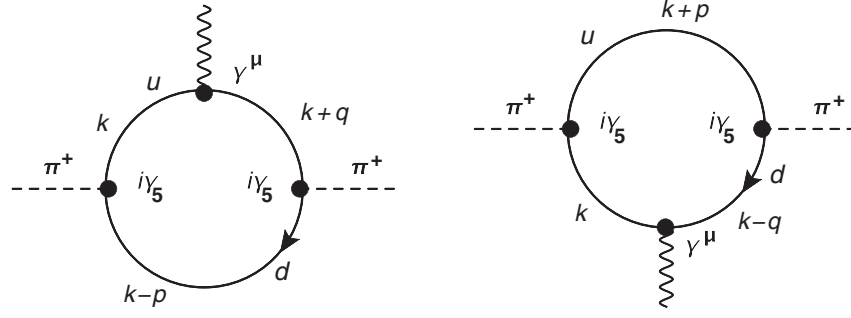


FIG. 1. Feynman diagrams for the one-loop evaluation of form factors F_3 and F_B for a charged pion in chiral quark models.

with the $+i\epsilon$ prescription in the propagators understood. The trace involves color and Dirac spaces. In the adopted convention, the $\sqrt{2}$ factors in the coupling follow from the isospin Clebsch-Gordan coefficients. The $F_{3,B}$ form factors are then evaluated according to Eq. (6) in a standard way.

Convenient algebraic *Mathematica* packages for this task are FeynCalc [24,25] and Package-X [26], which is also capable of evaluating the Passarino-Veltman (PaVe) one loop functions [27] (see Appendix A) up to three propagators. The result is

$$F_3^{\pi^+}(t) = \frac{2N_c \pi^2 g_{\pi^+ u \bar{d}}^2}{t - 4m_\pi^2} [(2\Delta^2 - 2m_\pi^2 + t)B_0^\Lambda(t, M_d^2, M_d^2) - 2(\Delta^2 + m_\pi^2)B_0^\Lambda(m_\pi^2, M_d^2, M_u^2) + 2(-\Delta^3 M_d - \Delta M_u(\Delta^2 - 2m_\pi^2 + t) + m_\pi^4)C_0^\Lambda(t, m_\pi^2, m_\pi^2, M_d^2, M_d^2, M_u^2)] + (u \leftrightarrow d, \Delta \rightarrow -\Delta), \quad (14)$$

$$F_B^{\pi^+}(t) = \frac{4\pi^2 g_{\pi^+ u \bar{d}}^2}{t - 4m_\pi^2} [-(2\Delta^2 - 2m_\pi^2 + t)B_0^\Lambda(t, M_d^2, M_d^2) - 2(-\Delta^3 M_d - \Delta M_u(\Delta^2 - 2m_\pi^2 + t) + m_\pi^4)C_0^\Lambda(t, m_\pi^2, m_\pi^2, M_d^2, M_d^2, M_u^2)] - (u \leftrightarrow d, \Delta \rightarrow -\Delta), \quad (15)$$

where B_0^Λ and C_0^Λ are the PaVe functions in the PV regularization, see Appendix A. Their explicit forms for general kinematics and quark masses are analytic, but lengthy. These exact formulas are used to obtain the results presented in the following sections.

On general symmetry grounds, as is also apparent from the explicit expressions in Appendix B, the quantities m_π^2 and $g_{\pi^+ u \bar{d}}^2$ appearing in (14), (15) are even under the exchange $u \leftrightarrow d$, hence are even functions of Δ . Therefore $F_3^{\pi^+}$ ($F_B^{\pi^+}$) is an even (odd) function of Δ and, correspondingly, a series expansion of $F_3^{\pi^+}$ ($F_B^{\pi^+}$) involves only even (odd) powers of Δ .

Much simplified formulas for the form factors follow in the chiral limit of $m_\pi^2 = 0$ and in the leading order in Δ , which we present for a better understanding of the following estimates. To the leading order in Δ and in the chiral limit the Goldberger-Treiman relation holds,

$$g_{\pi^+ u \bar{d}} = \frac{M}{f}, \quad (16)$$

where $f = 86$ MeV is the value of the pion weak-decay constant, F_π , in the chiral limit. With these simplifications we arrive at

$$F_3^{\pi^+}(t) = 1 + \frac{M^2 N_c}{4\pi^2 f^2} \left(2 - \sigma \log \left(\frac{\sigma + 1}{\sigma - 1} \right) \right) \Big|_{\text{reg}}, \quad (17)$$

$$F_B^{\pi^+}(t) = \frac{\Delta M^3}{2\pi^2 f^2 t} \left[\log^2 \left(\frac{\sigma + 1}{\sigma - 1} \right) - \frac{2}{\sigma} \log \left(\frac{\sigma + 1}{\sigma - 1} \right) \right] \Big|_{\text{reg}}, \quad (18)$$

where we have used the short-hand notation $\sigma = \sqrt{1 - 4M^2/t}$, and “reg” denotes the presence of regularization. Since $f \sim 1/\sqrt{N_c}$, we verify that $F_3 \sim 1$ and $F_B \sim 1/N_c$, as stated earlier.

It is instructive to look at the low- t expansion of the general formulas (14), (15) in the PV regularization, up to order Δ and m_π^2 . In doing so, we also expand to this order the coupling constant:

$$g_{\pi^+ u \bar{d}}^2 = \frac{M^2}{f^2} \left[1 - \frac{N_c \Lambda^4 m_\pi^2}{12\pi^2 f M (\Lambda^2 + M^2)^2} \right], \quad (19)$$

where here and below the higher order terms are dropped. The result is

$$\begin{aligned}
F_3^{\pi^+}(t) &= 1 + t \frac{N_c}{24\pi^2 f^2} \left[\frac{\Lambda^4}{(\Lambda^2 + M^2)^2} + \frac{\Lambda^4(\Lambda^2 + 3M^2)m_\pi^2}{5M^2(\Lambda^2 + M^2)^3} - \frac{N_c \Lambda^4 m_\pi^2}{12\pi^2 f^2 (\Lambda^2 + M^2)^2} \right] \\
&\stackrel{\Lambda \rightarrow \infty}{=} 1 + t \frac{N_c}{24\pi^2 f^2} \left[1 - \frac{2M^2}{\Lambda^2} + \frac{m_\pi^2}{5M^2} - \frac{N_c m_\pi^2}{12\pi^2 f^2} \right].
\end{aligned} \tag{20}$$

$$\begin{aligned}
F_B^{\pi^+}(t) &= t \frac{\Delta}{24\pi^2 f^2 M} \left[\frac{\Lambda^4(\Lambda^2 + 3M^2)}{(\Lambda^2 + M^2)^3} + \frac{4\Lambda^4(\Lambda^4 + 6M^4 + 4\Lambda^2 M^2)m_\pi^2}{15M^2(\Lambda^2 + M^2)^4} - \frac{N_c \Lambda^4 m_\pi^2}{12\pi^2 f^2 (\Lambda^2 + M^2)^2} \right] \\
&\stackrel{\Lambda \rightarrow \infty}{=} t \frac{\Delta}{24\pi^2 f^2 M} \left[1 - \frac{3M^4}{\Lambda^4} + \frac{4m_\pi^2}{15M^2} - \frac{N_c m_\pi^2}{12\pi^2 f^2} \right],
\end{aligned} \tag{21}$$

As typical constituent quark masses are $M \sim 300$ MeV, we infer from these formulas that the chiral corrections to the slopes are positive and small, at a level of a few percent.² Also, the results (20), (21) are not far from the infinite cutoff limit, since $\Lambda \sim 800$ MeV. The ratio of the ms radii is

$$\begin{aligned}
\frac{\langle r^2 \rangle_B^{\pi^+}}{\langle r^2 \rangle_3^{\pi^+}} &= \frac{\Delta(\Lambda^2 + 3M^2)}{N_c M(\Lambda^2 + M^2)} \left[1 + \frac{m_\pi^2(\Lambda^2 - 3M^2)}{15M^2(\Lambda^2 + 3M^2)} \right] \\
&\stackrel{\Lambda \rightarrow \infty}{=} \frac{\Delta}{N_c M} \left[1 + \frac{2M^2}{\Lambda^2} + \frac{m_\pi^2}{15M^2} \right].
\end{aligned} \tag{22}$$

We note that numerically, as follows from the fits shown in the proceeding sections, Δ/M is at the level of a few percent, and the higher-order terms in the expansion of F_B , starting at $(\Delta/M)^3$, are completely negligible.

IV. QUARK MASS SPLITTING

The formulas of the previous section show the *a priori* expected proportionality of F_B to the constituent mass splitting Δ . Clearly, to make numerical estimates we need the “physical” value for Δ , which is not a trivial matter. While for the current quark masses of all flavors, which are QCD parameters, we have available information from physical processes and perturbation theory, for the constituent quarks we need to rely on models. This is because the very notion of a constituent quark has a meaning only within a specified model, such as the NJL model in our case.

The first issue is the dependence of a constituent quark mass M_f on the current quark mass m_f . In NJL, the relation following from the self-consistent gap equation (see Appendix B and Fig. 4) is nearly linear at low m_f , namely $M_f \simeq M_f(m_f = 0) + \alpha m_f$, with $\alpha \sim 2$. So there is no naive

²Note, however, that our estimates are obtained from the one-quark-loop evaluation. Inclusion of pion loops would introduce effects relatively suppressed by $1/N_c$, but chirally dominant, as is the case of chiral perturbation theory.

additivity of the constituent and current quark masses³ (that would mean $\alpha = 1$), which is a feedback effect from the quark loop in the gap equation (see Appendix B).

The model value of m of Eq. (11) is obtained in the model by fitting the physical mass of the pion, m_π , which in the absence of EM effects is ~ 135 MeV.⁴ Our result, depending weakly on the adopted value of M , is $m \sim 7$ – 8 MeV. This value is about a factor of two larger than the value quoted by the PDG [7] at the scale $\mu = 2$ GeV in the $\overline{\text{MS}}$ renormalization scheme:

$$m(\mu = 2 \text{ GeV}) = 3.45_{-0.15}^{+0.55} \text{ MeV}. \tag{23}$$

The effect is due to the running of m with the scale. In perturbative QCD, at leading order (LO) in α_S , one has

$$\frac{m_f(\mu)}{m_f(\mu_0)} = \left(\frac{\alpha_S(\mu)}{\alpha_S(\mu_0)} \right)^{\frac{4}{\beta_0}}, \tag{24}$$

where with three flavors $\beta_0 = 9$, $\alpha_S = 4\pi/[\beta_0 \log(\mu^2/\Lambda_{\text{QCD}}^2)]$, and $\Lambda_{\text{QCD}} = 226$ MeV. Let μ_0 denote the quark-model scale, where $m(\mu_0) = 7$ MeV (model fit with $M = 300$ MeV) or 8 MeV (model fit with $M = 350$ MeV). Then, from Eq. (24) we can infer the value of μ_0 , which becomes

$$\begin{aligned}
\mu_0 &= 352_{-14}^{+68} \text{ MeV} \quad (\text{for } m(\mu_0) = 7 \text{ MeV}), \\
\mu_0 &= 314_{-10}^{+43} \text{ MeV} \quad (\text{for } m(\mu_0) = 8 \text{ MeV}),
\end{aligned} \tag{25}$$

where the errors reflect the uncertainty in Eq. (23). We thus see that the quark model scale μ_0 is very low.

Quite remarkably, the estimate of Eq. (25), especially for the case of 8 MeV, is very close to the value obtained with an altogether different method, i.e., by using the evolution of the valence quark fraction in the momentum sum rule for

³This is what was inaccurately assumed in [1], which lowered the NJL estimates presented there.

⁴The splitting $m_{\pi^+}^2 - m_{\pi^0}^2$ is mainly induced by $\mathcal{O}(\alpha_{\text{QED}})$ EM effects [28], while the quark mass splitting effect is negligible, as it appears at $\mathcal{O}(\delta^2)$ [29].

TABLE I. Parameters for the analysis of the pion baryonic form factor. The uncertainties in δ , Δ , and $\sqrt{\langle r^2 \rangle_B^{\pi^+}}$ follow from Eq. (27).

M [MeV]	Λ [MeV]	m [MeV]	δ [MeV]	Δ [MeV]	m_π [MeV]	F_π [MeV]	$\sqrt{\langle r^2 \rangle_B^{\pi^+}}$ [fm]
300	732	0	5.0(9)	9(2)	0	86	0.062(5)
300	830	7.0	5.0(9)	11(2)	135	93	0.062(5)
300	960	41.0	5.0(9)	13(2)	400	111	0.066(6)
280	870	6.5	4.7(7)	11(2)	135	93	0.067(5)
350	770	7.9	5.7(1.0)	9(1)	135	93	0.052(4)

the pion parton distribution function. There, one finds $\mu_0 = 313_{-10}^{+20}$ MeV [16,18]. One could of course argue here that the use of perturbative evolution down to such low scale is questionable at best, but the fact that the quark model scale is very low touches upon the essence of effective chiral quark models, where there are no explicit gluon degrees of freedom.

Actually, what we need in the present task is not a precise value of μ_0 , but only the fact that the $\overline{\text{MS}}$ evolution prescription preserves the ratios of the current quark masses of different flavor, which are independent of the scale:

$$\frac{m_f(\mu)}{m_{f'}(\mu)} = \frac{m_f(\mu_0)}{m_{f'}(\mu_0)}. \quad (26)$$

Therefore the ratio $m_d/m_u(\mu_0)$ which we should use in the model is exactly the same as the PDG [7] value at $\mu = 2$ GeV:

$$\frac{m_u}{m_d} = 0.47_{-0.07}^{+0.06}. \quad (27)$$

We can make another independent estimate of the splitting δ based on the difference of the K^0 and K^+ masses. First, one needs to subtract the EM effects, which contribute 2.6 MeV more to K^+ than to K^0 [28]. We thus need to adjust in the model the values of m_d and m_u such that the resulting splitting between K^0 and K^+ masses is the physical value plus the EM correction, $3.9 + 2.6 = 6.5$ [MeV]. With $m = 7$ MeV and $m_s = 181$ MeV (needed to fit the physical value of m_K) we get in the model $m_u/m_d \simeq 0.47$, a value in the center of (27), whereas with $m = 7$ MeV we find $m_u/m_d \simeq 0.52$, within the error of (27). To summarize, in our numerical studies presented below we will use the PDG ratio (27) as a credible estimate.

Finally, speaking of the QCD evolution, we recall that form factors corresponding to conserved currents are scale independent. Therefore both the charge and the baryonic form factors of the pion obtained at a given scale (here, at the quark model scale), are universal. Yet, to carry out the calculation, we need to know the pertinent quantities, such as m or δ , at the scale where the calculation is made.

V. RESULTS FOR THE PION

The parameters used in our estimates for the pion are collected in Table I. We study the dependence on m_π (the first three rows of the table), from the chiral limit, through the physical mass (without EM effects) of 135 MeV, up to a large value of 400 MeV, typical in some lattice QCD simulations. For the first three rows of Table I the value of M is fixed at 300 MeV, while Λ and m are fitted to the values of m_π and F_π (for the case $m_\pi = 400$ MeV we take $F_\pi = 110$ MeV, which is in the ball park of the range given in [30]).⁵ We note that the values of the baryonic msr are very stable, which simply reflects the fact that Δ changes very little. Rows 2, 4, and 5 compare the results at fixed $m_\pi = 135$ MeV, but for different values of M . Here we note some weak dependence of the values of msr, attributed to different values of Δ and M .

In Fig. 2, we show the dependence of the baryonic form factor of π^+ in the spacelike domain of $t \leq 0$, evaluated according to Eq. (15) with the parameters of Table I. To be less sensitive to the value of Δ , we plot $F_B^{\pi^+}(t)/\Delta$. The left panel shows the dependence on m_π and the right panel the dependence on M . We note very stable results, in particular at low $-t$. One should bear in mind that NJL is, by construction, a low-energy model, hence the results at $-t > \Lambda^2$ need not be credible.

Having the form factor in the momentum space, one may construct the transverse density [8–10] via a Fourier-Bessel transformation,

$$2\pi b \rho_B^{\pi^+}(b) = \int_0^\infty dQ Q b F_B^{\pi^+}(-Q^2) J_0(Qb). \quad (28)$$

This quantity is boost-invariant, hence free of ambiguities [31,32] present in the popular Breit-frame density in the radial coordinate. The result for π^+ is shown in the left panel of Fig. 3, solid line. We note the intuitive mechanistic interpretation, announced in the introduction. In π^+ , the heavier \bar{d} quark has a more compact distribution than the

⁵One can also use here the NLO χ PT that leads to similar numbers: $F_\pi = f[1 - 2\mu_\pi - \mu_K + \frac{4m_\pi^2}{f^2}(L_4^r + L_5^r) + \frac{8m_K^2}{f^2}L_4^r]$, with $\mu_P = m_P^2/(32\pi^2 f^2) \ln(m_P^2/\mu^2)$ and $L_{4(5)}^r = -0.3(1.4) \times 10^{-3}$ at $\mu = 0.77$ GeV.

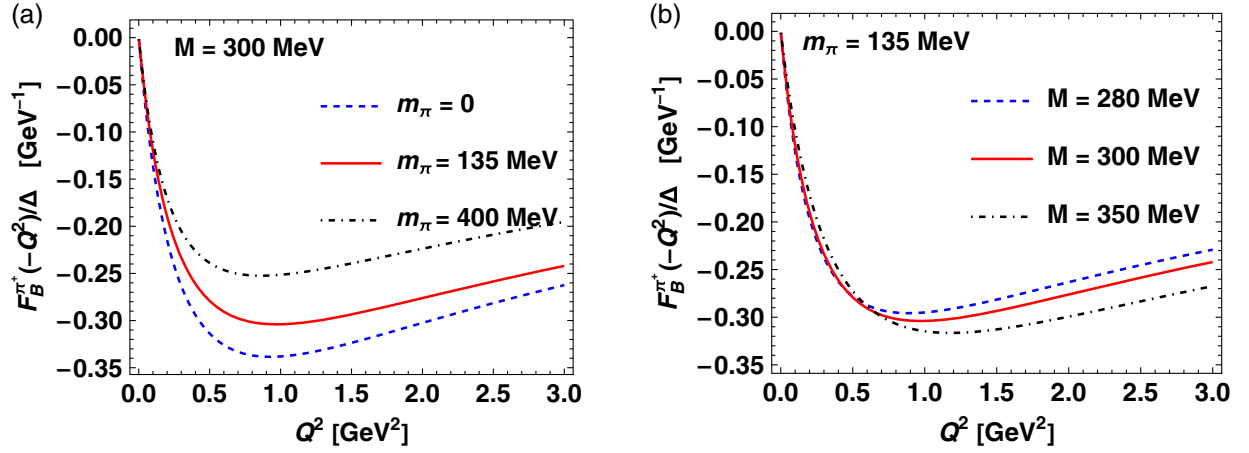


FIG. 2. Baryonic form factor of π^+ divided by $\Delta = M_d - M_u$, obtained from the NJL model with PV regularization. Parameters of the calculations are listed in Table I. Panel (a) compares the results for different values of m_π at a fixed constituent quark mass M , whereas panel (b) compares the results for different values of M at a fixed m_π .

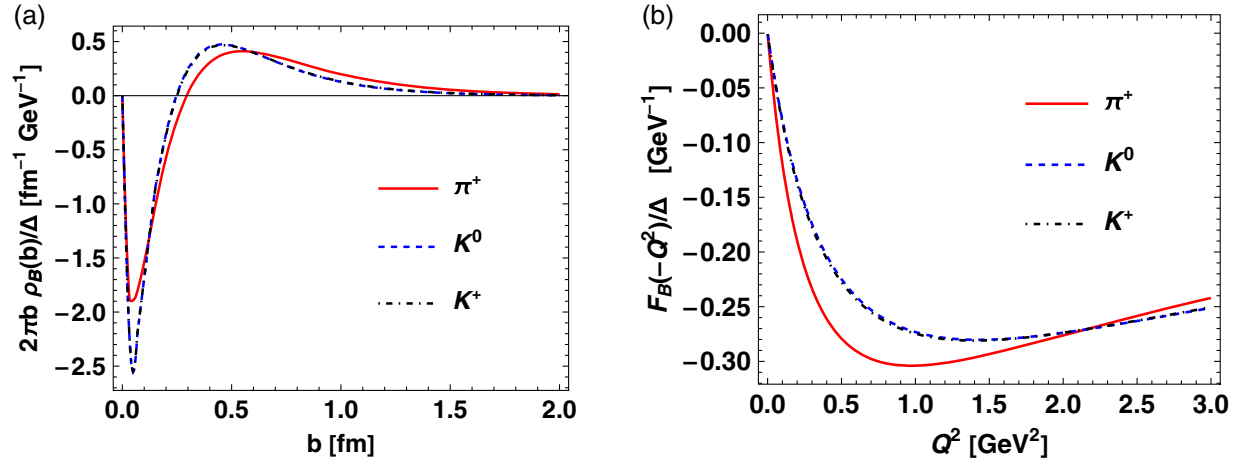


FIG. 3. The transverse baryonic density (a) and the baryonic form factor (b) for π^+ , K^0 , and K^+ , divided by, correspondingly, $\Delta = M_d - M_u$, $\Delta = M_s - M_d$, and $\Delta = M_s - M_u$. The curves for K^0 and K^+ nearly overlap.

lighter u quark, hence at low b there is an excess of antibaryon charge density. At large b the situation is opposite, such that the constraint of the vanishing total baryon number, or $\int db b \rho_B^{\pi^+}(b) = 0$, is satisfied.

For π^- , the corresponding plots of Figs. 2 and 3 are equal and opposite.

Using the relations

$$\begin{aligned} \langle r^2 \rangle_3^{\pi^+} &= \frac{1}{2} \langle r^2 \rangle_u^{\pi^+} + \frac{1}{2} \langle r^2 \rangle_{\bar{d}}^{\pi^+}, \\ \langle r^2 \rangle_B^{\pi^+} &= \frac{1}{3} \langle r^2 \rangle_u^{\pi^+} - \frac{1}{3} \langle r^2 \rangle_{\bar{d}}^{\pi^+}, \end{aligned} \quad (29)$$

we find in our model, with the physical pion mass and $M = 300$ MeV, the following msr of the u and \bar{d} constituent quarks in π^+ :

$$\begin{aligned} \langle r^2 \rangle_u^{\pi^+} &= 0.273(1) \text{ fm}^2 = (0.523(1) \text{ fm})^2, \\ \langle r^2 \rangle_{\bar{d}}^{\pi^+} &= 0.262(1) \text{ fm}^2 = (0.511(1) \text{ fm})^2, \end{aligned} \quad (30)$$

with the error reflecting the uncertainty in δ . This shows the advocated mechanistic feature that the heavier \bar{d} quark has a more compact distribution. In π^- , the above numbers hold with the replacement $u \rightarrow \bar{u}$ and $\bar{d} \rightarrow d$

VI. RESULTS FOR THE KAON

The case of the kaon is obtained directly from the pion with simple flavor substitutions. The case considered up to now has been $\pi^+ = u\bar{d}$. We can pass to $K^+ = u\bar{s}$ replacing $d \rightarrow s$, and to $K^0 = d\bar{s}$ replacing $u \rightarrow d$ and $\bar{d} \rightarrow \bar{s}$. All the

previously derived formulas then hold.⁶ The parameters used here are those of the second row of Table I, supplemented with the current strange quark mass at the scale μ_0 equal to $m_s = 180$ MeV, which fits the kaon mass. Note that the scale-independent ratio obtained that way, $m_s/m = 25.7$, is merely two standard deviations away from the PDG [7] value of $27.3^{+0.7}_{-1.3}$. The model in the strange sector is, however, not perfect, as the value of the kaon decay constant is $F_K = 100$ MeV, with experiment giving 110 (1) MeV. The issue may be improved by introducing more interaction terms in the NJL Lagrangian (see, e.g., [33–35]), but for our present exploratory study this problem is not critical.

The case of the baryonic form factor of the kaon is compared to the pion in the right panel of Fig. 3. The form factor in the figure is scaled with $\Delta = M_s - M_u$ for the case of K^+ and $\Delta = M_s - M_d$ for the case of K^0 , whereas for the pion $\Delta = M_d - M_u$. We note that the behavior of the kaon is similar to the pion. The scaled curves for K^+ and K^0 practically coincide.

The left panel of Fig. 3 shows the corresponding transverse densities. We note that the kaon curve is more compact, crossing zero at lower b . Similarly to the pion case, this feature is also in accordance to the “mechanistic” interpretation mentioned in the Introduction, as the constituent mass difference is larger in the kaon than in the pion.

The baryonic msr from the model are

$$\begin{aligned}\langle r^2 \rangle_B^{K^+} &= (0.24(1) \text{ fm})^2 = 0.056(1) \text{ fm}^2, \\ \langle r^2 \rangle_B^{K^0} &= (0.23(1) \text{ fm})^2 = 0.052(1) \text{ fm}^2.\end{aligned}\quad (31)$$

For structureless quarks, as in NJL in the large- N_c approximation,

$$\langle r^2 \rangle_B^{K^0} = -\langle r^2 \rangle_Q^{K^0}, \quad (32)$$

because the baryon number and charge of the s and d quarks are equal and opposite. PDG [7] quotes $\langle r^2 \rangle_Q^{K^0} = -(0.28(2) \text{ fm})^2 = -0.077(10) \text{ fm}^2$, 2.5 standard deviations larger than the value in (31). We note that the lattice QCD simulations [36] yield $\langle r^2 \rangle_Q^{K^0} = -0.055(15) \text{ fm}^2$, in agreement with the estimate (31) via Eq. (32).

The msr of the constituent quarks, analogous to Eq. (30), are

$$\begin{aligned}\langle r^2 \rangle_d^{K^0} &= 0.283(1) \text{ fm}^2 = (0.532(1) \text{ fm})^2, \\ \langle r^2 \rangle_{\bar{s}}^{K^0} &= 0.127(1) \text{ fm}^2 = (0.356(1) \text{ fm})^2, \\ \langle r^2 \rangle_u^{K^+} &= 0.295(1) \text{ fm}^2 = (0.543(1) \text{ fm})^2, \\ \langle r^2 \rangle_{\bar{s}}^{K^+} &= 0.127(1) \text{ fm}^2 = (0.356(1) \text{ fm})^2,\end{aligned}\quad (33)$$

⁶The symmetry arguments of Sec. II hold for the SU(3) case with the replacement of the I-spin with U-spin or V-spin.

with same numbers holding for \bar{K}^0 and K^- upon the replacement of quarks into antiquarks. Again, we note the mechanistic feature of the \bar{s} being significantly more compact, and the distribution of d in K^0 more compact than the distribution of u in K^+ .

VII. CONCLUSIONS

As we can see from Table I, the NJL model predictions for the baryonic msr of the pion are, taking into account various model parameters,

$$\langle r^2 \rangle_B^{\pi^+} = (0.06(1) \text{ fm})^2 = 0.004(1) \text{ fm}^2. \quad (34)$$

This is about a factor of 2 higher than the value extracted from the data, (1). Possible reasons for this discrepancy are as follows:

First, the NJL calculation takes into account only the one-quark-loop contribution, which is leading in N_c , but does not include the formally suppressed but potentially large chiral loops. For the case of the isospin msr of the pion, chiral loops contribute about 20% of the total result and are necessary to reproduce the data. In the present case we may expect a similar order on the effect.

Second, the extraction from the data (1) includes EM effects dressing the pion vertex (although the experimental procedure gets rid of the EM interactions in the initial and final states). These effects are not present in our calculation.

Finally, we comment on the lattice QCD prospects of measuring the baryonic msr of the pion and kaon. Comparing our numbers to the accuracy of the recent lattice QCD calculations of the charge form factor with physical quark masses, $\langle r^2 \rangle_Q^{\pi} = (0.648(15) \text{ fm})^2 = 0.42(2) \text{ fm}^2$ [37] and $\langle r^2 \rangle_Q^{\pi} = 0.430(5)(13) \text{ fm}^2$ [38], we remark that our effect (1) is an order of magnitude smaller, and the estimate (34)—a factor of 5 smaller than the lattice accuracy quoted above. However, on the lattice one may try to increase the value of the quark mass splitting up to the point where the signal is strong enough, and then extrapolate down to the physical point. It would be very interesting to see if with the present accuracy such a determination is possible.

For the case of the baryonic msr of the kaon (31), discussed in Sec. VI, the baryonic msr of K^0 may be considered to have already been measured on the lattice [36]. It satisfies identity (32), as no disconnected contributions have been accounted for in the simulations, which are estimated to be negligible.

Finally, we wish to underline that one may use different models (all one needs is current conservation and the ability to model the pion or kaon) to assess the size of the baryonic form factor. It would certainly be very interesting to have such independent estimates to confront them with the result (1) extracted from the experiment.

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APPENDIX A: REGULARIZED ONE-LOOP FUNCTIONS

All our results involve one loop functions, for which we use the Passarino-Veltman [27] convention (the $+i\epsilon$ prescription in the denominators is understood):

$$\begin{aligned} i\pi^2 A_0(M_1) &= \int d^4k \frac{1}{k^2 - M_1^2}, \\ i\pi^2 B_0(p^2, M_1, M_2) &= \int d^4k \frac{1}{[(k+p)^2 - M_1^2][k^2 - M_2^2]}, \\ i\pi^2 C_0(q^2, p^2, (p+q)^2, M_1, M_2, M_3) &= \int d^4k \frac{1}{[(k+p+q)^2 - M_1^2][(k+p)^2 - M_2^2][k^2 - M_3^2]}. \end{aligned} \quad (\text{A1})$$

The A_0 and B_0 functions are divergent and require regularization. The C_0 function is convergent, however, it still should be regularized to dispose of the high-momentum contributions in the quark-loop, which should not enter the low-energy model. The applied PV regularization prescription with two subtractions [11,15] amounts to the replacement

$$\begin{aligned} F^\Lambda(\{M_i^2\}) &= F(\{M_i^2\}) - F(\{M_i^2 + \Lambda^2\}) \\ &\quad + \Lambda^2 \frac{dF(\{M_i^2 + \Lambda^2\})}{d\Lambda^2}, \end{aligned} \quad (\text{A2})$$

where F is a the PaVe function (with arguments other than the masses suppressed). The chosen regularization is consistent with the symmetry requirements [11,15].

Explicitly

$$A_0^\Lambda(M_f) = -\frac{M_f^2 \log\left(\frac{M_f^2}{\Lambda^2 + M_f^2}\right) + \Lambda^2}{16\pi^4}. \quad (\text{A3})$$

Analytic but lengthy formulas for B_0^Λ and C_0^Λ can be obtained.

APPENDIX B: BASICS OF THE NAMBU-JONA-LASINIO MODEL

This Appendix presents for completeness the standard NJL formulas in the adopted notation and for the general case of unequal u and d quark masses, needed in our analysis. For explanations and the physics discussion the reader is referred to [11].

The gap equation for each flavor f has the form

$$M_f = m_f - 4\pi^2 GM_f N_c A_0^\Lambda(M_f), \quad (\text{B1})$$

where G is the NJL four-quark coupling constant (independent of flavor or the value of m_f). The quark condensate for a single flavor is

$$\langle \bar{q}_f q_f \rangle = 4\pi^2 M_f N_c A_0^\Lambda(M_f) = -\frac{M_f - m_f}{G}. \quad (\text{B2})$$

The mass of π^\pm , denoted as m_π^2 , is a root in p^2 of the denominator of the pion propagator, following from the Bethe-Salpeter equation,

$$4\pi^2(p^2 - \Delta^2) N_c B_0^\Lambda(p^2, M_u, M_d) = \frac{1}{G} \left(\frac{m_u}{M_u} + \frac{m_d}{M_d} \right), \quad (\text{B3})$$

The coupling constant of a charged pion to the u and d quarks is obtained from the residue of the Bethe-Salpeter amplitude at the pion pole,

$$\begin{aligned} \frac{1}{g_{\pi^+ u \bar{d}}^2} &= \frac{1}{g_{\pi^- d \bar{u}}^2} \\ &= 4\pi^2 \frac{d}{dp^2} [(p^2 - \Delta^2) N_c B_0^\Lambda(p^2, M_u, M_d)] \Big|_{p^2 = m_\pi^2}. \end{aligned} \quad (\text{B4})$$

The pion weak decay constant is

$$\begin{aligned} F_\pi &= \frac{2\pi^2 g_{\pi^+ u \bar{d}} N_c}{m_\pi^2} [2(m_\pi^2 - \Delta^2) M B_0^\Lambda(m_\pi^2, M_u, M_d) \\ &\quad - \Delta(A_0(M_u) - A_0(M_d))]. \end{aligned} \quad (\text{B5})$$

In the limit of $m_u = m_d = 0$ the Goldberger-Treiman relation (16) holds.

APPENDIX C: FIXING THE MODEL PARAMETERS AND Δ

The model has four parameters: G , m_u , m_d , and Λ , which with the expressions from the previous Appendix can be traded for M , Δ , m_π , and F_π . Fitting m_π and F_π to their physical values leaves two parameters: M and Δ . As usual in the NJL studies, we keep M free, whereas Δ can be related to the splitting of the *current* quark masses, $m_d - m_u$.

From the gap equation (B1), obviously, M_f is a function of m_f . We discuss this issue at a greater length, as it is not covered in detail in the literature, while we need it for the determination of the mass splitting Δ to be used in our estimates. Using Eq. (B2) in Eq. (B1) we arrive at

$$M_f(m_f) = M_f(0) \frac{\langle \bar{q}_f q_f \rangle(m_f)}{\langle \bar{q}_f q_f \rangle(0)} + m_f. \quad (\text{C1})$$

At small m_f we may expand the quark condensate,

$$\begin{aligned} \langle \bar{q}_f q_f \rangle(m_f) &= \langle \bar{q}_f q_f \rangle(0) + \chi_f m_f + \mathcal{O}(m_f^2), \\ \chi_f &= \langle \bar{q}_f q_f \rangle', \end{aligned} \quad (\text{C2})$$

where χ_f is the quark mass (or scalar) susceptibility in the chiral limit, and the prime denotes a derivative with respect to m_f at $m_f = 0$. Therefore

$$\begin{aligned} M_f(m_f) &= M_f(0) + \left[1 + M_f(0) \frac{\chi_f}{\langle \bar{q}_f q_f \rangle} \right] m_f + \mathcal{O}(m_f^2) \\ &= M_f(0) + \alpha m_f + \mathcal{O}(m_f^2), \end{aligned} \quad (\text{C3})$$

and in general $\alpha \neq 1$, in contrast to what one might naively assume. The above formula holds for a generic approach with the gap equation.

Explicitly, in the NJL model with PV regularization we find, by expanding Eq. (B1),

$$\begin{aligned} M'_f(0)m_f &= m_f - 4\pi^2 G N_c \frac{d}{dM_f} M_f A_0^\Lambda(M_f) \Big|_{M_f=M_f(0)} M'_f(0)m_f + \dots, \end{aligned} \quad (\text{C4})$$

from where we can evaluate $M'_f(0)$ and obtain

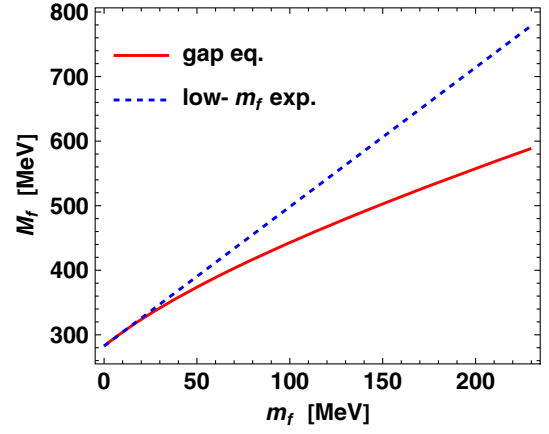


FIG. 4. Dependence of the constituent quark mass, M_f , on the current mass, m_f , in the NJL model with the PV regularization.

$$\begin{aligned} M_f(m_f) &= M_f(0) + \frac{m_f}{1 + 4\pi^2 G N_c \frac{d}{dM_f} M_f A_0^\Lambda(M_f) \Big|_{M_f=M_f(0)}} + \mathcal{O}(m_f^2). \end{aligned} \quad (\text{C5})$$

Eliminating G from Eq. (B1) at $m_f = 0$ we get

$$M_f(m_f) = M_f(0) - \frac{A_0^\Lambda[M_f(0)]m_f}{M_f \frac{d}{dM_f} A_0^\Lambda(M_f) \Big|_{M_f=M_f(0)}} + \mathcal{O}(m_f^2). \quad (\text{C6})$$

With the typically used parameters, the slope parameter is $\alpha \sim 2$.

Figure 4 shows that the small- m_f expansion works well in the range of the light current quark masses, but not for the higher values of the strange quark. Asymptotically, at very large m_f (and, therefore, large M_f) we have $A_0^\Lambda(M_f) \sim \Lambda^4 / (32\pi^4 M_f)^2$ and

$$M_f = m_f + \frac{G N_c \Lambda^4}{8\pi^2 m_f} + \mathcal{O}(1/m_f^2). \quad (\text{C7})$$

Thus, asymptotically, $M_f/m_f \rightarrow 1$.

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