Gauge symmetry breaking in flux compactification with a Wilson-line scalar condensate

Kento Akamatsu,[1] Takuya Hirose,[1] and Nobuhito Maru[1,2]

[1]*Department of Physics, Osaka Metropolitan University, Osaka 558-8585, Japan*
[2]*Nambu Yoichiro Institute of Theoretical and Experimental Physics (NITEP), Osaka Metropolitan University, Osaka 558-8585, Japan*

(Received 26 May 2022; accepted 12 August 2022; published 30 August 2022)

We discuss the gauge symmetry breaking of six-dimensional theories in flux compactification with a magnetic flux background and a constant vacuum expectation value (VEV) for the scalar fields, which are zero modes of extra spatial components of the gauge field. Although the effective potential for the scalar fields are known not to be generated classically and radiatively in a magnetic flux background only, the one-loop effective potential is shown to be generated by the effects of the nonzero constant VEV. As illustrations, we calculate the one-loop effective potential in SU(2) and SU(3) Yang-Mills theories. In both cases, we expect that the potential minimum is located at nonzero VEV and the gauge symmetry breaking takes place.

DOI: 10.1103/PhysRevD.106.035035

I. INTRODUCTION

Although the Standard Model (SM) has a successful theory, it still has some problems. Many attractive scenarios based on the higher-dimensional theory have been proposed as physics beyond the SM. In particular, flux compactification, which has been studied in string theory [1,2], has many attractive aspects: explanation of the generation number of the SM fermions [3,4] and computation of the Yukawa coupling [5–8].

Recently, it has been considered that the quantum corrections to the masses of the zero mode of the scalar field induced from extra components of a higher-dimensional gauge field [called the Wilson-line (WL) scalar field] are canceled [9–13] and are finite [14]. The reason why the quantum corrections are canceled is that the shift symmetry from translation in extra spaces forbids the mass term of the scalar field since the zero mode of the scalar field can be identified with the Nambu-Goldstone (NG) boson of spontaneously broken translational symmetry (or with the pseudo-NG boson in [14]). This cancellation mechanism may be applied to the hierarchy problem in the SM, which is the problem that the quantum corrections to the mass of the Higgs field are sensitive to the square of the ultraviolet cutoff scale of the theory. If we regard the Higgs field as the WL scalar field, which is an idea of gauge-Higgs unification [15–18], the quantum corrections to the mass of the Higgs field is canceled as mentioned above. In the gauge-Higgs unification, the finite Higgs mass is generated by the quantum corrections [18–22] controlled by the compactification scale. If the compactification scale is increased by the absence of the new physics discovery, the fine-tuning problem in the Higgs mass parameter is reintroduced. In flux compactification, however, if the translational symmetry in extra spaces is explicitly broken around TeV scale independent of the compactification scale, the light Higgs boson mass is radiatively generated. This logic is also applied to the potential of the WL scalar field.

In this paper, we investigate the gauge symmetry breaking in a higher-dimensional theory in flux compactification with a magnetic flux background and a constant WL scalar vacuum expectation value (VEV). First, we consider a six-dimensional SU(2) Yang-Mills theory compactified on a torus with a magnetic flux and a constant VEV. Calculating the Kaluza-Klein (KK) mass spectrum in the presence of both flux background and the constant VEV, we obtain the one-loop effective potential for the WL scalar field. Although the effective potential in the flux background only is not radiatively generated, the effective potential in both the flux background and the constant VEV is generated at one loop, and the potential minimum at nonvanishing constant VEV is expected. This concludes that gauge symmetry SU(2) is completely broken.

Next, we consider a six-dimensional SU(3) Yang-Mills theory compactified on a torus with a magnetic flux and a constant VEV. In this case, we consider two types of configurations where the flux background and the constant

VEV are developed. One is that the flux background and the constant VEV are in the eighth and first components of SU(3). The other is that the flux background and the constant VEV are in the eighth and sixth components of SU(3). Similarly calculating the one-loop effective potential as is done in SU(2) Yang-Mills theory, we find (expect) that the potential is minimized at nonvanishing constant VEV in the former (latter) case. In the former (latter) case, the gauge symmetry breaking $SU(3) \to U(1) \times U(1)$ [$SU(3) \to U(1)$] is found, respectively.

This paper is organized as follows. We give a setup of a six-dimensional SU(2) Yang-Mills theory with magnetic flux compactification and introduce the constant VEV in Sec. II. We furthermore consider a six-dimensional SU(3) Yang-Mills theory with magnetic flux compactification in Sec. III, where two types of configurations for the flux background and the constant VEV are taken. In both sections, the one-loop effective potential for the WL scalar field is calculated and the gauge symmetry breaking is discussed. The last section is devoted to our summary. In the Appendix, the calculation of the KK mass spectrum at second-order perturbation is summarized.

II. SU(2) YANG-MILLS THEORY

We consider a six-dimensional SU(2) Yang-Mills theory with two nontrivial backgrounds: a constant magnetic flux background and an ordinary constant vacuum expectation value.

A. Setup

Six-dimensional spacetime is $M^4 \times T^2$, where M^4 is the Minkowski spacetime and T^2 is a two-dimensional square torus. The Lagrangian of SU(2) Yang-Mills theory in six dimensions is

$$\mathcal{L}_6 = -\frac{1}{4} F^a_{MN} F^{aMN} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2} F^a_{\mu 5} F^{a\mu 5} - \frac{1}{2} F^a_{\mu 6} F^{a\mu 6} - \frac{1}{2} F^a_{56} F^{a56}, \tag{1}$$

where the field strength tensor and the covariant derivative are defined by

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M - ig[A_M, A_N]^a, \tag{2}$$

$$D_M A^a_N = D^{ac}_M A^c_N = (\delta^{ac} \partial_M + g\epsilon^{abc} A^b_M) A^c_N = \partial_M A^a_N - ig[A_M, A_N]^a. \tag{3}$$

The spacetime indices are $M, N = 0, 1, 2, 3, 5, 6$, $\mu, \nu = 0, 1, 2, 3$, $m, n = 5, 6$, and the gauge indices are $a, b, c = 1, 2, 3$. The metric convention $\eta_{MN} = \mathrm{diag}(-1, +1, \ldots, +1)$ is employed. ϵ^{abc} is a totally antisymmetric tensor of SU(2).

We discuss how the two backgrounds are introduced in our model. First, the constant magnetic flux is given by the VEV of the fifth and sixth component of the gauge fields $A^3_{5,6}$, which must satisfy their classical equation of motion:

$$D^m \langle F^a_{mn} \rangle = 0. \tag{4}$$

Second, the ordinary constant background is generated by the quantum correction in the sixth component of the gauge field A^1_6, for simplicity.[1] In this section, we choose a solution

$$\langle A^1_6 \rangle = v, \qquad \langle A^3_5 \rangle = -\frac{1}{2} f x_6, \qquad \langle A^3_6 \rangle = \frac{1}{2} f x_5, \quad \text{and} \quad \langle A^{1,2}_5 \rangle = \langle A^2_6 \rangle = 0. \tag{5}$$

$\langle A^3_{5,6} \rangle$ introduces a magnetic field parametrized by a constant f, namely, $\langle F^3_{56} \rangle = f$. Note that the flux background spontaneously breaks a translational invariance on the torus. The flux background breaks the gauge symmetry, which is broken to U(1) in this case. The flux is also associated with the degeneracy:

$$\frac{g}{2\pi} \int_{T^2} dx_5 dx_6 \langle F^3_{56} \rangle = \frac{g}{2\pi} L^2 f = N \in \mathbb{Z}, \tag{6}$$

where L^2 is an area of the torus. For simplicity, we set $L = 1$ from now on. It is useful to define ∂ and the scalar fields ϕ^a as

$$\partial \equiv \partial_z = \partial_5 - i\partial_6, \quad z \equiv \frac{1}{2}(x_5 + ix_6), \quad \phi^a = \frac{1}{\sqrt{2}}(A^a_6 + iA^a_5). \tag{7}$$

In these complex coordinates, the VEVs of $\phi^{1,3}$ are given by

$$\langle \phi^1 \rangle = \frac{v}{\sqrt{2}}, \qquad \phi^3 = \frac{f\bar{z}}{\sqrt{2}}, \tag{8}$$

and we expand ϕ^a around the backgrounds

$$\phi^a = \langle \phi^a \rangle + \varphi^a, \tag{9}$$

where φ^a are quantum fluctuations.

The Lagrangian (1) can be rewritten by using the new coordinates (7) as follows:

[1] In general, the constant background can be introduced by the fifth component of the gauge field A^1_5.

$$\begin{aligned}\mathcal{L}_6 = &-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \partial_\mu\bar{\phi}^a\partial^\mu\phi^a - \frac{1}{2}DA^a_\mu\overline{D}A^{a\mu}\\ &-\frac{i}{\sqrt{2}}(\partial_\mu\phi^a\bar{\partial}A^{a\mu} - \partial_\mu\bar{\phi}^a\partial A^{a\mu})\\ &+ig(\partial_\mu\phi^a[A^\mu,\bar{\phi}]^a + \partial^\mu\bar{\phi}^a[A_\mu,\phi]^a)\\ &-\frac{1}{4}(D\bar{\phi}^a + \overline{D}\phi^a + \sqrt{2}g[\phi,\bar{\phi}]^a)^2,\end{aligned} \tag{10}$$

where

$$\begin{cases} D\Phi^a \equiv (D_5 - iD_6)\Phi^a = \partial\Phi^a - \sqrt{2}g[\phi,\Phi]^a,\\ \bar{D}\Phi^a \equiv (D_5 + iD_6)\Phi^a = \bar{\partial}\Phi^a + \sqrt{2}g[\bar{\phi},\Phi]^a,\end{cases} \tag{11}$$

which express the covariant derivatives with respect to the complex coordinates in the torus. Φ^a denotes arbitrary fields in the adjoint representation. We can remove the mixing terms between the gauge and the scalar fields in the second line of Eq. (10) by introducing the gauge-fixing terms with a gauge parameter ξ:

$$\begin{aligned}\mathcal{L}_{g-f} \equiv &-\frac{1}{2\xi}(D_\mu A^{a\mu} + \xi\mathcal{D}_m A^{am})^2\\ = &-\frac{1}{2\xi}D_\mu A^{a\mu}D_\nu A^{a\nu} - \frac{g}{\sqrt{2}}(\partial\bar{\phi}^a[A_\mu,A^\mu]^a - \bar{\partial}\phi^a[A_\mu,A^\mu]^a)\\ &+\frac{\xi}{4}(\mathcal{D}\bar{\phi}^a - \overline{\mathcal{D}}\phi^a)^2 + \frac{i}{\sqrt{2}}(\partial_\mu\phi^a\bar{\partial}A^{a\mu} - \partial_\mu\bar{\phi}^a\partial A^{a\mu}).\end{aligned} \tag{12}$$

The new covariant derivatives $\mathcal{D},\overline{\mathcal{D}}$ are defined by replacing $\phi^a,\bar{\phi}^a$ in $D,\bar{D}$ with the VEVs $\langle\phi^a\rangle,\langle\bar{\phi}^a\rangle$, respectively. Because of the gauge fixing, we need to introduce the Faddeev-Popov ghost fields, and their Lagrangian is

$$\mathcal{L}_{\rm ghost} = -\bar{c}^a(D_\mu D^\mu + \xi D_m\mathcal{D}^m)c^a. \tag{13}$$

Then, the total Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\rm total} = &-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\xi}D_\mu A^{a\mu}D_\nu A^{a\nu} - \partial_\mu\bar{\phi}^a\partial^\mu\phi^a\\ &-\frac{1}{2}DA^a_\mu\bar{D}A^{a\mu} - \frac{g}{\sqrt{2}}(\partial\bar{\phi}^a[A_\mu,A^\mu]^a - \bar{\partial}\phi^a[A_\mu,A^\mu]^a)\\ &+\frac{\xi}{4}(\mathcal{D}\bar{\phi}^a - \bar{\mathcal{D}}\phi^a)^2 + ig(\partial_\mu\phi^a[A^\mu,\bar{\phi}]^a + \partial^\mu\bar{\phi}^a[A_\mu,\phi]^a)\\ &-\frac{1}{4}(D\bar{\phi}^a + \bar{D}\phi^a + \sqrt{2}g[\phi,\bar{\phi}]^a)^2\\ &-\bar{c}^a(D_\mu D^\mu + \xi D_m\mathcal{D}^m)c^a.\end{aligned} \tag{14}$$

For simplicity, we choose the Feynman gauge $\xi = 1$ throughout this paper.

B. The mass of the gauge fields A^a_μ

We will discuss mass eigenstates and eigenvalues of the fields A^a_μ, φ^a, c^a. In this subsection, we find mass eigenvalues and eigenstates of the gauge fields A^a_μ. The mass term of the gauge field corresponds to the background part of $-DA^a_\mu\bar{D}A^{a\mu}/2$:

$$\mathcal{L}_{AA} = -\frac{1}{2}\mathcal{D}A^a_\mu\bar{\mathcal{D}}A^{a\mu} = -\frac{1}{2}A^a_\mu[-\mathcal{D}\bar{\mathcal{D}}]A^{a\mu}. \tag{15}$$

We would like to regard the background covariant derivatives $\mathcal{D},\bar{\mathcal{D}}$ as creation and annihilation operators, respectively. In a matrix form, they are expressed as

$$\mathcal{D} = \begin{bmatrix} \partial & igf\bar{z} & 0\\ -igf\bar{z} & \partial & igv\\ 0 & -igv & \partial\end{bmatrix},\quad \bar{\mathcal{D}} = \begin{bmatrix} \bar{\partial} & -igfz & 0\\ igfz & \bar{\partial} & -igv\\ 0 & igv & \bar{\partial}\end{bmatrix}. \tag{16}$$

Diagonalizing them, we obtain

$$\begin{cases}\mathcal{D}_{\rm diag} = {\rm diag}(\partial, \partial + g\sqrt{f^2\bar{z}^2 + v^2}, \partial - g\sqrt{f^2\bar{z}^2 + v^2}),\\ \bar{\mathcal{D}}_{\rm diag} = {\rm diag}(\bar{\partial}, \bar{\partial} - g\sqrt{f^2z^2 + v^2}, \bar{\partial} + g\sqrt{f^2z^2 + v^2}).\end{cases} \tag{17}$$

Their commutation relation is

$$[\bar{\mathcal{D}}_{\rm diag}, \mathcal{D}_{\rm diag}]^{ac} = gf^2\left(\frac{z}{\sqrt{f^2z^2+v^2}} + \frac{\bar{z}}{\sqrt{f^2\bar{z}^2+v^2}}\right)\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}, \tag{18}$$

which depends on extra space coordinates. Therefore, $\mathcal{D},\bar{\mathcal{D}}$ cannot be identified with creation and annihilation operators.

Since the KK mass spectrum cannot be exactly solved by using the creation and annihilation operators, we would like to find them perturbatively by the expansion in v. In this expansion, $vL \ll 1$ or $v \ll 1$ in the present case is assumed. In other words, we consider the case where the compactification scale is much larger than the constant VEV v. From (16), we define the unperturbed parts $\mathcal{D}_3, \bar{\mathcal{D}}_3$ and the perturbed part V as

$$\mathcal{D}_3 \equiv \begin{bmatrix} \partial & igf\bar{z} & 0\\ -igf\bar{z} & \partial & 0\\ 0 & 0 & \partial\end{bmatrix},\quad \bar{\mathcal{D}}_3 \equiv \begin{bmatrix} \bar{\partial} & -igfz & 0\\ igfz & \bar{\partial} & 0\\ 0 & 0 & \bar{\partial}\end{bmatrix}, \tag{19}$$

and

$$V \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & igv \\ 0 & -igv & 0 \end{bmatrix}, \qquad (20)$$

respectively. In these notations, the covariant derivatives can be expressed as $\mathcal{D} = \mathcal{D}_3 + V$ and $\bar{\mathcal{D}} = \bar{\mathcal{D}}_3 + \bar{V}$. $\mathcal{D}_3$ and $\bar{\mathcal{D}}_3$ can be identified with creation and annihilation operators, which are diagonalized as follows:

$$\begin{cases} \mathcal{D}_{3,\mathrm{diag}} = \mathrm{diag}(\partial - gf\bar{z}, \partial + gf\bar{z}, \partial), \\ \bar{\mathcal{D}}_{3,\mathrm{diag}} = \mathrm{diag}(\bar{\partial} + gfz, \bar{\partial} - gfz, \bar{\partial}). \end{cases} \qquad (21)$$

Their diagonalizing unitary matrix U_3, which satisfies $U_3^{-1}\mathcal{D}_3 U_3 = \mathcal{D}_{3,\mathrm{diag}}$ and $U_3^{-1}\bar{\mathcal{D}}_3 U_3 = \bar{\mathcal{D}}_{3,\mathrm{diag}}$, is

$$U_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}. \qquad (22)$$

The commutation relation between $\mathcal{D}_{3,\mathrm{diag}}$ and $\bar{\mathcal{D}}_{3,\mathrm{diag}}$ is

$$[i\bar{\mathcal{D}}_{3,\mathrm{diag}}, i\mathcal{D}_{3,\mathrm{diag}}]^{ac} = 2gf \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (23)$$

The creation and annihilation operators are defined as

$$a = \frac{i}{\sqrt{\alpha_2}} \bar{\mathcal{D}}_{3,\mathrm{diag}}, \qquad a^\dagger = \frac{i}{\sqrt{\alpha_2}} \mathcal{D}_{3,\mathrm{diag}}, \qquad (24)$$

where $\alpha_2 \equiv 2gf$. The components of the creation and annihilation operators are summarized as follows:

$$\begin{cases} a_1 \equiv \frac{i}{\sqrt{\alpha_2}}(\bar{\partial} + gfz), \\ a_2 \equiv \frac{i}{\sqrt{\alpha_2}}(\bar{\partial} - gfz), \\ a_3 \equiv \frac{i}{\sqrt{\alpha_2}}\bar{\partial}, \end{cases} \quad \text{and} \quad \begin{cases} a_1^\dagger \equiv \frac{i}{\sqrt{\alpha_2}}(\partial - gf\bar{z}), \\ a_2^\dagger \equiv \frac{i}{\sqrt{\alpha_2}}(\partial + gf\bar{z}), \\ a_3^\dagger \equiv \frac{i}{\sqrt{\alpha_2}}\partial. \end{cases} \qquad (25)$$

We note that a_3 and $a_3^\dagger$ have no flux effects and play no role of creation or annihilation operators. $a_1^\dagger$ and a_2 are creation operators and a_1 and $a_2^\dagger$ are annihilation operators: The roles of the creation and annihilation operators for a_2 and $a_2^\dagger$ are inverted due to the commutation relation for the 2-direction $[a_2, a_2^\dagger] = -1$. The ground state mode functions are determined by

$$a_1 \xi_{0,j} = 0, \qquad a_2^\dagger \bar{\xi}_{0,j} = 0, \qquad (26)$$

where $\xi_{0,j}$ are the functions of z (for details, see [5]), and j labels the degeneracy of the ground state: $j = 0, \ldots, |N| - 1$.

Higher mode functions $\xi_{n,j}$ are constructed similar to the harmonic oscillator case [7]:

$$\xi_{n_1,j} = \frac{1}{\sqrt{n_1!}}(a_1^\dagger)^{n_1}\xi_{0,j},$$
$$\bar{\xi}_{n_2,j} = \frac{1}{\sqrt{n_2!}}(a_2)^{n_2}\bar{\xi}_{0,j} \quad (n_{1,2} \in \mathbb{Z}, n_{1,2} \geq 0). \qquad (27)$$

The functions $\xi_{n,j}$ satisfy the orthogonal conditions

$$\int_{T^2} d^2x \bar{\xi}_{n,j}\, \xi_{n',j'} = \delta_{n,n'}\delta_{j,j'}. \qquad (28)$$

To be operated by a or $a^\dagger$, we should define the states with the gauge indices:

$$\psi^1_{n_1,j} \equiv \begin{bmatrix} \xi_{n_1,j} \\ 0 \\ 0 \end{bmatrix}, \qquad \psi^2_{n_2,j} \equiv \begin{bmatrix} 0 \\ \bar{\xi}_{n_2,j} \\ 0 \end{bmatrix}. \qquad (29)$$

Moreover, using the periodic boundary condition of a torus, we define the eigenstates for the 3-direction

$$\psi^3_{l,m} \equiv \begin{bmatrix} 0 \\ 0 \\ \lambda_{l,m} \end{bmatrix} \quad (l, m \in \mathbb{Z}), \qquad (30)$$

where $\lambda_{l,m}$ are the functions of x_5 and x_6:

$$\lambda_{l,m} \equiv \lambda_{l,m}(x_5, x_6) = \exp[2\pi i(lx_5 + mx_6)]. \qquad (31)$$

These functions $\lambda_{l,m}$ also satisfy the orthogonal conditions

$$\int_{T^2} d^2x\, \bar{\lambda}_{l,m}\lambda_{l',m'} = \delta_{l,l'}\delta_{m,m'}. \qquad (32)$$

The eigenstates satisfy the following relations:

$$\begin{cases} a\psi^1_{n_1,j} = \sqrt{n_1}\psi^1_{n_1-1,j}, \\ a\psi^2_{n_2,j} = \sqrt{n_2+1}\psi^2_{n_2+1,j}, \\ a\psi^3_{l,m} = -\sqrt{\frac{4\pi^2}{\alpha_2}}(l+im)\psi^3_{l,m}, \end{cases} \quad \text{and}$$
$$\begin{cases} a^\dagger\psi^1_{n_1,j} = \sqrt{n_1+1}\psi^1_{n_1+1,j}, \\ a^\dagger\psi^2_{n_1,j} = \sqrt{n_2}\psi^2_{n_2-1,j}, \\ a^\dagger\psi^3_{l,m} = -\sqrt{\frac{4\pi^2}{\alpha_2}}(l-im)\psi^3_{l,m}. \end{cases} \qquad (33)$$

For convenience, we unify the labels of the eigenstates:

$$\psi^1_{\{n_1\}} \equiv \psi^1_{n_1,j}, \qquad \psi^2_{\{n_2\}} \equiv \psi^2_{n_2,j}, \qquad \psi^3_{\{n_3\}} \equiv \psi^3_{l,m}. \qquad (34)$$

Their products of eigenstates in the different directions are zero

$$(\psi^a_{\{n_a\}})^\dagger \psi^c_{\{n_c\}} = 0 \quad (a \neq c). \tag{35}$$

Then we define the unperturbed Hamiltonian H_0 and the perturbation V_1, V_2:

$$\begin{aligned}-\mathcal{D}\bar{\mathcal{D}} &= U_3[-\mathcal{D}_{3,\mathrm{diag}}\bar{\mathcal{D}}_{3,\mathrm{diag}} + \mathcal{D}_{3,\mathrm{diag}}U_3^{-1}VU_3 \\ &\quad -U_3^{-1}VU_3\bar{\mathcal{D}}_{3,\mathrm{diag}} + (U_3^{-1}VU_3)^2]U_3^{-1} \\ &\equiv U_3[H_0 + V_1 + V_1^\dagger + V_2]U_3^{-1} \\ &\equiv U_3 H U_3^{-1}.\end{aligned} \tag{36}$$

From the previous discussion, H_0 is expressed as

$$H_0 = \begin{bmatrix} \alpha_2 n_1 & 0 & 0 \\ 0 & \alpha_2(n_2+1) & 0 \\ 0 & 0 & 4\pi^2(l^2+m^2) \end{bmatrix}, \tag{37}$$

and its eigenstates of the gauge fields are defined by

$$\begin{aligned} A^{a'}_\mu U_3^{a'a} &\equiv \tilde{A}^a_\mu \equiv \sum_{\{n_a\}} \tilde{A}^a_{\mu,\{n_a\}} \psi^a_{\{n_a\}}, \\ (U_3^{-1})^{aa'} A^{a'\mu} &\equiv \tilde{A}^{a\mu} \equiv \sum_{\{n_a\}} \tilde{A}^{a\mu}_{\{n_a\}} \psi^a_{\{n_a\}}. \end{aligned} \tag{38}$$

The perturbations V_1 and V_2 are

$$\begin{aligned} V_1 &= \frac{gv}{\sqrt{2}} \begin{bmatrix} 0 & 0 & \partial - gf\bar{z} \\ 0 & 0 & i(\partial + gf\bar{z}) \\ \partial & -i\partial & 0 \end{bmatrix}, \\ V_2 &= \frac{g^2 v^2}{2} \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \end{aligned} \tag{39}$$

respectively. As was seen from the unperturbed Hamiltonian (37), we find that there are degeneracies: $\psi^1_{0,j}$ and $\psi^3_{l,m}$, $\psi^1_{n+1,j}$ and $\psi^2_{n,j}$. Thus, we should be careful when calculating their energies in perturbation.

For $\psi^1_{0,j}$ and $\psi^3_{l,m}$, the first-order perturbation energy from $V_1 + V_1^\dagger + V_2$, $E^{(1)}_{A,0}$ can be easily obtained

$$E^{(1)}_{A,n_1=0} = g^2v^2/2, \qquad E^{(1)}_{A,3} = g^2 v^2. \tag{40}$$

Note that we have to solve the secular equation for $\psi^1_{0,j}$ and $\psi^3_{l=0,m=0}$ because there exists the degeneracy, and the perturbation of $\psi^3_{l\neq 0,m\neq 0}$ can be obtained by V_2. The second-order perturbation energy $E^{(2)}_{A,0}$ for $\psi^1_{0,j}$ is shown in the Appendix.

For $\psi^1_{n+1,j}, \psi^2_{n,j} (n \geq 0)$ and $\psi^3_{l,m}$, the first-order perturbation energy from $V_1 + V_1^\dagger + V_2$, $E^{(1)}_A$ can be easily obtained

$$E^{(1)}_{A,1'} = 0, \qquad E^{(1)}_{A,2'} = g^2 v^2, \tag{41}$$

where the mode functions in the new direction $1'$ and $2'$ are defined as

$$\begin{aligned} \psi^{1'}_{n+1,j} &\equiv (i\psi^1_{n+1,j} + \psi^2_{n,j})/\sqrt{2}, \\ \psi^{2'}_{n+1,j} &\equiv (\psi^1_{n+1,j} + i\psi^2_{n,j})/\sqrt{2}. \end{aligned} \tag{42}$$

The second-order perturbation energies $E^{(2)}_{A,1'}$, $E_{A,2'}$, and $E^{(2)}_{A,3}$ are shown in the Appendix. Thus, we summarize the mass of the gauge fields as

$$\begin{cases} m^2_{A,n_1=0} = \frac{g^2v^2}{2} + E^{(2)}_{A,0}, \\ m^2_{A,1'} = \alpha_2(n+1) + E^{(2)}_{A,1'}, \\ m^2_{A,2'} = \alpha_2(n+1) + g^2v^2 + E^{(2)}_{A,2'}, \\ m^2_{A,3} = 4\pi^2(l^2+m^2) + g^2v^2 + E^{(2)}_{A,3}, \end{cases} \tag{43}$$

and we find that all of the gauge fields have nonzero mass if $v \neq 0$. Therefore, we conclude that the SU(2) gauge symmetry is completely broken. The fact that nonzero VEV v is realized will be seen in the potential analysis.

C. The mass of the scalar fields φ^a

The terms relevant to the scalar mass are

$$\begin{aligned} \mathcal{L} &\supset -\frac{1}{4}\{(\mathcal{D}\bar{\varphi} + \bar{\mathcal{D}}\varphi)^2 - 2\sqrt{2}g(\partial\langle\bar{\varphi}^a\rangle + \bar{\partial}\langle\varphi^a\rangle)[\varphi,\bar{\varphi}]^a\} \\ &\quad + \frac{1}{4}(\mathcal{D}\bar{\varphi} - \bar{\mathcal{D}}\varphi)^2 \\ &= -\bar{\tilde{\varphi}}(H + gf\,\mathrm{diag}(1,-1,0))\tilde{\varphi}, \end{aligned} \tag{44}$$

where

$$\begin{aligned} \bar{\varphi}^{a'} U_3^{a'a} &\equiv \bar{\tilde{\varphi}}^a \equiv \sum_{\{n_a\}} \bar{\tilde{\varphi}}^a_{\{n_a\}} \psi^a_{\{n_a\}}, \\ (U_3^{-1})^{aa'} \varphi^{a'} &\equiv \tilde{\varphi}^a \equiv \sum_{\{n_a\}} \tilde{\varphi}^a_{\{n_a\}} \psi^a_{\{n_a\}}. \end{aligned} \tag{45}$$

Since the energy eigenvalues for $\psi^1_{n,j}$ and $\psi^2_{n,j}$ are degenerate, we must solve the secular equation. We find the first-order perturbation energy from $V_1 + V_1^\dagger + V_2$ as

$$E^{(1)}_{\varphi,1''} = 0, \qquad E^{(1)}_{\varphi,2''} = g^2 v^2, \tag{46}$$

where the mode functions in new directions $1''$ and $2''$ are defined as

$$\psi^{1''}_{n,j} \equiv (i\psi^1_{n,j} + \psi^2_{n,j})/\sqrt{2},$$
$$\psi^{2''}_{n,j} \equiv (\psi^1_{n,j} + i\psi^2_{n,j})/\sqrt{2}. \tag{47}$$

The second-order perturbation energies $E^{(2)}_{\varphi,1''}$, $E^{(2)}_{\varphi,2''}$, and $E^{(2)}_{\varphi,3}$ are shown in the Appendix. Thus, the masses of the scalar fields are obtained as

$$\begin{cases} m^2_{\varphi,1''} = \alpha_2(n+\frac{1}{2}) + E^{(2)}_{\varphi,1''}, \\ m^2_{\varphi,2''} = \alpha_2(n+\frac{1}{2}) + g^2v^2 + E^{(2)}_{\varphi,2''}, \\ m^2_{\varphi,3} = 4\pi^2(l^2+m^2) + g^2v^2 + E^{(2)}_{\varphi,3}. \end{cases} \tag{48}$$

D. The mass of the ghost fields c^a

The terms relevant to the ghost mass are

$$\mathcal{L} \supset -\bar{c}^a(\mathcal{D}_m\mathcal{D}^m)^{ab}c^b \quad (m=5,6), \tag{49}$$

where

$$\begin{aligned}\mathcal{D}_m\mathcal{D}^m &= -(-\mathcal{D}\bar{\mathcal{D}}) - \frac{1}{2}[\mathcal{D},\bar{\mathcal{D}}] \\ &= -U_3\Big[H + \frac{1}{2}([\mathcal{D}_{3,\mathrm{diag}},\bar{\mathcal{D}}_{3,\mathrm{diag}}] \\ &\quad - V_1 - V_1^\dagger + V_3 + V_3^\dagger)\Big]U_3^{-1} \\ &= -U_3[H_0 + V_2 + gf\mathrm{diag}(1,-1,0) \\ &\quad + V_4 + V_4^\dagger]U_3^{-1}. \end{aligned} \tag{50}$$

V_3 and V_4 are defined as

$$V_3 \equiv U_3^{-1}VU_3\mathcal{D}_{3,\mathrm{diag}} \quad \text{and} \quad V_4 \equiv \frac{1}{2}(V_1+V_3). \tag{51}$$

Note that the first three terms in Eq. (50) are the same as those of the scalar fields. As in the previous section, we solve the secular equation and find the first-order perturbation energy from $V_2 + V_4 + V_4^\dagger$ as

$$E^{(1)}_{c,1''} = 0, \qquad E^{(1)}_{c,2''} = g^2v^2. \tag{52}$$

The second-order perturbation energies $E^{(2)}_{c,1''}$, $E^{(2)}_{c,2''}$, and $E^{(2)}_{c,3}$ are summarized in the Appendix.

Thus, the masses of the ghost fields are obtained as

$$\begin{cases} m^2_{c,1''} = \alpha_2(n+\frac{1}{2}) + E^{(2)}_{c,1''}, \\ m^2_{c,2''} = \alpha_2(n+\frac{1}{2}) + g^2v^2 + E^{(2)}_{c,2''}, \\ m^2_{c,3} = 4\pi^2(l^2+m^2) + g^2v^2 + E^{(2)}_{c,3}. \end{cases} \tag{53}$$

Note that the masses of the ghost fields are the same as that of the scalar mass at first order in the Feynman gauge $\xi = 1$. This fact greatly simplifies the potential analysis as will be discussed later.

E. The analysis of the effective potential

Since the potential of the constant WL scalar fields is not generated at tree level, we have to calculate the one-loop effective potential by use of the KK mass spectrum obtained in the previous subsections. We will show that the constant WL scalar VEV can be nonzero, and the gauge symmetry is broken.

In order to calculate the one-loop effective potential as general as possible, we parametrize the KK mass spectrum as follows:

$$\begin{aligned} &m_0^2 \equiv Ag^2v^2, \qquad m_n^2 \equiv \alpha(n+x) + Bg^2v^2, \\ &m^2_{l,m} \equiv 4\pi^2(l^2+m^2) + Cg^2v^2, \end{aligned} \tag{54}$$

where x is 0, 1, or 1/2 depending on the fields under consideration. We suppose that A, B, and C are positive constants. α means α_2 or α_3, which will be defined in the next section. Then, the typical forms of the one-loop effective potential can be written as

$$\nu_0(A) \equiv \int \frac{d^4p}{(2\pi)^4}\ln[p^2 + Ag^2v^2], \tag{55}$$

$$\nu_n(\alpha,B;x) \equiv \sum_{n=0}^{\infty}\int \frac{d^4p}{(2\pi)^4}\ln[p^2 + \alpha(n+x) + Bg^2v^2], \tag{56}$$

$$\nu_{l,m}(C) \equiv \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int \frac{d^4p}{(2\pi)^4}\ln[p^2 + 4\pi^2(l^2+m^2) + Cg^2v^2]. \tag{57}$$

Note that the second-order perturbations are neglected in these potentials. For an obvious reason, $\nu_0(A)$, $\nu_n(\alpha,B;x)$, and $\nu_{l,m}(C)$ will be referred to as the one-loop effective potential of zero-mode-type, with-flux-type, and without-flux-type, respectively.

1. The zero-mode-type ν_0

First, we consider the effective potential of zero-mode-type ν_0,

$$\begin{aligned}\nu_0(A) &= -\int \frac{d^4p}{(2\pi)^4}\int_0^\infty \frac{dt}{t}e^{-(p^2+Ag^2v^2)t} \\ &= -\frac{1}{16\pi^2}\int_0^\infty \frac{dt}{t^3}e^{-Ag^2v^2t}, \end{aligned}$$

where Schwinger's proper time integral is introduced in the first line, and the momentum integral is performed in the second line. Obviously, this integral diverges at $t = 0$, but we can extract a finite value from it. We will propose the idea later, and the regularized effective potential of zero-mode-type $\nu_{\mathrm{reg},0}$ is found as

$$\nu_{\mathrm{reg},0}(A) = \frac{(Ag^2v^2)^2}{576\pi^2}\left(\frac{9\zeta(3)}{\pi^2} - 1\right), \qquad (58)$$

where $\zeta(3)$ is Apéry's constant $\zeta(3) = 1.20205\cdots$ and $9\zeta(3)/\pi^2 - 1 = 0.0961381\cdots$.

2. *The with-flux-type ν_n*

Next, we consider the effective potential of with-flux-type ν_n,

$$\begin{aligned}\nu_n(\alpha, B; x) &= -\sum_n \int \frac{d^4p}{(2\pi)^4}\int_0^\infty \frac{dt}{t} e^{-(p^2+\alpha(n+x)+Bg^2v^2)t} \\ &= -\frac{1}{16\pi^2}\int_0^\infty \frac{dt}{t^3}\frac{e^{-\alpha(x+Bg^2v^2/\alpha-1)t}}{e^{\alpha t}-1} \\ &= -\frac{\alpha^2}{16\pi^2}\int_0^\infty \frac{dy}{y^{1-(-2)}}\frac{e^{-(x+Bg^2v^2/\alpha-1)y}}{e^y-1}. \qquad (59)\end{aligned}$$

To calculate this, we try to apply the integral representation of the Hurwitz ζ function [23]

$$\begin{aligned}\zeta(s,a) &= \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + \sum_{k=1}^{n}\frac{B_{2k}}{(2k)!}a^{1-s-2k}(s)_{2k-1} \\ &\quad + \frac{1}{\Gamma(s)}\int_0^\infty dy \frac{e^{-ay}}{y^{1-s}} \\ &\quad \times\left(\frac{1}{e^y-1} - \frac{1}{y} + \frac{1}{2} - \sum_{k=1}^{n}\frac{B_{2k}}{(2k)!}y^{2k-1}\right), \qquad (60)\end{aligned}$$

where n is a non-negative integer, B_n is the Bernoulli number, and $(s)_n$ is the Pochhammer symbol. Equation (60) is satisfied with the conditions

$$\mathrm{Re}\, s > -(2n+1) \quad (n \in \mathbb{Z}, n \geq 0), \quad s \neq 1, \quad \mathrm{Re}\, a > 0. \qquad (61)$$

Comparing the integral in (59) with the expression (60), we need to consider a case where $s = -2$. Therefore, it is enough to take $n = 1$ as follows:

$$\begin{aligned}\zeta(s,a) &= \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + \frac{sa^{-s-1}}{12} \\ &\quad + \frac{1}{\Gamma(s)}\int_0^\infty dy \frac{e^{-ay}}{y^{1-s}}\left(\frac{1}{e^y-1} - \frac{1}{y} + \frac{1}{2} - \frac{y}{12}\right), \qquad (62)\end{aligned}$$

where $(s)_1 = s$ and $B_2 = 1/6$ are used. The integral which includes a term of $1/(e^y - 1)$ corresponds to the with-flux-type ν_n. On the other hand, we know the form of $\zeta(-2, a)$ with elementary functions:

$$\begin{aligned}\zeta(-2,a) &= -\frac{1}{3}\sum_{k=0}^{3} {}_3\mathrm{C}_k B_{3-k} a^k \\ &= \frac{a^{-(-2)}}{2} + \frac{a^{1-(-2)}}{-2-1} + \frac{(-2)a^{-(-2)-1}}{12}, \qquad (63)\end{aligned}$$

and we find this to be equivalent to the first line of Eq. (62).

When we take $s = \epsilon - 2(\epsilon = (4-d)/2 \ll 1)$, Eq. (63) can be understood as the $\mathcal{O}(\epsilon^0)$ terms of both sides of Eq. (62). Therefore, the integral in the second line of (62) is understood to be $\mathcal{O}(\epsilon)$. Extracting the $\mathcal{O}(\epsilon)$ terms from Eq. (62), we obtain

$$\begin{aligned}&\int_0^\infty dy \frac{e^{-ay}}{y^3}\left(\frac{1}{e^y-1} - \frac{1}{y} + \frac{1}{2} - \frac{y}{12}\right) \\ &= \frac{1}{2}\left[\zeta^{(1,0)}(-2,a) - \frac{a}{12}(2\ln a + 1) \right.\\ &\quad \left. + \frac{a^2}{2}\ln a - \frac{a^3}{9}(3\ln a - 1)\right], \qquad (64)\end{aligned}$$

where we used the expansion of $\zeta(\epsilon - 2, a)$ in ϵ:

$$\zeta(\epsilon-2,a) = \zeta(-2,a) + \epsilon\zeta^{(1,0)}(-2,a) + \mathcal{O}(\epsilon^2). \qquad (65)$$

Although each integral of Eq. (64) diverges at $y = 0$, the right-hand side of (64) is finite. In other words, we can interpret that the last three terms of the left-hand side in (64) work as the regulators which extract a finite quantity from the divergent integral, and we are able to evaluate the potential of with-flux-type ν_n. Then, we obtain the regularized one-loop effective potential of with-flux-type $\nu_{\mathrm{reg},n}$:

$$\begin{aligned}\nu_{\mathrm{reg},n}(\alpha,B;x) = -\frac{\alpha^2}{32\pi^2}&\left[\zeta^{(1,0)}\left(-2, x+\frac{Bg^2v^2}{\alpha}-1\right) - \frac{1}{12}\left(x+\frac{Bg^2v^2}{\alpha}-1\right)\left\{2\ln\left(x+\frac{Bg^2v^2}{\alpha}-1\right)+1\right\}\right. \\ &\left. + \frac{1}{2}\left(x+\frac{Bg^2v^2}{\alpha}-1\right)^2\ln\left(x+\frac{Bg^2v^2}{\alpha}-1\right) - \frac{1}{9}\left(x+\frac{Bg^2v^2}{\alpha}-1\right)^3\left\{3\ln\left(x+\frac{Bg^2v^2}{\alpha}-1\right)-1\right\}\right]. \qquad (66)\end{aligned}$$

Note that the quantity $a = x + Bg^2v^2/\alpha - 1$ of Eq. (59) is not always positive: If a is zero or negative, it does not satisfy the condition (61). This situation occurs in the case of $x = 0$ or $1/2$. In that case, we separate the term of $n = 0$:

$$\begin{aligned} \nu_{\mathrm{reg},n}(\alpha, B; x < 1) &= -\int \frac{d^4p}{(2\pi)^4}\int_0^\infty \frac{dt}{t}e^{-(p^2+\alpha x+Bg^2v^2)t} \\ &\quad -\sum_{n=1}^{\infty}\int \frac{d^4p}{(2\pi)^4}\int_0^\infty \frac{dt}{t}e^{-(p^2+\alpha(n+x)+Bg^2v^2)t} \\ &= -\frac{\alpha^2}{16\pi^2}\int_0^\infty \frac{dy}{y^3}e^{-(x+Bg^2v^2/\alpha)y} \\ &\quad -\frac{\alpha^2}{16\pi^2}\int_0^\infty \frac{dy}{y^3}\frac{e^{-(x+Bg^2v^2/\alpha)y}}{e^y-1}. \end{aligned} \tag{67}$$

Although the first term in (67) is similarly divergent at $y = 0$, it can be finite by using (64) as

$$-\frac{1}{16\pi^2}\int_0^\infty \frac{dt}{t^3}e^{-(\alpha x+Bg^2v^2)t} \to \frac{\alpha^2}{576\pi^2}\left(x+\frac{Bg^2v^2}{\alpha}\right)^2\left(\frac{9\zeta(3)}{\pi^2}-1\right).$$

In this calculation, the terms, except for the third term of the left-hand side in (64), work as the regulators. Thus, the regularized one-loop effective potential of with-flux-type $\nu_{\mathrm{reg},n}(\alpha, B; x < 1)$ can be obtained

$$\begin{aligned} \nu_{\mathrm{reg},n}(\alpha, B; x<1) &= -\frac{\alpha^2}{32\pi^2}\left[\zeta^{(1,0)}\left(-2, x+\frac{Bg^2v^2}{\alpha}\right)\right. \\ &\quad -\frac{1}{12}\left(x+\frac{Bg^2v^2}{\alpha}\right)\left\{2\ln\left(x+\frac{Bg^2v^2}{\alpha}\right)+1\right\} \\ &\quad +\frac{1}{18}\left(x+\frac{Bg^2v^2}{\alpha}\right)^2\left\{9\ln\left(x+\frac{Bg^2v^2}{\alpha}\right)-\frac{9\zeta(3)}{\pi^2}+1\right\} \\ &\quad \left.-\frac{1}{9}\left(x+\frac{Bg^2v^2}{\alpha}\right)^3\left\{3\ln\left(x+\frac{Bg^2v^2}{\alpha}\right)-1\right\}\right]. \end{aligned} \tag{68}$$

3. The without-flux-type $\nu_{l,m}$

Finally, we consider the one-loop effective potential of without-flux-type $\nu_{l,m}$:

$$\begin{aligned} \nu_{l,m}(C) &= -\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_0^\infty \frac{dt}{t}\int \frac{d^4p}{(2\pi)^4}e^{-[p^2+4\pi^2(l^2+m^2)+Cg^2v^2]t} \\ &= -\frac{1}{16\pi^2}\sum_{l,m}\int_0^\infty \frac{dt}{t^3}e^{-[4\pi^2(l^2+m^2)+Cg^2v^2]t} \\ &= -\frac{1}{16\pi^2}\int_0^\infty du\, u e^{-Cg^2v^2/u}\sum_{l,m}e^{-4\pi^2(l^2+m^2)/u}. \end{aligned}$$

Using the Poisson resummation formula

$$\sum_{l=-\infty}^{\infty}\exp\left[-\frac{4\pi^2(l+v)^2}{u}\right] = \sqrt{\frac{u}{4\pi}}\sum_{r=-\infty}^{\infty}e^{-ur^2/4\pm 2\pi i r v} \tag{69}$$

with $v = 0$, we obtain

$$\begin{aligned} \nu_{l,m}(C) &= -\frac{1}{64\pi^3}\int_0^\infty du\, u^2 e^{-Cg^2v^2/u}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}e^{-u(r^2+s^2)/4} \\ &= -\frac{1}{64\pi^3}\sum_{r,s\neq 0}\left(\frac{4}{r^2+s^2}\right)^3\int_0^\infty du\, u^2 e^{-u} \\ &\quad \times\exp\left[-\frac{Cg^2v^2(r^2+s^2)}{4u}\right] \\ &\quad +\frac{1}{64\pi^3}\int_0^\infty dt\frac{e^{-Cg^2v^2t}}{t^3}\left(-\frac{1}{t}\right), \end{aligned} \tag{70}$$

where we note that a change of variable $t = (r^2 + s^2)/u$ is performed in the second line of (70) except for the $r = s = 0$ mode. For the first term of Eq. (70), we consider applying the modified Bessel function of the second kind,

$$\begin{aligned} K_\nu(z) &= \frac{1}{2}\left(\frac{z}{2}\right)^\nu\int_0^\infty dt\, t^{-\nu-1}\exp\left(-t-\frac{z^2}{4t}\right) \\ &= \frac{1}{2}\left(\frac{z}{2}\right)^{-\nu}\int_0^\infty du\, u^{\nu-1}\exp\left(-\frac{z^2}{4u}-u\right), \end{aligned} \tag{71}$$

where a change of variable $u = z^2/4t$ is performed in the second line. This is satisfied with the conditions $\mathrm{Re}\,\nu > -1/2, |\arg z| < \pi/4$. Thus, the first term of Eq. (70) becomes

$$\begin{aligned} &-\frac{1}{64\pi^3}\sum_{r,s\neq 0}\left(\frac{4}{r^2+s^2}\right)^3\int_0^\infty du\, u^2 e^{-u}\exp\left[-\frac{Cg^2v^2(r^2+s^2)}{4u}\right] \\ &= -\frac{C^{3/2}g^3v^3}{4\pi^3}\sum_{r,s\neq 0}\left(\frac{1}{r^2+s^2}\right)^{3/2}K_3\left(gv\sqrt{C(r^2+s^2)}\right). \end{aligned}$$

The second term of Eq. (70) is regularized by using Eq. (64):

$$\frac{1}{64\pi^3}\int_0^\infty dt\frac{e^{-Cg^2v^2t}}{t^3}\left(-\frac{1}{t}\right) \to \frac{(Cg^2v^2)^3}{4608\pi^3}\left(\frac{9\zeta(3)}{\pi^2}-1\right).$$

Therefore, the regularized one-loop effective potential of the without-flux-type is obtained

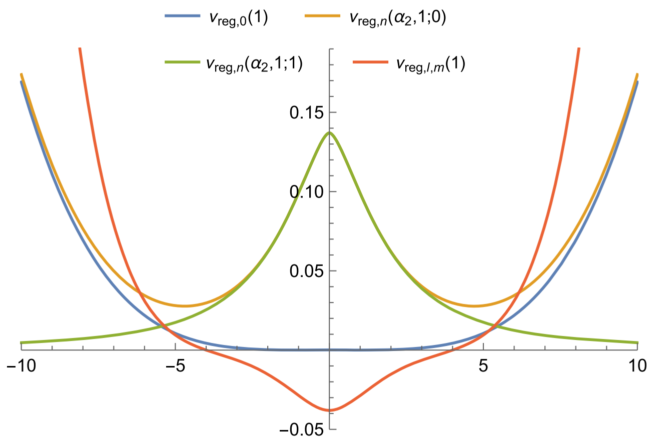


FIG. 1. An illustration of $\nu_{\mathrm{reg},0}(1), \nu_{\mathrm{reg},n}(\alpha_2,1;0), \nu_{\mathrm{reg},n}(\alpha_2,1;1)$, and $\nu_{\mathrm{reg},l,m}(1)$.

$$\nu_{\mathrm{reg},l,m}(C) = -\frac{(Cg^2v^2)^{3/2}}{4\pi^3}\sum_{r,s\neq 0}\left(\frac{1}{r^2+s^2}\right)^{3/2} \times K_3\left(gv\sqrt{C(r^2+s^2)}\right) + \frac{(Cg^2v^2)^3}{4608\pi^3}\left(\frac{9\zeta(3)}{\pi^2}-1\right). \tag{72}$$

The types of one-loop potentials $\nu_{\mathrm{reg},0}(1)$, $\nu_{\mathrm{reg},n}(\alpha_2,1;0)$, $\nu_{\mathrm{reg},n}(\alpha_2,1;1)$, and $\nu_{\mathrm{reg},l,m}(1)$ are shown in Fig. 1.

F. The one-loop effective potential of SU(2) Yang-Mills theory

We now apply the above results obtained in the previous subsections to SU(2) Yang-Mills theory. The effective potentials can be calculated by using the masses of the gauge fields A^a_μ, the scalar fields φ^a, and the ghost fields c^a,

$$V_A = \frac{3}{2}\frac{1}{(2\pi)^2}(\nu_0(1/2)+\nu_{l,m}(1)+\nu_n(\alpha_2,1;1)), \tag{73}$$

$$V_\varphi = \frac{1}{2}\frac{1}{(2\pi)^2}(\nu_{l,m}(1)+\nu_n(\alpha_3,1;1/2)), \tag{74}$$

$$V_c = -\frac{1}{2}\frac{1}{(2\pi)^2}(\nu_{l,m}(1)+\nu_n(\alpha_3,1;1/2)) = -V_\varphi. \tag{75}$$

Since these effective potentials are divergent, we extract the finite value from them. By using (58), (66), and (72), the regularized effective potential is expressed as

$$V_{\mathrm{reg},A} = \frac{3}{2}\frac{1}{(2\pi)^2}(\nu_{\mathrm{reg},0}(1/2)+\nu_{\mathrm{reg},l,m}(1)+\nu_{\mathrm{reg},n}(\alpha_2,1;1)), \tag{76}$$

$$V_{\mathrm{reg},\varphi} = \frac{1}{2}\frac{1}{(2\pi)^2}(\nu_{\mathrm{reg},l,m}(1)+\nu_{\mathrm{reg},n}(\alpha_3,1;1/2)), \tag{77}$$

$$V_{\mathrm{reg},c} = -\frac{1}{2}\frac{1}{(2\pi)^2}(\nu_{\mathrm{reg},l,m}(1)+\nu_{\mathrm{reg},n}(\alpha_3,1;1/2)) = -V_{\mathrm{reg},\varphi}. \tag{78}$$

Since V_φ and V_c cancel each other because of the same KK mass spectrum in the Feynman gauge, we have only to consider $V_{\mathrm{reg},A}$ as the total effective potential $V \equiv V_{\mathrm{reg},A} + V_{\mathrm{reg},\varphi} + V_{\mathrm{reg},c}$.[2] In detail, the total effective potential can be expressed as

$$\begin{aligned} V &= V_{\mathrm{reg},A} \\ &= \frac{(g^2v^2)^2}{6144\pi^4}\left(\frac{9\zeta(3)}{\pi}-1\right) - \frac{3(g^2v^2)^{3/2}}{32\pi^5}\sum_{r,s\neq 0}\left(\frac{1}{r^2+s^2}\right)^{3/2}K_3(gv\sqrt{r^2+s^2}) + \frac{(g^2v^2)^3}{12288\pi^5}\left(\frac{9\zeta(3)}{\pi}-1\right) \\ &\quad - \frac{3N^2}{16\pi^2}\left[\zeta^{(1,0)}\left(-2,\frac{g^2v^2}{4\pi N}\right) - \frac{1}{12}\left(\frac{g^2v^2}{4\pi N}\right)\left\{2\ln\left(\frac{g^2v^2}{4\pi N}\right)+1\right\} + \frac{1}{2}\left(\frac{g^2v^2}{4\pi N}\right)^2\ln\left(\frac{g^2v^2}{4\pi N}\right) \right. \\ &\quad \left. -\frac{1}{9}\left(\frac{g^2v^2}{4\pi N}\right)^3\left\{3\ln\left(\frac{g^2v^2}{4\pi N}\right)-1\right\}\right]. \end{aligned} \tag{79}$$

The one-loop effective potentials with $N = 3$ are shown in Fig. 2. The cancellation of the potentials between the scalar (blue line) and the ghost (green line) loop contributions can be explicitly verified. As can be seen from the total potential (red line) in Fig. 2, we can conclude that the origin of the potential is at least not a minimum, and the minimum of the effective potential is expected to locate at a nonzero constant VEV $v \neq 0$, and the gauge symmetry SU(2) is completely broken.

Here we should note that we cannot determine the location of the minimum since the location is expected to be beyond the range for which our perturbation is valid, as is shown in Fig. 2. Even if the value of the minimum is not determined, if we assume that the potential is bounded below, the pattern of gauge symmetry breaking can be confirmed from the gauge boson spectrum (43).

[2] If one does not choose Feynman gauge $\xi = 1$, the cancellation of $V_{\mathrm{reg},c}$ and $V_{\mathrm{reg},\varphi}$ does not occur. See [12] for this implication.

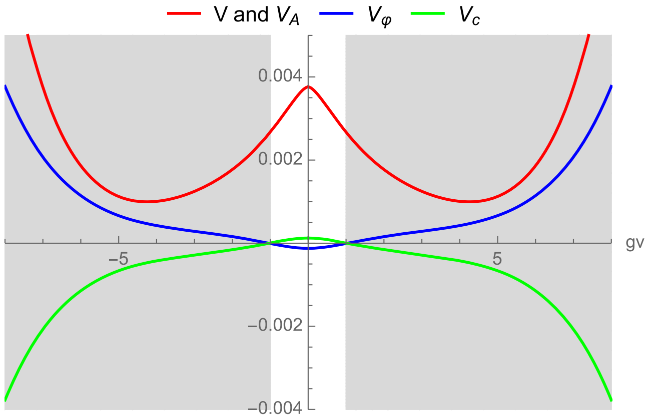


FIG. 2. One-loop effective potentials of SU(2) Yang-Mills theory. $N = 3$ is taken for simplicity. The shaded region represents a range where our perturbation analysis breaks down.

III. SU(3) YANG-MILLS THEORY

In the context of gauge-Higgs unification, SU(3) gauge symmetry is minimal to realize the zero mode of the WL scalar as an SU(2) Higgs doublet. In our case, if SU(3) is broken to $\mathrm{SU}(2)\times\mathrm{U}(1)$ by the VEV of the constant magnetic flux, the SU(2) Higgs doublet would appear, and the electroweak symmetry breaking can be discussed. As a first step toward the discussion, we now consider a six-dimensional SU(3) Yang-Mills theory. We consider two cases of directions where we introduce the VEVs

$$(1)\quad \langle\phi^1\rangle = \frac{v}{\sqrt{2}} \quad \text{and} \quad \langle\phi^8\rangle = \frac{f\bar{z}}{\sqrt{2}}, \tag{80}$$

$$(2)\quad \langle\phi^6\rangle = \frac{v}{\sqrt{2}} \quad \text{and} \quad \langle\phi^8\rangle = \frac{f\bar{z}}{\sqrt{2}}, \tag{81}$$

where the superscripts of ϕ denote the gauge indices of SU(3), and the structure constant is changed to that of SU(3) accordingly. Note that the flux background breaks the SU(3) gauge symmetry, which is broken $\mathrm{SU}(2)\times\mathrm{U}(1)$. Case (1) is not included in SU(2) theory, because two kinds of VEVs are both taken in the components of the unbroken symmetry. Other cases of developing the VEVs are reduced to the above two cases by the gauge rotations.

A. Case (1)

In this subsection, let us consider case (1) where the background covariant derivatives $\mathcal{D}, \bar{\mathcal{D}}$ are expressed as

$$\mathcal{D}^{ac} = \begin{bmatrix} \partial & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial & igv & 0 & 0 & 0 & 0 & 0 \\ 0 & -igv & \partial & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial & \frac{\sqrt{3}}{2}igf\bar{z} & 0 & \frac{1}{2}igv & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2}igf\bar{z} & \partial & -\frac{1}{2}igv & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}igv & \partial & \frac{\sqrt{3}}{2}igf\bar{z} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}igv & 0 & -\frac{\sqrt{3}}{2}igf\bar{z} & \partial & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial \end{bmatrix}, \tag{82}$$

$$\bar{\mathcal{D}}^{ac} = \begin{bmatrix} \bar\partial & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar\partial & -igv & 0 & 0 & 0 & 0 & 0 \\ 0 & igv & \bar\partial & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar\partial & -\frac{\sqrt{3}}{2}igfz & 0 & -\frac{1}{2}igv & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2}igfz & \bar\partial & \frac{1}{2}igv & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}igv & \bar\partial & -\frac{\sqrt{3}}{2}igfz & 0 \\ 0 & 0 & 0 & \frac{1}{2}igv & 0 & \frac{\sqrt{3}}{2}igfz & \bar\partial & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar\partial \end{bmatrix}. \tag{83}$$

Diagonalizing them, we obtain

$$\begin{cases} \mathcal{D}_{\text{diag}} = \text{diag}\Big(\partial, \partial - gv, \partial + gv, \partial - \frac{1}{2}gv - \frac{\sqrt{3}}{2}gf\bar{z}, \partial - \frac{1}{2}gv + \frac{\sqrt{3}}{2}gf\bar{z}, \partial + \frac{1}{2}gv - \frac{\sqrt{3}}{2}gf\bar{z}, \partial + \frac{1}{2}gv + \frac{\sqrt{3}}{2}gf\bar{z}, \partial\Big), \\ \bar{\mathcal{D}}_{\text{diag}} = \text{diag}\Big(\bar{\partial}, \bar{\partial} + gv, \bar{\partial} - gv, \bar{\partial} + \frac{1}{2}gv + \frac{\sqrt{3}}{2}gfz, \bar{\partial} + \frac{1}{2}gv - \frac{\sqrt{3}}{2}gfz, \bar{\partial} - \frac{1}{2}gv + \frac{\sqrt{3}}{2}gfz, \bar{\partial} - \frac{1}{2}gv - \frac{\sqrt{3}}{2}gfz, \bar{\partial}\Big). \end{cases} \tag{84}$$

Their diagonalizing unitary matrix U is given by

$$U = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2}i & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2}i & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & i & -1 & i & 0 \\ 0 & 0 & 0 & i & 1 & -i & 1 & 0 \\ 0 & 0 & 0 & 1 & -i & 1 & i & 0 \\ 0 & 0 & 0 & i & -1 & i & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}. \tag{85}$$

The commutation relation of $\mathcal{D}$ and $\bar{\mathcal{D}}$ becomes just a constant matrix:

$$[i\bar{\mathcal{D}}_{\text{diag}}, i\mathcal{D}_{\text{diag}}]^{ac} = \sqrt{3}gf\,\text{diag}(0,0,0,1,-1,1,-1,0), \tag{86}$$

where the creation and annihilation operators can be given as

$$a = \frac{i}{\sqrt{\alpha_3}}\bar{\mathcal{D}}_{\text{diag}}, \qquad a^\dagger = \frac{i}{\sqrt{\alpha_3}}\mathcal{D}_{\text{diag}}(\alpha_3 \equiv \sqrt{3}gf). \tag{87}$$

Defining further

$$a_{4,6} = \frac{i}{\sqrt{\alpha_3}}\left(\bar{\partial} + \frac{\sqrt{3}}{2}gfz\right), \qquad a_{4,6}^\dagger = \frac{i}{\sqrt{\alpha_3}}\left(\partial - \frac{\sqrt{3}}{2}gf\bar{z}\right), \tag{88}$$

$$a_{5,7} = \frac{i}{\sqrt{\alpha_3}}\left(\bar{\partial} - \frac{\sqrt{3}}{2}gfz\right), \qquad a_{5,7}^\dagger = \frac{i}{\sqrt{\alpha_3}}\left(\partial + \frac{\sqrt{3}}{2}gf\bar{z}\right) \tag{89}$$

in the matrix form of the creation and annihilation operators, the diagonalized part of $\mathcal{D}_{\text{diag}}$ and $\bar{\mathcal{D}}_{\text{diag}}$ can be expressed as

$$\begin{cases} (i\bar{\mathcal{D}}_{\text{diag}})^{44} = \sqrt{\alpha_3}a_{4,6} + \frac{1}{2}igv, \\ (i\bar{\mathcal{D}}_{\text{diag}})^{55} = \sqrt{\alpha_3}a_{5,7} + \frac{1}{2}igv, \\ (i\bar{\mathcal{D}}_{\text{diag}})^{66} = \sqrt{\alpha_3}a_{4,6} - \frac{1}{2}igv, \\ (i\bar{\mathcal{D}}_{\text{diag}})^{77} = \sqrt{\alpha_3}a_{5,7} - \frac{1}{2}igv, \end{cases} \quad \text{and}$$
$$\begin{cases} (i\mathcal{D}_{\text{diag}})^{44} = \sqrt{\alpha_3}a_{4,6}^\dagger - \frac{1}{2}igv, \\ (i\mathcal{D}_{\text{diag}})^{55} = \sqrt{\alpha_3}a_{5,7}^\dagger - \frac{1}{2}igv, \\ (i\mathcal{D}_{\text{diag}})^{66} = \sqrt{\alpha_3}a_{4,6}^\dagger + \frac{1}{2}igv, \\ (i\mathcal{D}_{\text{diag}})^{77} = \sqrt{\alpha_3}a_{5,7}^\dagger + \frac{1}{2}igv. \end{cases} \tag{90}$$

Note that the other components are just spatial derivatives and do not play any role of creation or annihilation operators.

1. The mass of the gauge fields A_μ^a

In the same way as the calculations in Sec. II B, we obtain the gauge mass matrix

$$-\mathcal{D}_{\text{diag}}\bar{\mathcal{D}}_{\text{diag}} \equiv m_A^2, \tag{91}$$

where the components are

$$\begin{cases} (m_A^2)^{11} = 4\pi^2(l_1^2 + m_1^2), & (m_A^2)^{22} = 4\pi^2\left\{l_2^2 + (m_2 - \frac{gv}{2\pi})^2\right\}, \\ (m_A^2)^{33} = 4\pi^2\left\{l_2^3 + (m_3 + \frac{gv}{2\pi})^2\right\}, & (m_A^2)^{44} = \alpha_3 n_4 + \frac{1}{4}g^2v^2, \\ (m_A^2)^{55} = \alpha_3(n_5 + 1) + \frac{1}{4}g^2v^2, & (m_A^2)^{66} = \alpha_3 n_6 + \frac{1}{4}g^2v^2, \\ (m_A^2)^{77} = \alpha_3(n_7 + 1) + \frac{1}{4}g^2v^2, & (m_A^2)^{88} = 4\pi^2(l_8^2 + m_8^2). \end{cases} \tag{92}$$

From the massless modes in (92), we find that the $\text{SU}(2) \times \text{U}(1)$ gauge symmetry is broken into $\text{U}(1) \times \text{U}(1)$ if $v \neq 0$.

2. The mass of the scalar fields φ^a

The scalar masses are calculated from the terms

$$2(-\mathcal{D}_{\text{diag}}\bar{\mathcal{D}}_{\text{diag}})^{ac} - 4igf(U^{-1})^{a'a}f^{8a'c'}U^{c'c} \equiv m_\varphi^2, \tag{93}$$

and the resulting KK mass spectra are found as

$$\begin{cases} (m_\varphi^2)^{11} = 4\pi^2(l_1^2+m_1^2), & (m_\varphi^2)^{22} = 4\pi^2\left\{l_2^2+(m_2-\frac{gv}{2\pi})^2\right\}, \\ (m_\varphi^2)^{33} = 4\pi^2\left\{l_2^3+(m_3+\frac{gv}{2\pi})^2\right\}, & (m_\varphi^2)^{44} = \alpha_3(n_4+\frac{1}{2})+\frac{1}{4}g^2v^2, \\ (m_\varphi^2)^{55} = \alpha_3(n_5+\frac{1}{2})+\frac{1}{4}g^2v^2, & (m_\varphi^2)^{66} = \alpha_3(n_6+\frac{1}{2})+\frac{1}{4}g^2v^2, \\ (m_\varphi^2)^{77} = \alpha_3(n_7+\frac{1}{2})+\frac{1}{4}g^2v^2, & (m_\varphi^2)^{88} = 4\pi^2(l_8^2+m_8^2). \end{cases} \tag{94}$$

3. The mass of the ghost fields c^a

The ghost masses are calculated from the terms

$$-\mathcal{D}_{\mathrm{diag}}\bar{\mathcal{D}}_{\mathrm{diag}}+\frac{\alpha_3}{2}[a,a^\dagger]\equiv m_c^2, \tag{95}$$

and the resulting KK spectra are obtained

$$\begin{cases} (m_c^2)^{11} = 4\pi^2(l_1^2+m_1^2), & (m_c^2)^{22} = 4\pi^2\left\{l_2^2+(m_2-\frac{gv}{2\pi})^2\right\}, \\ (m_c^2)^{33} = 4\pi^2\left\{l_3^2+(m_3+\frac{gv}{2\pi})^2\right\}, & (m_c^2)^{44} = \alpha_3(n_4+\frac{1}{2})+\frac{1}{4}g^2v^2, \\ (m_c^2)^{55} = \alpha_3(n_5+\frac{1}{2})+\frac{1}{4}g^2v^2, & (m_c^2)^{66} = \alpha_3(n_6+\frac{1}{2})+\frac{1}{4}g^2v^2, \\ (m_c^2)^{77} = \alpha_3(n_7+\frac{1}{2})+\frac{1}{4}g^2v^2, & (m_c^2)^{88} = 4\pi^2(l_8^2+m_8^2). \end{cases} \tag{96}$$

4. The potential calculation

The effective potential can be calculated by using the masses of the gauge fields A_μ^a, the scalar fields φ^a, the ghost fields c^a, and (56) and (57):

$$V_A = \frac{3}{2}\frac{1}{(2\pi)^2}(2\nu_{l,m}(0)+\nu_+(gv/2\pi)+\nu_-(gv/2\pi) + 2\nu_n(\alpha_3,1/4;0)+2\nu_n(\alpha_3,1/4;1)), \tag{97}$$

$$V_\varphi = \frac{1}{2}\frac{1}{(2\pi)^2}(2\nu_{l,m}(0)+\nu_+(gv/2\pi)+\nu_-(gv/2\pi) + 4\nu_n(\alpha_3,1/4;1/2)), \tag{98}$$

$$V_c = -\frac{1}{2}\frac{1}{(2\pi)^2}(2\nu_{l,m}(0)+\nu_+(gv/2\pi)+\nu_-(gv/2\pi) + 4\nu_n(\alpha_3,1/4;1/2)) = -V_\varphi, \tag{99}$$

where $\nu_\pm(V)$ is defined as

$$\nu_\pm(V)\equiv\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int\frac{d^4p}{(2\pi)^4}\ln[p^2+4\pi^2\{l^2+(m\pm V)^2\}]. \tag{100}$$

The terms which do not contain the constant VEV v, $\nu_{l,m}(0)$ are irrelevant to determine the potential minimum.

We consider here new one-loop effective potentials of without-flux-type $\nu_\pm$,

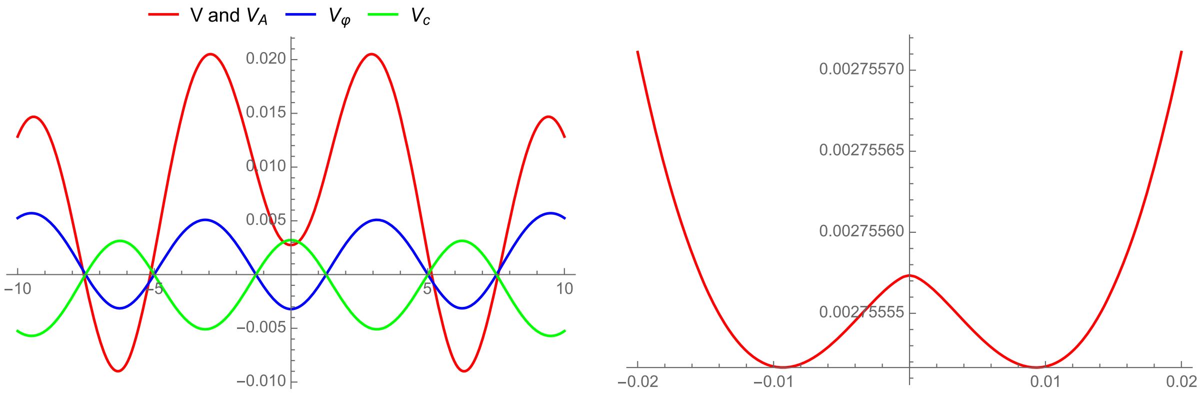
$$\begin{aligned}\nu_\pm(V) &= -\sum_{l,m}\int_0^\infty\frac{dt}{t}\int\frac{d^4p}{(2\pi)^4} \times\exp(-[p^2+4\pi^2\{l^2+(m\pm V)\}t]) \\ &= -\frac{1}{16\pi^2}\sum_{l,m}\int_0^\infty\frac{dt}{t^3}\exp(-4\pi^2\{l^2+(m\pm V)\}t) \\ &= -\frac{1}{16\pi^2}\int_0^\infty duu\sum_{l,m}\exp(-4\pi^2\{l^2+(m\pm V)^2\}/u).\end{aligned}$$

Using the Poisson resummation formula (69), $\nu_\pm$ becomes

$$\nu_\pm(V) = -\frac{1}{64\pi^3}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}e^{\pm2\pi iVs}\int_0^\infty duu^2e^{-u(r^2+s^2)/4}.$$

Since the term with $s=0$ has no dependence on v, it may be removed, and we obtain the regularized one-loop effective potentials of without-flux-type $\nu_{\mathrm{reg},\pm}$,

$$\begin{aligned}\nu_{\mathrm{reg},\pm}(V) &= -\frac{1}{64\pi^3}\sum_r\sum_{s\neq0}e^{\pm2\pi iVs}\int_0^\infty duu^2e^{-u(r^2+s^2)/4} \\ &= -\frac{4}{\pi^3}\sum_{r=-\infty}^{\infty}\sum_{s=1}^{\infty}\frac{1}{(r^2+s^2)^3}\cos(2\pi Vs).\end{aligned} \tag{101}$$



(a) The effective potential V.

(b) A close-up of the effective potential.

FIG. 3. A picture of the effective potential V with $N = 3$ and its parts V_A, V_φ, and V_c. (b) is a close-up of the effective potential (a).

Note that this periodic potential often appears in gauge-Higgs unification [22,24].

By using (66), (68), (72), and (101), each regularized effective potential is expressed as

$$V_{\mathrm{reg},A} = \frac{3}{2}\frac{1}{(2\pi)^2}(2\nu_{\mathrm{reg},l,m}(0) + 2\nu_{\mathrm{reg},\pm}(gv/2\pi) + 2\nu_{\mathrm{reg},n}(\alpha_3, 1/4; 0) + 2\nu_{\mathrm{reg},n}(\alpha_3, 1/4; 1)), \tag{102}$$

$$V_{\mathrm{reg},\varphi} = \frac{1}{2}\frac{1}{(2\pi)^2}(2\nu_{\mathrm{reg},l,m}(0) + 2\nu_{\mathrm{reg},\pm}(gv/2\pi) + 4\nu_{\mathrm{reg},n}(\alpha_3, 1/4; 1/2)), \tag{103}$$

$$V_{\mathrm{reg},c} = -\frac{1}{2}\frac{1}{(2\pi)^2}(2\nu_{\mathrm{reg},l,m}(0) + 2\nu_{\mathrm{reg},\pm}(gv/2\pi) + 4\nu_{\mathrm{reg},n}(\alpha_3, 1/4; 1/2)) = -V_{\mathrm{reg},\varphi}. \tag{104}$$

Because of $V_{\mathrm{reg},c} = -V_{\mathrm{reg},\varphi}$, we consider $V_{\mathrm{reg},A}$ as the total effective potential $V \equiv V_{\mathrm{reg},A} + V_{\mathrm{reg},\varphi} + V_{\mathrm{reg},c}$. In detail, the total one-loop effective potential can be expressed as

$$\begin{aligned} V = V_{\mathrm{reg},A} \supset & -\frac{3}{\pi^5}\sum_{r=-\infty}^{\infty}\sum_{s=1}^{\infty}\frac{1}{(r^2+s^2)^3}\cos(gvs) \\ & -\frac{9N^2}{16\pi^2}\left[\zeta^{(1,0)}\left(-2, \frac{g^2v^2}{8\sqrt{3}\pi N}\right)\right. \\ & -\frac{1}{12}\left(\frac{g^2v^2}{8\sqrt{3}\pi N}\right)\left(2\ln\frac{g^2v^2}{8\sqrt{3}\pi N} + 1\right) \\ & +\frac{1}{36}\left(\frac{g^2v^2}{8\sqrt{3}\pi N}\right)^2\left\{18\ln\frac{g^2v^2}{8\sqrt{3}\pi N} - \frac{9\zeta(3)}{2\pi^2} + 1\right\} \\ & \left.-\frac{1}{9}\left(\frac{g^2v^2}{8\sqrt{3}\pi N}\right)^3\left(3\ln\frac{g^2v^2}{8\sqrt{3}\pi N} - 1\right)\right]. \end{aligned} \tag{105}$$

In (105), the $2\nu_{\mathrm{reg},l,m}(0)$ term was dropped because it is irrelevant to finding the potential minimum.

The effective potential (105) of SU(3) Yang-Mills theory with $N = 3$ is shown in Fig. 3. Note that V_φ and V_c cancel each other, and V_A itself becomes the total potential V. It seems that there is a local minimum at $gv = 0$, but it is not correct. The behavior of V around $gv = 0$ is shown in Fig. 3(b), and we find that there are two local minima at $gv \neq 0$. The reason why V becomes convex upward at $gv = 0$ is due to the existence of the term $g^2v^2 \ln g^2v^2$ in V_A: It becomes dominant as gv gets close to zero. Furthermore, we emphasize that the logarithmic term is originated from the potential of the with-flux-type. If the magnetic flux is absent, we have no such contribution to the potential. Then, the potential is a periodic one as seen in the gauge-Higgs unification, where the origin of the potential becomes convex downward, which implies the origin can be a local minimum. The effects from the potential of the with-flux-type are very crucial in our analysis of gauge symmetry breaking. Thus, we find that the minimum of the effective potential has a nonzero VEV $v \neq 0$, and the gauge symmetry $\mathrm{SU}(2)\times \mathrm{U}(1)$ is broken into $\mathrm{U}(1)\times \mathrm{U}(1)$.

B. Case (2)

In this section, we consider case (2)

$$\langle\phi^6\rangle = \frac{v}{\sqrt{2}}, \qquad \langle\phi^8\rangle = \frac{f\bar{z}}{\sqrt{2}}. \tag{106}$$

In this case, the components of the VEV where the constant WL scalar develops are in the broken generators under $\mathrm{SU}(3) \to \mathrm{SU}(2)\times \mathrm{U}(1)$. This case is similar to Sec. II. In this situation, the background covariant derivatives $\mathcal{D}$, $\bar{\mathcal{D}}$ are expressed as

$$\mathcal{D}^{ac}=\begin{bmatrix} \partial & 0 & 0 & 0 & -\frac{1}{2}igv & 0 & 0 & 0 \\ 0 & \partial & 0 & \frac{1}{2}igv & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial & 0 & 0 & 0 & \frac{1}{2}igv & 0 \\ 0 & -\frac{1}{2}igv & 0 & \partial & \frac{\sqrt{3}}{2}igf\bar{z} & 0 & 0 & 0 \\ \frac{1}{2}igv & 0 & 0 & -\frac{\sqrt{3}}{2}igf\bar{z} & \partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial & \frac{\sqrt{3}}{2}igf\bar{z} & 0 \\ 0 & 0 & -\frac{1}{2}igv & 0 & 0 & -\frac{\sqrt{3}}{2}igf\bar{z} & \partial & \frac{\sqrt{3}}{2}igv \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2}igv & \partial \end{bmatrix}, \quad (107)$$

$$\bar{\mathcal{D}}^{ac}=\begin{bmatrix} \bar{\partial} & 0 & 0 & 0 & \frac{1}{2}igv & 0 & 0 & 0 \\ 0 & \bar{\partial} & 0 & -\frac{1}{2}igv & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\partial} & 0 & 0 & 0 & -\frac{1}{2}igv & 0 \\ 0 & \frac{1}{2}igv & 0 & \bar{\partial} & -\frac{\sqrt{3}}{2}igfz & 0 & 0 & 0 \\ -\frac{1}{2}igv & 0 & 0 & \frac{\sqrt{3}}{2}igfz & \bar{\partial} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\partial} & -\frac{\sqrt{3}}{2}igfz & 0 \\ 0 & 0 & \frac{1}{2}igv & 0 & 0 & \frac{\sqrt{3}}{2}igfz & \bar{\partial} & -\frac{\sqrt{3}}{2}igv \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2}igv & \bar{\partial} \end{bmatrix}. \quad (108)$$

If we diagonalize them, the eigenvalues are the same form as (17). Therefore, we apply the perturbation theory as we have seen in Sec. II B. Defining the unperturbed parts to be $\mathcal{D}_8$, $\bar{\mathcal{D}}_8$ and the perturbation part V such as (19) and (20), respectively, the covariant derivatives can be represented as $\mathcal{D}=\mathcal{D}_8+V$ and $\bar{\mathcal{D}}=\bar{\mathcal{D}}_8+\bar{V}$. Diagonalizing $\mathcal{D}_8$, $\bar{\mathcal{D}}_8$, we obtain the eigenvalues (84) with $v=0$. Their diagonalizing unitary matrix is given by

$$U_8=\frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}, \quad (109)$$

which is different from U in Sec. III 1. We can apply the discussion in Sec. II B by replacing U_3 in Sec. II B with U_8. According to (36), we define $V_1\equiv\mathcal{D}_{8,\mathrm{diag}}U_8^{-1}VU_8$ and $V_2\equiv(U_8^{-1}VU_8)^2$. In particular, we focus on V_2:

$$V_2=\frac{g^2v^2}{4}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2i & 0 \\ 0 & 0 & 0 & 0 & 0 & 2i & 2 & 0 \\ 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 3 \end{bmatrix}. \quad (110)$$

From (92) with $v=0$ and V_2, we find that the pairs $\psi^3_{l,m}$, $\psi^8_{l,m}$ and $\psi^6_{n+1,j}$, $\psi^7_{n,j}$ are degenerate, and we must solve the secular equations. As a result, the first-order perturbation energy $E^{(1)}$ has

$$E^{(1)}_{3'}=g^2v^2,\quad E^{(1)}_{8'}=0,\quad E^{(1)}_{6'}=0,\quad E^{(1)}_{7'}=g^2v^2, \quad (111)$$

where the mode functions in new directions $3'$ and $8'$ are defined as

$$\psi^{3'}_{l,m}=(-\psi^3_{l,m}+\sqrt{3}\psi^8_{l,m})/2,\quad \psi^{8'}_{l,m}=(\sqrt{3}\psi^3_{l,m}+\psi^8_{l,m})/2, \quad (112)$$

$$\psi^{6'}_{n,j}=(i\psi^6_{n+1,j}+\psi^7_{n,j})/\sqrt{2},\quad \psi^{7'}_{n,j}=(\psi^6_{n+1,j}+i\psi^7_{n,j})/\sqrt{2}. \quad (113)$$

Thus, the masses of the gauge fields can be obtained as

$$-\mathcal{D}_{\rm diag}\bar{\mathcal{D}}_{\rm diag} \equiv m_A^2, \tag{114}$$

$$\begin{cases} (m_A^2)^{11} = 4\pi^2(l_1^2+m_1^2)+\frac{1}{4}g^2v^2, & (m_A^2)^{22} = 4\pi^2(l_2^2+m_2^2)+\frac{1}{4}g^2v^2, \\ (m_A^2)^{3'3'} = 4\pi^2(l_{3'}^2+m_{3'}^2)+g^2v^2, & (m_A^2)^{44} = \alpha_3 n_4+\frac{1}{4}g^2v^2, \\ (m_A^2)^{55} = \alpha_3(n_5+1)+\frac{1}{4}g^2v^2, & (m_A^2)^{6'6'} = \frac{1}{2}g^2v^2(n_6=0),\ \alpha_3(n_{6'}+1)\,(n_{6'}\geq 0), \\ (m_A^2)^{7'7'} = \alpha_3(n_{7'}+1)+g^2v^2, & (m_A^2)^{8'8'} = 4\pi^2(l_{8'}^2+m_{8'}^2), \end{cases} \tag{115}$$

where we ignore the second-order perturbation energy.

In the previous discussion, the total one-loop effective potential has been expressed by $V = V_{{\rm reg},A}$. From (115), we have

$$V = \frac{3}{2}\frac{1}{(2\pi)^2}(2\nu_{{\rm reg},l,m}(1/4)+\nu_{{\rm reg},l,m}(1)+\nu_{{\rm reg},n}(\alpha_3,1/4;0) + \nu_{{\rm reg},n}(\alpha_3,1/4;1)+\nu_{{\rm reg},n}(\alpha_3,1;1)+\nu_{{\rm reg},0}(1/2)). \tag{116}$$

The effective potential of SU(3) Yang-Mills theory with $N=3$ is shown in Fig. 4. The shape of the potential in Fig. 4 is similar to that (the red line) in Fig. 2. The minimum of the effective potential might have a nonzero constant VEV $v\neq 0$, and the gauge symmetry $SU(2)\times U(1)$ is broken to U(1).

We should comment here again as was done in the SU(2) case that we cannot determine the location of the minimum since the location is expected to be beyond the range which our perturbation is valid, as shown in Fig. 4. Even if the value of the minimum is not determined, if we assume that the potential is bounded below, the pattern of gauge symmetry breaking can be confirmed from the gauge boson spectrum (92).

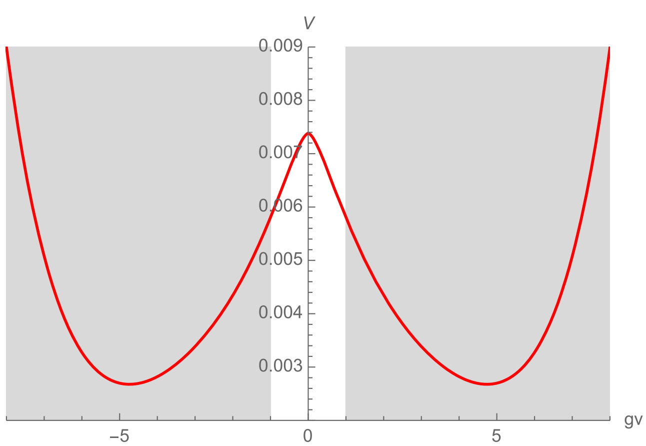


FIG. 4. An illustration of the effective potential with $N=3$. The shaded region represents a range where our perturbation analysis breaks down.

IV. SUMMARY

In this paper, we have studied six-dimensional Yang-Mills theories compactified on a torus with a magnetic flux and a constant VEV. Before constructing realistic models, we have discussed simple models of SU(2) and SU(3) Yang-Mills theories to understand the basic structures of the gauge symmetry breaking.

We have first given a setup of the SU(2) model and derived the KK masses in terms of perturbation theory in quantum mechanics. By using the KK masses, we have calculated the one-loop effective potential. In those computations, we have focused on the integral representation of the Hurwitz ζ function, and the regularized effective potential has been obtained. From the obtained one-loop effective potential and the mass of the gauge fields, we have seen that the SU(2) gauge symmetry is completely broken because of the flux background and the constant VEV.

Next, we have considered an SU(3) model where two types of directions have been used to introduce the flux background and the constant VEV. The extension to SU(3) is necessary for WL scalar fields to be an SU(2) doublet in the SM. In the case of Sec. III 1, the flux background and the constant VEV have been introduced in the eighth and first components of SU(3) gauge symmetry, respectively. The one-loop effective potential has been calculated, and it has been found that the potential has a nonzero VEV $v\neq 0$, which implies that the SU(3) gauge symmetry is broken to $U(1)\times U(1)$ by the flux background and the constant VEV. On the other hand, in the case of Sec. III 2, the flux background and the constant VEV have been introduced in the eighth and sixth components of SU(3) gauge symmetry, respectively. The case corresponds to gauge-Higgs unification in that the constant VEV is taken in one of the components of the broken generators of the original symmetry $SU(3)/(SU(2)\times U(1))$ in our model. The one-loop effective potential has been found to expect a nonzero VEV $v\neq 0$, and the SU(3) gauge symmetry is broken to U(1) by the flux background and the constant VEV.

Although the results obtained in this paper are very interesting, they are not realistic as they stand. If we identify the WL scalar field with the SM SU(2) Higgs doublet in our SU(3) model, the gauge symmetry breaking pattern SU(3) → SU(2) × U(1) → U(1) [or U(1) × U(1)] is not a correct pattern of the electroweak symmetry breaking SU(2) × U(1) → U(1). In order to realize such an electroweak symmetry breaking, it would be interesting to take into account fermion field contributions to the one-loop effective potential.

In some of the models discussed in this paper, the location of the potential minimum could not be determined in a parameter region where our perturbation is valid. It would be desirable to obtain the value of the potential minimum in a perturbative region by extending our analysis to the models with fermions. These issues are left for our future work.

ACKNOWLEDGMENTS

This work is supported by JSPS KAKENHI Grant No. JP22J15562 (T. H.).

APPENDIX: THE SECOND-ORDER PERTURBATION ENERGY

In this appendix, we represent the second-order perturbation energy. Since V_2 has an order of $\mathcal{O}(g^2v^2)$, the second-order perturbation energy from V_2 has an order of $\mathcal{O}((g^2v^2)^2)$ and we neglect it.

1. The mass of the gauge field

The second-order perturbation energy from $V_1+V_1^\dagger$ for $\psi^1_{0,j}$ is

$$\begin{aligned} E^{(2)}_{A,0} &= -\sum_{(l,m)\neq(0,0)} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_1+V_1^\dagger)\psi^1_{0,j}|^2}{4\pi^2(l^2+m^2)} \\ &= -\frac{g^2v^2}{2}\left[\sum_{(l,m)\neq(0,0)} \frac{4\pi^2(l^2+m^2)}{4\pi^2(l^2+m^2)}\left|\int_{T^2} d^2x\bar{\lambda}_{l,m}\xi_{0,j}\right|^2\right] \\ &\equiv -\frac{g^2v^2}{2}\left[\sum_{(l,m)\neq(0,0)} |C_{l,m,0,j}|^2\right]. \end{aligned} \tag{A1}$$

We do not describe the integrals $C_{l,m,0,j}$ in detail.

The second-order perturbation energy from $V_1+V_1^\dagger$ for $\psi^{1'}_{n+1,j}, \psi^{2'}_{n+1,j}, \psi^3_{l=0,m=0}$, and $\psi^3_{l\neq0,m\neq0}$ is

$$E^{(2)}_{A,1'} = -\sum_{l,m} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_1+V_1^\dagger)\psi^{1'}_{n+1,j}|^2}{4\pi^2(l^2+m^2)-\alpha_2(n+1)}, \tag{A2}$$

$$E^{(2)}_{A,2'} = -\sum_{l,m} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_1+V_1^\dagger)\psi^{2'}_{n+1,j}|^2}{4\pi^2(l^2+m^2)-\alpha_2(n+1)}, \tag{A3}$$

$$\begin{aligned} E^{(2)}_{A,3,l=0,m=0} &= \sum_{n,j} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_1+V_1^\dagger)\psi^{1'}_{n+1,j}|^2}{4\pi^2(l^2+m^2)-\alpha_2(n+1)} \\ &\quad + \sum_{n,j} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_1+V_1^\dagger)\psi^{2'}_{n+1,j}|^2}{4\pi^2(l^2+m^2)-\alpha_2(n+1)}, \end{aligned} \tag{A4}$$

and

$$\begin{aligned} E^{(2)}_{A,3,l\neq0,m\neq0} &= \frac{g^2v^2}{2}\sum_j |C_{l,m,0,j}|^2 \\ &\quad + \sum_{n,j} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_1+V_1^\dagger)\psi^{1'}_{n+1,j}|^2}{4\pi^2(l^2+m^2)-\alpha_2(n+1)} \\ &\quad + \sum_{n,j} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_1+V_1^\dagger)\psi^{2'}_{n+1,j}|^2}{4\pi^2(l^2+m^2)-\alpha_2(n+1)}. \end{aligned} \tag{A5}$$

2. The mass of the scalar field

The second-order perturbation energy from $V_1+V_1^\dagger$, $E^{(2)}_\varphi$ is

$$E^{(2)}_{\varphi,1''} = -\sum_{l,m} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_1+V_1^\dagger)\psi^{1''}_{n,j}|^2}{4\pi^2(l^2+m^2)-\alpha_2(n+1/2)}, \tag{A6}$$

$$E^{(2)}_{\varphi,2''} = -\sum_{l,m} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_1+V_1^\dagger)\psi^{2''}_{n,j}|^2}{4\pi^2(l^2+m^2)-\alpha_2(n+1/2)}, \tag{A7}$$

and

$$\begin{aligned} E^{(2)}_{\varphi,3} &= \sum_{n,j} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_1+V_1^\dagger)\psi^{1''}_{n,j}|^2}{4\pi^2(l^2+m^2)-\alpha_2(n+1/2)} \\ &\quad + \sum_{n,j} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_1+V_1^\dagger)\psi^{2''}_{n,j}|^2}{4\pi^2(l^2+m^2)-\alpha_2(n+1/2)}. \end{aligned} \tag{A8}$$

3. The mass of the ghost field

The second-order perturbation energy by $V_4+V_4^\dagger$, $E^{(2)}_4$ is

$$E^{(2)}_{c,1''} = -\sum_{l,m} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_4+V_4^\dagger)\psi^{1''}_{n,j}|^2}{4\pi^2(l^2+m^2)-\alpha_2(n+1/2)}, \tag{A9}$$

$$E^{(2)}_{c,2''} = -\sum_{l,m} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_4+V_4^\dagger)\psi^{2''}_{n,j}|^2}{4\pi^2(l^2+m^2)-\alpha_2(n+1/2)}, \tag{A10}$$

and

$$\begin{aligned} E^{(2)}_{c,3} &= \sum_{n,j} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_4+V_4^\dagger)\psi^{1''}_{n,j}|^2}{4\pi^2(l^2+m^2)-\alpha_2(n+1/2)} \\ &\quad + \sum_{n,j} \frac{|\int_{T^2} d^2x(\psi^3_{l,m})^\dagger(V_4+V_4^\dagger)\psi^{2''}_{n,j}|^2}{4\pi^2(l^2+m^2)-\alpha_2(n+1/2)}. \end{aligned} \tag{A11}$$

[1] R. Blumenhagen, B. Kors, D. Lust, and S. Stieberger, Four-dimensional string compactifications with D-branes, orientifolds and fluxes, Phys. Rep. **445**, 1 (2007).

[2] L. E. Ibanez and A. M. Uranga, *String Theory and Particle Physics: An Introduction to String Theory* (Cambridge University Press, Cambridge, England, 2012).

[3] E. Witten, Some properties of O(32) superstrings, Phys. Lett. **149B**, 351 (1984).

[4] H. Abe, K. S. Choi, T. Kobayashi, and H. Ohki, Three generation magnetized orbifold models, Nucl. Phys. **B814**, 265 (2009).

[5] D. Cremades, L. E. Ibanez, and F. Marchesano, Computing Yukawa couplings from magnetized extra dimensions, J. High Energy Phys. 05 (**2004**) 079.

[6] H. Abe, K.-S. Choi, T. Kobayashi, and H. Ohki, Higher order couplings in magnetized brane models, J. High Energy Phys. 06 (2009) 080.

[7] Y. Hamada and T. Kobayashi, Massive modes in magnetized brane models, Prog. Theor. Phys. **128**, 903 (2012).

[8] Y. Matsumoto and Y. Sakamura, Yukawa couplings in 6D gauge-Higgs unification on T^2/Z_N with magnetic fluxes, Prog. Theor. Exp. Phys. **2016**, 053B06 (2016).

[9] W. Buchmuller, M. Dierigl, E. Dudas, and J. Schweizer, Effective field theory for magnetic compactifications, J. High Energy Phys. 04 (2017) 052.

[10] D. M. Ghilencea and H. M. Lee, Wilson lines and UV sensitivity in magnetic compactifications, J. High Energy Phys. 06 (2017) 039.

[11] W. Buchmuller, M. Dierigl, and E. Dudas, Flux compactifications and naturalness, J. High Energy Phys. 08 (2018) 151.

[12] T. Hirose and N. Maru, Cancellation of one-loop corrections to scalar masses in Yang-Mills theory with flux compactification, J. High Energy Phys. 08 (2019) 054.

[13] M. Honda and T. Shibasaki, Wilson-line scalar as a Nambu-Goldstone boson in flux compactifications and higher-loop corrections, J. High Energy Phys. 03 (2020) 031.

[14] T. Hirose and N. Maru, Nonvanishing finite scalar mass in flux compactification, J. High Energy Phys. 06 (2021) 159.

[15] N. S. Manton, A new six-dimensional approach to the Weinberg-Salam model, Nucl. Phys. **B158**, 141 (1979).

[16] Y.Hosotani, Dynamical mass generation by compact extra dimension, Phys. Lett. **126B**, 309 (1983).

[17] Y.Hostonani, Dynamical of non-integrable phases and gauge symmetry breaking, Ann. Phys. (N.Y.) **190**, 233 (1989).

[18] H. Hatanaka, T. Inami, and C. S. Lim, The gauge hierarchy problem and higher dimensional gauge theories, Mod. Phys. Lett. A **13**, 2601 (1998).

[19] C. S. Lim, N. Maru, and K. Hasegawa, Six dimensional gauge-Higgs unification with an extra space S^2 and the hierarchy problem, J. Phys. Soc. Jpn. **77**, 074101 (2008).

[20] N. Maru and T. Yamashita, Two-loop calculation of Higgs mass in gauge-Higgs unification: 5D massless QED compactified on S^1, Nucl. Phys. **B754**, 127 (2006).

[21] Y. Hosotani, N. Maru, K. Takenaga, and T. Yamashita, Two loop finiteness of Higgs mass and potential in the gauge-Higgs unification, Prog. Theor. Phys. **118**, 1053 (2007).

[22] I. Antoniadis, K. Benakli, and M. Quiros, Finite Higgs mass without supersymmetry, New J. Phys. **3**, 20 (2001).

[23] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions*, 2nd ed. (Springer, Berlin, 2012).

[24] Masahiro Kubo, C. S. Lim, and Hiroyuki Yamashita, The Hosotani mechanism in bulk gauge theories with an orbifold extra space S^1/Z_2, Mod. Phys. Lett. A **17**, 2249 (2002).