4D modular flavor symmetric models inspired by a higher-dimensional theory

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We study a scenario to derive four-dimensional modular flavor symmetric models from a higher-dimensional theory by assuming the compactification consistent with the modular symmetry. In our scenario, wave functions in extra-dimensional compact space are modular forms. That leads to constraints on combinations between modular weights and Γ_N (Γ'_N) representations of matter fields. We also present illustrating examples.

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I. INTRODUCTION

The supersymmetric (SUSY) modular invariant theories give us an attractive framework to address the flavor problem of quarks and leptons. Indeed, finite modular flavor symmetric models have been presented for years [1–13]. The homogeneous modular group $\Gamma = SL(2,\mathbb{Z})$ and inhomogeneous modular group $\bar{\Gamma} = SL(2,\mathbb{Z})/\mathbb{Z}_2$ include S_3, A_4, S_4, A_5 as finite subgroups [14]. Indeed, the quotients $\Gamma_N = \bar{\Gamma}/\Gamma(N)$ are isomorphic to $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$, and $\Gamma_5 \simeq A_5$, while $\Gamma/\Gamma(2) \simeq S_3$, where $\Gamma(N)$ are principle congruence subgroups. These non-Abelian flavor symmetries such as S_3, A_4, S_4, A_5 were often used to derive quark and lepton mass matrices successfully in flavor model building before the studies of modular flavor models [15–24].

In modular flavor models, Yukawa couplings are modular forms depending on the modulus τ, and are certain representations under Γ_N and their covering groups Γ'_N. We assign modular weights and Γ_N (Γ'_N) representations to matter fields as well as Higgs fields, although Higgs fields are assigned to a Γ_N (Γ'_N) trivial singlet in most of the modular flavor models. Then, the structure of quark and lepton mass matrices is given by certain modular forms under the assumption that the Yukawa coupling terms (in the superpotential) as well as mass terms are invariant under the modular symmetry. By taking these modular flavor symmetric mass matrices, one can realize realistic quark and lepton masses and mixing angles by fixing the modulus τ. The CP violation and related phenomena have been also studied [25–34]. Besides mass matrices of quarks and leptons, related topics such as grand unified theory, leptogenesis, dark matter, etc., have been discussed in many works [35–94]. It is also remarked that the standard model effective field theory (SMEFT) has been studied in the modular symmetry [95,96]. Theoretical investigations have also been proceeded [97–110]. Various combinations of matter modular weights and Γ_N (Γ'_N) representations have been studied in order to lead to phenomenologically interesting results. On the other hand, the modular symmetry is the geometrical symmetry of compact spaces such as T^2 and the orbifold $T^2/\mathbb{Z}_2$. Thus, four-dimensional modular flavor symmetric models could be derived from a higher-dimensional theory such as a superstring theory. For example, flavor transformations under the modular symmetry were studied in heterotic orbifold models [111–113] and magnetized D-brane models [4,114–119]. Furthermore, Calabi-Yau compactifications have many moduli, and they have larger geometrical symmetries, i.e., symplectic modular symmetries $Sp(g,\mathbb{Z})$ [120–123]. However, in most four-dimensional (4D) modular flavor models, their relations with a higher-dimensional theory are not clear: How do 4D modular flavor symmetric models appear as a 4D low-energy effective field theory from a higher-dimensional theory? Our purpose in this paper is to propose a scenario to derive 4D modular flavor symmetric models from a higher-dimensional theory. We do not specify its compactification, but we assume generic compactification consistent with the modular symmetry. We study the Kaluza-Klein decompositions in a modular-symmetric way. In this scenario, wave functions in extra-dimensional compact space can be written by modular forms. Such a scenario leads to constraints of 4D modular flavor symmetric models. Modular weights and representations of matter fields are constrained.

This paper is organized as follows. In Sec. II, we give a brief review of the modular symmetry and modular forms. We also study the structure of $\Gamma(3)$ modular forms. In Sec. III, we study a scenario to derive 4D modular flavor symmetric models from a higher-dimensional theory with modular-symmetric compactification. In Sec. IV, we study illustrating examples with A_4 modular flavor symmetry. Section V is our conclusion. In the Appendix, we show an example to project wave functions with Γ_N reducible representations to an irreducible one.

II. MODULAR SYMMETRY AND MODULAR FORMS

A. Modular symmetry

Here, we briefly review the modular symmetry and modular forms. The $SL(2,\mathbb{Z})=\Gamma$ group is a group of the following 2×2 matrices:

$$\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1}$$

where a, b, c, d are integers and $ad-bc=1$. The $SL(2,\mathbb{Z})$ group is generated by S and T,

$$S=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

They satisfy the following algebraic relations:

$$S^4=1, \qquad (ST)^3=1. \tag{3}$$

The modulus τ transforms as

$$\tau\to\gamma\tau=\frac{a\tau+b}{c\tau+d} \tag{4}$$

under the modular symmetry. The generators S and T satisfy the following algebraic relations on τ:

$$S^2=1, \qquad (ST)^3=1, \tag{5}$$

i.e., $PSL(2,\mathbb{Z})=SL(2,Z)/\mathbb{Z}_2=\bar{\Gamma}$.

The modular forms are described by a holomorphic function $f_i(\tau)$, which transforms under the modular symmetry as

$$f_i(\gamma\tau)=(c\tau+d)^k\rho(\gamma)_{ij}f_j(\tau) \tag{6}$$

with k and $\rho(\gamma)_{ij}$ being the modular weight and unitary matrices, respectively.

Here, we introduce the principal congruence subgroups

$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z}), \begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N\right\}. \tag{7}$$

The $\Gamma(N)$ modular forms satisfy

$$f_i(\gamma\tau)=(c\tau+d)^k f_i(\tau) \tag{8}$$

for $\gamma\in\Gamma(N)$. Thus, the unitary matrices are representations of quotients $\Gamma_N=\bar{\Gamma}/\Gamma(N)$. Interestingly, the quotients Γ_N with $N=3,4,5$ are isomorphic to A_4, S_4, A_5, respectively. In addition, Γ_8 and Γ_{16} include $\Delta(96)$ and $\Delta(384)$ [4]. These are finite modular subgroups including irreducible triplet representations. Moreover, the quotient $\Gamma_2=\Gamma/\Gamma(2)$ is isomorphic to S_3.

Since $S^2=1$ in $\bar{\Gamma}$ on the modulus τ, the modular weight k must be even. The dimensions $d_k(\Gamma(N))$ of modular forms of weights k and levels N are determined mathematically and shown in Table I. These modular forms are d_k representations of Γ_N. In general, they are reducible representations and can be decomposed to irreducible representations as shown in the next subsection for $N=3$.

The above modular forms can be extended to $\Gamma=SL(2,\mathbb{Z})$, which is the double covering group of $\bar{\Gamma}$. For this group, the modular weights can be odd integers, and $\rho(\gamma)_{ij}$ are representations of the double covering groups of Γ_N, Γ'_N. Furthermore, we can extend the double covering group of $\Gamma=SL(2,\mathbb{Z})$. The modular weights can be half-integers, and $\rho(\gamma)_{ij}$ are representations of the double covering groups of Γ'_N. For example, such modular forms with half-integers are obtained in magnetized D-brane models on T^2 and $T^2/\mathbb{Z}_2$ [118].

B. $\Gamma(3)$ modular forms

Here, we show explicitly $\Gamma(3)$ modular forms and their A_4 representations. The A_4 group has four irreducible representations, $\mathbf{3},\mathbf{1},\mathbf{1}',\mathbf{1}''$. Their tensor products are obtained as

TABLE I. Dimensions of modular forms of the level N and weight k.

N	$d_k(\Gamma(N))$	Γ_N
2	$k/2+1$	S_3
3	$k+1$	A_4
4	$2k+1$	S_4
5	$5k+1$	A_5

$$\mathbf{3}\times\mathbf{3}=\mathbf{3}_s+\mathbf{3}_a+\mathbf{1}+\mathbf{1}'+\mathbf{1}'',$$
$$\mathbf{3}\times\mathbf{1}=\mathbf{3}\times\mathbf{1}'=\mathbf{3}\times\mathbf{1}''=\mathbf{3}, \tag{9}$$

where $\mathbf{3}_s$ and $\mathbf{3}_a$ are symmetric and antisymmetric, respectively, and

$$\mathbf{1}_m\times\mathbf{1}_n=\mathbf{1}_\ell, \tag{10}$$

where $\ell=m+n$ (mod 3), $\mathbf{1}_0=\mathbf{1}$, $\mathbf{1}_1=\mathbf{1}'$, and $\mathbf{1}_2=\mathbf{1}''$.

The $\Gamma(3)$ modular forms of weight $k=2$ have dimension $d_2=3$, and they are the A_4 triplet. Their explicit forms are written by [1]

$$Y_{\mathbf{3}}^{(2)}(\tau)=\begin{pmatrix}Y_1(\tau)\\Y_2(\tau)\\Y_3(\tau)\end{pmatrix}, \tag{11}$$

$$\begin{aligned}Y_1(\tau)&=\frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)}+\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)}+\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}-\frac{27\eta'(3\tau)}{\eta(3\tau)}\right),\\
Y_2(\tau)&=\frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)}+\omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)}+\omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right),\\
Y_3(\tau)&=\frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)}+\omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)}+\omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right),\end{aligned} \tag{12}$$

where $\eta(\tau)$ is the Dedekind eta function,

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n),\, q=\exp(2\pi i\tau). \tag{13}$$

The modular forms of higher weights are obtained by the tensor products of $Y_{\mathbf{3}}^{(2)}(\tau)$. The modular forms of weight $k=4$ have dimension $d_4=5$. They decompose to $\mathbf{3}$, $\mathbf{1}$, and $\mathbf{1}'$ and are written explicitly by

$$Y_{\mathbf{3}}^{(4)}(\tau)=\begin{pmatrix}Y_1^2-Y_2Y_3\\Y_3^2-Y_1Y_2\\Y_2^2-Y_1Y_3\end{pmatrix},$$
$$Y_{\mathbf{1}}^{(4)}=Y_1^2+2Y_2Y_3,\qquad Y_{\mathbf{1}'}^{(4)}=Y_3^2+2Y_1Y_2. \tag{14}$$

The modular form corresponding to the nontrivial singlet $\mathbf{1}''$ vanishes identically $Y_{\mathbf{1}''}^{(4)}=Y_2^2+2Y_1Y_3=0$ [1]. Also, the modular form corresponding to $\mathbf{3}_a$ vanishes.

Similarly, modular forms of higher weights are constructed (see, e.g., Refs. [54,57]). The modular forms of weight $k=6$ have dimension $d_6=7$. They decompose to $\mathbf{3}+\mathbf{3}+\mathbf{1}$, and are written explicitly by

$$Y_{\mathbf{3},\mathbf{1}}^{(6)}(\tau)=Y_{\mathbf{1}}^{(4)}Y_{\mathbf{3}}^{(2)}(\tau)=(Y_1^2+2Y_2Y_3)\begin{pmatrix}Y_1\\Y_2\\Y_3\end{pmatrix},$$
$$Y_{\mathbf{3},\mathbf{2}}^{(6)}(\tau)=Y_{\mathbf{1}'}^{(4)}Y_{\mathbf{3}}^{(2)}(\tau)=(Y_3^2+2Y_1Y_2)\begin{pmatrix}Y_3\\Y_1\\Y_2\end{pmatrix},$$
$$Y_{\mathbf{1}}^{(6)}(\tau)=(Y_{\mathbf{3}}^{(4)}Y_{\mathbf{3}}^{(2)})_{\mathbf{1}}=Y_1^3+Y_2^3+Y_3^3-3Y_1Y_2Y_3, \tag{15}$$

where $(Y_{\mathbf{3}}^{(4)}Y_{\mathbf{3}}^{(2)})_{\mathbf{1}}$ is the trivial singlet projection of the tensor product $Y_{\mathbf{3}}^{(4)}Y_{\mathbf{3}}^{(2)}$.

The modular forms of weight $k=8$ have dimension $d_8=9$. They decompose to $\mathbf{3}+\mathbf{3}+\mathbf{1}+\mathbf{1}'+\mathbf{1}''$, and are written explicitly by

$$Y_{\mathbf{3},\mathbf{1}}^{(8)}(\tau)=Y_{\mathbf{1}}^{(4)}Y_{\mathbf{3}}^{(4)}=(Y_1^2+2Y_2Y_3)\begin{pmatrix}Y_1^2-Y_2Y_3\\Y_3^2-Y_1Y_2\\Y_2^2-Y_1Y_3\end{pmatrix},$$
$$Y_{\mathbf{3},\mathbf{2}}^{(8)}(\tau)=Y_{\mathbf{1}'}^{(4)}Y_{\mathbf{3}}^{(4)}=(Y_3^2+2Y_1Y_2)\begin{pmatrix}Y_2^2-Y_1Y_3\\Y_1^2-Y_2Y_3\\Y_3^2-Y_1Y_2\end{pmatrix},$$
$$\begin{aligned}Y_{\mathbf{1}}^{(8)}(\tau)&=Y_{\mathbf{1}}^{(4)2}=(Y_1^2+2Y_2Y_3)^2,\\
Y_{\mathbf{1}'}^{(8)}(\tau)&=Y_{\mathbf{1}}^{(4)}Y_{\mathbf{1}'}^{(4)}=(Y_1^2+2Y_2Y_3)(Y_3^2+2Y_1Y_2),\\
Y_{\mathbf{1}''}^{(8)}(\tau)&=Y_{\mathbf{1}'}^{(4)}Y_{\mathbf{1}'}^{(4)}=(Y_3^2+2Y_1Y_2)^2.\end{aligned} \tag{16}$$

Note that the nontrivial singlet $\mathbf{1}''$ appears when the weight $k=8$.

The modular forms of weight $k=10$ have dimension $d_{10}=11$. They decompose to $\mathbf{3}+\mathbf{3}+\mathbf{3}+\mathbf{1}+\mathbf{1}'$, and are written explicitly by

TABLE II. A_4 representations for each weight k.

k	d_k	A_4 representations
2	3	$\mathbf{3}$
4	5	$\mathbf{3}+\mathbf{1}+\mathbf{1}'$
6	7	$\mathbf{3}+\mathbf{3}+\mathbf{1}$
8	9	$\mathbf{3}+\mathbf{3}+\mathbf{1}+\mathbf{1}'+\mathbf{1}''$
10	11	$\mathbf{3}+\mathbf{3}+\mathbf{3}+\mathbf{1}+\mathbf{1}'$

$$Y^{(10)}_{\mathbf{3},\mathbf{1}}(\tau) = Y^{(8)}_{\mathbf{1}} Y^{(2)}_{\mathbf{3}}(\tau) = (Y_1^2 + 2Y_2Y_3)^2 \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix},$$

$$Y^{(10)}_{\mathbf{3},\mathbf{2}}(\tau) = Y^{(8)}_{\mathbf{1}''} Y^{(2)}_{\mathbf{3}}(\tau) = (Y_3^2 + 2Y_1Y_2)^2 \begin{pmatrix} Y_2 \\ Y_3 \\ Y_1 \end{pmatrix},$$

$$Y^{(10)}_{\mathbf{3},\mathbf{3}}(\tau) = Y^{(8)}_{\mathbf{1}'} Y^{(2)}_{\mathbf{3}}(\tau) = (Y_1^2 + 2Y_2Y_3)(Y_3^2 + 2Y_1Y_2) \begin{pmatrix} Y_3 \\ Y_1 \\ Y_2 \end{pmatrix},$$

$$Y^{(10)}_{\mathbf{1}}(\tau) = Y^{(4)}_{\mathbf{1}} Y^{(6)}_{\mathbf{1}} = (Y_1^2 + 2Y_2Y_3)(Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3),$$

$$Y^{(10)}_{\mathbf{1}'}(\tau) = Y^{(4)}_{\mathbf{1}'} Y^{(6)}_{\mathbf{1}} = (Y_3^2 + 2Y_1Y_2)(Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3). \tag{17}$$

Table II shows the A_4 representations for the modular forms of each weight k.

III. 4D LOW-ENERGY EFFECTIVE FIELD THEORY FROM A HIGHER-DIMENSIONAL THEORY

We study a scenario to derive 4D modular flavor symmetric models from a $(4+d)$-dimensional theory by compactification. We assume that the modulus τ describes geometrical characters of d-dimensional compact space such as shape, although the compact space may have other moduli. On top of that, we assume that the compact space has the modular symmetry on τ. This originates from the symmetry of τ, that is, the geometric symmetry of extra-dimensional spaces. For instance, if extra-dimensional spaces include T^2 or its orbifold as subspace, such backgrounds enjoy the $SL(2,\mathbb{Z})$ modular symmetry when the modulus τ is identified with the complex structure modulus or the Kähler modulus [4,114–119]. Furthermore, the $Sp(2h,\mathbb{Z})$ symplectic modular symmetry also appears in toroidal orbifolds with multimoduli [107] and Calabi-Yau backgrounds [120–123]. Our discussion can be applied for such a compact space. In this section, we study the modular-symmetric theory without specifying the extra-dimensional space. Most 4D modular flavor symmetric models are constructed within the framework of a (global) supersymmetric theory. Hence, we assume that our compactification preserves 4D $\mathcal{N}=1$ supersymmetry.

We denote coordinates of 4D spacetime and d-dimensional compact space by x and y, respectively. Bosonic fields $\Phi(x,y)$ and spinor fields $\Psi(x,y)$ in a higher-dimensional theory are written by Kaluza-Klein decomposition as

$$\Phi(x,y) = \sum_i \phi_i(x)\varphi_i(y) + \cdots,$$

$$\Psi(x,y) = \sum_i \psi_i(x)\chi_i(y) + \cdots. \tag{18}$$

Bosonic fields $\Phi(x,y)$ correspond to scalars or vectors in $(4+d)$ dimensions, but vector fields with vector indices along extra-dimensional space are 4D scalars. The first terms on the rhs are massless modes (zero modes) and the others are massive modes. Here, we focus on massless modes. In general, there is more than one zero mode, which is labeled by the index i. The zero-mode index i corresponds to the flavor index in the 4D low-energy effective field theory. Hereafter, we often omit this index i.

It is noted that $\phi(x)$ and $\psi(x)$ are 4D fields, while $\varphi(y)$ and $\chi(y)$ are wave functions in extra dimensions. The wave functions $\varphi(y)$ and $\chi(y)$ depend on the modulus τ, namely, metric deformations of extra-dimensional space. Since this theory is assumed to be modular symmetric, these wave functions are modular forms.[1] Thus, for fixed modular weight k, Γ_N representations of wave functions $\varphi(y)$ and $\chi(y)$ are constrained as in the previous section. For example, for $N=3$, the wave functions $\varphi(y)$ and $\chi(y)$ of weight $k=2$ are only the A_4 triplet, but not singlets. When $k=4$, wave functions $\varphi(y)$ and $\chi(y)$ can correspond to either of $\mathbf{3}$, $\mathbf{1}$, $\mathbf{1}'$, but not $\mathbf{1}''$. The singlet $\mathbf{1}''$ can appear for $k=8$ and higher weights. Hence, we have constraints on weights k and Γ_N representations in a higher-dimensional theory. Note that the value of level N depends on models in a higher-dimensional theory. For instance, the Γ_3 representations appear in twisted modes of the heterotic $T^2/\mathbb{Z}_3$ orbifold [111–113]. Other representations are also possible on heterotic orbifolds and magnetized D-brane models. The value of N depends on the geometric structure of extra-dimensional spaces as well as background sources. It is important to reveal the geometric meaning of N, but we leave these issues for future research.

[1]For example, wave functions in magnetized D-brane models are modular forms [4,114–119].

For fixed weight k, there are $d_k(\Gamma(N))$ dimensions of modular forms as shown in Table I. Hereafter, we assume that not all modular forms, but one or more irreducible representations of Γ_N, appear as zero-mode wave functions of matter fields. Such a projection from $d_k(\Gamma(N))$ dimensions to irreducible representations would be possible by imposing certain boundary conditions. In the Appendix, we show an example to project out some of the reducible representations in zero-mode wave functions so as to obtain irreducible representations. Here, we show that such projections can be consistent with the modular symmetry. We assume that a wave function $\varphi_1(y)$ corresponding to d_k-dimensional modular forms of $\Gamma(N)$ satisfies a zero-mode equation with a specific boundary condition. When those d_k-dimensional wave functions are a reducible representation of Γ_N, the unitary matrix $\rho(\gamma)_{ij}$ is represented by

$$\rho(\gamma)_{ij}\begin{pmatrix}\varphi_1(y)\\ \vdots\\ \vdots\end{pmatrix}=\begin{pmatrix}\rho^{(1)}(\gamma)_{ij} & & \\ & \rho^{(2)}(\gamma)_{ij} & \\ & & \ddots\end{pmatrix}\begin{pmatrix}\varphi_1(y)\\ \vdots\\ \vdots\end{pmatrix}. \tag{19}$$

That implies that the modular transformation of $\varphi_1(y)$ is closed in the irreducible representation corresponding to $\rho^{(1)}(\gamma)_{ij}$ but does not transform other representations such as $\rho^{(2)}(\gamma)_{ij}$. Thus, it is consistent with the modular symmetry to pick up an irreducible representation from d_k-dimensional modular forms.

The minimal SUSY model has one pair of Higgs modes, i.e., the up-sector and down-sector Higgs fields. They must be Γ_N singlets. Thus, it is reasonable to assign the modular weight $k=0$ to the Higgs modes.

It is natural to normalize the wave function of the weight k as

$$\int d^dy\sqrt{g}|\varphi(y)|^2=\frac{1}{(2\mathrm{Im}(\tau))^k} \tag{20}$$

with g being the determinant of the metric of extra-dimensional space. This normalization is consistent with the modular symmetry. Indeed, the left-hand side of the above expressions transforms as

$$\int d^dy\sqrt{g}|\varphi(y)|^2\to|c\tau+d|^{2k}\int d^dy\sqrt{g}|\varphi(y)|^2, \tag{21}$$

taking into account the modular transformation of $\varphi(y)$:

$$\varphi(y)\to\rho(\gamma)(c\tau+d)^k\varphi(y). \tag{22}$$

Note that $\int d^dy\sqrt{g}$ is invariant under the modular transformation, i.e., the coordinate transformation. Thus, it is consistent with the modular transformation of τ:

$$(2\mathrm{Im}(\tau))^{-k}\to|c\tau+d|^{2k}(2\mathrm{Im}(\tau))^{-k}. \tag{23}$$

We start with the following canonical kinetic term,

$$\partial_M\Phi^*\partial^M\Phi, \tag{24}$$

in a higher-dimensional theory.[2] Then, we carry out dimensional reduction by the use of the above normalization so as to obtain the following Kähler potential of matter fields:

$$K=\frac{1}{(2\mathrm{Im}(\tau))^k}|\phi(x)|^2. \tag{25}$$

We require that the 4D effective field theory is invariant under the modular transformation. The 4D matter fields must have the modular weights $-k$, which have opposite signs compared with the wave function weights k. Here and hereafter, we use the notation that the same letter is used for both the superfield and its lowest scalar component.

Next, we study the Yukawa coupling terms in the superpotential. For example, suppose that the Yukawa coupling terms in the 4D effective theory originate from the following terms in a higher-dimensional theory:

$$y\bar{\Psi}_e\Phi_H^*\Psi_L, \tag{26}$$

where Φ_H^* is the higher-dimensional field corresponding to the 4D down-sector Higgs field H_d, and Ψ_L, Ψ_e are higher-dimensional fields corresponding to left-handed and right-handed leptons in the 4D effective theory. Then, we integrate the extra dimensions y so as to derive the Yukawa coupling terms in the 4D superpotential,

$$W=Y_e(\tau)LH_de^c. \tag{27}$$

Here, the 4D Yukawa coupling $Y_e(\tau)$ is obtained by

$$Y_e(\tau)=y\int d^dy\sqrt{g}\chi_{e^c}(y)\chi_L(y)\varphi_H^*(y). \tag{28}$$

The 4D modes L and e^c have modular weights $-k_L$ and $-k_e$, respectively, while H_d has vanishing modular weight. The 4D Yukawa coupling $Y_e(\tau)$ has modular weight k_L+k_e because the product of wave functions in the extra dimension has weight k_L+k_e. Then, the above superpotential is invariant under the modular transformation. Indeed, the modular transformation of matter fields (22) induces the correct modular transformation of the Yukawa couplings:

[2]Note that other higher-dimensional higher-derivative and interaction terms will provide the correction terms in the Kähler potential, but they would be suppressed by the compactification scale.

$$Y_e(\tau) \to (c\tau + d)^{k_L + k_e}\rho(\gamma)Y_e(\tau). \tag{29}$$

The same result can be derived another way as follows. The product of wave functions $\chi_{e^c}(y)\chi_L(y)$ can be expanded by all the Kaluza-Klein (KK) wave functions Φ_H,

$$\chi_L(y)\chi_{e^c}(y) = Y_e(\tau)\varphi_H(y) + \cdots, \tag{30}$$

since all the KK wave functions are a complete set. The first term on the rhs corresponds to the massless mode, while the others are massive modes.[3] The expansion coefficient $Y_e(\tau)$ corresponds to the 4D Yukawa coupling. Both sides must have the same modular weight, and φ_H has vanishing weight. Thus, the 4D Yukawa coupling $Y_e(\tau)$ has modular weight $k_L + k_e$. Thus, we can derive the 4D modular flavor symmetric model from a higher-dimensional theory. In this scenario, we have the constraint on combinations between modular weights and Γ_N representations of matter fields, although one has assigned modular weights and Γ_N representations to matter fields without such a constraint in modular flavor models, which have been constructed so far. For example, one cannot assign odd weights to matter fields in modular Γ_N flavor models. The modular Γ'_N flavor symmetry is then required to assign odd weights to matter fields. In the next section, we show A_4 modular flavor models as illustrating examples.

IV. EXAMPLES IN A_4 MODULAR FLAVOR MODELS

In the previous section, we have studied a scenario to derive 4D modular flavor symmetric models from a higher-dimensional theory. In this scenario, we have the constraint on combinations of modular weights and Γ_N representations for matter fields. For example, matter fields must have even modular weights in A_4 modular flavor models. The matter fields with modular weight $-k = -2$ must be A_4 triplet, but other representations cannot be allowed. The nontrivial A_4 singlet $\mathbf{1}''$ cannot be assigned to the matter fields with modular weight $-k = -2, -4, -6$, but can be assigned to the matter fields with $-k = -8$. We present A_4 models.

In many A_4 models, three generations of lepton doublets L are assigned to the A_4 triplet $\mathbf{3}$, and three generations of right-handed charged leptons e^c are assigned to three A_4 singlets, $\mathbf{1}, \mathbf{1}'', \mathbf{1}'$. In order to use such assignments of A_4 representations, we study the model that three generations of lepton doublets L_i have modular weight $-k = -2$ and three generations of right-handed charged leptons e^c_i have modular weight $-k = -8$. Such an assignment is summarized in Table III.

[3]In specific higher-dimensional theories, the production of massless wave functions can be expanded only by massless modes [124–126].

TABLE III. Assignment of A_4 representations and weights.

	L_i	e^c_i	H_d
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$
A_4	$\mathbf{3}$	$\mathbf{1}, \mathbf{1}'', \mathbf{1}'$	$\mathbf{1}$
k	-2	-8	0

The A_4 modular invariant superpotential relevant to the lepton sector can be written by

$$W = \sum_{\mathbf{r}} \alpha_{\mathbf{r}} (Y^{(10)}_{\mathbf{r}} L)_{\mathbf{1}} H_d e^c_1 + \sum_{\mathbf{r}} \beta_{\mathbf{r}} (Y^{(10)}_{\mathbf{r}} L)_{\mathbf{1}'} H_d e^c_{1''} + \sum_{\mathbf{r}} \gamma_{\mathbf{r}} (Y^{(10)}_{\mathbf{r}} L)_{\mathbf{1}''} H_d e^c_{1'} + \sum_{\mathbf{r}} g_{\mathbf{r}} \frac{Y^{(4)}_{\mathbf{r}}}{\Lambda} LHLH. \tag{31}$$

We set $\alpha_{\mathbf{r}} = \beta_{\mathbf{r}} = \gamma_{\mathbf{r}} = 0$ except $\mathbf{r} = (\mathbf{3}, \mathbf{1})$ of Eq. (17), and the dimensionful parameter Λ is set to obtain the correct scale of neutrino masses. On the other hand, nonvanishing $g_{\mathbf{r}}$'s are given for $\mathbf{r} = \mathbf{3}, \mathbf{1}, \mathbf{1}'$ of Eq. (14). Then, this superpotential is quite similar to the one in Ref. [31]. Indeed, we set

$$\tau = 0.0796 + 1.0065i, \quad g_{\mathbf{1}}/g_{\mathbf{3}} = 0.124, \quad g_{\mathbf{1}'}/g_{\mathbf{3}} = -0.802,$$
$$\alpha_{\mathbf{3},\mathbf{1}}/\gamma_{\mathbf{3},\mathbf{1}} = 6.82 \times 10^{-2}, \quad \beta_{\mathbf{3},\mathbf{1}}/\gamma_{\mathbf{3},\mathbf{1}} = 1.02 \times 10^{-3}, \tag{32}$$

so as to realize realistic values of charged lepton mass ratios and neutrino mass squared differences. The obtained mixing angles are

$$\sin^2\theta_{12} = 0.294, \quad \sin^2\theta_{23} = 0.563, \quad \sin^2\theta_{13} = 0.0226, \tag{33}$$

which are within a 1σ error bar of observed values [127]. Thus, we can construct the modular flavor symmetric models, which are consistent with our scenario and can derive realistic results.

Although the above model is a simple model, we may be able to study other assignments consistent with our scenario. For example, we assign A_4 representations and modular weights to three generations of e^c_i as $\mathbf{1}$ (weight -4), $\mathbf{1}''$ (weight -8), and $\mathbf{1}'$ (weight -4), while we use the same assignment for L_i and H_d. Then, the modular A_4 invariant superpotential can be written by

$$W = \sum_{\mathbf{r}} \alpha_{\mathbf{r}} (Y^{(6)}_{\mathbf{r}} L)_{\mathbf{1}} H_d e^c_1 + \sum_{\mathbf{r}} \beta_{\mathbf{r}} (Y^{(10)}_{\mathbf{r}} L)_{\mathbf{1}'} H_d e^c_{1''} + \sum_{\mathbf{r}} \gamma_{\mathbf{r}} (Y^{(6)}_{\mathbf{r}} L)_{\mathbf{1}''} H_d e^c_{1'} + \sum_{\mathbf{r}} g_{\mathbf{r}} \frac{Y^{(4)}_{\mathbf{r}}}{\Lambda} LHLH. \tag{34}$$

We also set $\alpha_{\mathbf{r}} = \beta_{\mathbf{r}} = \gamma_{\mathbf{r}} = 0$ except $\mathbf{r} = (\mathbf{3}, \mathbf{1})$ of Eqs. (15) and (17). By using proper values of the parameters, we can realize almost the same results of the lepton masses and mixing angles as the previous model. In this model, three generations of e_i^c have two different modular weights, -4 and -8. Thus, these three generations may originate from not a single field $\Phi(x,y)$ in a higher-dimensional theory, but at least two fields $\Phi(x,y)$ and $\Phi'(x,y)$ where one field corresponds to modular weight -4 and the other corresponds to weight -8.

As another model, we assign three generations of e_i^c as $\mathbf{1}$ (weight -4), $\mathbf{1}'$ (weight -4), and $\mathbf{1}'$ (weight -4), while we use the same assignment for L_i and H_d. Then, the modular A_4 invariant superpotential can be written by

$$W = \sum_{\mathbf{r}} \alpha_{\mathbf{r}} (Y_{\mathbf{r}}^{(6)} L)_1 H_d e c_1 + \sum_{\mathbf{r}} \beta_{\mathbf{r}} (Y_{\mathbf{r}}^{(6)} L)_{1''} H_d e^c_{1'} + \sum_{\mathbf{r}} \gamma_{\mathbf{r}} (Y_{\mathbf{r}}^{(6)} L)_{1''} H_d e^c_{1'} + \sum_{\mathbf{r}} g_{\mathbf{r}} \frac{Y_{\mathbf{r}}^{(4)}}{\Lambda} LHLH. \tag{35}$$

Taking $\alpha_{\mathbf{r}} = \gamma_{\mathbf{r}} = 0$ except $\mathbf{r} = (\mathbf{3},\mathbf{1})$ and $\beta_{\mathbf{r}} = 0$ except $\mathbf{r} = (\mathbf{3},\mathbf{2})$ of Eq. (15), we can also realize almost the same results of the lepton masses and mixing angles as the previous model. Since three generations of e_i^c have the same modular weight in this model, they can originate from a single field $\Phi(x,y)$ in a higher-dimensional theory. In this model, two modes have the same A_4 representation $\mathbf{1}'$ and the same weight -4. They may have different properties on boundary conditions in extra dimensions, e.g., Z_N twist eigenvalues. Alternatively, we assign three generations of e_i^c as $\mathbf{1}$ (weight -4), $\mathbf{1}'$ (weight -4), and $\mathbf{1}$ (weight -4).

These models are consistent with our scenario and can lead to realistic lepton masses and mixing angles. One of the important issues is to study their difference in particle phenomenology, i.e., how to distinguish these models. The first model in Table III and the second model have different modular weights for the matter fields with the representation $\{\mathbf{1},\mathbf{1}'\}$. Here, we give a comment on the phenomenological difference due to the modular weights.

Within the framework of supergravity theory, soft scalar masses m_i with the moduli-dependent Kähler metric $K_{i\bar{i}}$ are given as [128]

$$m_i = m_{3/2}^2 - \sum_X |F^X|^2 \partial_X \partial_{\bar{X}} \ln K_{i\bar{i}}, \tag{36}$$

when F-terms F^X of the moduli X develop their vacuum expectation values. Suppose that the F-term F^τ of the modulus τ develops its vacuum expectation value. Then, the Kähler metric in Eq. (25) leads to the soft masses [76]

$$m_i^2 = m_{3/2}^2 - k_i \frac{|F^\tau|^2}{(2\mathrm{Im}\tau)^2}. \tag{37}$$

The first model in Table III leads to degenerate soft masses in three generations of right-handed leptons as well as left-handed leptons. In the second model, the right-handed lepton with the representation $\mathbf{1}''$ has a modular weight different from the other. Thus, their slepton masses are not degenerate.

Similarly, in the third model, three generations of slepton masses are degenerate. We may have a difference between the first and third models in higher-dimensional operators in the SMEFT [96].

V. CONCLUSION

We have studied the scenario to derive 4D modular flavor symmetric models from a higher-dimensional theory. In our scenario, wave functions in extra dimensions are modular forms. That leads to the constraints on combinations between modular weights and Γ_N (Γ'_N) representations, which have not been considered in the bottom-up approach. As illustrating examples, we have shown explicit A_4 models, taking into account the constraints from a higher-dimensional theory. We have found that realistic results on lepton masses and mixing angles are realized. Our discussions can also be applied to the quark sector, though we do not discuss them here. We can extend them to other Γ_N models and their covering groups. Further studies along our scenario would be important to connect 4D flavor models with a higher-dimensional theory such as a superstring theory.

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APPENDIX: PROJECTION BY BOUNDARY CONDITION

Here, we show an example to project wave functions with Γ_N reducible representations to an irreducible one by imposing further boundary conditions.

Suppose that the following five wave functions satisfy the same zero-mode equations in some compactification with the complex coordinate $z = y_1 + \tau y_2$, which may have other dimensional coordinates,

$$\begin{aligned}
\chi_1(z,\tau) &\equiv (\psi^{0,2}(z,\tau))^4 + (\psi^{1,2}(z,\tau))^4, \\
\chi_2(z,\tau) &\equiv 2\sqrt{3}(\psi^{0,2}(z,\tau))^2(\psi^{1,2}(z,\tau))^2, \\
\chi_3(z,\tau) &\equiv (\psi^{0,2}(z,\tau))^4 - (\psi^{1,2}(z,\tau))^4, \\
\chi_4(z,\tau) &\equiv 2((\psi^{1,2}(z,\tau))^3\psi^{1,2}(z,\tau) + \psi^{0,2}(z,\tau)(\psi^{1,2}(z,\tau))^3), \\
\chi_5(z,\tau) &\equiv 2((\psi^{1,2}(z,\tau))^3\psi^{1,2}(z,\tau) - \psi^{0,2}(z,\tau)(\psi^{1,2}(z,\tau))^3),
\end{aligned} \tag{A1}$$

where

$$\psi^{j,M}(z,\tau) \equiv \left(\frac{M}{\mathcal{A}^2}\right)^{1/4} e^{\pi i M z \frac{\mathrm{Im} z}{\mathrm{Im}\tau}} \vartheta \begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix} (Mz, M\tau), \quad j \in \mathbb{Z}/M\mathbb{Z}, \tag{A2}$$

and ϑ denotes the Jacobi-theta function defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\nu, \tau) = \sum_{\ell \in \mathbb{Z}} e^{\pi i (a+\ell)\tau} e^{2\pi i (a+\ell)(\nu + b)}. \tag{A3}$$

These wave functions transform each other under the modular symmetry. Under the S transformation with $z \to -z/\tau$, these wave functions transform

$$\chi_i(z,\tau) \to (-\tau)^2 \rho(S)_{ij} \chi_j(z,\tau), \tag{A4}$$

where

$$\rho(S)_{ij} = \begin{pmatrix} \rho^{(1)}(S)_{ij} & 0 \\ 0 & \rho^{(2)}(S)_{ij} \end{pmatrix}, \tag{A5}$$

$$\rho^{(1)}(S) = -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad \rho^{(2)}(S) = -\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A6}$$

The above behavior implies that the above wave functions have modular weight 2. Under the T transformation, these wave functions transform

$$\chi_i(z,\tau) \to \rho(T)_{ij} \chi_j(z,\tau), \tag{A7}$$

where

$$\rho(T)_{ij} = \begin{pmatrix} \rho^{(1)}(T)_{ij} & 0 \\ 0 & \rho^{(2)}(T)_{ij} \end{pmatrix}, \tag{A8}$$

$$\rho^{(1)}(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho^{(2)}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}. \tag{A9}$$

$\rho(S)$ and $\rho(T)$ are representations of $\Gamma_4 \simeq S_4$. In particular, χ_i are reducible representations. (χ_1, χ_2) correspond to the doublet $\mathbf{2}$ of S_4, while (χ_3, χ_4, χ_5) correspond to the triplet $\mathbf{3}'$.

In addition to the above compactification, we impose further boundary conditions. We study the shifts of the coordinate

$$z \to z + (m + n\tau)/2, \quad (m,n) = (1,0), (0,1), (1,1). \tag{A10}$$

The modes (χ_1, χ_2) are invariant under all of these shifts. On the other hand, the modes (χ_3, χ_4, χ_5) transform

$$\chi_i \to e^{\pi i Q^i_{(m,n)}} \chi_i, \tag{A11}$$

where

$$Q^3_{(m,n)} = (0,1,1), \quad Q^4_{(m,n)} = (1,0,1), \quad Q^5_{(m,n)} = (1,1,0) \tag{A12}$$

for $(m,n) = (1,0), (0,1), (1,1)$, respectively. Thus, if we require the shift invariance of wave unctions, we can project the five wave functions χ_i with the representations $\mathbf{2} + \mathbf{3}'$ to the irreducible representation $\mathbf{2}$, (χ_1, χ_2). (For shift invariance, see Refs. [116,118,129].)

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