Timelike behavior of the pion electromagnetic form factor in the functional formalism

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The electromagnetic form factor of the pion is calculated within the use of functional formalism. We develop integral representation for the minimal set of Standard Model Green's functions and derive the dispersion relation for the form factor in the two-flavor QCD isospin limit $m_u = m_d$. We use the dressed quark propagator as obtained from the gap equation in Minkowski space and within the Dyson-Schwinger equations formalism to derive the approximate dispersion relation for the form factor for the first time. We evaluate the form factor for the spacelike as well as for the timelike momentum in the presented formalism. A new Nakanishi-like form of integral representation is proved on the basis of the vector Bethe-Salpeter equation for the quark-photon vector with a ladder-rainbow kernel. The gauge technique turns out to be a part of the entire structure of the vertex. In the analytic approach presented here, it is shown that a large amount of the ρ-meson peak in the cross section $e^+e^- \to \pi^+\pi^-$ is governed by the gauge invariance of QED/QCD—i.e., by a gauge-technique-constructed quark-photon vertex. This approximation naturally explains the broad shape of the ρ-meson peak.

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I. INTRODUCTION

The understanding of QCD as a quantum field theory would not be complete without mastering all domains—the perturbative calculation of high-energy processes—as well as by achieving success in nonperturbative evaluations of low-energy hadronic production processes. In the latter case, the timelike character of transferred momenta is an additional challenge. Developing a new nonperturbative technique opens new prospects in this last and not yet well theoretically mastered area. In this respect, the charged meson form factors and the transition meson form factors constitute rather precise data on the electromagnetic structure of the light meson. Simultaneously, both processes are simple enough for theoretical description based on quark and gluon degrees of freedom. The pion form factor $F(s)$ in a timelike region of momenta ($s = q^2 > 0$) carries nontrivial information in amplitudes for the production of the two lightest hadrons: a $\pi_+\pi_-$ pair. It has been measured with unprecedented accuracy (0.5% [1]) in the domain of the appearance of a striking resonant structure. Using a phenomenological description based on Breit-Wigner fits, the resonant structure corresponds to a 775 MeV heavy ρ meson, where very nearby at $s = 780$ MeV, a small ($\simeq$5%) admixture of very narrow ω resonance is also observed [2]. The physical neutral ρ, having charged partners also, is hence effectively considered as a neutral component of the vector isotriplet, while the physical ω is, within a good approximation, a strong isosinglet. Moreover, vector mesons could be practically untouched by the QCD non-Abelian anomaly, and both aforementioned light electrically neutral vectors could be practically identical in the isospin limit defined as $m_u = m_d$. However, light vector meson couplings to other mesons are quite different, providing their cross-section shapes differ very dramatically in various processes. We brought some new hints, supported by quantitative results, which offer the fact that various components of QCD/QED vertices contribute very differently in different processes. More concretely, facing the amount of the pion peak obtained, it is very likely that seven out of the eight components of transverse quark-photon vertices could contribute by only a limited amount, suppressed in the vicinity of the ρ-meson peak, and its shape is dictated by the gauge invariance more then we expected.

In a spacelike region $[F(t), t = -q^2; t > 0]$, the available data [3–7] are represented by a smooth decreasing curve for which the perturbative QCD prediction [8–12] reads

$$F_\pi(t) \to \frac{64\pi^2 f_\pi^2}{(11 - 2/3n_f)tL(t)} \times [1 + B_2 L_t^{-\frac{50}{581}} + B_4 L_t^{-\frac{364}{405}} + O(L_t^{-1})]^2, \quad (1.1)$$

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where $L_t = \ln(t/\Lambda^2_{\rm QCD})$ and the coefficients B_i are related to the nonperturbative part—the pion distribution and the light-cone Bethe-Salpeter wave function. The actual asymptotic predictions within today's available experimental range $Q^2 \simeq 100\ \mathrm{GeV}^2$ moves the validity of perturbative QCD predictions more toward the deep spacelike scale.

New interesting resonant structures—e.g., the deep dip at 1.5 GeV—and other heavier resonances have been found in the shape of F by using the initial photon state method in the *BABAR* 2012 experiment [7]. With increasing energy, a theory like vector meson dominance rapidly becomes a tautology of what is observed in the experiment: the experimental masses of ground-state and excited mesons become mass parameters of the theory, while the widths of resonances very much reflect the introduced effective couplings among various mesons. Chiral perturbation theory [13] has calculated the electromagnetic form factors near the threshold in various approximations [14–25], while the evaluation at higher energy, $Q > 0.5$ GeV, is out of convergence with the theory, and further phenomenological degrees of freedom need to be added in order to continue to higher energy [26]. The functional approach provides good results for spacelike mesonic form factors [27–37], where the approach connects all lengths naturally: it is nonperturbative at low Q^2 where QCD is strong, and it complies with perturbation theory at spacelike asymptotic region of momenta. Due to known limitations and obstacles, only a few studies [38–40] based on the quantum field theory functional formalism offer a result for the function F in the entire Minkowski space. A new approach [37] for the evaluation of F based on the integral representation of Bethe-Salpeter functions is employed at the level of the constituent quark model (in this approximation, the running of quark masses, as well as the momentum dependence of the quark renormalization function, is ignored), and the authors restrict themselves to only the spacelike argument of photon momenta. In the presented study, we follow similar lines as the authors in Ref. [37], but we take the momentum dependence in the quark propagator into account. Consequently, for the first time, we calculate the electromagnetic pion form factor in the entire Minkowski space.

The distinct shapes of ρ and ω resonances appearing in processes where they dominate [say, the former in the function $F(s)$ and the latter in the 3π production, for instance] are a known striking feature. The ρ meson is a broad resonance, while the ω peak is 20 times more narrow. The 70% contribution of the two-pion production cross section $\sigma(ee\to\pi\pi)$ to the muon anomalous magnetic momentum a_μ is an integral quantitative expression of the above statement. Single-pion or three-pion productions dominated by the exchange of ω (and not ρ) mesons in e^+e^- collisions contribute to a_μ by a remarkably smaller amount. To explain this, new terms with new couplings related with ρ-ω-π mixing are incorporated in the effective theories of QCD [26,41–44]. These new effective couplings ensure the tree-level decay of both mesons: the decay of $\rho\to\pi\pi$ happens at a point, while the decay of ω happens through the radiation of pions and the subsequent decay of virtual $\rho\to\pi\pi$, so ρ participates virtually, and its propagation slows the decay of the ω meson. Although less effective and more demanding in practice, it is also worthwhile to understand this origin from a microscopic explanation based on the quark and gluon degrees of freedom. An understanding of the detailed shape of ρ-meson resonance with no more than QCD Lagrangian parameters is certainly not equivalent to an empirical introduction of different phenomenological couplings between light vectors and pseudoscalars.

It is useful [12,19–25,45–48] to consider the pion form factor as the boundary value of an analytical function which has a cut on the timelike axis of the q^2 variable, which starts in the branch point $s_{\rm th} = q^2_{\rm th} = 4m^2_\pi$, the production threshold. Thus, the electromagnetic form factor $F_\pi(t)$ and the production form factor $F(s)$ can be evaluated from the dispersion relation for F:

$$F(q^2) = \int_0^\infty d\omega \frac{g(\omega)}{q^2-\omega+i\epsilon}, \tag{1.2}$$

with the unique spectral function g, which represents the imaginary part of $F(s)$ itself, $\Im F(s) = -\pi g(s)$, and which vanishes below $s_{\rm th}$, provided that $F(s)$ is the real function there. We will simply write $F_\pi(x)$ for any momentum, either for spacelike $x<0$ or for the timelike argument $x=s>0$, and for the Euclidean scalar product we have $q^2_E = -t$ in the convention used in this paper.

We present the technique, which within the use of quark and gluon degrees of freedom, leads to the form of the dispersion relation in Eq. (1.2). It does not use predetermined properties of vector mesons; they appear as a solution of Schwinger-Dyson equations for propagators and vertices. Vector meson masses are not an input anymore; furthermore, the ρ meson is not taken as a stable hadron—it has no associated real pole in the S-matrix, and therefore it does not come from the solution of the homogeneous bound state BSE at all. The function g in Eq. (1.2) is then given by a multidimensional integral over the spectral functions of quark propagators and Nakanishi weight functions for the Bethe-Salpeter pion vertex function, as well as over the weight functions which appear in the integral representation for the quark-photon vertex.

We will report on an exploratory study of the timelike pion electromagnetic form factor using integral representations (IRs) of QCD Green's functions, which are derived from nonperturbative truncation of QCD/QED Dyson-Schwinger equations (DSEs). The IR is introduced in Sec. IV, and the proof is relegated to the appendixes. Proposed IRs for vertices instantly offer the analytical continuation of Euclidean solutions for QCD Green's functions, as well as for hadronic form factors. In order to get the necessary functions for calculation of the pion form factor F, we use the solution of a combination of DSEs and the Bethe-Salpeter equation (BSE), which was employed recently for the purpose of calculation of the pion transition form factor [39] and hadron vacuum polarization [38]. Furthermore, we derive a formula for the form factor F in this limit and calculate the integral in Eq. (1.2) numerically.

II. THE ELECTROMAGNETIC PION FORM FACTOR FOR TIMELIKE ARGUMENTS AND THE MINIMAL SET OF EQUATIONS OF MOTION

The evaluation of the pion form factor is a typical quantum field theory problem which involves bound states as final or initial states. How to calculate such a transition in the BSE approach is generally known [49]. Since we deal with gauge theory, which has the additional approximate global symmetries, the Green's function we used as a building block should respect the vectorial as well as axial Ward identities as a constraint. In the case of electromagnetic form factors, the working expansion is known [29,50], and here we will consider only the first term, which defines the so-called (dressed) relativistic impulse approximation (RIA). This matrix element reads

$$\begin{aligned}\mathcal{J}^\mu(p,Q) &= eF_\pi(Q^2)p^\mu\\ &= \frac{2N_c}{3}ie\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}[G^\mu_{EM,u}(k+Q/2,k-Q/2)\Gamma_\pi(k_{r\pi_-},p+Q/2)S_d(k+p)\tilde{\Gamma}_\pi(k_{r\pi_+},Q/2-p)]\\ &\quad+\frac{2N_c}{3}ie\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}[G^\mu_{EM,u}(k-Q/2,k+Q/2)\Gamma_\pi(k_{r\pi_-},p+Q/2)S_d(k-p)\tilde{\Gamma}_\pi(k_{r\pi_+},Q/2-p)]\\ &\quad+\frac{N_c}{3}ie\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}[G^\mu_{EM,d}(k_+Q/2,k_-Q/2)\tilde{\Gamma}_\pi(k_{r\pi_-},Q/2+p)S_u(k+p)\Gamma_\pi(k_{r\pi_-},Q/2-p)+\cdots],\\ &\quad+\frac{N_c}{3}ie\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}[G^\mu_{EM,d}(k_+Q/2,k_-Q/2)\tilde{\Gamma}_\pi(k_{r\pi_-},Q/2+p)S_u(k+p)\Gamma_\pi(k_{r\pi_-},Q/2-p)+\cdots],\end{aligned}\tag{2.1}$$

where the expressions in the first (second) two lines correspond with diagrams where the photon with momentum Q couples to the up (down) quark with electric charge $2/3e$ $(1/3e)$. In Eq. (2.1), S_u stands for the up-quark propagator, Q is the photon momentum, and $\Gamma_\pi(a,b)$ is the pion vertex function, with $a(b)$ being the relative (total) momentum of the quark-antiquark pair. The second line represents the triangle diagram, which has the opposite circulation of momentum (compared to the first one, and we also flip the sign by taking $k\to k$). Although we write them explicitly here, it is not difficult to show they contribute equivalently, being individually proportional to the relative momentum of the pionic pair p and the pionic form factor F. Thus, up to the charge prefactor, there are four identical contributions in the isospin limit, for which the propagators of light quarks are equal by definition, $S_u=S_d$. All propagators and vertices are dressed. For a diagrammatic representation of the above, see for instance Ref. [37]. The bare BSE vertices are solutions of BSEs with the vertex function on the right-hand side of the BSE, being in fact identical to the BSE vertex in the approximation employed here.

The matrix G^μ_{EM} at each line in Eq. (2.1) is the quark-photon semiamputated vertex defined as

$$G^\mu_{EM,q}(k_-,k_+)=S_q(k_-)\Gamma^\mu_{EM,q}(k,Q)S_q(k_+),\tag{2.2}$$

where $k_\pm=k\pm Q/2$ stands for the momenta of fermionic lines, and where the proper vertex Γ^μ_{EM} is determined by its own inhomogeneous BSE, which reads

$$\Gamma^\mu_{EM}(k,P)=\gamma^\mu+i\int\frac{d^4l}{(2\pi)^4}S(l_+)\Gamma^\mu_{EM}(l,P)S(l_-)K(l,k,P),\tag{2.3}$$

where we have omitted the quark flavor q. Different flavor combinations enter various form factors of the meson, and since flavor is not mixed by our choice of interacting kernels K, we will always mean a single-quark flavor quark-photon vertex.

Thus, in order to evaluate the form factor in Eq. (2.1), one needs to know the quark propagator S, the pion Bethe-Salpeter vertex function Γ_π, as well as the quark-photon vertex [Eq. (2.3)]. In the isospin limit, the propagators of u and d as well as the quark-photon vertices of the u and d quarks are identical, and by applying charge conjugation, one can show that the second line, up to a different prefactor, which turns out to be $+1/3e$, is equal to the first one. The approximated set of equations for the pion vertex and the quark propagator we used here were obtained in Refs. [38,39] and will be described in the following section.

III. LIGHT QUARK PROPAGATORS AND THE PION VERTICES

To get the solution for the functions $S(k)$ and $\Gamma_\pi(k,P)$, we use the simultaneous solution of a DSE for the quark and a BSE for the pion, and thus we follow quite a common practice used in Refs. [38,51–66].

The BSE for the vertex function Γ_π reads

$$\Gamma_\pi(p,P) = i\int \frac{d^4k}{(2\pi)^4}\gamma_\mu S_q(k_+)\Gamma_\pi(k,P)S_q(k_-) \times \gamma_\nu\left[-g^{\mu\nu}V_g(q) - C_\Gamma \frac{q^\mu q^\nu}{(q^2)^2}\right], \quad (3.1)$$

where the momentum $q = k - p$, and we label $C_\Gamma = 4/3\xi g^2$, being thus identified with the longitudinal part of the gluon propagator in a class of linear covariant gauges. The first term should not be confused with the propagator at all, albeit its momentum dependence of the above kernel was chosen to mimic the so-called ladder-rainbow approximation with a one-gluon exchange and reads

$$V_g(q) = \int d\omega \frac{\rho_g(\omega)}{q^2-\omega+i\epsilon}, \quad \rho_g(\omega) = c_g[\delta(\omega - m_g^2) - \delta(\omega - m_L^2)], \quad (3.2)$$

and it is inspired by a solution of DSEs for the gluon DSE in the Landau gauge [67]. This model was found particularly useful for a relatively large region of nontrivial couplings C_Γ requiring a certain departure from the popular Landau gauge, which is our convenient strategy. Obviously, for nontrivial ξ, the effective kernel V_g is gauge-fixing dependent.

We found that the presence of nontrivial longitudinal modes improves the convergence of the solution in the form of the integral representation (IR) for the quark propagator. This IR reads

$$S(k) = \int_0^\infty dx \frac{\not{k}\rho_v(x) + \rho_s(x)}{k^2 - s + i\epsilon}, \quad (3.3)$$

where two functions ρ_v and ρ_s fully characterize the quark propagator.

Notably, the longitudinal part of the kernel is the only source of UV divergence in the presented model, which was removed by dimensional renormalization.

In Eq. (3.1), P is the total momentum of the meson satisfying $P^2 = M^2, M = 140$ MeV for the ground state, and the arguments in the quark propagator are $k_\pm = k \pm P/2$. The DSE/BSE system provides a precise solution for the quark propagator calculated in the gauge $C_\Gamma/(4\pi)^2 = 0.18$, and the kernel couplings (3.2) $c_g/(4\pi)^2 = -1.8$ and $m_g^2/m_L^2 = 2/7.5$, where m_g in physical units is $m_g = 556$ MeV.

The pion BSE vertex function $\Gamma_\pi(P,p)$ is composed from the four scalar functions:

$$\Gamma_\pi(P,p) = \gamma_5(\Gamma_E(P,p) + \not{p}\Gamma_F(P,p) + \not{P}\Gamma_G(P,p) + [\not{p},\not{P}]\Gamma_H(P,p)), \quad (3.4)$$

where all of them are used to determine the pion mass, and all of them contribute to the electromagnetic form factor. In our exploratory study presented here, we simplify and use only the first component formally.

IV. IRs DERIVED FROM DSEs AND THEIR USE IN CALCULATION OF THE FUNCTION F_π

A sort of Nakanishi IRs, originally developed for scalar theories [68], is slowly getting more use in nonperturbative settings of QCD [37–39,69,70]; needless to say, a certain controversy on existing actual analytical forms exists [71]. Independently of the detailed form of IRs for Green's functions in QCD and the Standard Model, their important property is their great role in the performance of analytical integration in momentum space.

IRs for Green's functions in quantum field theory play an important role, since they allow analytical integration. We perform momentum integration in Eq. (2.1) analytically by using the well-known formula for the Euclidean space momentum integral. For this purpose, we employ IRs for all functions needed; more concretely, we use the IR for the quark propagators in Eq. (3.3) with the solution for $\rho_{v,s}$ as obtained, for instance, in Ref. [38]. This is motivated by the following chiral limit Goldberger-Treiman-like identity:

$$\Gamma_E(0,p) = \frac{B(p)}{f_\pi}, \quad (4.1)$$

where the scalar function B appears in the inverse of the quark propagator:

$$S(p)^{-1} = \not{p}A(p) - B(p). \quad (4.2)$$

Hence, we use a simplified version of the BS vertex, which reads

$$\Gamma_\pi(p,P) = \gamma_5 \frac{1}{\mathcal{N}}\int_0^\infty do \frac{\rho_B(o)}{p^2 - o + i\epsilon} \quad (4.3)$$

and was used with $\mathcal{N}$ being the normalization factor satisfying approximately $\mathcal{N} = f_\pi$, with its exact value dictated by the canonical normalization of the BSE vertex.

The last missing ingredient is the quark-antiquark-photon vertex $\Gamma^\mu_{EM,f}$, for which we derive its own IR in Appendix A. The version for the semiamputated vertex [Eq. (2.2)] reads

$$G^{\mu}_{EM}(p_-,p_+)=\sum_{i=1}^{8}V_i^{\mu}T_i(p^2,p.Q,Q^2)+\int_0^{\infty}d\omega\int_{-1}^{1}dz\frac{\rho_v(\omega)[\not{p}_-\gamma^{\mu}\not{p}_++\omega\gamma^{\mu}]+\rho_s(\omega)[\not{p}_-\gamma^{\mu}+\gamma^{\mu}\not{p}_+]}{[p^2+p.Qz+Q^2/4-\omega+i\epsilon]^2}, \quad (4.4)$$

where, as we show in Appendix A, the second line is in fact equivalent to the gauge technique ansatz, and the first line completes the entire expression by adding all transverse components independently. The eight transverse components satisfy the condition of transversality $V.Q=0$, and their concrete form is a matter of convention. Their convenient representation can be chosen in the following way:

$$\begin{aligned} V_1^{\mu}&=\gamma_T^{\mu}, & V_5&=p_T^{\mu},\\ V_2^{\mu}&=p_T^{\mu}\not{p}, & V_6&=[\gamma_T^{\mu},\not{p}],\\ V_3^{\mu}&=p_T^{\mu}\not{Q}, & V_7&=[\gamma_T^{\mu},\not{Q}],\\ V_4^{\mu}&=\gamma_T^{\mu}[\not{Q},\not{p}], & V_8&=p_T^{\mu}\not{p}\not{Q}, \end{aligned} \quad (4.5)$$

and the associated scalar functions T_i satisfy the three-dimensional integral representation

$$T_i(p^2,p.Q,Q^2)=\int_0^{\infty}d\omega\int_1^{\infty}d\alpha \times\int_{-1}^{1}dz\frac{\rho_{i,[2]}(\omega,\alpha,z)}{[p^2+p.Qz+\frac{Q^2}{4}\alpha-\omega+i\epsilon]^2}. \quad (4.6)$$

We recall that all functions T_i are for a given gauge uniquely determined by the theory (by the solution of DSEs) through the solution for ρ_i. Also, note that somehow arbitrary momentum-dependent prefactors used elsewhere in decomposition [Eq. (4.4)] are not allowed here, unless they fulfill the herein proposed IR.

There exist obviously a set of equivalent choices, depending on which part of the transverse components is added to the term which is fixed by gauge covariance. Other definitions of IR are possible, and even the single longitudinal component $\gamma_{\mu}\to\frac{Q^{\mu}\not{Q}}{Q^2}$ can be used to express the IR, which is fixed by Ward identities. We do not know yet which choice is more advantageous from other perspectives—e.g., which is more suited for numerical solution. As shown in the Appendixes, we have chosen the gauge-technique-inspired form as a tribute to the first nonperturbative solution of DSE for the gauge vertex appearing in the literature [72–75]. Another advantage is that the gauge technique reduces the proper vertex to the γ matrix in the limit of vanishing gauge couplings (all of them in our case).

The value $N=2$ was chosen to derive the form of IR [Eq. (4.4)] from the DSE [Eq. (2.3)]. Since the DSE is the equation for the proper vertex, the appropriate IR for this vertex is derived in the first step. Only then is it shown that the derived IR for the proper function Γ^{μ}_{EM} is equivalent to the proposed IR [Eq. (4.4)] for the semiamputated vertex G^{μ}_{EM}.

A. Calculation of F

In order to get the form factor, we use a quite primitive, albeit not easy approach, and as we use the IRs for all vertices, we match their denominators by using Feynman parametrization. This allows the shifting of the loop integration momentum, and we integrate analytically in momentum space. After the momentum integration, the resulting integral involves nine dimensional integral over the variables of various IRs. The integrand is highly singular for $Q^2>0$, thus being not useful in its instant form that we arrive in after momentum integration. Hence, in order to reduce the number of numerical integrations, we use a further "gauge technique approximation," which reduces identical pairs of IR weight functions to the same number of single functions. Thus, for instance,

$$\int dadb\rho_v(a)\rho_v(b)\to\int da\rho_v(a), \quad (4.7)$$

with all b's replaced by a's in the integral kernel. It allows further integration over auxiliary Feynman variables, provided we are left with a five-dimensional integral at the end. The appropriate derivation is related in the Appendixes. Furthermore, since the weight of the BSE vertex function is

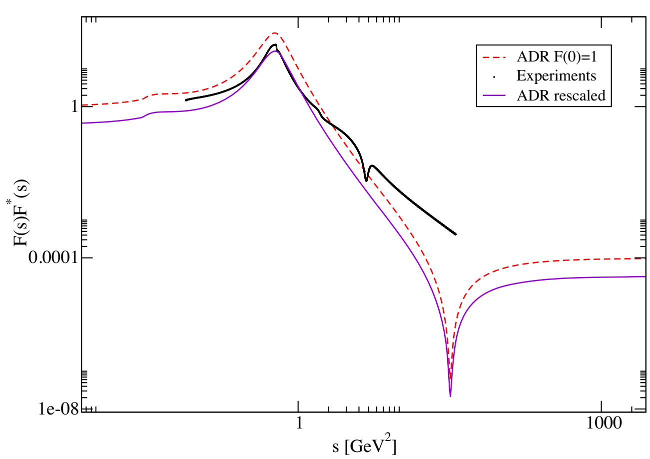

FIG. 1. Calculated magnitude of the pion electromagnetic form factor for $Q^2>0$ and comparison with experiments. The error bars are not shown; they are within the visible size of the line and are much smaller than the deviation of presented calculations. The solid line is rescaled by a constant, as described in the text.

much less known then the spectral function of the quarks, we ignore its details and use the integral reduction further for the purpose of numerical evaluation here. The numerical results are presented in Fig. 2 for spacelike momentum, where we compare with the experiment. The systematic error is estimated to be around a few percentage at a few GeV; however, adjusting $F(0) = 1$ is needed, as the proper renormalization does not lead to the correct value automatically.

Using some further approximations, we derive the dispersion relation (DR) [Eq. (1.2)] and provide the first estimate for the resulting spectral functions of the pion electromagnetic form factor. The result is valid for low momentum, and it consists of the following two terms:

$$F(q^2) = \int_0^\infty d\omega \frac{g_1(\omega)}{q^2 - \omega + i\epsilon} + q^2 \int_0^\infty d\omega \frac{g_2(\omega)}{q^2 - \omega + i\epsilon}, \tag{4.8}$$

where the first term could be responsible for correct normalization $F(0) = 1$ if no approximation (linearization) is made. The functions g_1 and g_2 are given by an expression involving only a single integration due to the approximation employed. The result is not exact, and it suffers from systematic error due to some ignored terms; however, it is enough to show that the form factor develops the ρ-meson peak. In fact, the gauge-technique-approximated vertex is enough to get almost the entire structure of the ρ-meson peak, and we ignore all other transverse vertices at this stage. In order to support this statement quantitatively, the dominant contribution to $F_\pi(Q^2)$ has been calculated numerically, and its square is compared with world averaged experimental data in Fig. 1. The averaged data of the *BABAR*, BESS, CMD/SND, and KLOE experiments [1] were fitted as described in Ref. [76], noting that there is negligibly small experimental error 0.5% on the ρ-meson peak.

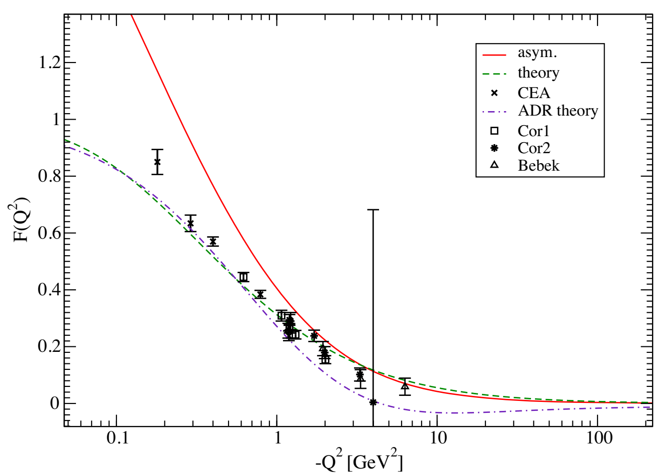

FIG. 2. Calculated pion electromagnetic form factor for $Q^2 < 0$ and comparison with experiment and asymptotic prediction. The line labeled by ADR stays for evaluation based on further approximations needed to evaluate the spectral function in the dispersion relation for F. For the asymptotic prediction (upper red line), we have chosen the function $F_{\rm asym}(t) = \frac{64\pi^2 f_\pi^2}{9tL_a}(1 + 0.1L_a^{-0.1})^2$, $L_a = \ln(e + t/\Lambda^2 {\rm QCD})$, $\Lambda_{\rm QCD} = 250$ MeV, which obviously has a correct asymptotic, given in Eq. (1.1). The sources for the experimental points are Ref. [77] for crosses, Ref. [78] for squares, Ref. [79] for stars, and Ref. [80] for triangles.

Within the used approximation, the derivation of the desired dispersion relation [Eq. (4.8)] is quite straightforward, albeit a bit lengthy, and it is delegated to Appendix C of this work.

The phase δ of the form factor $F = |F|e^{i\delta}$ is shown in Fig. 3. It overestimates the phase obtained by other methods, but it is still a satisfactory representative in our initial study. From the obtained phase, we estimate that the systematical error can be as much as 30%, which is caused by linearization and other cruel approximations we made. We assume the magnitude posses the same systematics and that it overestimates the experimentally measured magnitude F, if a naive condition $F(0) = 1$ were imposed on the approximated form factor. We lower F by a scale factor $\sqrt{3}$ for the purpose of better comparison. Hence, there are two lines representing the identical result obtained by our approximate dispersion relations: the dashed line corresponds to the standard normalization $F(0) = 1$, and the solid line represents the same calculated result, but shifted down due to rescaling. For higher Q^2, the result obtained from the dispersion relation becomes untrustworthy due to our pure approximation. Obviously, the first line is correct at zero momenta, while the second one reasonably approximates the peak. The difference is systematic error, which is quite large in the present example. This error suppression is an open task and remains a future challenge.

Furthermore, we use the approximated DR and evaluate the form factor in the spacelike domain as well. We recall

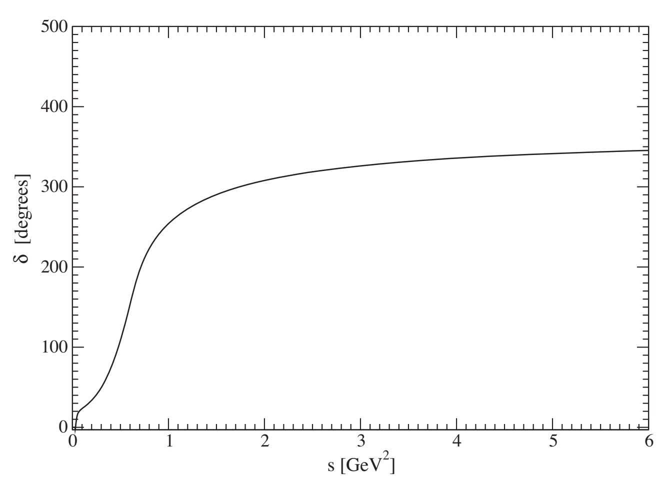

FIG. 3. Phase of the pion form factor F as obtained here.

that this approximation differs with the previous one, and we add this result to Fig. 2 for comparison.

Needless to say, the inclusion of transverse quark-photon form factors could improve the picture, and going beyond isospin approximation could leave some nontrivial imprints on the form factor shape. Some part of the systematical error could be due to this missing contribution; however, the missing off-peak contribution is difficult to explain as due only to the absence of transverse quark-photon components. To get rid of this uncertainty, the developed IR in the previous section could be used. We expect that the first interesting solution for vertices will be found in the next decade. Beyond our isospin approximation, further integrations (at least two) should appear in practice due to the necessity to use a more sophisticated, but unluckily also a more dimensional, IR [69,70] in order to evaluate the electromagnetic form factor in the nonsymmetric case.

We do not use IRs for the transverse vertices in the part devoted to the numerical study of F due to our simplified approximation. However, the revelation of their entire structure represents an important theoretical hint for future studies.

B. Renormalization within IRs

The use of the proposed IRs allows the dimensional regularization to be used to regularize four-dimensional momentum integrals when they show UV divergence.

Regarding the renormalization of the vertex, the only allowed UV infinities could be associated with γ matrix structure, since transverse form factors have no associated terms in the Lagrangian of the Standard Model. However, note that the second and the eighth transverse components in the list [Eq. (4.5)] can, in principle, spoil the renormalization properties for our (till now preferred) choice of power $N = 2$ in the denominator of the IR. Actually, such naive UV divergences appear, and it turns out that they cancel neither mutually nor against the UV term generated by the gauge technique.

One possibility to get rid of UV divergences from the beginning is that a more general N can be equivalently considered. Assuming a different N is allowed and describes the same form factor:

$$T_j(k^2, k.Q, Q^2) = \int_0^\infty d\omega \int_1^\infty d\alpha \times \int_{-1}^{1} dz \frac{\rho_{j,[N_j]}(\omega,\alpha,z)}{[k^2 + k.Qz + \frac{Q^2}{4}\alpha - \omega + i\epsilon]^{N_i}}. \tag{4.9}$$

Then Nakanishi's distributions ρ_N with a different integer parameter N are related through the following relation:

$$\rho_{T[N-1]}(\omega,\alpha,z) = \frac{-1}{(N-1)} \frac{d\rho_{T[N]}(\omega,\alpha,z)}{d\omega},$$
$$\rho_{T[N+1]}(\omega,\alpha,z) = -N \int_0^\omega do \rho_{T[N]}(o,\alpha,z), \tag{4.10}$$

where we assume that the Nakanishi weights vanish at boundaries.

Higher values of N are formally allowed; however, they would complicate the future evaluation of the hadronic form factor, so we stay with $N = 2$ here. To make our calculation meaningful for such a low N, we need to prevent this theory from unwanted UV divergences another way. For this purpose, one needs to impose the following sum rules:

$$\int_\Gamma d(\omega,\alpha,z)\rho_{2,[2]}(\omega,\alpha,z) = \int_{\Gamma_3} d(\omega,\alpha,z)\rho_{8,[2]}(\omega,\alpha,z) = 0, \tag{4.11}$$

for two weight functions of potentially dangerous transverse form factors.

In Eq. (4.11), we have introduced the abbreviation for the three-dimensional integration

$$\int_{\Gamma_3} d(\omega,\alpha,z) f \equiv \int_0^\infty d\omega \int_1^\infty d\alpha \int_{-1}^{1} dz f, \tag{4.12}$$

which will be used for the purpose of brevity.

Actually, the combination of T_2 components with the gauge term of the interaction kernel then produces UV divergence, which is proportional to γ_T^μ—i.e., to the first component of the proper vertex. Similarly, the eighth component, which is quadratic in the relative momentum p of the quark-antiquark pair ($\simeq p^\mu \not{p} \not{Q}$), provides linear divergence in the proper vertex. In the dimensional regularization scheme, it has the form

$$\frac{\xi g^2}{12\pi^2}(\gamma^\mu \not{Q} - Q^\mu)(\epsilon^{-1} + \text{finite}) \int_{\Gamma_3} d(\omega,\alpha,z)\rho_{8,[2]}(\omega,\alpha,z). \tag{4.13}$$

The derivation of the entire contribution to the quark-photon vertex due to the gauge interaction is shown in the Appendixes. Quite generally, within the condition (4.11), one makes our vertex DSE finite within all transverse components properly accounted for.

V. SUMMARY AND DISCUSSION

Our results are a strong hint that there exists a consistent integral representation of QCD Green's functions. If so, it is of great interest to explore the physical predictions or their use to calculate physical processes that were already experimentally measured, but were beyond theoretical capabilities due to the timelike character of momenta in the nonperturbative low-energy strong regime of QCD. We have already derived integral representations for the quark-photon gauge vertex showing it contains a part which is identical with the gauge technique. Within two approximations, we obtained the result for the pion form factor, yet without the inclusion of other transverse components of the vertex. The spectral function of the pion electromagnetic form factor has been obtained from the applications of integral representation to Dyson-Schwinger equations for the first time. Up to the norm, the form factor agrees with the experimental data at low Q^2 in both the spacelike as well as the timelike domain of momenta. To this point, let us mention that the gauge technique was used in the so-called spectral quark model studies in Refs. [81–83], wherein no further transverse vertices were needed to describe the broad shape of the ρ-meson peak in calculated electromagnetic pion form factors. Although the spectral model does not solve the equations of motion for propagators, nor does it use the lattice predictions for this purpose, nevertheless a prognostic feature of spectral models was that the simple vertex solely dictated by the Abelian gauge invariance could be enough for a gross description. In this paper, we extend the study of Ref. [81] in the sense that the quark propagators were calculated from the set of QCD DSEs, and within a certain ambiguity we confirm that the gauge technique is enough to provide the gross shape of the pion form factor.

Of course, deficiencies are due to the missing ω meson and due to the absence of an isospin-symmetry-violating contribution. Further shortcomings—e.g., the incorrect rate $F_\pi(0)/F_{pi}(m_\rho)$—appear due to the approximations—e.g., due to the linearization we have used at this stage. Also, the phase follows the Watson theorem very freely. In our case, we get $\delta = 250$ at 1 GeV, which overestimates the values of others ($\delta = 150$). Actually, the number of numerical integrations required for the evaluation of any hadronic form factor is the main weakness of the proposed method. It is not the nonperturbative evaluation of building blocks: QCD vertices and propagators where the calculations is stuck, but the evaluation of form factors, where a large number of entering Green's functions limit the evaluation. Further improvement of calculation technology—e.g., avoiding a cumbersome number of auxiliary Feynman integrations till now needed for the evaluation of hadronic form factors—is a great theoretical challenge for the future. Perhaps a possible generalization of old fashioned Cutkosky rules would be a promising theoretical direction to deal with the problem more efficiently.

We expect an improvement after the smooth and more realistic version of the kernel [Eq. (3.2)] is used. More improvements can be achieved when a correct weight function $\rho_\pi(a)$, or rather its two-dimensional form $\rho_\pi(a, z)$, of the pion BS vertex IR

$$\Gamma_E(p, P) = \gamma_5 \frac{1}{\mathcal{N}} \int_0^\infty do \int_{-1}^1 dz \times \frac{\rho_E(o, z, m_\pi)}{p^2 + p.Pz + m_\pi^2/4 - o + i\epsilon}, \tag{5.1}$$

is included.

To get the desired analytical form factor in the Minkowski space, recall that at least quark propagator needs to satisfy a standard two body dispersion—a generalized Källén-Lehmann representation, although the quark spectral function does need to be positive definitive function. Most importantly, no other singularities but the single cut is allowed. Nontrivially, here we achieve this goal by our choice of the quark-antiquark interaction kernel.

In this respect, for many other DSE/BSE studies presented in the literature [51–58,60,61,63,64,84], which are based on the popular version of the Maris-Tandy (MT) interaction kernel introduced in Ref. [85], the proof of the dispersion relation could be more complicated. And at least the derivation of desired dispersion relation used here would invalidate at very beginning.

Recall, due to the Gaussian kernel used in MTs, the interaction strength of the MT BSE kernel blows up at timelike infinity. This leads to the known behavior: the analytical continuation of quark propagators exhibits an infinite number of complex conjugated poles [55,59,62,86]. Such propagators are not analytic in the domain required for the existence of the IR [Eq. (3.3)], and one can repeat again, the derivation presented here would invalidate from the very beginning.

Our modeled DSE/BSE interacting kernel is certainly very primitive, but it includes the important ingredient: purely longitudinal interaction. While there should not be too much interesting physics contained in it, its numerical presence ensures that the ladder-rainbow approximation will work in the entire domain of Minkowski momentum space. Identifying a concrete numerical value of the gauge parameter requires further knowledge about other QCD vertices, which is out of model reach. However newly, in order to exhibit approximate gauge-fixing independence of the presented model, we have changed the gauge-fixing parameters (the entire coupling C_Γ). Thus, we solve the system numerically in a new gauge once again, determine a gauge-dependent coupling c_g, and in a new gauge we calculate the function F. In all cases, only this single parameter was varied in order to meet the pionic observables: the pion mass and pionic weak decay constant. The shape of the function F has been reproduced in several different gauges, showing that the model is actually the model of quantum gauge theory: the QCD. We plan to perform a similar study within an improved setup of BSE vertices.

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APPENDIX A: INTEGRAL REPRESENTATION FOR QUARK-PHOTON VERTICES

The form of the IR for the proper and semiamputated photon-quark vertex is derived in this appendix. Both integral representations are related, and the appropriate forms are derived as a self-consistent solution of the DSE for the vertex [Eq. (2.3)]. The reason to keep the IR for both the proper as well as for the semiamputated vertex is theoretical and practical. While the DSEs are more conveniently solved in terms of the proper Green's function, the hadronic form factors are more easily evaluated in terms of semiamputated vertices.

Using the labeling of momentum as described in the main text, the form of integral representation we are going to derive for the proper vertex reads as follows:

$$\Gamma^{\mu}_{\rm EM}(p,Q) = C\gamma_{\mu} + \Gamma^{\mu}_{EM,L}(p,Q) + \Gamma^{\mu}_{EM,T}(p,Q),$$
$$\Gamma^{\mu}_{EM,L}(p,Q) = \sum_{i=1}^{4} W^{\mu}_{i} L_i(p^2, p.Q, Q^2),$$
$$\Gamma^{\mu}_{EM,T}(p,Q) = \sum_{i=1}^{8} V^{\mu}_{i} T_i(p^2, p.Q, Q^2),$$
$$T_i(p^2,p.Q,Q^2) = \int_0^{\infty} d\omega \int_1^{\infty} d\alpha \int_{-1}^{1} dz \frac{\tau_{i,[1]}(\omega,\alpha,z)}{F(p,Q;\omega,\alpha,z)}, \tag{A1}$$

where

$$F(p,Q;\omega,\alpha,z) = p^2 + p.Qz + \frac{Q^2}{4}\alpha - \omega + i\epsilon \tag{A2}$$

for short, and the individual quark momenta associated with the quark legs are $p_{\pm} = p \pm Q/2$, which is the variable used to label the semiamputated vertex in the main text. Here, W_i are longitudinal matrices chosen as $1Q^{\mu}$, $Q^{\mu}\not{p}$, $Q^{\mu}\not{Q}$, and $Q^{\mu}(\not{p}\not{Q} - \not{Q}\not{p})$, respectively. The capital letter V_i stands for the transverse matrix satisfying $V.Q = 0$, and γ^{μ} has been taken out for calculation convenience. The IR for scalar form factors L_i satisfies exactly the same IR as the one for T_i, but with the distribution τ replaced by its own Nakanishi weight function, say λ.

The bracketed index "[1]" means that the first power of the denominator appearing in the last line in Eq. (A1) has been chosen, and if it is not different, the label will be omitted. T and L are scalar form factors, while V are for times-four matrices, wherein their Dirac index is not shown for brevity, and the unit (i.e., $\delta_{\alpha,\beta}$) in the case of the component V_5 will not be shown either. Recall that τ, λ are distributions; they may involve the product of smooth functions with delta functions.

1. IR based on DSEs and the relation with Nakanishi's PTIR

For pedagogical reasons, we mention the connection with the PTIR [68] and the IR used herein. First of all, let us recall here that the PTIR has been derived inductively by using perturbation theory from Feynman rules for scalar theories, and for various forms of PTIR we refer to Nakanishi's original textbook.

Since the form of IR for a Feynman diagram is dictated by the structure of denominators, it is very natural to assume that a sort of PTIR does exist for any renormalizable quantum field theory in $3+1$ dimensions. Furthermore, it is useful to assume (at least for a while) that the only difference is that there are as many various Nakanishi weight functions as the number of independent vector/tensor matrices needed to describe a given Feynman diagram. In our case of a triple fermion-gauge vertex, there can be as many as 12 such Nakanishi weight functions ρ_i. Thus, for each single component, the associated form factor T or L in 4.4 could satisfy the following PTIR:

$$T,L(p,Q) = \int_0^{\infty} d\omega \int dx_1 dx_2 dx_3 \times \frac{\rho(x_1,x_2,x_3,\omega)}{[p^2 x_1 + p.Qx_2 + Q^2 x_3 - \omega + i\epsilon]}. \tag{A3}$$

The polynomial structure and matrices which appear in the numerator of any Feynman diagrams for the gauge theory vertex are dictated by Lorentz invariance and are crucial for the number of components, but not for the number of integral variables. Three x variables are known to be bounded as the Nakanishi weight function carries the delta function $\delta(1-\sum_i x_i)$, which is almost the entirety of the information we can get from analyses of Feynman diagrams in general. Obviously, by dividing by x_1 in the kernel and defining a new variable, our proposed IR [Eq. (4.4)] is included in the PTIR-inspired form of the gauge vertex; however, this is far from saying that it is derivable from PTIR.

In this respect, one can only say that the form of Nakanishi weight functions—i.e., the 12 distributions of τ and λ—can be inspected from the perturbation theory expansion by studying each individual Feynman diagram in separation. There would be a very limited benefit of doing so in a strong coupling theory like QCD. Therefore, our proof of the IR [Eq. (A1)] does not follow from perturbation theory, but relies on the self-consistent solution of DSE within the suggested form of IR implemented. This, when embedded into the rhs of the DSE for the vertex [Eq. (2.3)], after the integration over the momentum in the Euclidean space, reappears on the lhs of the DSE again and has an exactly identical form that has entered—i.e., the form of IR [Eq. (A1)]. The set of weight functions ρ_i (or equivalently,

τ_i and λ_i) must obey certain conditions: they satisfy a new coupled set of integrodifferential equations into which the vertex DSE [Eq. (2.3)] is transformed. These equations do not depend on the momenta, but on the three spectral/integral variables ω, α, and z, with their domain self-consistently determined by the DSEs for vertices and propagators.

For clarity, we should mention here that the gauge technique form of the vertices [72–75], which was employed in the calculation of meson form factors [38,39,81], represents an approximate subset of IR [Eq. (4.4)]. While gauge technique vertices are derived Ward identities, they are not fully self-consistent, since they, at any known approximation of DSE, do generate richer structure involving longitudinal as well as transverse vertices. Their entire form is captured by three parametric IRs [Eq. (4.4)].

We do not know yet whether the new integrodifferential equations for Nakanishi weights provide a unique solution; however, we assume it is the case. We do not even know whether functions ρ_i exist at all, since the numerical solution is yet out of our reach at the moment. However, when keeping the solution at hand, as has been already checked in the case of more simple truncation of DSE systems [87,88], the consistency with the standard Euclidean formulation can be straightforwardly inspected by the comparison.

In the next subsection of this appendix, we will illustrate the proof on the example of the contribution stemming from the product of the gauge technique vertex and the gauge part of the propagator, as well as deriving the IR [Eq. (4.4)] for the particular example of the T_5 component. In the subsequent subsection, we write down the relation between IR for proper and semiamputated vertices, which closes the proof. We do not provide the entire list of all contributions, since we do not need them in our approximation. Note especially that the conversion of transverse pieces of G^{μ}_{EM} is a quite straightforward task and is illustrated enough in the single-component example.

2. IR for proper vertices

The IR has two pieces: the first is governed by gauge covariance, and the second involves all transverse components independently. Here we show that both terms give rise to an IR for the proper vertex with the same structure of transverse components, as well as giving rise to longitudinal components in Γ. Four later longitudinal terms are in fact completely dictated by the gauge term. We begin with transverse vertices due to the calculation simplicity. In the second part, we derive IR for Γ as it follows from our DSEs.

a. Contributions to and due to the transverse vertices

Contributions from transverse vertices are exemplified for the most important cases. These are the ones due to the first and the fifth components; the latter is known to be dominant, at least in the Landau gauge. Thus, in fact, here we show their contributions due to other gauges.

Further, for the purpose of discussion of the renormalization, we also review the contribution due to the second as well as due to the eighth component of the transverse part of the vertex.

V5 (g): We start with the contribution governed by the fifth component in Eq. (4.4)—i.e., by k_T, where the momentum k is the relative momentum of the produced quark-antiquark pair. The contribution due to the metric tensor $\gamma \times \gamma$ part of the kernel [and hence due to the $\gamma^{\mu}_{ab} \times \gamma_{\mu;cd}/(q^2-\mu^2)$ matrices] and the contribution due to the gauge part proceed similarly, and we will describe the details for the first example. The first contribution to the proper vertex in our DSE thus reads

$$-ic_g \int \frac{d^4k}{(2\pi)^4} \int_{\Gamma_3} d(\omega,\alpha,z) \frac{\rho_5(\omega,\alpha,z)\gamma_\nu (k^\mu - \frac{Q^\mu k.Q}{Q^2})\gamma^\nu}{[F(k,Q;\omega,\alpha,z)]^2 (q^2-\mu_g^2+i\epsilon)} - \cdots, \tag{A4}$$

where three dots represent the identical integral with the phenomenological parameter μ_g replaced as $\mu_g \to \Lambda_g$. Using the Feynman variable x to match two denominators, making a standard square completion, shifting the integral variable and integrating over the momentum, we get, after some algebra, for the considered contribution

$$\frac{4c_g}{(4\pi)^2} \int_{\Gamma_3} d(\omega,\alpha,z) \int_0^1 dx \frac{\rho_5(\omega,\alpha,z) p_T^\mu}{p^2 + p.Qz + \frac{Q^2}{4}\frac{\alpha - z^2 x}{1-x} - \frac{\omega}{1-x} - \frac{\mu_g^2}{x}} - \cdots, \tag{A5}$$

where we have factorized the prefactor $x(1-x)$ out of the numerator and canceled it against the same factor in the numerator, and where the meaning of the three dots is just as above in Eq. (A4). We do not write the Dirac index, and we also omit the explicit writing of the Feynman infinitesimal term $i\epsilon$ in most denominators for the purpose of brevity.

In what follows, we perform the substitution $\omega \to \tilde{\omega}$ and then $x \to \tilde{\alpha}$, such that

$$\tilde{\omega} = \frac{\omega}{1-x} + \frac{\mu_g^2}{x}, \tag{A6}$$

$$\tilde{\alpha} = \frac{\alpha - z^2 x}{1-x}, \tag{A7}$$

which provides the following result for the contribution of Eq. (A4):

$$p_T^\mu \frac{4c_g}{(4\pi)^2} \int_{-1}^1 dz \int_1^\infty d\alpha \int_\alpha^\infty d\tilde{\alpha} \int_{\frac{\mu_g^2}{x}}^\infty d\tilde{\omega} \times \frac{x(1-x)^2}{\tilde{\alpha}-\alpha} \frac{\rho_5[(\tilde{\omega} - \frac{\mu_g^2}{x})(1-x),\alpha,z]}{F(p,Q;\tilde{\omega},\tilde{\alpha},z)} - \cdots, \tag{A8}$$

where the ordering of integrals is important. To avoid complicated explicit notation, in case the measure dx is not explicitly written, the letter x will be kept for the following function:

$$x = \frac{\tilde{\alpha} - \alpha}{\tilde{\alpha} - z^2}. \tag{A9}$$

Also, as follows from the inverse transformation [Eq. (A7)], the variable $\omega_g = \omega$, which reads

$$\omega_g = \left(\tilde{\omega} - \frac{\mu_g^2}{x}\right)(1 - x), \tag{A10}$$

will be kept for purpose of brevity.

In order to get the desired form of IR for the proper vertex, we need to change the integration ordering. Also, it is convenient to send the information about integration volume into the kernel by using the Heaviside step function. Performing this correctly, one can write for the contribution of Eq. (A4) the resulting IR:

$$p_T^\mu \int_{\Gamma_3} d(\tilde{\omega}, \tilde{\alpha}, z) \frac{\tau_{5a}(\tilde{\omega}, \tilde{\alpha}, z)}{F(p, Q; \tilde{\omega}, \tilde{\alpha}, z)}$$
$$\tau_{5a}(\tilde{\omega}, \tilde{\alpha}, z) = \frac{4c_g}{(4\pi)^2} \int_1^{\tilde{\alpha}} d\alpha \frac{x(1-x)^2 \theta(\tilde{\omega} - \frac{\mu_g^2}{x})}{\tilde{\alpha} - \alpha} \rho_5[\omega_g, \alpha, z] - \cdots. \tag{A11}$$

where the three dots remind us that we should change μ_g into the parameter Λ appropriately—i.e., one should introduce a new variable

$$\omega_L = \left(\tilde{\omega} - \frac{\Lambda^2}{x}\right)(1 - x) \tag{A12}$$

to define the new variable ω_L in the function $\rho_5[\omega_L, \alpha, z]$.

Here we could stress the difference from the Nakanishi derivation of PTIR. Blindly following Nakanishi's derivation would mean the use of the variable x to give rise to our variable ω (or $\tilde{\omega}$), and we have used a slightly different strategy here. In our approach here, we avoid numerically inconvenient square roots otherwise presented in the kernel (see toy models without confinement [89,90]). Recall that the trick we use here would be impossible without using the fact that the quark propagator is entirely described by a continuous spectral function. In fact, this is the issue of confinement, which allows us to write the simple equation for IR.

V5 (L): We continue with the contribution coming from the transverse vertex V_5 matched with the gauge longitudinal interaction part of the kernel K. This particularly simple contribution reads

$$-iC_\Gamma \int \frac{d^4k}{(2\pi)^4} \int_{\Gamma_3} d(\omega, \alpha, z) \rho_5[\omega, \alpha, z] \frac{(k^\mu - \frac{Q^\mu k.Q}{Q^2})}{[F(k, Q; \omega, \alpha, z)]^2 q^2}. \tag{A13}$$

After a few steps sketched in previous cases, this relation can be converted into the following IR:

$$p_T^\mu \int_{\Gamma_3} d(\tilde{\omega}, \tilde{\alpha}, z) \frac{\tau_{5b}[\tilde{\omega}, \tilde{\alpha}, z]}{F(p, Q; \tilde{\omega}, \tilde{\alpha}, z)}$$
$$\tau_{5b}[\tilde{\omega}, \tilde{\alpha}, z] = \frac{C_\Gamma}{(4\pi)^2} \int_1^{\tilde{\alpha}} d\alpha \frac{\alpha - z^2}{(z^2 - \tilde{\alpha})^2} \rho_5\left[\frac{z^2 - \alpha}{z^2 - \tilde{\alpha}} \omega, \alpha, z\right]. \tag{A14}$$

Thus, for the resulting total contribution due to the fifth component, one just needs to sum up

$$\Delta\tau_5 = \tau_{5a} + \tau_{5b}. \tag{A15}$$

Amazingly, due to its simplicity, the IR exhibits a self-reproducing property: the contribution to the fifth component $\Delta\tau_5$ is given by the integral over the function ρ_5, and no other component is generated. However, the contribution is not complete, the contribution is not entire, and other components (e.g., $p^\mu \not{p}\not{Q}$) can contribute as well. Of course, one should keep in mind that ρ is the Nakanishi weight distribution for the semiamputated vertex, while τ is for the proper vertex. Hence, the relation between proper and semiamputated vertices needs to be established. This is the subject of the second part of this appendix. Before that, we review other important contributions.

The transformation of contributions from terms which involve combinations of momenta is straightforward, albeit quite involved. For the purpose of brevity, we write the results in the form of fractions which include the second power of F in the numerator, and we also leave polynomial momentum structure in the numerator (it can be absorbed by *per partes* integration, which is not shown here).

V1: In the next part, we will inspect the contribution which arises due to the first component of the transverse vertex. As with the others, it is combined with the gauge part, as well as with the metric phenomenological interaction in the DSE for the vertex. Explicitly written, the first contribution reads

$$-iC_\Gamma \int \frac{d^4k}{(2\pi)^4} \int_{\Gamma_3} d(\omega,\alpha,z) \rho_1[\omega,\alpha,z] \frac{\not{q}(\gamma^\mu - \frac{Q^\mu \not{Q}}{Q^2})\not{q}}{[F(k,Q;\omega,\alpha,z)]^2 (q^2)^2}, \tag{A16}$$

which after repeating similar steps as for the V_5 component above, leads, after some summations and trivial algebra, to the following form:

$$-\gamma_T^\mu \int_{\Gamma_3} d(\tilde{\omega},\tilde{\alpha},z) \int_1^{\tilde{\alpha}} d\alpha \frac{C_\Gamma}{(4\pi)^2} \frac{x(1-x)(2-x)}{(\tilde{\alpha}-\alpha)} \frac{\rho_1[\tilde{\omega}(1-x),\alpha,z]}{F(p,Q;\tilde{\omega},\tilde{\alpha},z)}$$
$$- p_T^\mu \int_{\Gamma_3} d(\tilde{\omega},\tilde{\alpha},z) \int_1^{\tilde{\alpha}} d\alpha \frac{C_\Gamma}{(4\pi)^2} \frac{\rho_1[\tilde{\omega},\alpha,z]}{[F(p,Q;\tilde{\omega},\tilde{\alpha},z)]^2} \frac{x^2(1-x)(\not{Q}z+2\not{p})}{(\tilde{\alpha}-\alpha)}, \quad \text{(A17)}$$

noting the second power of F in the second line.

Assuming the boundary condition $\rho_1(0,\alpha,z)=0$, we use *per partes* integration with respect to $\tilde{\omega}$, which increases the power of F by a unit. The final three-component IR of the desired form [Eq. (4.4)] then reads

$$\int_{\Gamma_3} d(\tilde{\omega},\tilde{\alpha},z) \frac{C_\Gamma}{(4\pi)^2 F(p,Q;\tilde{\omega},\tilde{\alpha},z)} \times \int_1^{\tilde{\alpha}} d\alpha \Bigg[-\gamma_T^\mu \frac{x(1-x)(2-x)\rho_1[\tilde{\omega}(1-x),\alpha,z]}{(\tilde{\alpha}-\alpha)} - p_T^\mu \frac{x^2(1-x)\frac{d}{d\tilde{\omega}}\rho_1(\tilde{\omega}(1-x),\alpha,z)](\not{Q}z+2\not{p})}{(\tilde{\alpha}-\alpha)} \Bigg]. \quad \text{(A18)}$$

V1 g: To calculate the contribution due to the interaction kernel with the metric tensor is more simple. The appropriate contribution reads

$$-iC_\Gamma \int \frac{d^4k}{(2\pi)^4} \int_{\Gamma_3} d(\omega,\alpha,z)\rho_1[\omega,\alpha,z] \times \frac{\gamma_\beta(\gamma^\mu - \frac{Q^\mu \not{Q}}{Q^2})\gamma^\beta}{[F(k,Q;\omega,\alpha,z)]^2(q^2-\mu_g^2)} - \cdots. \quad \text{(A19)}$$

Repeating basically the same steps which were used to transform the T_5 contribution, one gets for Eq. (A19) the following result:

$$-\gamma_T^\mu \int_{\Gamma_3} \frac{d(\tilde{\omega},\tilde{\alpha},z)}{F(p,Q;\tilde{\omega},\tilde{\alpha},z)} \frac{C_\Gamma}{(4\pi)^2} \times \int_1^{\tilde{\alpha}} d\alpha \frac{2x(1-x)\rho_1[\omega_g,\alpha,z]\theta(\tilde{\omega}-\frac{\mu_g^2}{x})}{\tilde{\alpha}-\alpha}, \quad \text{(A20)}$$

where the substitution of Eq. (A7) was made at the end of the transformation [x stands for the fraction in Eq. (A9), and ω_g is defined by Eq. (A10)].

V2: The contribution to the proper quark-photon vertex due to the second transverse component (i.e., $k_T^\mu \not{k}$) and due to the gauge interaction kernel takes the form

$$-iC_\Gamma \int \frac{d^4k}{(2\pi)^4} \int_{\Gamma_3} d(\omega,\alpha,z)\rho_2[\omega,\alpha,z] \times \frac{\not{q}(k^\mu - \frac{Q^\mu k.Q}{Q^2})\not{k}\not{q}}{[F(k,Q;\omega,\alpha,z)]^2(q^2)^2}, \quad \text{(A21)}$$

which can be readily transformed into the following form:

$$\gamma_T^\mu \frac{C_\Gamma}{4(4\pi)^2}(1/\epsilon_d - \gamma_E) \int_{\Gamma_3} d(\omega,\alpha,z)\rho_2[\omega,\alpha,z]$$
$$+ \gamma_T^\mu \frac{C_\Gamma}{2(4\pi)^2} \int_{\Gamma_3} d(\tilde{\omega},\tilde{\alpha},z) \int_1^{\tilde{\alpha}} d\alpha \frac{x^2(1-x)}{\tilde{\alpha}-\alpha} \frac{(R_2[\omega(1-x),\alpha,z] - (1-x)(Q.pz+2p^2)\rho_2[\tilde{\omega}(1-x),\alpha,z])}{F(p,Q;\tilde{\omega},\tilde{\alpha},z)}$$
$$- p_T^\mu \not{p} \frac{C_\Gamma}{(4\pi)^2} \int_{\Gamma_3} d(\tilde{\omega},\tilde{\alpha},z) \int_1^{\tilde{\alpha}} d\alpha \frac{x(1-x)^2(1+x)}{\tilde{\alpha}-\alpha} \frac{\rho_2[\omega(1-x),\alpha,z]}{F(p,Q;\tilde{\omega},\tilde{\alpha},z)}$$
$$+ 2p_T^\mu \frac{C_\Gamma}{(4\pi)^2} \int_{\Gamma_3} d(\tilde{\omega},\tilde{\alpha},z) \int_1^{\tilde{\alpha}} d\alpha \frac{x^2(1-x)^2}{\tilde{\alpha}-\alpha} \frac{\rho_2[\omega(1-x),\alpha,z](-Q.pz/2 - Q^2\tilde{\alpha}/4 + \tilde{\omega})(\not{Q}z/2+\not{p})}{[F(p,Q;\tilde{\omega},\tilde{\alpha},z)]^2}. \quad \text{(A22)}$$

We plan to publish the details of derivation and further useful relations in separate Supplemental Material.

The Euler constant γ_E, which arises in the equation above, is due to the standard dimensional regularization and should not be confused with projected gamma matrices. As we can see, the second component of T gives rise to nontrivial contributions to the four different components, including the second component itself. The elimination of momentum from the numerator and adjusting the power of F to the desired value N_2 is matter of standard practice. Furthermore, let us mention that due to the z dependence, many terms turn out to be zero, and the equation above is already in a form suited for the first calculation. Note also that there is a single term proportional to γ_T, which is divergent in the limit $d \to 4; (\epsilon_d \to 0)$.

b. Contributions due to the gauge technique vertex

In the following part, we will inspect the proper vertex contribution due to the gauge technique IR. As a part of the proof, we will show that the gauge technique is equivalent to the proposed IR for a semiamputated vertex. Then we will calculate its contribution to a proper vertex for its combination with the gauge—i.e., the purely longitudinal part of the interaction kernel. As is clear from the previous part devoted to the transverse components, deriving the IR due to the interaction with a metric tensor is a matter of practice, where several well-controlled changes in derivation cannot violate the resulting functional form of the IR.

Considering the DSE with aforementioned inputs on the rhs of the DSE [Eq. (2.3)] means to evaluate the following contribution:

$$-iC_\Gamma \int \frac{d^4k}{(2\pi)^4} \not{q} G^\mu_{GT}(k_-,k_+) \frac{\not{q}}{(q^2)^2}, \tag{A23}$$

where again we label $C_\Gamma = e_q g^2 \xi T_a^2$, and the momentum associated with the internal gluon line is $q = p - k$.

In the first step, we will show that the gauge technique vertex, in its conventional form

$$G^\mu_{GT}(k_-,k_+) = \int_{-\infty}^{\infty} dx \frac{\rho(x)\gamma^\mu}{[\not{k}_- - x + i\epsilon][\not{k}_+ - x + i\epsilon]} \tag{A24}$$

(see Refs. [72–75]), is identical to the second line of IR for a semiampuated vertex [Eq. (4.4)]. As we prefer to work with two quark propagator spectral functions ρ_v and ρ_s, we rewrite the above expression into the less familiar form

$$G^\mu_{GT}(k_-,k_+) = \int_0^\infty da \frac{\rho_v(a)[\not{k}_-\gamma^\mu\not{k}_+ + a\gamma^\mu] + \rho_s(a)[\not{k}_-\gamma^\mu + \gamma^\mu\not{k}_+]}{(k_-^2 - a + i\epsilon)(k_+^2 - a + i\epsilon)}, \tag{A25}$$

where two functions ρ_v and ρ_s are defined on R^+, and they are related with the single function ρ in the following manner:

$$\rho_v(a) = \frac{\rho(\sqrt{a}) + \rho(-\sqrt{a})}{2\sqrt{a}}, \quad \rho_s(a) = \frac{\rho(\sqrt{a}) - \rho(-\sqrt{a})}{2}. \tag{A26}$$

The advantage of our choice is that the function on the lhs takes a nontrivial value at the positive real axis, which simplifies many manipulations we will perform and show in this appendix. In addition, we use the identity

$$\frac{1}{k_-^2 - a}\frac{1}{k_+^2 - a} = \int_{-1}^{+1} dz \frac{1}{[k^2 + k.Qz + \frac{Q^2}{4} - a]^2} \tag{A27}$$

in order to match the denominators in Eq. (A25), getting thus the desired form that corresponds to the second line of the entire IR [Eq. (4.4)].

In this way, we have proved that the gauge technique is a part of the proposed IR in addition to converting G_{GT} into a form suited for other evaluation. Substituting Eq. (A25) into the formula (A23), we get at this stage

$$\sum_{v=a,b,c,d} \Gamma^\mu_v(k,P) = -iC_\Gamma \sum_{v=a,b,c,d} \int \frac{d^4k}{(2\pi)^4} \int_0^\infty da \int_{-1}^{+1} dz \times \frac{N^\mu_v(k,p,Q;a)}{[k^2 + k.Qz + \frac{Q^2}{4} - a]^2 (q^2)^2}, \tag{A28}$$

$$N^\mu_a = \rho_v(a)\not{q}(\not{k} - \not{Q}/2)\gamma^\mu(\not{k} + \not{Q}/2)\not{q}; \quad N^\mu_b = \rho_v(a)\not{q}\gamma^\mu\not{q}, \tag{A29}$$

$$N^\mu_c = 2\rho_s(a)k^\mu q^2; \qquad N^\mu_d = \not{q}[\gamma^\mu, \not{Q}]\not{q}. \tag{A30}$$

From this point, up to the different matrix structure of the numerator, the treatment of the rest is easy, as in the case of the previous study of the transverse contribution. Of course, the IR is two instead of three dimensional; thus, aside from a completely continuous part, one can expect the delta function when one uses three-dimensional write-up $[\delta(\alpha - 1)]$. Nevertheless, even so, the IR for the proper function turns out to be three-dimensional.

To derive the IR, we will use the variable y to match the result with q^2 in the denominator, which leads to the following result:

$$\Gamma^\mu_v(k,P) = -i \int \frac{d^4k}{(2\pi)^4} \int_0^\infty \int_{-1}^1 dz \int_0^1 dy \frac{3C_\Gamma y(1-y)N^\mu_v}{[(k + \frac{Q}{2}zy - p(1-y))^2 - (\frac{Q}{2}zy - p(1-y))^2 + p^2(1-y) + \frac{Q^2}{4}y - ay]^4} \tag{A31}$$

for each term defined in Eq. (A28).

The rest of the transformation is quite universal, and we will not repeat it for all terms individually, but we illustrate it for two scalar cases. The first is the integral where we replace the function N^μ_v with $N_1 = f(a)$, where $f(a)$ stands for some continuous functions. Let us label such an auxiliary scalar vertex by the index 1. Integrating over the momentum, we directly get

$$\Gamma_1(k,P) = \frac{C_\Gamma}{(4\pi)^2} \int_{-1}^1 dz \int_0^\infty da f(a) \int_0^1 dy \times \frac{[y(1-y)]^{-1}}{[p^2 + Q.pz + \frac{Q^2}{4}\frac{1-z^2y}{1-y} - \frac{a}{1-y}]^2}. \tag{A32}$$

For completeness, we repeat the entire transformation here.

Let us perform the substitution $y \to \alpha$ such that

$$\alpha = \frac{1 - z^2 y}{1 - y}, \qquad y = \frac{\alpha - 1}{\alpha - z^2}. \tag{A33}$$

And then, the second substitution $a \to \omega$, such that

$$\omega = \frac{a}{1 - y}, \tag{A34}$$

obtaining thus for Eq. (A32) the following expression:

$$\Gamma_1(k, P) = \frac{C_\Gamma}{(4\pi)^2} \int_{-1}^{1} dz \int_1^\infty d\alpha \int_0^\infty d\omega \times \frac{\frac{1-z^2}{(1-\alpha)(z^2-\alpha)} f[\omega \frac{1-z^2}{\alpha - z^2}]}{[p^2 + Q.pz + \frac{Q^2}{4}\alpha - \omega + i\epsilon]^2}. \tag{A35}$$

As in the previous part, here the variable y, if used without integral measure in any expression, will be kept even after the substitutions are performed for the purpose of brevity. Its meaning will be unique throughout this paper and is always given by the second equation in (A33).

The scalar function Γ_1 is not yet in the desired form, and for this purpose we perform *per partes* integration with respect to the variable ω. Doing this, we can write

$$\Gamma_1(k, P) = \int_{\Gamma_3} d(\omega, \alpha, z) \frac{\frac{C_\Gamma (1-z^2)}{(4\pi)^2 (1-\alpha)(\alpha - z^2)} \frac{d}{d\omega} f[\omega \frac{1-z^2}{\alpha - z^2}]}{F(p, Q; \omega, \alpha, z)}, \tag{A36}$$

where we have assumed the function f is vanishing at boundaries. Recall that within numerical accuracy, this is certainly true for the quark spectral function, and we will repeatably exploit the fact that $[\rho_{v,s}(0) = \rho_{v,s}(\infty) = 0]$.

The assiduous reader can note that there is an infrared log divergence involved in the α integral in Eq. (A36). These are standard IF divergences due to massless gauge boson modes, and actually, similar divergences appear in Γ^μ and make the associated form factor procedure dependent. If this appears numerically, it could be used to cancel against similar divergences due to the emission of soft real photons in the physical cross section. This fact, however, does not bother us yet, since we are not going to solve the DSEs system in this paper.

Such IF divergence does not appear for a less divergent kernel. Therefore, by replacing, for instance, the $1/q^4$ kernel in Eq. (A32) with $1/q^2$, one can get the IR in the following regular form:

$$\int_{\Gamma_3} d(\omega, \alpha, z) \frac{\frac{C_\Gamma}{(4\pi)^2} f[\omega \frac{1-z^2}{\alpha - z^2}]}{F(p, Q; \omega, \alpha, z)}. \tag{A37}$$

C: Repeating the game for our vertex in Eq. (A23) is relatively straightforward. The only complication is that one is faced with a larger number of momentum integrations over various tensors. Here we start with the simplest case, say the N_c^μ term in Eq. (A28). Taking changes into account, one gets two vector contributions: the first contributes to the transverse component V_5 (and by the same amount to the longitudinal counterpartner), and the second is purely longitudinal. Explicitly, it reads

$$\Gamma_c^\mu(p, Q) = \frac{C_\Gamma}{(4\pi)^2} \int_0^\infty d\omega \int_1^\infty d\alpha \int_{-1}^{1} dz \frac{(1 - z^2)}{(\alpha - z^2)^3} \times \rho_s \left[\omega \frac{1 - z^2}{\alpha - z^2}\right] \frac{[2p^\mu (1 - z^2) + Q^\mu z(1 - \alpha)]}{F(p, Q; \omega, \alpha, z)}. \tag{A38}$$

A: The conversion of Γ_a is technically the most demanding—not only does this piece involve UV divergence, but a double *per partes* integration is needed to convert this part into the desired IR. Hence, we will comment on some steps in more detail.

Here, the UV divergent terms, which stem from the first terms of the numerator expansions

$$\not{q}\not{Q}\gamma^\mu \not{Q}\not{q} = \gamma^\mu Q^2 q^2 + 4q^\mu Q.q\not{q} - 2Q^\mu q^2 \not{Q} - 2q^\mu Q^2 \not{Q},$$
$$\not{q}\not{k}\gamma^\mu \not{k}\not{q} = \gamma^\mu k^2 q^2 + \cdots \tag{A39}$$

will be concerned here. We will not list all IRs stemming from other contributions, which are relatively easy to evaluate; we will publish them when an actual numerical solution is available.

In order to see how individual terms arise during the derivation, we will write down a few intermediate steps. Summing the first terms in the expansions above, we get after Feynman parametrization [i.e., before the transformation in Eq. (A34)] the following result:

$$-iC_\Gamma \int da \rho_v(a) \int_0^1 dx \int_{-1}^{1} dz \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu [k^2 - \frac{Q^2}{4}] x \Gamma(3)}{[\tilde{k}^2 + p^2(1-x)x + \frac{Q^2}{4} x(1 - z^2 x) + p.Qzx(1-x) - ax]^3}, \tag{A40}$$

where $\tilde{k} = k + QZx/2 - p(1 - x)$, and we omit some prefactors for the purpose of brevity.

We will use the dimensional regularization; thus, we label $\epsilon^{-1} = 4 - d$ as the divergent constant in four dimensions. After the usual shift, the term proportional to $\tilde{k}^2$ gives

$$\int_0^1 dx \frac{2\gamma^\mu x}{(4\pi)^2}\left[-\frac{2}{\epsilon} - \gamma_E + \ln(1-x)x + \ln F\left(p, Q; \frac{a}{1-x}, \frac{1-z^2 x}{1-x}, z\right)\right], \tag{A41}$$

where we omit some unimportant prefactors.

After the substitution of Eq. (A34) (with x instead of y), we get the following entire expression:

$$\Gamma_c^\mu(p,Q) = \gamma^\mu \mathrm{Const} + \gamma^\mu \frac{C_\Gamma}{(4\pi)^2}\int_{\Gamma_3} d(\omega,\alpha,z)\frac{2y(1-y)(1-z^2)}{(z^2-\alpha)^2}\rho_v[\omega(1-y)]\ln[F(p,Q;\omega,\alpha,z)]$$
$$+\gamma^\mu \frac{C_\Gamma}{(4\pi)^2}\int_{\Gamma_3} d(\omega,\alpha,z)\frac{(1-z^2)}{(z^2-\alpha)^2}\rho_v[\omega(1-y)]\frac{\frac{Q^2}{4}(z^2y^2-1)+p^2(1-y)y - Q.pz(1-y)y}{F(p,Q;\omega,\alpha,z)}, \tag{A42}$$

where in order to avoid cluttering notation, we remind the reader here that the letter y is simply Eq. (A33) (since x is reserved for a different function in our notational convention). In this process, a constant term (UV divergent) γ^μ Const. appears, into which we also sent some constant pieces which have been generated during the derivation. It should be kept in mind that the entire vertex—i.e., the finite as well the infinite part—could be consistent with the renormalization of the quark DSE due to the WTI.

To transform the first line to the desired IR we use *per partes* integration with respect to the variable ω. For this purpose, we use the following expression for the primitive function in the numerator:

$$R_v[\omega,\alpha,z] = \int_0^\omega du \rho_v[u(1-y)]. \tag{A43}$$

Further, irrespective of the value of the boundary term, we send it into the constant term. The remainder of the first line then reads

$$\gamma_\mu\left[\mathrm{Const} + \int_{\Gamma_3} d(\omega,\alpha,z)\frac{\rho_{\log}(\omega,\alpha,z)}{F(p,Q;\omega,\alpha,z)}\right], \tag{A44}$$

where the contribution to the vertex weight function is

$$\rho_{\log}(\omega,\alpha,z) = 2C_\Gamma \frac{y(1-y)(1-z^2)}{(\alpha-z^2)^2}R_v[\omega,\alpha,z]. \tag{A45}$$

To convert the second line in Eq. (A42), one can divide the term with p^2 as the first step. Then the term in the numerator, which is proportional to the variable Q^2, can be treated by *per partes* integration to cancel it with the price; we get $\ln(J)$ instead of J^{-1}. In order to get J back in the denominator, one can integrate *per partes*, but now with respect to the variable ω. The terms involving the scalar product $p.Q$ in the numerator can be treated analogously, but instead of the variable α, one needs to use the variable z. The single resting term has already been derived in this form of IR. The entire result for the second line is then given by Eq. (A44), where instead of $\rho_{\log}$ we have the following function:

$$C_\Gamma \frac{d}{dz}\left[\frac{(1-z^2)}{(z^2-\alpha)^2}z(1-y)(2y-1)R_v[\omega,\alpha,z]\right]$$
$$+C_\Gamma \frac{d}{d\alpha}\left[\frac{(1-z^2)}{(z^2-\alpha)^2}[1-z^2y^2+\alpha(1-y)^2]R_v[\omega,\alpha,z]\right]$$
$$-C_\Gamma \omega \frac{(1-z^2)}{(z^2-\alpha)^2}(1-y)^2\rho_v[\omega(1-y)]. \tag{A46}$$

D: Repeating the game for the last numerator in Eq. (A28) gives us

$$\Gamma_d^\mu(p,P) = \int_{-1}^1 dz \int_1^\infty d\alpha \int_0^\infty d\omega \frac{C_\Gamma}{(4\pi)^2}$$
$$\times \frac{(1-\alpha)(1-z^2)}{(z^2-\alpha)^3}\frac{M^\mu \rho_s(\omega\frac{1-z^2}{\alpha-z^2})}{F(p,Q;\omega,\alpha,z)^2}$$
$$M^\mu = (\not p + \not Q/2)[\gamma_\mu, \not Q](\not p + \not Q/2), \tag{A47}$$

where we do not write the Dirac index for brevity.

Using the identity

$$M^\mu = p^2[\gamma^\mu, \not Q] + 2p^\mu[\not p, \not Q] + 2Q.p[\not p, \gamma^\mu] - \frac{z^2}{4}Q^2[\gamma^\mu, \not Q]$$
$$+ z(Q^\mu[\not Q, \not p] + Q^2[\not p, \gamma^\mu]), \tag{A48}$$

one can immediately recognize various components of the quark-photon vertex, and thus, for instance, the last line when implemented in Eq. (A47) gives $\simeq z(-1)Q^2 V_6$ in the numerator. The last step to get the desired IR is the *per partes* contribution with respect to the variable α, which lowers the power of F in the denominator and cancels out the unwanted presence of Q^2 in the numerator. To convert other terms of M into IR is a matter of simple algebra and the repeated use of *per partes* integration. The result, together with the numerical solutions, will be published in the future.

3. Integral representation for a semiamputated vertex

Using an accepted form of the semiamputated vertex (SAV), we have shown that the proper vertex Γ^μ satisfies the integral representation, which up to the power of the denominator has an identical form to the assumed form of the semiamputated vertex itself. What remains is to show that the IR for the SAV is consistent with the obtained IR for the proper vertex from the DSE solution.

The first power of F in the denominator of the IR is the preferable choice for this purpose, as it simplifies some parts of the calculation. However, recall that $N = 2$ was the preferred choice in the preceding sections. Here, we should therefore note that these two weight functions are simply related.

Thus, we are going to find a relation between the IRs of the left and right sides of the following definition:

$$G^{\mu}_{EM}(k^+, k^-) = S(k^-)\Gamma^{\mu}_{EM,T}(k, Q)S(k^+), \tag{A49}$$

with all functions on the rhs expressed through their own IR. By plugging the IR for the proper vertex, which we recall here as

$$\Gamma^{\mu}_{EM,T}(k, Q) = \sum_{i=1}^{8} V^{\mu}_i T_{\Gamma,i}(k^2, k.Q, Q^2),$$
$$T_{\Gamma,i}(k^2, k.Q, Q^2) = \int_0^\infty d\omega_\Gamma \int_0^\infty d\alpha \int_{-1}^{1} dz_\Gamma \times \frac{\tau_i(\omega, \alpha, z)}{[F(k, Q; \omega_\Gamma, \alpha, z)]}, \tag{A50}$$

together with spectral representations for the quark propagators S, into the rhs of the SAV definition,

$$G^{\mu}_{EM}(k^+, k^-) = S(k^-)\Gamma^{\mu}_{EM,T}(k, Q)S(k^+), \tag{A51}$$

we are prepared to convert the resulting expression,

$$\int_0^\infty dadbd\omega d\alpha \int_{-1}^{1} dz_\Gamma \frac{\not{k}_-\rho_v(a) + \rho_s(a)}{(k_-^2 - a + i\epsilon)} \times \frac{\tau^i_T(\omega_\Gamma, \alpha, z_\Gamma)V^{\mu}_i}{(k^2 + k.Qz_\Gamma + \frac{Q^2}{4}\alpha - \omega_\Gamma + i\epsilon)} \frac{(\not{k}_+\rho_v(b) + \rho_s(b))}{(k_+^2 - b + i\epsilon)}, \tag{A52}$$

into the form of the suggested IR [Eq. (4.4)] for the lhs of the definition of the SAV.

Let us first briefly describe the core of the proof. As a first step, we commute all V's from the middle position into the front, and we use the Feynman rules for denominators to match the propagators S and the proper vertex together. This gives us the IR with some additional presence of the scalar product of the momenta in the numerator. If a given term in the numerator belongs to T_i, we need only adjust a proper denominator $N = 2$. If there is additional momentum dependence in the prefactor, we use the *per partes* integration to remove it with a simultaneous change of power of the numerator. At the end, we adjust the power of the denominator to $N = 2$ by *per partes* integration with respect to the newly defined variable ω_{new}. From all of V, we choose the Dirac γ_T only; the other terms proceed similarly. In what follows, we will not write the Feynman $i\epsilon$; its presence is assumed implicitly.

Using the formula

$$\frac{1}{k_-^2 - a}\frac{1}{k_+^2 - b} = \int_1^{-1} dz_G \frac{1}{[k^2 + k.Qz_G + \frac{Q^2}{4} - \omega_G]^2},$$
$$\omega_G \equiv \frac{a}{2}(1 - z_G) + \frac{b}{2}(1 + z_G) \tag{A53}$$

and further matching with the denominator of the proper vertex in Eq. (A52)

$$\frac{1}{[k^2 + k.Qz_G + \frac{Q^2}{4} - \omega_G]^2}\frac{1}{[k^2 + k.Qz_\Gamma + \frac{Q^2}{4}\alpha - \omega_\Gamma]} = \int_0^1 dx \frac{2x}{[k^2 + k.Q(z_G x + z_\Gamma(1-x) + \frac{Q^2}{4}(x + \alpha(1-x)) - \omega_G x - \omega_\Gamma(1-x) + i\epsilon]^3}, \tag{A54}$$

one can write the result

$$G^{\mu}_{EM}(k, Q) = \int_0^\infty d\alpha dadb \int_0^1 dx \int_{-1}^{1} dz_\Gamma dz_G \frac{-2x[\rho_v(a)\rho_v(b)\gamma^\mu(Q^2/4 - k^2) + R^\mu]}{[k^2 + k.Q(z_G x + z_\Gamma(1-x) + \frac{Q^2}{4}(x + \alpha(1-x)) - \omega_G x - \omega_\Gamma(1-x)]^3},$$
$$R^\mu = (2k^\mu + Q^\mu)(\not{k} - \not{Q}/2) + \gamma^\mu/2[\not{k}, \not{Q}] + \rho_s(a)\rho_s(b)\gamma^\mu + \rho_v(a)\rho_s(b)(2k^\mu + Q^\mu) - \rho_v(a)\rho_s(b)\gamma^\mu)(\not{k} + \not{Q}/2) + \rho_s(a)\rho_v(b)\gamma^\mu)(\not{k} - \not{Q}/2). \tag{A55}$$

As a next step, we perform the following substitutions:

$$\tilde{\alpha} = x + \alpha(1-x),$$
$$\tilde{z} = z_G x + z_\Gamma(1-x),$$
$$\tilde{\omega} = \omega_G x + \omega_\Gamma(1-x), \tag{A56}$$

such that $\alpha \to \tilde{\alpha}$, $z_G \to \tilde{z}$, and $x \to \tilde{\omega}$; thus, we get the wanted form of the denominator. Doing this explicitly, we can write for Eq. (A52)

$$G^\mu_{\rm EM}(k,Q) = \int_0^\infty d\tilde{\omega} d\tilde{\alpha} \int_{-1}^1 d\tilde{z} \frac{I}{k^2 + k.Q\tilde{z} + \frac{Q^2}{4}\tilde{\alpha} - \tilde{\omega}},$$
$$I = \int_0^\infty da db d\omega_\Gamma \int_{-1}^1 dz_\Gamma \times \frac{-2x \prod \theta[\rho_v(a)\rho_v(b)\gamma^\mu(Q^2/4 - k^2) + R^\mu]}{(1-x)[\frac{a}{2}(1+z_\Gamma) + \frac{b}{2}(1 - z_\Gamma - \omega_\Gamma]}, \tag{A57}$$

where x is the solution of Eq. (A56)—i.e., it is a function $x(a, b, z_\Gamma, \tilde{z}, \tilde{\alpha}, \omega_\Gamma)$. We label $\prod \theta$ the product of step Heaviside functions which define the integration domain and straightforwardly stem from the transformation in Eq. (A56). They ensure that the numerator is zero in the boundaries of three integrals appearing in Eq. (A57). To get the form of IR with the desired power of the numerator, one only needs to employ *per partes* integrations.

APPENDIX B: EVALUATING THE PION FORM FACTOR WITHIN THE GAUGE TECHNIQUE APPROXIMATION

The function $F(Q^2)$ due to the gauge technique (GT) vertex in two approximations is derived in this appendix. Both are based on the GT-like linearization, which reduces the number of numerical integrations. Further approximation is made to arrive at the dispersion relation for the form factor $F(Q)$. Both approximations split at the very end. To begin, we substitute the IR for propagators and vertices, and by changing the ordering of integrations, we perform momentum integrations following standard procedures known from perturbation theory.

Thus, after the Feynman parametrization, we can perform integration over the momentum exactly in a way known for the evaluation of Feynman integrals in perturbation theory. As we are not in perturbation theory, we remain with a number of integrals over the weight functions of all integral representations, as well as all those with three new auxiliary integrals. For the latter, we use the Feynman variable x to mach the denominators of two IRs for the BSE vertices; then we use the variable y to match the result with the denominator of the quark propagator, which connects these two vertices, and at the end we will use the variable t in order to match the result of previous matching with the denominator of the IR for the quark-photon vertex.

The above described steps [for the entire matrix element $\mathcal{J}^\mu(Q|GT)$] read explicitly

$$\mathcal{J}^\mu(Q) = -i2N_c \int \frac{d^4k}{(2\pi)^4} \int_s \frac{\Gamma(5) y t^2 (1-t) U^\mu}{[(k+l)^2 + J]^5}, \tag{B1}$$

$$U^\mu = 4\rho_v(\gamma)\rho_v(\omega)\left[\left(\omega - k^2 + \frac{Q^2}{4}\right)(p^\mu - k^\nu) + 2k^\mu(k.p - k^2) + \frac{1}{2}Q^\mu k.Q\right] 8\rho_s(\gamma)\rho_s(\omega)k^\mu,$$
$$J = -l^2 + \frac{Q^2}{4}(1-t) - \omega(1-t) + p^2(1-y)t - \gamma(1-y)t + \left[\frac{p^2}{4} + \frac{Q^2}{16} - ax - b(1-x)\right] yt,$$
$$l^2 = \frac{Q^2}{4} o^2 + p^2\left(-1 + \frac{y}{2}\right)^2 t^2,$$
$$o = z(1-t) - \frac{1-2x}{2} yt, \tag{B2}$$

where we have used some shorthand notations: mainly, we have also factorized the weight functions ρ_π of the pion vertex functions into the overall measure—for this, we use the abbreviation

$$\int_s = \int_0^1 dx dy dt \int_0^\infty d\omega \int_{-1}^1 dz \int da \int db \rho_\pi(a)\rho_\pi(b), \tag{B3}$$

but we omit all trivial terms which were proportional to the product of external momenta $Q.p = 0$. However, we keep the p^2 variable for the purpose of easier tracking of the presented derivation. We will use the fact that pions are on shell—i.e., the equation $p^2 = m_\pi^2 - Q^2/4$ from the following lines.

For the purpose of integration over the momentum, we perform the standard shift $k \to k - l$, where $l = \frac{Q}{2} o + p(-1 + y/2)t$, with the polynomial function o defined by Eq. (B2):

$$o = z(1-t) + \frac{1-x}{2} yt, \tag{B4}$$

which after the integration over the momentum provides the nontrivial part of our matrix element in the form

$$\mathcal{J}^\mu(p,Q) = F(Q^2) p^\mu, \tag{B5}$$

where the pion form factor F is proportional to the following expression:

$$F(Q^2) = \frac{2N_c}{(4\pi)^2)} \int_s yt^2(1-t) \frac{4\rho_v(\gamma)\rho_v(\omega)}{(4\pi)^2 J^3} \times \left[(1-f)\omega + f\left(-\frac{Q^2}{4}(1+o^2) + 2p^2 f - p^2 f^2 \right) \right] + \frac{4\rho_s(\gamma)\rho_s(\omega)}{(4\pi)^2 J^2} + \frac{4\rho_v(\gamma)\rho_v(\omega)}{(4\pi)^2 J^3}(1-3f), \tag{B6}$$

where we have labeled

$$f = \left(1 - \frac{y}{2}\right)t. \tag{B7}$$

The individual prefactors in Eq. (B6) follow from the standard evaluation performed in Euclidean space, although we come back to the Minkowski metric convention immediately. We just remind the reader with the example

$$-i \int \frac{d^4k}{(2\pi)^4} \frac{\Gamma(5)k^2}{(k^2+J)^5} = \frac{2}{(4\pi)^2 J^2}. \tag{B8}$$

Note here that for purpose of consistency, the variable p^2 was also kept Euclidean for a while, and the on-shell condition $p^2 = m_\pi^2 - Q^2/4$ is imposed only afterward.

For positive timelike Q^2, the real part of the denominator J passes zero value, and albeit not written explicitly, the presence of an infinitesimal Feynman imaginary part is assumed.

In what follows, it is convenient to split the denominator J, getting

$$J = \frac{Q^2}{4}\Box - \Delta,$$
$$\Delta = m_\pi^2\left[\left(1 - \frac{3}{4}y\right)t - f^2\right] - \omega(1-t) - \gamma(1-y)t - (ax + b(1-x))yt,$$
$$\Box = -o^2 + 1 - t - (1-y)t + f^2. \tag{B9}$$

In Eq. (B6), we do not write trivial terms, including also those, which are proportional linearly to the variable o. These terms are zero, as can be inspected by the substitutions $z \to -z$ and $x \to 1-x$ with simultaneous interchange of the pion spectral function arguments $a \to b$. In this way, one gets the identical expression for the appropriate contributions to the form factor, but with opposite sign; hence, it is zero. This, together with the on-shell condition $p.Q = 0$, causes the term proportional to the total momentum Q^μ to be absent for each diagram individually, and the matrix element has an identical Lorentz structure to a charged pointlike scalar particle.

Before evaluating singular and hence more complicated Minkowski expressions, we derive the formula suited for the numerical integrations for spacelike momentum Q. For this purpose, we integrate over the variable z analytically. To proceed, furthermore, we use the "gauge technique" trick again and make linearization in ρ_π, which allows us to reduce the number of integrations further. As a consequence, $xa + (1-x)b \to b$, and the integration over the variable x can be done analytically in closed form. For the purpose of numeric integration, the same is done for the product of the quark spectral function integrals, where after matching by the virtue of gauge technique linearization, we make linearization in ρ_v (ρ_s) such that $\omega(1-t) - \gamma(1-y)t \to \gamma(1-yt)$. In what follows, we will write $\tilde{\gamma} = \gamma(1-yt) + byt$.

There are only two necessary integrals for evaluation of the function F for the spacelike momentum $Q_E^2 = -Q^2$. The first we show here:

$$\int_{-1}^{1} dz \int_0^1 dx J^{-3} = \frac{\theta(a)}{\frac{Q_E^2}{4}(1-t)ty} D_x\left[\frac{1}{4a(x^2+a)} - \frac{3x\arctan[\frac{x}{\sqrt{a}}]}{4a^{5/2}}\right] + \frac{\theta(-a)}{\frac{Q_E^2}{4}(1-t)ty} D_x\left[\frac{1}{4a(x^2+a)} - \frac{3x\tanh^{-1}[\frac{x}{\sqrt{-a}}]}{4(-a)^{5/2}}\right], \tag{B10}$$

with the function a_d defined as

$$a_d = -\frac{Q_E^2}{4}[(1-t) - (1-y)t + f^2] + (1 - 3/4y - f^2)m_\pi^2 - \tilde{\gamma}. \tag{B11}$$

The second required integral reads

$$\int_{-1}^{1} dz \int_0^1 dx J^{-2} = \frac{-\theta(a)}{\frac{Q_E^2}{4}(1-t)ty} D_x\left[x\frac{\arctan[\frac{x}{\sqrt{a}}]}{a^{3/2}}\right] - \frac{\theta(-a)}{\frac{Q_E^2}{4}(1-t)ty} D_x\left[x\frac{\tanh^{-1}[\frac{x}{\sqrt{-a}}]}{(-a)^{3/2}}\right], \tag{B12}$$

where we have introduced the abbreviations

$$D_x[h(x)] = h(x_u) - h(x_d),$$
$$x_u = Q_E/2[1 - t - yt/2],$$
$$x_d = Q_E/2[1 - t + yt/2] \tag{B13}$$

for some function $[h(x)]$.

Using the above integrals in Eq. (B6) constitutes the final expression that we have used for the numerical evaluation for the spacelike value of Q^2.

APPENDIX C: DERIVATION OF THE DISPERSION RELATION

The final expression for F based on the formulas derived above is still not yet in a form suited for numerical evaluation in the region of Minkowski momentum $Q^2 > 0$. Recalling the presence of the small Feynman factor $i\epsilon$, the log in inverse hyperbolical tangents, as well as the function a, is badly singular near the real axis of momentum Q, and the expression is numerically ill. Since complete analytical integration is still out of our reach, we make a further simplification. For this purpose, we go back into the expression and ignore the presence of the o^2 term in the integrand, which allows the conversion of all terms into the desired dispersion relation. We show the derivation for most singular $1/J^3$ term in Eq. (B6); the conversion of other terms is straightforward within the method used.

Ignoring o^2 terms, as well as ignoring small terms proportional to m_π, we can integrate over the variables x and z. The result simply means to replace the integration symbols $\int dxdz$ with a factor 2. The remaining relevant integral we need to evaluate reads

$$\int_0^1 dt \int_0^1 dy \frac{(1-y/2)^2 y t^2 (1-t)}{J^3}, \qquad \text{(C1)}$$

where the denominator reduces as

$$J = \Box \frac{Q^2}{4} - \tilde{\gamma}. \qquad \text{(C2)}$$

To proceed further, we perform the last linearization

$$\int d\gamma \rho_{v,s}(\gamma) \int db \rho_\pi(b) \to \int d\gamma \tilde{\rho}_{v,s}(\gamma), \qquad \text{(C3)}$$

where we assume that new functions $\tilde{\rho}$ on the rhs of Eq. (C3) are such that the resulting form factor F remains unchanged when taking $\tilde{\gamma} \to \gamma$ in the denominator—i.e., from now,

$$J = \Box \frac{Q^2}{4} - \gamma,$$
$$\Box = 1 - t - (1-y)t + (1-y/2)^2 t^2, \qquad \text{(C4)}$$

and we also assume $\tilde{\rho} \simeq \rho + \delta_\rho$, with the function δ_ρ representing corrections.

In addition, we introduce the unit in the form

$$1 = \int_0^\infty d\alpha \delta(\alpha - \gamma/\Box) \qquad \text{(C5)}$$

into the expression (C1) and integrate over the variable t. After that we get for Eq. (C1) the following expression:

$$\int_0^1 dy \int_0^\infty d\alpha \frac{\alpha}{\gamma^2} \frac{y t_-^2 (1-t_-)\theta(t_-)\theta(1-t_-)}{2(1-(1-y/2)t_-)[\frac{Q^2}{4} - \alpha + i\epsilon]^3}, \qquad \text{(C6)}$$

where t_- is the root of the equation $\Box\alpha - \gamma = 0$. Explicitly, it reads

$$t_- = \frac{1-\sqrt{\gamma/\alpha}}{1-y/2}, \qquad \text{(C7)}$$

noting that since $\alpha > 0$, the step function can be equivalently taken as $\theta(\alpha-\gamma)\theta(4\frac{\gamma}{y^2} - \alpha)$. Note that the contribution from the second root $t_+ = 1 + /\ldots$ is trivial, since $t_+ > 1$, being thus always outside of the interval for the original integral variable y.

Let us change the ordering of the integrations and integrate over the variable y. Theta functions presented in the kernel imply

$$\int_0^1 dy \int_0^\infty d\alpha \to \int_\gamma^\infty d\alpha \int_0^{2\sqrt{\frac{\gamma}{\alpha}}} dy. \qquad \text{(C8)}$$

After the integration, we get

$$\int_0^\infty d\alpha \frac{4}{\gamma} \frac{\theta(1-\sqrt{\frac{\gamma}{\alpha}})\theta(\alpha-\gamma)}{[\frac{Q^2}{4} - \alpha + i\epsilon]^3}$$
$$\times \left[-2 - \left(2\sqrt{\frac{\alpha}{\gamma}} - 1\right) \ln\left(1 - \sqrt{\frac{\gamma}{\alpha}}\right)\right] \left(\sqrt{\frac{\alpha}{\gamma}} - 1\right)^2. \qquad \text{(C9)}$$

After that, we perform double *per partes* integration with respect to the variable α, such that we get the desired dispersion relation

$$F(Q^2) = \int_0^\infty d\alpha \frac{g(\alpha)}{[\frac{Q^2}{4} - \alpha + i\epsilon]},$$
$$g(\alpha) = \frac{4N_c}{(4\pi^2)} \int_0^\alpha d\gamma \frac{2\hat{\rho}_v(\gamma)}{\gamma} K(\alpha,\gamma) + \cdots,$$
$$K(\alpha,\gamma) = \frac{BA^2 + A + B - 1/2}{2\alpha^{3/2}\gamma^{1/2}} - \frac{2 + A.B}{\alpha\gamma} - \frac{B}{\alpha^{1/2}\gamma^{3/2}} + \frac{1}{2\alpha^2}, \qquad \text{(C10)}$$

where

$$A = 1 - \sqrt{\frac{\alpha}{\gamma}}, \qquad B = \ln\left(1 - \sqrt{\frac{\gamma}{\alpha}}\right), \qquad \text{(C11)}$$

and where the dots represent remaining and not shown contributions (stemming also from the integration over the function J^{-2}; we found that these terms can be safely neglected in the approximation employed here).

Our approximation leads to some systematical error: it smoothly overestimates the form factor at medium timelike Q^2, and the dispersion relation does not provide correct form factor for $|Q^2| > 2$ GeV^2; hence, we call the form factor calculated on the relation (C10) the approximated dispersion relation (ADR) result. Nevertheless, it offers reasonable comparison with the approximation derived in the previous section. Hence, we guess that our ADR does not cripple the function F below 1 GeV too much, keeping the shape of ρ-meson resonance not distorted much.

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