$\mathcal{N}=4$ supersymmetric linear $W_\infty[\lambda]$ algebra

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(Received 10 May 2022; accepted 22 June 2022; published 21 July 2022)

From the recently known $\mathcal{N}=2$ supersymmetric linear $W_\infty^{K,K}[\lambda]$ algebra where K is the dimension of the fundamental (or antifundamental) representation of the bifundamental $\beta\gamma$ and bc ghost system, we determine its $\mathcal{N}=4$ supersymmetric enhancement at $K=2$. We construct the $\mathcal{N}=4$ stress energy tensor, the first $\mathcal{N}=4$ multiplet, and their operator product expansions (OPEs) in terms of the above bifundamentals. We show that the OPEs between the first $\mathcal{N}=4$ multiplet and itself are the same as the corresponding ones in the $\mathcal{N}=4$ coset $\frac{SU(N+2)}{SU(N)}$ model under the large (N,k) 't Hooft–like limit with fixed $\lambda_{co}\equiv\frac{(N+1)}{(k+N+2)}$, up to two central terms. The two parameters are related to each other, $\lambda=\frac{1}{2}\lambda_{co}$. We also provide other OPEs by considering the second, the third, and the fourth $\mathcal{N}=4$ multiplets in the $\mathcal{N}=4$ supersymmetric linear $W_\infty[\lambda]$ algebra.

DOI: 10.1103/PhysRevD.106.026008

I. INTRODUCTION

The free field construction in two-dimensional conformal field theory is useful to study the extension of the conformal symmetries in string theory. Because their operator product expansions (OPEs) take the simple form in the sense that the right-hand sides of OPEs do not contain the fields, contrary to the affine Kac-Moody algebra, it is straightforward to determine the conserved currents of any (conformal) weights in terms of the quadratic free fields with multiple derivatives. To describe the supersymmetric theory, the fermionic free field is necessary to describe the symmetries as well as the bosonic free field. Depending on the weights of the bosonic and fermionic fields, the weights of the currents we can make by using them are determined naturally from a simple counting of weights. The central charge of the Virasoro algebra consisting of the stress energy tensor of weight-2 is fixed by the number of (bosonic and fermionic) free fields. Usually, the bosonic field has the weight-1 (or zero) while the fermionic field has the weight-$\frac{1}{2}$.

More generally, the above weights of the bosonic and fermionic fields can be deformed by a parameter λ [1]. Although the weights of each bosonic and fermionic field depend on this λ parameter explicitly, due to the plus and minus signs in the coefficients in front of λ of the weights, the weights of particular composite combinations of these free fields do not contain the parameter λ. Then we can construct the currents of integer (or half integer) weights in terms of free fields as mentioned before by considering that the weights of each term should not depend on the λ. Of course, the λ dependence in the coefficients in front of free fields in the expression of the currents occurs in a very nontrivial way [2,3]. This is a new feature because the structure constants of the resulting algebra contain the λ dependence explicitly, compared to the ones in the previous paragraph.

So far, we have two bosonic and two fermionic free fields. There exist two fundamental OPEs between them. We can introduce the multiple bosonic and fermionic fields which transform as bifundamentals. Because they are independent fields and the multiple defining OPEs satisfy independently, all the previous analysis can be generalized to describe the symmetries easily. For example, the central charge of the Virasoro algebra is simply a sum over each contribution from bosonic and fermionic free fields. For each current of weight-h, there exist multicomponent generators. The corresponding W_∞ algebras (without any deformation parameter λ) are obtained in [4–6]. By construction, because there are many fermionic currents, there is more room for the supersymmetric theory we would like to obtain.

Then it is natural to consider the multiple bosonic and fermionic free fields together with the deformation of λ. Recently, in [7], the $\mathcal{N}=2$ supersymmetric linear $W_\infty^{K,K}[\lambda]$ algebra is obtained by analyzing the currents from the multiple bosonic and fermionic free fields with derivatives. Here K is the dimension of fundamental representation of the above bifundamentals. The number of bosonic currents

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of each weight ($h = 1, 2, \ldots$) is given by $2K^2$ which is equal to the number of fermionic currents of each weight ($h = \frac{3}{2}, \frac{5}{2}, \ldots$). The factor of 2 appears because we are considering the complex free fields. Among $2K^2$-fermionic currents, two of them play the role of $\mathcal{N} = 2$ supersymmetry generators. There are also K^2-fermionic currents of weight-$\frac{1}{2}$, and this fact will affect the structure of the $\mathcal{N} = 4$ superconformal algebra.[1]

In this paper, we would like to construct the $\mathcal{N} = 4$ supersymmetric linear $W_\infty^{2,2}[\lambda]$ algebra by focusing on the $K = 2$ case, which is very special in the sense that only this $K = 2$ will provide the supersymmetric theory we want to obtain. Then among eight fermionic currents of weight-$\frac{3}{2}$, half of them play the role of $\mathcal{N} = 4$ supersymmetry generators. The remaining half of them belong to the first $\mathcal{N} = 4$ multiplet. Moreover, the lowest fermionic currents of weight-$\frac{1}{2}$ can join the generators of the $\mathcal{N} = 4$ superconformal algebra. For the weight-1 currents, the seven of them play the role of the bosonic generators of the $\mathcal{N} = 4$ superconformal algebra and the remaining one will appear in the lowest operator in the first $\mathcal{N} = 4$ multiplet. For the weight-2, one of them is given by the stress energy tensor, six of them will appear in the generators of the first $\mathcal{N} = 4$ multiple, and the remaining one will arise in the lowest operator in the second $\mathcal{N} = 4$ multiplet. For the weights greater than 2, we can analyze similarly, and they can be placed into the corresponding $\mathcal{N} = 4$ multiplets appropriately, according to $SO(4)$ indices of $\mathcal{N} = 4$ superspace.

We would like to determine the explicit algebra for how the above analysis on the weight contents fits in the $\mathcal{N} = 4$ supersymmetric linear $W_\infty[\lambda]$ algebra.[2]

In this paper, we determine the $\mathcal{N} = 4$ stress energy tensor, the first, the second, the third, and the fourth $\mathcal{N} = 4$ multiplets in the $\mathcal{N} = 4$ supersymmetric $W_\infty[\lambda]$ algebra by using the $\beta\gamma$ and bc ghost systems. We calculate the various OPEs between them, where the sum of two (super) weights appearing on the left-hand side is less than or equal to 4 ($h_1 + h_2 \leq 4$), in $\mathcal{N} = 4$ superspace. As in the abstract, the case of $h_1 = 1 = h_2$ reproduces the corresponding one [10,11] in the $\mathcal{N} = 4$ coset model under the large (N, k) 't Hooft–like limit.

In Sec. II, we review the $\beta\gamma$ and bc ghost systems, and the bosonic and fermionic currents can be written in terms of these fields.

In Sec. III, we determine the $\mathcal{N} = 4$ stress energy tensor, the first $\mathcal{N} = 4$ multiplet, and their OPEs explicitly.

In Sec. IV, we summarize what we have obtained in this paper, and future directions are described.

In the appendixes, some of the details in Sec. III are presented explicitly.

We are heavily using the Thielemans package [12] with the help of *Mathematica* [13].

II. REVIEW

A. The fundamental OPEs

The bosonic $\beta\gamma$ and fermionic bc ghost systems satisfy the following OPEs [7,8]:

$$\gamma^{i,\bar{a}}(z)\beta^{\bar{j},b}(w) = \frac{1}{(z-w)}\delta^{i\bar{j}}\delta^{\bar{a}b} + \cdots,$$
$$c^{i,\bar{a}}(z)b^{\bar{j},b}(w) = \frac{1}{(z-w)}\delta^{i\bar{j}}\delta^{\bar{a}b} + \cdots. \tag{2.1}$$

The fundamental indices a, b run over $a, b = 1, 2$ and the antifundamental indices $\bar{a}, \bar{b}$ run over $\bar{a}, \bar{b} = 1, 2$. The fundamental indices i, j of $SU(N)$ run over $i, j = 1, 2, \ldots, N$, and the antifundamental indices $\bar{i}, \bar{j}$ of $SU(N)$ run over $\bar{i}, \bar{j} = 1, 2, \ldots, N$.

B. The quadratic bosonic and fermionic operators

We can construct the bosonic and fermionic operators (or currents) by taking the quadratic expressions of above $\beta\gamma$ and bc ghost systems in the presence of various holomorphic derivatives as follows [7,8]:

$$V^{(s)+}_{\lambda,\bar{a}b} = \sum_{i=0}^{s-1} a^i(s,\lambda)\partial^{s-1-i}((\partial^i\beta^{\bar{l}b})\delta_{l\bar{l}}\gamma^{l\bar{a}})$$
$$+ \sum_{i=0}^{s-1} a^i\left(s,\lambda+\frac{1}{2}\right)\partial^{s-1-i}((\partial^i b^{\bar{l}b})\delta_{l\bar{l}}c^{l\bar{a}}),$$
$$V^{(s)-}_{\lambda,\bar{a}b} = -\frac{(s-1+2\lambda)}{(2s-1)}\sum_{i=0}^{s-1} a^i(s,\lambda)\partial^{s-1-i}((\partial^i\beta^{\bar{l}b})\delta_{l\bar{l}}\gamma^{l\bar{a}})$$
$$+ \frac{(s-2\lambda)}{(2s-1)}\sum_{i=0}^{s-1} a^i\left(s,\lambda+\frac{1}{2}\right)\partial^{s-1-i}((\partial^i b^{\bar{l}b})\delta_{l\bar{l}}c^{l\bar{a}}),$$
$$Q^{(s)+}_{\lambda,\bar{a}b} = \sum_{i=0}^{s-1} \alpha^i(s,\lambda)\partial^{s-1-i}((\partial^i\beta^{\bar{l}b})\delta_{l\bar{l}}c^{l\bar{a}})$$
$$- \sum_{i=0}^{s-2} \beta^i(s,\lambda)\partial^{s-2-i}((\partial^i b^{\bar{l}b})\delta_{l\bar{l}}\gamma^{l\bar{a}}),$$
$$Q^{(s)-}_{\lambda,\bar{a}b} = \sum_{i=0}^{s-1} \alpha^i(s,\lambda)\partial^{s-1-i}((\partial^i\beta^{\bar{l}b})\delta_{l\bar{l}}c^{l\bar{a}})$$
$$+ \sum_{i=0}^{s-2} \beta^i(s,\lambda)\partial^{s-2-i}((\partial^i b^{\bar{l}b})\delta_{l\bar{l}}\gamma^{l\bar{a}}). \tag{2.2}$$

The first two operators of weight s are bosonic and the last two operators of weight $(s-\frac{1}{2})$ are fermionic. Each term on the right-hand sides has the summation over the indices l and $\bar{l}$ of $SU(N)$. Each operator has four components,

[1] There are also similar constructions in [8,9].

[2] The terminology of $W_\infty[\lambda]$ algebra is used here instead of using the previous terminology of $W_\infty^{2,2}[\lambda]$ algebra for simplicity.

11,12,21, and 22 in the indices $\bar{a}$ and b. Each coefficient on the right-hand sides depends on the weight s [or $(s-\frac{1}{2})$] and the λ. They can be summarized by [2,3]

$$\begin{aligned} a^i(s,\lambda) &\equiv \binom{s-1}{i}\frac{(-2\lambda-s+2)_{s-1-i}}{(s+i)_{s-1-i}}, \qquad 0\le i\le (s-1),\\ \alpha^i(s,\lambda) &\equiv \binom{s-1}{i}\frac{(-2\lambda-s+2)_{s-1-i}}{(s+i-1)_{s-1-i}}, \qquad 0\le i\le (s-1),\\ \beta^i(s,\lambda) &\equiv \binom{s-2}{i}\frac{(-2\lambda-s+2)_{s-2-i}}{(s+i)_{s-2-i}}, \qquad 0\le i\le (s-2). \end{aligned} \tag{2.3}$$

The parentheses in (2.3) stand for the binomial coefficients, and the $(a)_n$ symbols stand for the rising Pochhammer symbol $(a)_n \equiv a(a+1)\cdots(a+n-1)$. There are nontrivial relations between these coefficients [2,3].

C. The λ-dependent currents

We can split the above bosonic operators into the one written in terms of bosonic fields and the other written in terms of fermionic fields by simple linear combinations. For the fermionic operators, we also split them in terms of the one having only one kind of fermion fields and the other having only the other kind of fermion fields. Then we obtain the following operators with the explicit bifundamental indices [7]:

$$\begin{aligned} W^{\lambda,\bar{a}b}_{F,h} &= \frac{n_{W_{F,h}}}{q^{h-2}}\frac{(-1)^h}{\sum_{i=0}^{h-1}a^i(h,\frac{1}{2})}\left[\frac{(h-1+2\lambda)}{(2h-1)}V^{(h)+}_{\lambda,\bar{a}b}+V^{(h)-}_{\lambda,\bar{a}b}\right],\\ W^{\lambda,\bar{a}b}_{B,h} &= \frac{n_{W_{B,h}}}{q^{h-2}}\frac{(-1)^h}{\sum_{i=0}^{h-1}a^i(h,0)}\left[\frac{(h-2\lambda)}{(2h-1)}V^{(h)+}_{\lambda,\bar{a}b}-V^{(h)-}_{\lambda,\bar{a}b}\right],\\ Q^{\lambda,\bar{a}b}_{h+\frac{1}{2}} &= \frac{1}{2}\frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}}\frac{(-1)^{h+1}h}{\sum_{i=0}^{h-1}\beta^i(h+1,0)}[Q^{(h+1)-}_{\lambda,\bar{a}b}-Q^{(h+1)+}_{\lambda,\bar{a}b}],\\ \bar{Q}^{\lambda,b\bar{a}}_{h+\frac{1}{2}} &= \frac{1}{2}\frac{n_{W_{Q,h+\frac{1}{2}}}}{q^{h-1}}\frac{(-1)^{h+1}}{\sum_{i=0}^{h}\alpha^i(h+1,0)}[Q^{(h+1)-}_{\lambda,\bar{a}b}+Q^{(h+1)+}_{\lambda,\bar{a}b}]. \end{aligned} \tag{2.4}$$

The first two operators of weight h are bosonic and the last two operators of weight $(h+\frac{1}{2})$ are fermionic.[3] The overall coefficients do not depend on the λ. We list the explicit expressions for low weights by substituting (2.3) and (2.5) into (2.4) as follows [7]:

$$\begin{aligned} W^{\lambda,\bar{a}b}_{F,1} &= -\frac{1}{4}(V^{(1)-}_{\lambda,\bar{a}b}+2\lambda V^{(1)+}_{\lambda,\bar{a}b}),\\ W^{\lambda,\bar{a}b}_{F,2} &= \left(V^{(2)-}_{\lambda,\bar{a}b}+\frac{1}{3}(1+2\lambda)V^{(2)+}_{\lambda,\bar{a}b}\right),\\ W^{\lambda,\bar{a}b}_{F,3} &= -4\left(V^{(3)-}_{\lambda,\bar{a}b}+\frac{1}{5}(2+2\lambda)V^{(3)+}_{\lambda,\bar{a}b}\right),\\ W^{\lambda,\bar{a}b}_{F,4} &= 16\left(V^{(4)-}_{\lambda,\bar{a}b}+\frac{1}{7}(3+2\lambda)V^{(4)+}_{\lambda,\bar{a}b}\right),\\ W^{\lambda,\bar{a}b}_{B,1} &= -\frac{1}{4}(-V^{(1)-}_{\lambda,\bar{a}b}+(1-2\lambda)V^{(1)+}_{\lambda,\bar{a}b}),\\ W^{\lambda,\bar{a}b}_{B,2} &= \left(-V^{(2)-}_{\lambda,\bar{a}b}+\frac{1}{3}(2-2\lambda)V^{(2)+}_{\lambda,\bar{a}b}\right),\\ W^{\lambda,\bar{a}b}_{B,3} &= -4\left(-V^{(3)-}_{\lambda,\bar{a}b}+\frac{1}{5}(3-2\lambda)V^{(3)+}_{\lambda,\bar{a}b}\right),\\ W^{\lambda,\bar{a}b}_{B,4} &= 16\left(-V^{(4)-}_{\lambda,\bar{a}b}+\frac{1}{7}(4-2\lambda)V^{(4)+}_{\lambda,\bar{a}b}\right),\\ Q^{\lambda,\bar{a}b}_{\frac{3}{2}} &= \frac{1}{\sqrt{2}}(Q^{(2)-}_{\lambda,\bar{a}b}-Q^{(2)+}_{\lambda,\bar{a}b}),\\ Q^{\lambda,\bar{a}b}_{\frac{5}{2}} &= -2\sqrt{2}(Q^{(3)-}_{\lambda,\bar{a}b}-Q^{(3)+}_{\lambda,\bar{a}b}),\\ Q^{\lambda,\bar{a}b}_{\frac{7}{2}} &= 8\sqrt{2}(Q^{(4)-}_{\lambda,\bar{a}b}-Q^{(4)+}_{\lambda,\bar{a}b}),\\ Q^{\lambda,\bar{a}b}_{\frac{9}{2}} &= -32\sqrt{2}(Q^{(5)-}_{\lambda,\bar{a}b}-Q^{(5)+}_{\lambda,\bar{a}b}),\\ \bar{Q}^{\lambda,b\bar{a}}_{\frac{1}{2}} &= -\frac{1}{2\sqrt{2}}(\bar{Q}^{(1)-}_{\lambda,\bar{a}b}+\bar{Q}^{(1)+}_{\lambda,\bar{a}b}),\\ \bar{Q}^{\lambda,b\bar{a}}_{\frac{3}{2}} &= \frac{1}{\sqrt{2}}(\bar{Q}^{(2)-}_{\lambda,\bar{a}b}+\bar{Q}^{(2)+}_{\lambda,\bar{a}b}),\\ \bar{Q}^{\lambda,b\bar{a}}_{\frac{5}{2}} &= -2\sqrt{2}(\bar{Q}^{(3)-}_{\lambda,\bar{a}b}+\bar{Q}^{(3)+}_{\lambda,\bar{a}b}),\\ \bar{Q}^{\lambda,b\bar{a}}_{\frac{7}{2}} &= 8\sqrt{2}(\bar{Q}^{(4)-}_{\lambda,\bar{a}b}+\bar{Q}^{(4)+}_{\lambda,\bar{a}b}),\\ \bar{Q}^{\lambda,b\bar{a}}_{\frac{9}{2}} &= -32\sqrt{2}(\bar{Q}^{(5)-}_{\lambda,\bar{a}b}+\bar{Q}^{(5)+}_{\lambda,\bar{a}b}),\cdots. \end{aligned} \tag{2.6}$$

We can easily see that the normalization for the overall factor is increased by -4 when we increase the weight. Note that the lowest weight for the bosonic operators is given by 1 and the one for the fermionic operators is given by $\frac{1}{2}$. The $Q^{\lambda,\bar{a}b}_{\frac{1}{2}}$ is identically zero.

Then we have eight bosonic currents for the weight $h=1,2,\ldots$ and eight fermionic currents for the weight $h+\frac{1}{2}=\frac{3}{2},\frac{5}{2},\ldots$ in (2.4). In next section, we will construct the $\mathcal{N}=4$ multiplets as well as the $\mathcal{N}=4$ stress energy tensor by using (2.4).

[3] The normalizations are given by [14]

$$n_{W_{F,h}} = \frac{2^{h-3}(h-1)!}{(2h-3)!!}q^{h-2}, \qquad n_{W_{B,h}} = \frac{2^{h-3}h!}{(2h-3)!!}q^{h-2},$$
$$n_{Q_{h+\frac{1}{2}}} = \frac{2^{h-\frac{1}{2}}h!}{(2h-1)!!}q^{h-1} = n_{\bar{Q}_{h+\frac{1}{2}}}. \tag{2.5}$$

Then the q dependence in (2.4) disappears.

III. THE $\mathcal{N}=4$ SUPERSYMMETRIC LINEAR $W_\infty[\lambda]$ ALGEBRA

A. The construction of the $\mathcal{N}=4$ stress energy tensor

1. The λ-dependent quasiprimary stress energy tensor

The stress energy tensor of weight-2 from (2.6) is given by [2,3]

$$L=(W^{\lambda,11}_{\mathrm{B},2}+W^{\lambda,22}_{\mathrm{B},2}+W^{\lambda,11}_{\mathrm{F},2}+W^{\lambda,22}_{\mathrm{F},2}), \tag{3.1}$$

which is equal to $V^{(2)+}_{\lambda,\bar{a}b}\delta_{b\bar{a}}$. The central charge, which is the same as the fourth order pole of the OPE $L(z)L(w)$ times two, is

$$c=6N(1-4\lambda). \tag{3.2}$$

At $\lambda=0$, the central charge becomes $6N$. We will observe that the remaining seven weight-2 operators appear in the first and the second $\mathcal{N}=4$ multiplets.

2. The λ-dependent weight-$\frac{3}{2}$ primary supersymmetry currents

It is natural to consider that the weight-$\frac{3}{2}$ operators, which depend on the λ, can be obtained from the corresponding ones at $\lambda=0$. Our starting point is the following ansatz for the weight-$\frac{3}{2}$ operators:

$$\begin{aligned}
G^1&=-\frac{1}{2}(Q^{\lambda,11}_{\frac{3}{2}}+i\sqrt{2}Q^{\lambda,12}_{\frac{3}{2}}+2i\sqrt{2}Q^{\lambda,21}_{\frac{3}{2}}-2Q^{\lambda,22}_{\frac{3}{2}}\\
&\quad-2\bar{Q}^{\lambda,11}_{\frac{3}{2}}-2i\sqrt{2}\bar{Q}^{\lambda,12}_{\frac{3}{2}}-i\sqrt{2}\bar{Q}^{\lambda,21}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}}),\\
G^2&=\frac{i}{2}(Q^{\lambda,11}_{\frac{3}{2}}+2i\sqrt{2}Q^{\lambda,21}_{\frac{3}{2}}-2Q^{\lambda,22}_{\frac{3}{2}}-2\bar{Q}^{\lambda,11}_{\frac{3}{2}}\\
&\quad-2i\sqrt{2}\bar{Q}^{\lambda,12}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}}),\\
G^3&=\frac{i}{2}(Q^{\lambda,11}_{\frac{3}{2}}+i\sqrt{2}Q^{\lambda,12}_{\frac{3}{2}}-2Q^{\lambda,22}_{\frac{3}{2}}-2\bar{Q}^{\lambda,11}_{\frac{3}{2}}\\
&\quad-i\sqrt{2}\bar{Q}^{\lambda,21}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}}),\\
G^4&=\frac{1}{2}Q^{\lambda,11}_{\frac{3}{2}}+Q^{\lambda,22}_{\frac{3}{2}}+\bar{Q}^{\lambda,11}_{\frac{3}{2}}+\frac{1}{2}\bar{Q}^{\lambda,22}_{\frac{3}{2}}.
\end{aligned} \tag{3.3}$$

We can check that the third order pole of $G^i(z)G^j(w)$ is given by $\frac{2}{3}c\delta^{ij}$ with (3.2). We have seen half of the weight-$\frac{3}{2}$ operators in (3.3), and the remaining ones will be given in the next subsection for the first $\mathcal{N}=4$ multiplet.

From now on, we construct the remaining operators in the $\mathcal{N}=4$ superconformal algebra based on the explicit expressions of (3.3).

3. The λ-independent weight-1 primary operators

From the defining equations of the second order pole in the OPE $G^i(z)G^j(w)$, which are given by $-2i(T^{ij}+\frac{1}{2}(1-4\lambda)\epsilon^{ijkl}T^{kl})(w)$, we can determine the following six weight-1 operators which do not depend on the λ[4]:

$$\begin{aligned}
T^{12}&=-i(2iW^{\lambda,11}_{\mathrm{B},1}-\sqrt{2}W^{\lambda,12}_{\mathrm{B},1}-2iW^{\lambda,22}_{\mathrm{B},1}+2iW^{\lambda,11}_{\mathrm{F},1}\\
&\quad-2\sqrt{2}W^{\lambda,12}_{\mathrm{F},1}-2iW^{\lambda,22}_{\mathrm{F},1}),\\
T^{13}&=-i(-2iW^{\lambda,11}_{\mathrm{B},1}+4\sqrt{2}W^{\lambda,21}_{\mathrm{B},1}+2iW^{\lambda,22}_{\mathrm{B},1}-2iW^{\lambda,11}_{\mathrm{F},1}\\
&\quad+2\sqrt{2}W^{\lambda,21}_{\mathrm{F},1}+2iW^{\lambda,22}_{\mathrm{F},1}),\\
T^{14}&=-i(2W^{\lambda,11}_{\mathrm{B},1}+i\sqrt{2}W^{\lambda,12}_{\mathrm{B},1}+4i\sqrt{2}W^{\lambda,21}_{\mathrm{B},1}-2W^{\lambda,22}_{\mathrm{B},1}\\
&\quad-2W^{\lambda,11}_{\mathrm{F},1}-2i\sqrt{2}W^{\lambda,12}_{\mathrm{F},1}-2i\sqrt{2}W^{\lambda,21}_{\mathrm{F},1}+2W^{\lambda,22}_{\mathrm{F},1}),\\
T^{23}&=-i(-2W^{\lambda,11}_{\mathrm{B},1}-i\sqrt{2}W^{\lambda,12}_{\mathrm{B},1}-4i\sqrt{2}W^{\lambda,21}_{\mathrm{B},1}+2W^{\lambda,22}_{\mathrm{B},1}\\
&\quad-2W^{\lambda,11}_{\mathrm{F},1}-2i\sqrt{2}W^{\lambda,12}_{\mathrm{F},1}-2i\sqrt{2}W^{\lambda,21}_{\mathrm{F},1}+2W^{\lambda,22}_{\mathrm{F},1}),\\
T^{24}&=-i(-2iW^{\lambda,11}_{\mathrm{B},1}+4\sqrt{2}W^{\lambda,21}_{\mathrm{B},1}+2iW^{\lambda,22}_{\mathrm{B},1}+2iW^{\lambda,11}_{\mathrm{F},1}\\
&\quad-2\sqrt{2}W^{\lambda,21}_{\mathrm{F},1}-2iW^{\lambda,22}_{\mathrm{F},1}),\\
T^{34}&=-i(-2iW^{\lambda,11}_{\mathrm{B},1}+\sqrt{2}W^{\lambda,12}_{\mathrm{B},1}+2iW^{\lambda,22}_{\mathrm{B},1}+2iW^{\lambda,11}_{\mathrm{F},1}\\
&\quad-2\sqrt{2}W^{\lambda,12}_{\mathrm{F},1}-2iW^{\lambda,22}_{\mathrm{F},1}).
\end{aligned} \tag{3.4}$$

Note that the right-hand sides of (3.4) are proportional to the expressions of $\Phi^{(1),ij}_1$ at $\lambda=0$ when we replace the weight-2 in the $W^{\bar{a}b}_{\mathrm{B},2}$ and $W^{\bar{a}b}_{\mathrm{F},2}$ [10] with the weight-1 by generalizing to the λ dependent ones. We can check that the first order pole of the OPE $G^i(z)G^j(w)$ provides the correct quasiprimary operator $L(w)$ and the corresponding descendants of (3.4).

4. The λ-independent weight-$\frac{1}{2}$ primary operators

Again, the defining equation of the second order pole of the OPE $G^i(z)T^{jk}(w)$, $-(\epsilon^{ijkl}\Gamma^l+(1-4\lambda)(\delta^{ik}\Gamma^j-\delta^{ij}\Gamma^k))(w)$ allows us to determine the following weight-$\frac{1}{2}$ operators:

$$\begin{aligned}
\Gamma^1&=-\frac{i}{2}(-2\bar{Q}^{\lambda,11}_{\frac{1}{2}}-2i\sqrt{2}\bar{Q}^{\lambda,12}_{\frac{1}{2}}-i\sqrt{2}\bar{Q}^{\lambda,21}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}}),\\
\Gamma^2&=-\frac{1}{2}(-2\bar{Q}^{\lambda,11}_{\frac{1}{2}}-2i\sqrt{2}\bar{Q}^{\lambda,12}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}}),\\
\Gamma^3&=-\frac{1}{2}(-2\bar{Q}^{\lambda,11}_{\frac{1}{2}}-i\sqrt{2}\bar{Q}^{\lambda,21}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}}),\\
\Gamma^4&=i\left(\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\frac{1}{2}\bar{Q}^{\lambda,22}_{\frac{1}{2}}\right).
\end{aligned} \tag{3.5}$$

Note that the right-hand sides of (3.5) can be obtained from (3.3) by replacing the weight-$\frac{3}{2}$ with the weight-$\frac{1}{2}$ with an

[4]We can easily see that the parameter $\alpha=\frac{1}{2}\frac{(k^+-k^-)}{(k^++k^-)}$ with $k^+=k+1$ and $k^-=N+1$ in the $\mathcal{N}=4$ coset model becomes $\alpha=\frac{1}{2}(1-2\lambda_{co})=\frac{1}{2}(1-4\lambda)$ under the large (N,k) limit with fixed $\lambda_{co}\equiv\frac{(N+1)}{(k+N+2)}$ [10,11,15]. See also [16].

overall factor i. Further analysis for the first order pole of the OPE $G^i(z)T^{jk}(w)$ provides the correct descendants of (3.5) and the primary operators of weight-$\frac{3}{2}$ in (3.3). These are not dependent on the λ because the lowest fermionic operators $\bar{Q}^{\lambda,a\bar{b}}_{\frac{1}{2}}$ do not contain the λ from (2.2) and (2.4).

5. The λ-independent weight-1 quasiprimary operator

For the final weight-1 operator, we can use the defining equation for the first order pole of the OPE $G^i(z)\Gamma^j(w)$ which is equal to $(-\varepsilon^{ijkl}T^{kl}+i\delta^{ij}U)(w)$. It turns out that

$$U=2[(W^{\lambda,11}_{\mathrm{F},1}+W^{\lambda,22}_{\mathrm{F},1})+(W^{\lambda,11}_{\mathrm{B},1}+W^{\lambda,22}_{\mathrm{B},1})], \quad (3.6)$$

which does not contain the λ. Therefore, the eight independent weight-1 operators from $W^{\bar{a}b}_{B,2}$ and $W^{\bar{a}b}_{F,2}$ are given by (3.4) and (3.6). The remaining one will be given in the next subsection for the first $\mathcal{N}=4$ multiplet.

In the next subsection, we will describe whether the above five kinds of operators will produce the known $\mathcal{N}=4$ superconformal algebra or not.

B. The OPEs between the $\mathcal{N}=4$ stress energy tensor and itself

We calculate the OPEs between the five operators in the $\mathcal{N}=4$ stress energy tensor and the weight-1 operator in that multiplet.

1. The OPE between the weight-1, $\frac{1}{2}$, 1 operators and the weight-1 operator

From the explicit expressions (3.6)–(3.4), (2.6), and (2.1), we can check the following OPEs:

$$\begin{aligned}U(z)U(w)&=+\cdots,\\ \Gamma^i(z)U(w)&=+\cdots,\\ T^{ij}(z)U(w)&=+\cdots.\end{aligned} \quad (3.7)$$

The last two are the standard results [11,15] while the first one is a rather trivial result. This is due to the fact that the expression for the U in (3.6) has the same relative coefficients. The standard result for the first one leads to the nontrivial second order pole, which is given by a central term. We expect that the OPE between $L(z)U(w)$ contains the third order pole because the OPE $\partial U(z)U(w)$ has no singular term.

2. The OPE between the weight-$\frac{3}{2}$ operators and the weight-1 operator

Similarly, by using (3.3), (3.6), and previous defining equations, we obtain the following OPE:

$$G^i(z)U(w)=-\frac{1}{(z-w)^2}[i\Gamma^i](w)-\frac{1}{(z-w)}[i\partial\Gamma^i](w)+\cdots. \quad (3.8)$$

The weight-1 operator U plays the role of keeping the structure of the weight-$\frac{3}{2}$ operator on the left-hand side with the weight reduced to $\frac{1}{2}$. The relative coefficient for the descendant can be seen from the standard conformal field theory analysis.[5]

3. The OPE between the weight-2 operator and the weight-1 operator

Let us consider the final nontrivial OPE with (3.1) and (3.6). It turns out that there exists

$$L(z)U(w)=-\frac{1}{(z-w)^3}[N]+\frac{1}{(z-w)^2}[U](w)+\frac{1}{(z-w)}[\partial U](w)+\cdots. \quad (3.10)$$

Compared to the standard result [11,15], the above OPE contains the central term, as mentioned before. This is due to the fact that the expression of (3.6) has particular relative coefficients.[6]

4. The $\mathcal{N}=4$ supersymmetric OPE in the $\mathcal{N}=4$ superspace

We can put the operators of $\mathcal{N}=4$ superconformal algebra found in the previous section into a superfield in the $\mathcal{N}=4$ superspace.[7] Then we have the $\mathcal{N}=4$ stress energy tensor [11,15]

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[5]There is

$$(G^i-i(1-4\lambda)\partial\Gamma^i)(z)U(w)=-\frac{1}{(z-w)^2}[i\Gamma^i](w)-\frac{1}{(z-w)}[i\partial\Gamma^i](w)+\cdots, \quad (3.9)$$

due to one of the relations in (3.7). This will be used in the $\mathcal{N}=4$ superspace description.

[6]We have

$$\left(L-\frac{1}{2}(1-4\lambda)\partial U\right)(z)U(w)=-\frac{1}{(z-w)^3}N+\frac{1}{(z-w)^2}U(w)+\frac{1}{(z-w)}\partial U(w)+\cdots, \quad (3.11)$$

which will be used later.

[7]The coordinates of $\mathcal{N}=4$ superspace can be described as $(Z,\bar{Z})$ where $Z=(z,\theta^i)$, $\bar{Z}=(\bar{z},\bar{\theta}^i)$, and the $SO(4)$-vector index i runs over $i=1,\ldots,4$. The left covariant spinor derivative is given by $D^i=\theta^i\frac{\partial}{\partial z}+\frac{\partial}{\partial\theta^i}$ with nontrivial anticommutators $\{D^i,D^j\}=2\delta^{ij}\frac{\partial}{\partial z}$. The simplified notation θ^{4-0} stands for $\theta^1\theta^2\theta^3\theta^4$. The complement $4-i$ is defined such that $\theta^1\theta^2\theta^3\theta^4=\theta^{4-i}\theta^i$ [11,15].

$$\mathbf{J} = -\Delta + i\theta^j\Gamma^j - i\theta^{4-jk}T^{jk} - \theta^{4-j}(G^j - i(1-4\lambda)\partial\Gamma^j) + \theta^{4-0}\left(2L - \frac{1}{2}(1-4\lambda)\partial^2\Delta\right)$$
$$\equiv \left(-\Delta, i\Gamma^i, -iT^{ij}, -(G^i - i(1-4\lambda)\partial\Gamma^i), 2\left(L - \frac{1}{2}(1-4\lambda)\partial^2\Delta\right)\right), \tag{3.12}$$

where the lowest component has the following relation: $-\partial\Delta \equiv U$. We will use the operator U rather than the operator Δ. The precise relations between the components and its superfields in (3.12) at vanishing fermionic coordinates can be summarized by [11,15]

$$U \leftrightarrow \partial\mathbf{J}, \qquad \Gamma^i \leftrightarrow -iD^i\mathbf{J}, \qquad T^{ij} \leftrightarrow -\frac{i}{2!}\varepsilon^{ijkl}D^kD^l\mathbf{J},$$
$$G^i \leftrightarrow -\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{J} - (1-4\lambda)\partial D^i\mathbf{J},$$
$$L \leftrightarrow \frac{1}{2\cdot 4!}\varepsilon^{ijkl}D^iD^jD^kD^l\mathbf{J} - \frac{1}{2}(1-4\lambda)\partial^2\mathbf{J}. \tag{3.13}$$

Because of the extra terms in the fourth and fifth elements of the $\mathcal{N}=4$ stress energy tensor, there are extra terms in the corresponding expressions of (3.13).

Then we can write down the previous equations (3.7), (3.8) [or (3.9)], and (3.10) [or (3.11)] including other OPEs in the component approach in terms of the following single OPE in the $\mathcal{N}=4$ superspace:

$$\mathbf{J}(Z_1)\mathbf{J}(Z_2) = -\frac{\theta_{12}^{4-0}}{z_{12}^2}N + \frac{\theta_{12}^{4-i}}{z_{12}}D^i\mathbf{J}(Z_2) + \frac{\theta_{12}^{4-0}}{z_{12}}2\partial\mathbf{J}(Z_2) + \cdots. \tag{3.14}$$

Compared to the standard expression [11,15], there is no $\log(z_{12})$ term.[8]

In Appendix A, all the component OPEs are summarized explicitly.[9] It is straightforward to obtain these OPEs from (3.14) by using the various superderivatives with the relations (3.13).

[8]We have the corresponding OPE

$$\mathbf{J}(Z_1)\partial\mathbf{J}(Z_2) = -\frac{\theta_{12}^{4-0}}{z_{12}^3}2N + \frac{\theta_{12}^{4-i}}{z_{12}^2}D^i\mathbf{J}(Z_2) + \frac{\theta_{12}^{4-0}}{z_{12}^2}2\partial\mathbf{J}(Z_2) + \frac{\theta_{12}^{4-i}}{z_{12}}\partial D^i\mathbf{J}(Z_2) + \frac{\theta_{12}^{4-0}}{z_{12}}2\partial^2\mathbf{J}(Z_2) + \cdots. \tag{3.15}$$

From (3.15), which is more relevant to the previous three component results [(3.7), (3.9), and (3.11)], we obtain (3.14) after the integrations.

[9]Compared to the construction in [10], the presence of $W_{B,1}^{\lambda,\bar{a}b}$ and $\bar{Q}_{\frac{1}{2}}^{\lambda,b\bar{a}}$ in (3.4), (3.5), and (3.6) is new, and these will change the structure of the algebra.

C. The construction of the first $\mathcal{N}=4$ multiplet

1. The λ-dependent weight-1 primary operator

Let us start with the final weight-1 primary operator of the $\mathcal{N}=2$ superconformal algebra [7,17]

$$\Phi_0^{(1)} = 4[(1-2\lambda)(W_{\mathrm{F},1}^{\lambda,11} + W_{\mathrm{F},1}^{\lambda,22}) - 2\lambda(W_{\mathrm{B},1}^{\lambda,11} + W_{\mathrm{B},1}^{\lambda,22})]. \tag{3.16}$$

The field contents of (3.16) are the same as that in (3.6). At $\lambda=0$, only the first two terms in (3.16) contribute to the final expression and reproduce the one in [10]. Compared to the previous construction on the weight-1 operator, the λ-dependent coefficients appear in the above.

2. The λ-dependent weight-$\frac{3}{2}$ primary operators

From the defining equation [11,15,18] of

$$G^i(z)\Phi_0^{(1)}(w)\Big|_{\frac{1}{(z-w)}} = -\Phi_{\frac{1}{2}}^{(1),i}(w), \tag{3.17}$$

we can determine the following primary [under the stress energy tensor (3.1)] operators of weight-$\frac{3}{2}$

$$\Phi_{\frac{1}{2}}^{(1),1} = \frac{1}{2}(Q_{\frac{3}{2}}^{\lambda,11} + i\sqrt{2}Q_{\frac{3}{2}}^{\lambda,12} + 2i\sqrt{2}Q_{\frac{3}{2}}^{\lambda,21} - 2Q_{\frac{3}{2}}^{\lambda,22} + 2\bar{Q}_{\frac{3}{2}}^{\lambda,11} + 2i\sqrt{2}\bar{Q}_{\frac{3}{2}}^{\lambda,12} + i\sqrt{2}\bar{Q}_{\frac{3}{2}}^{\lambda,21} - \bar{Q}_{\frac{3}{2}}^{\lambda,22}),$$
$$\Phi_{\frac{1}{2}}^{(1),2} = -\frac{i}{2}(Q_{\frac{3}{2}}^{\lambda,11} + 2i\sqrt{2}Q_{\frac{3}{2}}^{\lambda,21} - 2Q_{\frac{3}{2}}^{\lambda,22} + 2\bar{Q}_{\frac{3}{2}}^{\lambda,11} + 2i\sqrt{2}\bar{Q}_{\frac{3}{2}}^{\lambda,12} - \bar{Q}_{\frac{3}{2}}^{\lambda,22}),$$
$$\Phi_{\frac{1}{2}}^{(1),3} = -\frac{i}{2}(Q_{\frac{3}{2}}^{\lambda,11} + i\sqrt{2}Q_{\frac{3}{2}}^{\lambda,12} - 2Q_{\frac{3}{2}}^{\lambda,22} + 2\bar{Q}_{\frac{3}{2}}^{\lambda,11} + i\sqrt{2}\bar{Q}_{\frac{3}{2}}^{\lambda,21} - \bar{Q}_{\frac{3}{2}}^{\lambda,22}),$$
$$\Phi_{\frac{1}{2}}^{(1),4} = -\frac{1}{2}Q_{\frac{3}{2}}^{\lambda,11} - Q_{\frac{3}{2}}^{\lambda,22} + \bar{Q}_{\frac{3}{2}}^{\lambda,11} + \frac{1}{2}\bar{Q}_{\frac{3}{2}}^{\lambda,22}. \tag{3.18}$$

The field contents of (3.18) are the same as the ones in (3.3). The only difference appears in the minus signs of $Q_{\frac{3}{2}}^{\lambda,\bar{a}b}$. Then we have the complete weight-$\frac{3}{2}$ operators in (3.3) and (3.18). Compared to the $\lambda=0$ case in [10], the generalization of the fermionic operators to the nonzero λ case in (2.4) provides the exact relative coefficients in (3.18). In other words, for the expressions in [10] at $\lambda=0$, a simple generalization of (2.4) leads to the above result in (3.18). This is also true for other remaining operators of weights-2, $\frac{5}{2}$, 3.

3. The λ-dependent weight-2 primary operators

By using the following defining equation [11]:

$$G^i(z)\Phi_{\frac{1}{2}}^{(1),j}(w)\Big|_{\frac{1}{(z-w)}} = -\left[\delta^{ij}\partial\Phi_0^{(1)} - \frac{1}{2}\varepsilon^{ijkl}\Phi_1^{(1),kl}\right](w), \tag{3.19}$$

we obtain the weight-2 primary operators, by taking two different indices, as follows:

$$\begin{aligned}
\Phi_1^{(1),12} &= 2iW_{\mathrm{B},2}^{\lambda,11} - \sqrt{2}W_{\mathrm{B},2}^{\lambda,12} - 2iW_{\mathrm{B},2}^{\lambda,22} + 2iW_{\mathrm{F},2}^{\lambda,11} \\
&\quad - 2\sqrt{2}W_{\mathrm{F},2}^{\lambda,12} - 2iW_{\mathrm{F},2}^{\lambda,22}, \\
\Phi_1^{(1),13} &= -2iW_{\mathrm{B},2}^{\lambda,11} + 4\sqrt{2}W_{\mathrm{B},2}^{\lambda,21} + 2iW_{\mathrm{B},2}^{\lambda,22} - 2iW_{\mathrm{F},2}^{\lambda,11} \\
&\quad + 2\sqrt{2}W_{\mathrm{F},2}^{\lambda,21} + 2iW_{\mathrm{F},2}^{\lambda,22}, \\
\Phi_1^{(1),14} &= 2W_{\mathrm{B},2}^{\lambda,11} + i\sqrt{2}W_{\mathrm{B},2}^{\lambda,12} + 4i\sqrt{2}W_{\mathrm{B},2}^{\lambda,21} - 2W_{\mathrm{B},2}^{\lambda,22} \\
&\quad - 2W_{\mathrm{F},2}^{\lambda,11} - 2i\sqrt{2}W_{\mathrm{F},2}^{\lambda,12} - 2i\sqrt{2}W_{\mathrm{F},2}^{\lambda,21} + 2W_{\mathrm{F},2}^{\lambda,22}, \\
\Phi_1^{(1),23} &= -2W_{\mathrm{B},2}^{\lambda,11} - i\sqrt{2}W_{\mathrm{B},2}^{\lambda,12} - 4i\sqrt{2}W_{\mathrm{B},2}^{\lambda,21} + 2W_{\mathrm{B},2}^{\lambda,22} \\
&\quad - 2W_{\mathrm{F},2}^{\lambda,11} - 2i\sqrt{2}W_{\mathrm{F},2}^{\lambda,12} - 2i\sqrt{2}W_{\mathrm{F},2}^{\lambda,21} + 2W_{\mathrm{F},2}^{\lambda,22}, \\
\Phi_1^{(1),24} &= -2iW_{\mathrm{B},2}^{\lambda,11} + 4\sqrt{2}W_{\mathrm{B},2}^{\lambda,21} + 2iW_{\mathrm{B},2}^{\lambda,22} + 2iW_{\mathrm{F},2}^{\lambda,11} \\
&\quad - 2\sqrt{2}W_{\mathrm{F},2}^{\lambda,21} - 2iW_{\mathrm{F},2}^{\lambda,22}, \\
\Phi_1^{(1),34} &= -2iW_{\mathrm{B},2}^{\lambda,11} + \sqrt{2}W_{\mathrm{B},2}^{\lambda,12} + 2iW_{\mathrm{B},2}^{\lambda,22} + 2iW_{\mathrm{F},2}^{\lambda,11} \\
&\quad - 2\sqrt{2}W_{\mathrm{F},2}^{\lambda,12} - 2iW_{\mathrm{F},2}^{\lambda,22}.
\end{aligned} \tag{3.20}$$

We observe, as described before, that by taking the corresponding expressions for the weight-1 operators at $\lambda=0$ and replacing them with the ones in (2.4), the above results can be obtained. So far, we have obtained the seven weight-2 operators consisting of (3.1) and (3.20), and the remaining one will appear in the lowest component of the second $\mathcal{N}=4$ multiplet.

4. The λ-dependent weight-$\frac{5}{2}$ quasiprimary operators

From the defining equation [11] of

$$\begin{aligned}
&G^i(z)\Phi_1^{(1),jk}(w)\Big|_{\frac{1}{(z-w)}} \\
&= -\left[(\delta^{ij}\Phi_{\frac{3}{2}}^{(1),k} - \delta^{ik}\Phi_{\frac{3}{2}}^{(1),j}) + \varepsilon^{ijkl}\partial\Phi_{\frac{1}{2}}^{(1),l}\right](w),
\end{aligned} \tag{3.21}$$

we can determine the following weight-$\frac{5}{2}$ quasiprimary operators:

$$\begin{aligned}
\tilde{\Phi}_{\frac{3}{2}}^{(1),1} &\equiv \Phi_{\frac{3}{2}}^{(1),1} - \frac{1}{3}(1-4\lambda)\partial\Phi_{\frac{1}{2}}^{(1),1} \\
&= -\frac{1}{2}(Q_{\frac{5}{2}}^{\lambda,11} + i\sqrt{2}Q_{\frac{5}{2}}^{\lambda,12} + 2i\sqrt{2}Q_{\frac{5}{2}}^{\lambda,21} - 2Q_{\frac{5}{2}}^{\lambda,22} \\
&\quad - 2\bar{Q}_{\frac{5}{2}}^{\lambda,11} - 2i\sqrt{2}\bar{Q}_{\frac{5}{2}}^{\lambda,12} - i\sqrt{2}\bar{Q}_{\frac{5}{2}}^{\lambda,21} + \bar{Q}_{\frac{5}{2}}^{\lambda,22}), \\
\tilde{\Phi}_{\frac{3}{2}}^{(1),2} &\equiv \Phi_{\frac{3}{2}}^{(1),2} - \frac{1}{3}(1-4\lambda)\partial\Phi_{\frac{1}{2}}^{(1),2} \\
&= \frac{i}{2}(Q_{\frac{5}{2}}^{\lambda,11} + 2i\sqrt{2}Q_{\frac{5}{2}}^{\lambda,21} - 2Q_{\frac{5}{2}}^{\lambda,22} - 2\bar{Q}_{\frac{5}{2}}^{\lambda,11} \\
&\quad - 2i\sqrt{2}\bar{Q}_{\frac{5}{2}}^{\lambda,12} + \bar{Q}_{\frac{5}{2}}^{\lambda,22}), \\
\tilde{\Phi}_{\frac{3}{2}}^{(1),3} &\equiv \Phi_{\frac{3}{2}}^{(1),3} - \frac{1}{3}(1-4\lambda)\partial\Phi_{\frac{1}{2}}^{(1),3} \\
&= \frac{i}{2}(Q_{\frac{5}{2}}^{\lambda,11} + i\sqrt{2}Q_{\frac{5}{2}}^{\lambda,12} - 2Q_{\frac{5}{2}}^{\lambda,22} - 2\bar{Q}_{\frac{5}{2}}^{\lambda,11} \\
&\quad - i\sqrt{2}\bar{Q}_{\frac{5}{2}}^{\lambda,21} + \bar{Q}_{\frac{5}{2}}^{\lambda,22}), \\
\tilde{\Phi}_{\frac{3}{2}}^{(1),4} &\equiv \Phi_{\frac{3}{2}}^{(1),4} - \frac{1}{3}(1-4\lambda)\partial\Phi_{\frac{1}{2}}^{(1),4} \\
&= \frac{1}{2}(Q_{\frac{5}{2}}^{\lambda,11} + 2Q_{\frac{5}{2}}^{\lambda,22} + 2\bar{Q}_{\frac{5}{2}}^{\lambda,11} + \bar{Q}_{\frac{5}{2}}^{\lambda,22}).
\end{aligned} \tag{3.22}$$

Note that by starting with (3.18) with minus signs in $Q_{\frac{3}{2}}^{\lambda,\bar{a}b}$ and increasing the weights by one, we reproduce the above results (3.22). For the case of $\lambda=0$ in [10], the weight-$\frac{5}{2}$ operators are primary but at nonzero λ, and the above operators (3.22) are quasiprimary under the stress energy tensor (3.1), although we are using the same notation.[10] We expect that the half of other weight-$\frac{5}{2}$ operators will appear in the second $\mathcal{N}=4$ multiplet.

5. The λ-dependent weight-3 quasiprimary operator

Finally, by using the following defining equation [11]:

$$G^i(z)\Phi_{\frac{3}{2}}^{(1),j}(w)\Big|_{\frac{1}{(z-w)}} = -[\partial\Phi_1^{(1),ij} + \delta^{ij}\Phi_2^{(1)}](w), \tag{3.23}$$

we obtain the weight-3 quasiprimary operator, by taking two equal indices,

$$\begin{aligned}
\tilde{\Phi}_2^{(1)} &\equiv \Phi_2^{(1)} - \frac{1}{3}(1-4\lambda)\partial^2\Phi_0^{(1)} \\
&= -2(W_{\mathrm{B},3}^{\lambda,11} + W_{\mathrm{B},3}^{\lambda,22} + W_{\mathrm{F},3}^{\lambda,11} + W_{\mathrm{F},3}^{\lambda,22}).
\end{aligned} \tag{3.24}$$

We observe that by increasing the weight by one from the stress energy tensor (3.1), the above expression can be seen with the overall factor. Compared to the $\lambda=0$ case in [10] where the corresponding operator is primary, the above operator is quasiprimary.[11] The remaining seven other

[10]The operators $\Phi_{\frac{3}{2}}^{(1),i}$, which are the components of $\mathcal{N}=4$ superfields later, are not quasiprimary. See also Appendix B.

[11]The operator $\Phi_2^{(1)}$ is not quasiprimary, and see also Appendix B.

weight-3 operators will appear in the next $\mathcal{N}=4$ multiplets. Six of them appear in the second $\mathcal{N}=4$ multiplet, and one of them appears in the third $\mathcal{N}=4$ multiplet.

D. The OPEs between the $\mathcal{N}=4$ stress energy tensor and the first $\mathcal{N}=4$ multiplet

We calculate the OPEs between the five kinds of operators in the $\mathcal{N}=4$ stress energy tensor and the lowest weight-1 operator in the first $\mathcal{N}=4$ multiplet.

1. The OPE between the weight-1 operator and the weight-1 operator

From the explicit expressions in (3.6) and (3.16), the following OPE can be obtained:

$$U(z)\Phi_0^{(1)}(w) = \frac{1}{(z-w)^2}[N] + \cdots. \tag{3.25}$$

Compared to the standard result [11] which is trivial, the above OPE has a singular term on the right-hand side. This is due to the fact that this weight-1 operator has the particular coefficients.

2. The OPE between the weight-$\frac{1}{2}$ operators and the weight-1 operator

Similarly, we obtain the following OPE from (3.5) and the previous weight-1 operator;

$$\Gamma^i(z)\Phi_0^{(1)}(w) = -\frac{1}{(z-w)}[\Gamma^i](w) + \cdots. \tag{3.26}$$

This implies that the weight-1 operator preserves the structure of the weight-$\frac{1}{2}$ operator on the left-hand side, and this is new, compared to the standard result [11].

3. The OPE between the weight-1 operators and the weight-1 operator

For the weight-1 operator (3.4), we obtain the following trivial result:

$$T^{ij}(z)\Phi_0^{(1)}(w) = +\cdots. \tag{3.27}$$

4. The OPE between the weight-$\frac{3}{2}$ operators and the weight-1 operator

By using (3.3) and (3.16), the following OPE is satisfied:

$$G^i(z)\Phi_0^{(1)}(w) = -\frac{1}{(z-w)}\Phi_{\frac{1}{2}}^{(1),i}(w) + \cdots. \tag{3.28}$$

Under the action of the weight-1 operator, the numerical coefficients appearing in the weight-$\frac{3}{2}$ operators are shifted to the ones in the weight-$\frac{3}{2}$ operators appearing on the right-hand side.[12]

5. The OPE between the weight-2 operator and the weight-1 operator

Finally, the last fundamental OPE from the stress energy tensor (3.1) can be summarized by

$$L(z)\Phi_0^{(1)}(w) = \frac{1}{(z-w)^2}\Phi_0^{(1)}(w) + \frac{1}{(z-w)}\partial\Phi_0^{(1)}(w) + \cdots. \tag{3.30}$$

This implies that the weight-1 operator is primary.[13]

6. The $\mathcal{N}=4$ supersymmetric OPE in the $\mathcal{N}=4$ superspace

As before, we write down each component operator of the first $\mathcal{N}=4$ multiplet in the $\mathcal{N}=4$ superspace as follows [11,15,18]:

$$\begin{aligned}\mathbf{\Phi}^{(1)} &= \Phi_0^{(1)} + \theta^i\Phi_{\frac{1}{2}}^{(1),i} + \theta^{4-ij}\Phi_1^{(1),ij} + \theta^{4-i}\Phi_{\frac{3}{2}}^{(1),i} + \theta^{4-0}\Phi_2^{(1)} \\ &\equiv (\Phi_0^{(1)}, \Phi_{\frac{1}{2}}^{(1),i}, \Phi_1^{(1),ij}, \Phi_{\frac{3}{2}}^{(1),i}, \Phi_2^{(1)}), \quad i,j = 1,\ldots,4.\end{aligned} \tag{3.32}$$

The precise relations between the components and its superfields in (3.32) can be described by [11,15], similar to (3.13),

$$\begin{aligned}&\Phi_0^{(1)} \leftrightarrow \mathbf{\Phi}^{(1)}, \quad \Phi_{\frac{1}{2}}^{(1),i} \leftrightarrow D^i\mathbf{\Phi}^{(1)}, \\ &\Phi_1^{(1),ij} \leftrightarrow -\frac{1}{2!}\varepsilon^{ijkl}D^kD^l\mathbf{\Phi}^{(1)}, \quad \Phi_{\frac{3}{2}}^{(1),i} \leftrightarrow \frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{\Phi}^{(1)}, \\ &\Phi_2^{(1)} \leftrightarrow \frac{1}{4!}\varepsilon^{ijkl}D^iD^jD^kD^l\mathbf{\Phi}^{(1)}.\end{aligned} \tag{3.33}$$

[12] We have the OPE

$$\begin{aligned}(G^i - i(1-4\lambda)\partial\Gamma^i)(z)\Phi_0^{(1)}(w) &= -\frac{1}{(z-w)^2}i(1-4\lambda)\Gamma^i(w) \\ &\quad -\frac{1}{(z-w)}\Phi_{\frac{1}{2}}^{(1),i}(w) + \cdots,\end{aligned} \tag{3.29}$$

which will be used in the $\mathcal{N}=4$ superspace description.

[13] Similarly, the following OPE can be obtained:

$$\begin{aligned}&\left(L + \frac{1}{2}(1-4\lambda)\partial U\right)(z)\Phi_0^{(1)}(w) \\ &= -\frac{1}{(z-w)^3}N(1-4\lambda) + \frac{1}{(z-w)^2}\Phi_0^{(1)}(w) + \frac{1}{(z-w)}\partial\Phi_0^{(1)}(w) \\ &\quad + \cdots,\end{aligned} \tag{3.31}$$

which will be used in the $\mathcal{N}=4$ superspace description.

In other words, by taking the fermionic coordinates on the right-hand sides to zero, we obtain the corresponding operators on the left-hand sides.

Therefore, the previous Eqs. (3.25)–(3.28) [or (3.29)], and (3.30) [or (3.31)], including other various OPEs, can be rewritten as

$$\begin{aligned}\mathbf{J}(Z_1)\mathbf{\Phi}^{(1)}(Z_2) = \Bigg[&-\frac{\theta_{12}^{4-0}}{z_{12}^3}2N(1-4\lambda)\\&+\frac{\theta_{12}^{4-i}}{z_{12}^2}(1-4\lambda)D^i\mathbf{J}(Z_2)-\frac{\theta_{12}^i}{z_{12}}D^i\mathbf{J}-\frac{1}{z_{12}}N\Bigg]\\&+\frac{\theta_{12}^{4-0}}{z_{12}^2}2\mathbf{\Phi}^{(1)}(Z_2)+\frac{\theta_{12}^{4-i}}{z_{12}}D^i\mathbf{\Phi}^{(1)}(Z_2)\\&+\frac{\theta_{12}^{4-0}}{z_{12}}2\partial\mathbf{\Phi}^{(1)}(Z_2)+\cdots. \qquad (3.34)\end{aligned}$$

This implies that the first $\mathcal{N}=4$ multiplet is not a primary operator under the $\mathcal{N}=4$ supersymmetry because there are the first four terms in (3.34).

In Appendix B, we present all the component OPEs explicitly. As before, these can be checked by using the superderivatives in [11].

E. The OPEs between the first $\mathcal{N}=4$ multiplet and itself

We calculate the OPEs between the five kinds of operators in the first $\mathcal{N}=4$ multiplet and the lowest weight-1 operator in that multiplet.

1. The OPE between the weight-1 operator and itself

By using Eqs. (3.16), (2.6), (2.2), and (2.1), we obtain the following OPE:

$$\Phi_0^{(1)}(z)\Phi_0^{(1)}(w) = \frac{1}{(z-w)^2}[2N(1-4\lambda)]+\cdots. \qquad (3.35)$$

Note the presence of the factor $(1-4\lambda)$ in the above.

2. The OPE between the weight-$\frac{3}{2}$ operators and the weight-1 operator

With the explicit expressions (3.18) and the previous defining relations, we obtain the following OPEs

$$\Phi_{\frac{1}{2}}^{(1),i}(z)\Phi_0^{(1)}(w) = -\frac{1}{(z-w)}[G^i](w)+\cdots. \qquad (3.36)$$

This implies that the role of the weight-1 operator $\Phi_0^{(1)}(w)$ in this OPE changes the signs of $\bar{Q}_{\frac{3}{2}}^{\lambda,\bar{a}b}$. This leads to the first order pole on the right-hand side of the above OPE.

3. The OPE between the weight-2 operators and the weight-1 operator

With the help of (3.20), we determine the following OPEs:

$$\begin{aligned}\Phi_1^{(1),ij}(z)\Phi_0^{(1)}(w) = &\frac{1}{(z-w)^2}[2i(1-4\lambda)T^{ij}+i\varepsilon^{ijkl}T^{kl}](w)\\&+\frac{1}{(z-w)}[2i(1-4\lambda)\partial T^{ij}\\&+i\varepsilon^{ijkl}\partial T^{kl}](w)m+\cdots. \qquad (3.37)\end{aligned}$$

In this case, the role of $\Phi_0^{(1)}(w)$ in this OPE decreases the weight of the weight-2 operator on the left-hand side by one, and it turns out that the second order pole of above OPE is a linear combination of (3.4). We have mentioned that there are some similarities in the weight-2 operator $\Phi_1^{(1),ij}$ and the weight-1 operator T^{ij}. Furthermore, the relative coefficients 1 and 1 between the second and first order poles on the right-hand side can be understood from the property of the standard conformal field theory analysis based on the weights of $\Phi_1^{(1),ij}$, $\Phi_0^{(1)}$, and T^{ij} which are primary under the stress energy tensor.

4. The OPE between the weight-$\frac{5}{2}$ operators and the weight-1 operator

From the expressions of (3.22), we determine the following OPEs:

$$\begin{aligned}&\Phi_{\frac{3}{2}}^{(1),i}(z)\Phi_0^{(1)}(w)\\&=\frac{1}{(z-w)^3}[16i\lambda(1-2\lambda)\Gamma^i](w)\\&\quad+\frac{1}{(z-w)^2}[32i\lambda(1-2\lambda)\partial\Gamma^i+3(1-4\lambda)G^i](w)\\&\quad+\frac{1}{(z-w)}\Bigg[24i\lambda(1-2\lambda)\partial^2\Gamma^i\\&\quad+\frac{8}{3}(1-4\lambda)\partial G^i+\frac{1}{2}\Phi_{\frac{1}{2}}^{(2),i}\Bigg](w)+\cdots. \qquad (3.38)\end{aligned}$$

Note that the weight-$\frac{5}{2}$ operator on the left-hand side is not a quasiprimary operator, as mentioned before. Therefore, the coefficients of the descendants in the above OPE are not known in general. It turns out that according to the realization of the $\beta\gamma$ and bc ghost system, we obtain the above result. If we use the quasiprimary weight-$\frac{5}{2}$ operator $\tilde{\Phi}_{\frac{3}{2}}^{(1),i}$, then the coefficient of the G^i in the second order pole is given by $\frac{8}{3}(1-4\lambda)$ and others remain the same. The contributions from the extra terms in the $\tilde{\Phi}_{\frac{3}{2}}^{(1),i}$ can be used from (3.36). The first order pole in the above does not change when we use the weight-$\frac{5}{2}$ operator $\Phi_{\frac{3}{2}}^{(1),i}$ or $\tilde{\Phi}_{\frac{3}{2}}^{(1),i}$.

After subtracting the descendants in the first order pole, we are left with a new quasiprimary operator which cannot be written in terms of the previously known operators. It turns out that there are

$$\begin{aligned}
\Phi_{\frac{1}{2}}^{(2),1} &= -2\times\left[\frac{1}{2}(Q_{\frac{5}{2}}^{\lambda,11}+i\sqrt{2}Q_{\frac{5}{2}}^{\lambda,12}+2i\sqrt{2}Q_{\frac{5}{2}}^{\lambda,21}-2Q_{\frac{5}{2}}^{\lambda,22}+2\bar{Q}_{\frac{5}{2}}^{\lambda,11}+2i\sqrt{2}\bar{Q}_{\frac{5}{2}}^{\lambda,12}+i\sqrt{2}\bar{Q}_{\frac{5}{2}}^{\lambda,21}-\bar{Q}_{\frac{5}{2}}^{\lambda,22})\right],\\
\Phi_{\frac{1}{2}}^{(2),2} &= -2\times\left[-\frac{i}{2}(Q_{\frac{5}{2}}^{\lambda,11}+2i\sqrt{2}Q_{\frac{5}{2}}^{\lambda,21}-2Q_{\frac{5}{2}}^{\lambda,22}+2\bar{Q}_{\frac{5}{2}}^{\lambda,11}+2i\sqrt{2}\bar{Q}_{\frac{5}{2}}^{\lambda,12}-\bar{Q}_{\frac{5}{2}}^{\lambda,22})\right],\\
\Phi_{\frac{1}{2}}^{(2),3} &= -2\times\left[-\frac{i}{2}(Q_{\frac{5}{2}}^{\lambda,11}+i\sqrt{2}Q_{\frac{5}{2}}^{\lambda,12}-2Q_{\frac{5}{2}}^{\lambda,22}+2\bar{Q}_{\frac{5}{2}}^{\lambda,11}+i\sqrt{2}\bar{Q}_{\frac{5}{2}}^{\lambda,21}-\bar{Q}_{\frac{5}{2}}^{\lambda,22})\right],\\
\Phi_{\frac{1}{2}}^{(2),4} &= -2\times\left[-\frac{1}{2}Q_{\frac{5}{2}}^{\lambda,11}-Q_{\frac{5}{2}}^{\lambda,22}+\bar{Q}_{\frac{5}{2}}^{\lambda,11}+\frac{1}{2}\bar{Q}_{\frac{5}{2}}^{\lambda,22}\right]. \qquad (3.39)
\end{aligned}$$

We realize that this looks similar to the ones in (3.18) in the sense that we obtain the above expressions by increasing the weight of the right-hand sides of (3.18) by one and multiplying the overall factor −2. At this moment, it is not clear how the numerical factor $\frac{1}{2}$ in the coefficient $\frac{1}{2}$ of $\Phi_{\frac{1}{2}}^{(2),i}$ in the first order pole appears.

5. The OPE between the weight-3 operator and the weight-1 operator

By using the expression in (3.24), we obtain

$$\begin{aligned}
\Phi_2^{(1)}(z)\Phi_0^{(1)}(w) &= \frac{1}{(z-w)^4}[4N(1-12\lambda+24\lambda^2)] + \frac{1}{(z-w)^3}[32\lambda(1-2\lambda)U](w)\\
&+\frac{1}{(z-w)^2}\left[48\lambda(1-2\lambda)\partial U+2\left(\Phi_0^{(2)}-\frac{8}{3}(1-4\lambda)L\right)\right](w)\\
&+\frac{1}{(z-w)}\left[32\lambda(1-2\lambda)\partial^2 U+2\left(\partial\Phi_0^{(2)}-\frac{8}{3}(1-4\lambda)\partial L\right)\right](w)+\cdots. \qquad (3.40)
\end{aligned}$$

Similarly, for the quasiprimary weight-3 operator $\tilde{\Phi}_2^{(1)}(z)$, the corresponding OPE has the fourth order term as $-16N\lambda(1-2\lambda)$ where the relation (3.35) is used and the remaining singular terms remain the same as above. Again, after subtracting the descendant in the second order pole, there exists a new quasiprimary operator of weight-2 which cannot be written in terms of previously known operators.

It turns out that we obtain the lowest operator in the second $\mathcal{N}=4$ multiplet shifted by the stress energy tensor

$$\begin{aligned}
\Phi_0^{(2)}-\frac{8}{3}(1-4\lambda)L = -2\times[4((1-2\lambda)(W_{\mathrm{F},2}^{\lambda,11}+W_{\mathrm{F},2}^{\lambda,22})\\
-2\lambda(W_{\mathrm{B},2}^{\lambda,11}+W_{\mathrm{B},2}^{\lambda,22}))]. \qquad (3.41)
\end{aligned}$$

The structure of the right-hand side looks similar to the one in (3.16), and we obtain the above expression by increasing the weight of the right-hand side of (3.16) by one and multiplying the overall factor of −2. Also it is not clear how the numerical factor of 2 in front of the second terms in the second order pole appears.[14]

6. The $\mathcal{N}=4$ supersymmetric OPE in the $\mathcal{N}=4$ superspace

Now we would like to construct the single $\mathcal{N}=4$ supersymmetric OPE in the $\mathcal{N}=4$ superspace based on the previous component results. The precise relations between the components and its superfields can be summarized by (3.13) and (3.33).

Then we eventually determine the following $\mathcal{N}=4$ super OPE, after putting the above five fundamental OPEs (3.35)–(3.40) into the corresponding singular terms in the $\mathcal{N}=4$ superspace

[14]We can write down the weight-2 operator from (3.41) as

$$\begin{aligned}
\Phi_0^{(2)} = -\frac{16}{3}(1-\lambda)(W_{\mathrm{F},2}^{\lambda,11}+W_{\mathrm{F},2}^{\lambda,22})\\
+\frac{8}{3}(1+2\lambda)(W_{\mathrm{B},2}^{\lambda,11}+W_{\mathrm{B},2}^{\lambda,22}). \qquad (3.42)
\end{aligned}$$

$$
\begin{aligned}
\mathbf{\Phi}^{(1)}(Z_1)\mathbf{\Phi}^{(1)}(Z_2) = {} & \frac{\theta_{12}^{4-0}}{z_{12}^4} 4N(1-12\lambda+24\lambda^2) + \frac{\theta_{12}^{4-i}}{z_{12}^3}[16\lambda(1-2\lambda)D^i\mathbf{J}](Z_2) + \frac{\theta_{12}^{4-0}}{z_{12}^3}[32\lambda(1-2\lambda)\partial\mathbf{J}](Z_2) + \frac{1}{z_{12}^2}2N(1-4\lambda) \\
& + \frac{\theta_{12}^{4-ij}}{z_{12}^2}\left[2(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}D^kD^l\mathbf{J} + \frac{1}{2!}\varepsilon^{ijkl}\varepsilon^{klmn}D^mD^n\mathbf{J}\right](Z_2) \\
& + \frac{\theta_{12}^{4-i}}{z_{12}^2}\left[32\lambda(1-2\lambda)\partial D^i\mathbf{J} - 3(1-4\lambda)\left(\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{J} - (1-4\lambda)\partial D^i\mathbf{J}\right)\right](Z_2) \\
& + \frac{\theta_{12}^{4-0}}{z_{12}^2}\left[48\lambda(1-2\lambda)\partial^2\mathbf{J} + 2\left(\mathbf{\Phi}^{(2)} - \frac{8}{3}(1-4\lambda)\left(\frac{1}{2}\frac{1}{4!}\varepsilon^{ijkl}D^iD^jD^kD^l\mathbf{J} - \frac{1}{2}(1-4\lambda)\partial^2\mathbf{J}\right)\right)\right](Z_2) \\
& + \frac{\theta_{12}^{i}}{z_{12}}\left[\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{J} + (1-4\lambda)\partial D^i\mathbf{J}\right](Z_2) \\
& + \frac{\theta_{12}^{4-ij}}{z_{12}}\left[2(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}\partial D^kD^l\mathbf{J} + \frac{1}{2!}\varepsilon^{ijkl}\varepsilon^{klmn}\partial D^mD^n\mathbf{J}\right](Z_2) \\
& + \frac{\theta_{12}^{4-i}}{z_{12}}\left[24\lambda(1-2\lambda)\partial^2D^i\mathbf{J} + \frac{1}{2}D^i\mathbf{\Phi}^{(2)} - \frac{8}{3}(1-4\lambda)\left(\frac{1}{3!}\varepsilon^{ijkl}\partial D^jD^kD^l\mathbf{J} - (1-4\lambda)\partial^2D^i\mathbf{J}\right)\right](Z_2) \\
& + \frac{\theta_{12}^{4-0}}{z_{12}}\left[32\lambda(1-2\lambda)\partial^3\mathbf{J} + 2\left(\partial\mathbf{\Phi}^{(2)} - \frac{8}{3}(1-4\lambda)\left(\frac{1}{2}\frac{1}{4!}\varepsilon^{ijkl}\partial D^iD^jD^kD^l\mathbf{J} - \frac{1}{2}(1-4\lambda)\partial^3\mathbf{J}\right)\right)\right](Z_2) \\
& + \cdots.
\end{aligned}
\tag{3.43}
$$

The λ dependence appears as $(1-4\lambda)$ or $\lambda(1-2\lambda)$ except the first central term of (3.43). We have checked that the above OPE, except the two central terms appearing in the first two lines of (3.43), is the same as the one [10] under the large (N,k) limit.

In Appendix C, we write down all the component OPEs explicitly for convenience. Equivalently, all the OPEs in (C1) can be obtained from (3.43) by acting on the various superderivatives on both sides and putting the fermionic coordinates to zero. In Appendix D, the five fundamental OPEs in the $\mathcal{N}=4$ coset model under the large (N,k) limit are given explicitly in (D1). We observe that the previous OPEs (3.35)–(3.38) and (3.40) are identical to the ones in (D1) together with $\lambda=\frac{1}{2}\lambda_{co}$ in footnote 4. Furthermore, other OPEs appearing in (C1) are identified with the ones in [10] we do not present in this paper.

F. The OPEs between the other $\mathcal{N}=4$ multiplets

From the defining equation [11,15,18] of

$$G^i(z)\Phi_0^{(2)}(w)\Big|_{\frac{1}{(z-w)}} = -\Phi_{\frac{1}{2}}^{(2),i}(w), \tag{3.44}$$

which is obtained from the relation (3.17) by changing the weight properly, we can determine the quasiprimary operators of weight-$\frac{5}{2}$ appearing in (3.39) by using (3.44).

By using the following defining equation [11]:

$$G^i(z)\Phi_{\frac{1}{2}}^{(2),j}(w)\Big|_{\frac{1}{(z-w)}} = -\left[\delta^{ij}\partial\Phi_0^{(2)} - \frac{1}{2}\varepsilon^{ijkl}\Phi_1^{(2),kl}\right](w), \tag{3.45}$$

coming from (3.19), we obtain the weight-3 quasiprimary operators, by taking two different indices in (3.45), as follows:

$$
\begin{aligned}
\Phi_1^{(2),12} &= -2\times[2iW_{\mathrm{B},3}^{\lambda,11} - \sqrt{2}W_{\mathrm{B},3}^{\lambda,12} - 2iW_{\mathrm{B},3}^{\lambda,22} + 2iW_{\mathrm{F},3}^{\lambda,11} \\
&\quad - 2\sqrt{2}W_{\mathrm{F},3}^{\lambda,12} - 2iW_{\mathrm{F},3}^{\lambda,22}], \\
\Phi_1^{(2),13} &= -2\times[-2iW_{\mathrm{B},3}^{\lambda,11} + 4\sqrt{2}W_{\mathrm{B},3}^{\lambda,21} + 2iW_{\mathrm{B},3}^{\lambda,22} - 2iW_{\mathrm{F},3}^{\lambda,11} \\
&\quad + 2\sqrt{2}W_{\mathrm{F},3}^{\lambda,21} + 2iW_{\mathrm{F},3}^{\lambda,22}], \\
\Phi_1^{(2),14} &= -2\times[2W_{\mathrm{B},3}^{\lambda,11} + i\sqrt{2}W_{\mathrm{B},3}^{\lambda,12} + 4i\sqrt{2}W_{\mathrm{B},3}^{\lambda,21} - 2W_{\mathrm{B},3}^{\lambda,22} \\
&\quad - 2W_{\mathrm{F},3}^{\lambda,11} - 2i\sqrt{2}W_{\mathrm{F},3}^{\lambda,12} - 2i\sqrt{2}W_{\mathrm{F},3}^{\lambda,21} + 2W_{\mathrm{F},3}^{\lambda,22}], \\
\Phi_1^{(2),23} &= -2\times[-2W_{\mathrm{B},3}^{\lambda,11} - i\sqrt{2}W_{\mathrm{B},3}^{\lambda,12} - 4i\sqrt{2}W_{\mathrm{B},3}^{\lambda,21} \\
&\quad + 2W_{\mathrm{B},3}^{\lambda,22} - 2W_{\mathrm{F},3}^{\lambda,11} - 2i\sqrt{2}W_{\mathrm{F},3}^{\lambda,12} \\
&\quad - 2i\sqrt{2}W_{\mathrm{F},3}^{\lambda,21} + 2W_{\mathrm{F},3}^{\lambda,22}], \\
\Phi_1^{(2),24} &= -2\times[-2iW_{\mathrm{B},3}^{\lambda,11} + 4\sqrt{2}W_{\mathrm{B},3}^{\lambda,21} + 2iW_{\mathrm{B},3}^{\lambda,22} \\
&\quad + 2iW_{\mathrm{F},3}^{\lambda,11} - 2\sqrt{2}W_{\mathrm{F},3}^{\lambda,21} - 2iW_{\mathrm{F},3}^{\lambda,22}], \\
\Phi_1^{(2),34} &= -2\times[-2iW_{\mathrm{B},3}^{\lambda,11} + \sqrt{2}W_{\mathrm{B},3}^{\lambda,12} + 2iW_{\mathrm{B},3}^{\lambda,22} + 2iW_{\mathrm{F},3}^{\lambda,11} \\
&\quad - 2\sqrt{2}W_{\mathrm{F},3}^{\lambda,12} - 2iW_{\mathrm{F},3}^{\lambda,22}].
\end{aligned}
\tag{3.46}
$$

From the defining equation [11] of

$$
\begin{aligned}
G^i(z)\Phi_1^{(2),jk}(w)\Big|_{\frac{1}{(z-w)}} &= -[(\delta^{ij}\Phi_{\frac{3}{2}}^{(2),k} - \delta^{ik}\Phi_{\frac{3}{2}}^{(2),j}) \\
&\quad + \varepsilon^{ijkl}\partial\Phi_{\frac{1}{2}}^{(2),l}](w),
\end{aligned}
\tag{3.47}
$$

which can be obtained from (3.21), we can determine the following weight-$\frac{7}{2}$ quasiprimary operators by taking the appropriate indices in (3.47):

$$\tilde{\Phi}_{\frac{3}{2}}^{(2),1} \equiv \Phi_{\frac{3}{2}}^{(2),1} - \frac{1}{5}(1-4\lambda)\partial\Phi_{\frac{1}{2}}^{(2),1}$$
$$= -2\times\left[-\frac{1}{2}(Q_{\frac{7}{2}}^{\lambda,11} + i\sqrt{2}Q_{\frac{7}{2}}^{\lambda,12} + 2i\sqrt{2}Q_{\frac{7}{2}}^{\lambda,21} - 2Q_{\frac{7}{2}}^{\lambda,22} - 2\bar{Q}_{\frac{7}{2}}^{\lambda,11} - 2i\sqrt{2}\bar{Q}_{\frac{7}{2}}^{\lambda,12} - i\sqrt{2}\bar{Q}_{\frac{7}{2}}^{\lambda,21} + \bar{Q}_{\frac{7}{2}}^{\lambda,22})\right],$$

$$\tilde{\Phi}_{\frac{3}{2}}^{(2),2} \equiv \Phi_{\frac{3}{2}}^{(2),2} - \frac{1}{5}(1-4\lambda)\partial\Phi_{\frac{1}{2}}^{(2),2}$$
$$= -2\times\left[\frac{i}{2}(Q_{\frac{7}{2}}^{\lambda,11} + 2i\sqrt{2}Q_{\frac{7}{2}}^{\lambda,21} - 2Q_{\frac{7}{2}}^{\lambda,22} - 2\bar{Q}_{\frac{7}{2}}^{\lambda,11} - 2i\sqrt{2}\bar{Q}_{\frac{7}{2}}^{\lambda,12} + \bar{Q}_{\frac{7}{2}}^{\lambda,22})\right],$$

$$\tilde{\Phi}_{\frac{3}{2}}^{(2),3} \equiv \Phi_{\frac{3}{2}}^{(2),3} - \frac{1}{5}(1-4\lambda)\partial\Phi_{\frac{1}{2}}^{(2),3}$$
$$= -2\times\left[\frac{i}{2}(Q_{\frac{7}{2}}^{\lambda,11} + i\sqrt{2}Q_{\frac{7}{2}}^{\lambda,12} - 2Q_{\frac{7}{2}}^{\lambda,22} - 2\bar{Q}_{\frac{7}{2}}^{\lambda,11} - i\sqrt{2}\bar{Q}_{\frac{7}{2}}^{\lambda,21} + \bar{Q}_{\frac{7}{2}}^{\lambda,22})\right],$$

$$\tilde{\Phi}_{\frac{3}{2}}^{(2),4} \equiv \Phi_{\frac{3}{2}}^{(2),4} - \frac{1}{5}(1-4\lambda)\partial\Phi_{\frac{1}{2}}^{(2),4}$$
$$= -2\times\left[\frac{1}{2}(Q_{\frac{7}{2}}^{\lambda,11} + 2Q_{\frac{7}{2}}^{\lambda,22} + 2\bar{Q}_{\frac{7}{2}}^{\lambda,11} + \bar{Q}_{\frac{7}{2}}^{\lambda,22})\right]. \quad (3.48)$$

Finally, by using the following defining equation [11]:

$$G^i(z)\Phi_{\frac{3}{2}}^{(2),j}(w)\Big|_{\frac{1}{(z-w)}} = -[\partial\Phi_1^{(2),ij} + \delta^{ij}\Phi_2^{(2)}](w), \quad (3.49)$$

which is obtained from (3.23), we obtain the weight-4 quasiprimary operator, by taking two equal indices in (3.49),

$$\tilde{\Phi}_2^{(2)} \equiv \Phi_2^{(2)} - \frac{1}{5}(1-4\lambda)\partial^2\Phi_0^{(2)}$$
$$= -2\times[-2(W_{\mathrm{B},4}^{\lambda,11} + W_{\mathrm{B},4}^{\lambda,22} + W_{\mathrm{F},4}^{\lambda,11} + W_{\mathrm{F},4}^{\lambda,22})]. \quad (3.50)$$

Therefore, the second $\mathcal{N}=4$ multiplet by considering the super weight-2 in (3.32) is given by (3.42), (3.39), (3.46), (3.48), and (3.50).

We can further analyze the OPEs between the next $\mathcal{N}=4$ multiplets. By using the previous relations (3.16), (3.18), (3.20), (3.22), and (3.24) and the relation (3.42), the OPEs between the $\mathcal{N}=4$ stress energy tensor and the second $\mathcal{N}=4$ multiplet are given in Appendix E in $\mathcal{N}=4$ superspace explicitly. The OPEs between the first and the second $\mathcal{N}=4$ multiplets can be obtained and are given in Appendix F in $\mathcal{N}=4$ superspace explicitly. Moreover, the OPEs between the second $\mathcal{N}=4$ multiplet and itself can be determined, and they are given in Appendix G. Then we can determine the third $\mathcal{N}=4$ multiplet in (F2) and the fourth $\mathcal{N}=4$ multiplet in (G2). In Appendixes H–J, the OPEs between the $\mathcal{N}=4$ stress energy tensor and the third $\mathcal{N}=4$ multiplet, the OPEs between the $\mathcal{N}=4$ stress energy tensor and the fourth $\mathcal{N}=4$ multiplet, and the OPEs between the first $\mathcal{N}=4$ multiplet and the third $\mathcal{N}=4$ multiplet are presented, respectively.

IV. CONCLUSIONS AND OUTLOOK

By using the $\beta\gamma$ and bc ghost systems explicitly, we have constructed the generators in the $\mathcal{N}=4$ supersymmetric linear $W_\infty[\lambda]$ algebra: the $\mathcal{N}=4$ stress energy tensor, the first $\mathcal{N}=4$ multiplet, and the second $\mathcal{N}=4$ multiplet (and the third and fourth $\mathcal{N}=4$ multiplets). Moreover, their algebras between these generators are determined, and in particular, the OPEs between the first $\mathcal{N}=4$ mutiplet and itself are equivalent to the corresponding ones in the $\mathcal{N}=4$ coset model under the large (N,k) limit. Contrary to the findings in [19], the modes of the currents in the present results are not restricted to the wedges but can have any integers or half integers because our construction is based on the OPEs between the currents.

So far, we have considered the OPEs between the $\mathcal{N}=4$ stress energy tensor and the first $\mathcal{N}=4$ multiplet (and other OPEs in Appendixes E–J). Then it is natural to ask what are the OPEs between the h_1th $\mathcal{N}=4$ multiplet and the h_2th $\mathcal{N}=4$ multiplet for any weights h_1 and h_2. In the analysis of (G2), we can figure out the explicit form for the five kinds of currents for general weight-h. The lowest component can be obtained easily up to the overall normalization. The remaining components can also be determined with the weight dependent overall factors. Then the question is how we can write down the OPEs between the currents appearing in (2.4) for each component in terms of the λ dependent structure constants introduced in [19]. It would be interesting to rewrite all the structure constants obtained in this paper in terms of previously known ones presented in [7]. This will give us some hints to figure out their behaviors for generic weights.[15]

[15]For example, the structure constant appearing in $W_{F,2}^{\lambda,\bar{a}b}\delta_{b\bar{a}}(w)$ of the OPE between $W_{F,4}^{\lambda,\bar{a}b}\delta_{b\bar{a}}(z)$ and $W_{F,4}^{\lambda,\bar{a}b}\delta_{b\bar{a}}(w)$ is given by $\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3)$ around Eq. (3.18) in [7]. This λ dependent function is related to $p_{F,4}^{4,4}(m,n,\lambda)$ appearing in that paper. By realizing that we can extract $W_{F,4}^{\lambda,\bar{a}b}\delta_{b\bar{a}}$ from the present context and we have $W_{F,4}^{\lambda,\bar{a}b}\delta_{b\bar{a}} = \frac{1}{28}(3+2\lambda)\Phi_2^{(2)} - \frac{1}{384}\Phi_0^{(4)} + \frac{1}{140}(3+2\lambda)(-1+4\lambda)\partial^2\Phi_0^{(2)}$, we can check that the sixth order pole in the OPE between these three terms and itself reproduces the above structure constant. Furthermore, by using the weight-4 current $W_{B,4}^{\lambda,\bar{a}b}\delta_{b\bar{a}} = \frac{1}{14}(2-\lambda)\Phi_2^{(2)} + \frac{1}{384}\Phi_0^{(4)} - \frac{1}{70}(-2+\lambda)(-1+4\lambda)\partial^2\Phi_0^{(2)}$ and the weight-3 current $W_{B,3}^{\lambda,\bar{a}b}\delta_{b\bar{a}} = \frac{1}{10}(-3+2\lambda)\Phi_2^{(1)} - \frac{1}{48}\Phi_0^{(3)} + \frac{1}{30}(-3+2\lambda)(-1+4\lambda)\partial^2\Phi_0^{(1)}$, we obtain the sixth order pole in the OPE between them, which is equal to $512(\lambda-1)\lambda(2\lambda-3)(2\lambda+1)$. This structure constant appears in Eq. (B.2) of [7] and is related to $p_{B,4}^{4,3}(m,n,\lambda)$. In these examples, the bifundamental indices are contracted with each other. However, we should also obtain the OPEs between the currents with free bifundamental indices for generic weights.

Because the second $\mathcal{N}=4$ multiplet from the free field approach is not directly related to the corresponding $\mathcal{N}=4$ multiplet from the coset fields in [10] (for example, the OPE between the $\Phi_0^{(1)}$ and $\Phi_0^{(2)}$ in the former does not vanish while that in the latter does vanish), it would be interesting to obtain the correct second $\mathcal{N}=4$ multiplet in the coset model at finite (N,k) as a first step. Note that according to the free field approach in this paper, the currents (3.42), (3.39), (3.46), (3.48), and (3.50) are the quasiprimary operators under the stress energy tensor. We need to find out the correct basis where the corresponding currents in the $\mathcal{N}=4$ coset model should reflect this quasiprimary condition at least by calculating all the nonlinear terms. After that, we also expect that under the large (N,k) limit, for example, the OPEs between the first $\mathcal{N}=4$ multiplet and the second $\mathcal{N}=4$ multiplet in the coset model will produce the ones in Appendix F.

In the context of celestial holography [20], we have seen that the wedge subalgebra of $w_{1+\infty}$ algebra [21] provides the symmetries on the celestial sphere [22]. See also [23]. Moreover, the analysis for the $\mathcal{N}=1$ supersymmetric $w_{1+\infty}$ algebra is obtained in [24,25]. In the present context, the above $w_{1+\infty}$ algebra is related to the OPEs between the currents $W_{F,h}^{\lambda,\bar{a}b}\delta_{b\bar{a}}$ with the structure constants $p_{F,h}^{h_1,h_2}(m,n,\lambda)$ described in footnote 15. In the context of $\mathcal{N}=4$ supersymmetric linear $W_\infty[\lambda]$ algebra, the currents of weight-h are made of (1) the lowest current in the hth $\mathcal{N}=4$ multiplet, (2) the middle current in the $(h-1)$th $\mathcal{N}=4$ multiplet, and (3) the highest current (and the lowest current with two derivatives) in the $(h-2)$th $\mathcal{N}=4$ multiplet. It would be interesting to observe whether the corresponding supersymmetric Einstein-Yang-Mills theory at nonzero deformation parameters λ (or q) reveals the OPEs we have obtained in this paper or not.

In [2,3], the explicit representation of the corresponding algebra is given by the differential operators in terms of commuting parameters $\Lambda^{\pm}(z)$ and anticommuting parameters $\Theta^{\pm}(z)$.[16] Under the symmetry generated by the currents in this paper, the transformation of any fields (or operators) is given by the following contour integrals over z with the OPEs between the currents and the fields (along the lines of [26])

$$\begin{aligned}\delta_{\Lambda^{\pm}_{\bar{a}b}}f(w) &= \frac{1}{2\pi i}\oint_{C_w} dz\Lambda^{\pm}_{\bar{a}b}(z)V^{(s)\pm}_{\lambda,\bar{a}b}(z)f(w)\\ &\equiv \left[\frac{1}{2\pi i}\oint_{C_w} dz\Lambda^{\pm}_{\bar{a}b}(z)V^{(s)\pm}_{\lambda,\bar{a}b}(z), f(w)\right],\\ \delta_{\Theta^{\pm}_{\bar{a}b}}f(w) &= \frac{1}{2\pi i}\oint_{C_w} dz\Theta^{\pm}_{\bar{a}b}(z)Q^{(s)\pm}_{\lambda,\bar{a}b}(z)f(w)\\ &\equiv \left[\frac{1}{2\pi i}\oint_{C_w} dz\Theta^{\pm}_{\bar{a}b}(z)Q^{(s)\pm}_{\lambda,\bar{a}b}(z), f(w)\right\}, \qquad (4.1)\end{aligned}$$

[16] We thank the referee for pointing out the questions raised in the remaining paragraphs.

where the contour C_w surrounds the point z and there are no summations over the indices $\bar{a}$ and b on the right-hand sides of (4.1). We can describe the corresponding (anti)commutator relations between the "charges" and the fields.

By using the result of the following OPE:

$$V^{(s)+}_{\lambda,\bar{a}b}(z)\beta^{\bar{m}c}(w) = \delta_{c\bar{a}}\sum_{i=0}^{s-1}a^i(s,\lambda)\left(\partial_z^{s-1-i}\frac{1}{(z-w)}\right)\partial^i\beta^{\bar{m}b}(w) + \cdots, \qquad (4.2)$$

and substituting this (4.2) into (4.1), we obtain

$$\delta_{\Lambda^{+}_{\bar{a}b}}\beta^{\bar{m}c}(w) = \left(\delta_{c\bar{a}}\sum_{i=0}^{s-1}a^i(s,\lambda)(-1)^{s-1-i}(s-1-i)! \times(\partial^{s-1-i}\Lambda^{(s)+}_{\bar{a}b}(w))\partial^i\right)\beta^{\bar{m}c}(w). \qquad (4.3)$$

This implies that we realize that there exist the corresponding linear differential operators appearing inside the bracket in (4.3). Therefore, the nontrivial differential operators occur only when the first element $\bar{a}$ of the currents $V^{(s)+}_{\lambda,\bar{a}b}(z)$ and the second element c of the operators $\beta^{\bar{m}c}(w)$ are equal to each other.

Similarly, we use the above currents on the $b^{\bar{j}c}(w)$. From the result of

$$V^{(s)+}_{\lambda,\bar{a}b}(z)b^{\bar{m}c}(w) = \delta_{c\bar{a}}\sum_{i=0}^{s-1}a^i\left(s,\lambda+\frac{1}{2}\right)\left(\partial_z^{s-1-i}\frac{1}{(z-w)}\right) \times\partial^i b^{\bar{m}b}(w) + \cdots, \qquad (4.4)$$

we determine the following transformation with (4.4):

$$\delta_{\Lambda^{+}_{\bar{a}b}}b^{\bar{j}c}(w) = \left(\delta_{c\bar{a}}\sum_{i=0}^{s-1}a^i\left(s,\lambda+\frac{1}{2}\right)(-1)^{s-1-i}(s-1-i)! \times(\partial^{s-1-i}\Lambda^{(s)+}_{\bar{a}b}(w))\partial^i\right)b^{\bar{j}c}(w). \qquad (4.5)$$

We observe that there exist the corresponding linear differential operators appearing inside the brackets in (4.5).

Because of the multiple derivatives of $\beta^{\bar{j}b}(w)$ in the currents $V^{(s)+}_{\lambda,\bar{a}b}(z)$, the next OPE is rather complicated and it turns out that

$$V^{(s)+}_{\lambda,\bar{a}b}(z)\gamma^{j\bar{c}}(w) = \delta_{b\bar{c}}\sum_{i=0}^{s-1}a^i(s,\lambda)(-1)^s i!\sum_{t=0}^{i+1}(i+1-t)_{s-1-i} \times\frac{1}{t!}\frac{1}{(z-w)^{s-t}}\partial^t\gamma^{j\bar{a}}(w) + \cdots. \qquad (4.6)$$

Note that there is a summation over t and its maximum number is given by $(i+1)$. There is also a summation over i.

Then we obtain the following result by using the relation (4.6):

$$\delta_{\Lambda^+_{\bar{a}b}}\gamma^{j\bar{c}}(w) = \left(\delta_{b\bar{c}}\sum_{i=0}^{s-1} a^i(s,\lambda)(-1)^s i! \sum_{t=0}^{i+1}(i+1-t)_{s-1-i}\frac{1}{t!}(\partial^{s-1-t}\Lambda^{(s)+}_{\bar{a}b}(w))\partial^t\right)\gamma^{j\bar{c}}(w). \quad (4.7)$$

There exist the corresponding linear differential operators with the double summations appearing inside the brackets in (4.7).

Again, because of the multiple derivatives of $b^{\bar{j}b}(w)$ in the currents $V^{(s)+}_{\lambda,\bar{a}b}(z)$, the next OPE is complicated and it turns out that

$$V^{(s)+}_{\lambda,\bar{a}b}(z)c^{j\bar{c}}(w) = \delta_{b\bar{c}}\sum_{i=0}^{s-1} a^i\left(s,\lambda+\frac{1}{2}\right)(-1)^s i! \sum_{t=0}^{i+1}(i+1-t)_{s-1-i}\frac{1}{t!}\frac{1}{(z-w)^{s-t}}\partial^t c^{j\bar{c}}(w)+\cdots. \quad (4.8)$$

The corresponding transformation, with the help of (4.8), can be written as

$$\delta_{\Lambda^+_{\bar{a}b}}c^{j\bar{c}}(w) = \left(\delta_{b\bar{c}}\sum_{i=0}^{s-1} a^i\left(s,\lambda+\frac{1}{2}\right)(-1)^s i! \sum_{t=0}^{i+1}(i+1-t)_{s-1-i}\frac{1}{t!}(\partial^{s-1-t}\Lambda^{(s)+}_{\bar{a}b}(w))\partial^t\right)c^{j\bar{c}}(w). \quad (4.9)$$

The corresponding linear differential operators with the double summations appearing inside the bracket in (4.9) occur.

Now we can consider the symmetry generated by fermionic currents, and they can be described as follows:

$$\begin{aligned}
\delta_{\Theta^+_{\bar{a}b}}\beta^{\bar{m}c}(w) &= -\left(\delta_{c\bar{a}}\sum_{i=0}^{s-2}\beta^i(s,\lambda)(-1)^{s-2-i}(s-2-i)!(\partial^{s-2-i}\Theta^{(s)+}_{\bar{a}b}(w))\partial^i\right)b^{\bar{m}c}(w),\\
\delta_{\Theta^+_{\bar{a}b}}b^{\bar{j}c}(w) &= \left(\delta_{c\bar{a}}\sum_{i=0}^{s-1}\alpha^i(s,\lambda)(-1)^{s-1-i}(s-1-i)!(\partial^{s-1-i}\Theta^{(s)+}_{\bar{a}b}(w))\partial^i\right)\beta^{\bar{j}c}(w),\\
\delta_{\Theta^+_{\bar{a}b}}\gamma^{j\bar{c}}(w) &= \left(\delta_{b\bar{c}}\sum_{i=0}^{s-1}\alpha^i(s,\lambda)(-1)^s i!\sum_{t=0}^{i+1}(i+1-t)_{s-1-i}\frac{1}{t!}(\partial^{s-1-t}\Theta^{(s)+}_{\bar{a}b}(w))\partial^t\right)c^{j\bar{c}}(w),\\
\delta_{\Theta^+_{\bar{a}b}}c^{j\bar{c}}(w) &= \left(\delta_{b\bar{c}}\sum_{i=0}^{s-2}\beta^i(s,\lambda)(-1)^{s-1}i!\sum_{t=0}^{i+1}(i+1-t)_{s-2-i}\frac{1}{t!}(\partial^{s-2-t}\Theta^{(s)+}_{\bar{a}b}(w))\partial^t\right)\gamma^{j\bar{c}}(w). \quad (4.10)
\end{aligned}$$

Therefore, we have the transformations of the $\beta\gamma$ and bc ghost systems under the bosonic and fermionic currents, summarized by (4.3), (4.5), (4.7), (4.9), and (4.10).

Furthermore, for the remaining bosonic and fermionic currents, we can calculate the corresponding OPEs with the $\beta\gamma$ and bc ghost systems, and we summarize the following results:

$$\begin{aligned}
\delta_{\Lambda^-_{\bar{a}b}}\beta^{\bar{m}c}(w) &= -\left(\delta_{c\bar{a}}\frac{(s-1+2\lambda)}{(2s-1)}\sum_{i=0}^{s-1}a^i(s,\lambda)(-1)^{s-1-i}(s-1-i)!(\partial^{s-1-i}\Lambda^{(s)-}_{\bar{a}b}(w))\partial^i\right)\beta^{\bar{m}c}(w),\\
\delta_{\Lambda^-_{\bar{a}b}}b^{\bar{j}c}(w) &= \left(\delta_{c\bar{a}}\frac{(s-2\lambda)}{(2s-1)}\sum_{i=0}^{s-1}a^i\left(s,\lambda+\frac{1}{2}\right)(-1)^{s-1-i}(s-1-i)!(\partial^{s-1-i}\Lambda^{(s)-}_{\bar{a}b}(w))\partial^i\right)b^{\bar{j}c}(w),\\
\delta_{\Lambda^-_{\bar{a}b}}\gamma^{j\bar{c}}(w) &= \left(\delta_{b\bar{c}}\frac{(s-2\lambda)}{(2s-1)}\sum_{i=0}^{s-1}a^i(s,\lambda)(-1)^s i!\sum_{t=0}^{i+1}(i+1-t)_{s-1-i}\frac{1}{t!}(\partial^{s-1-t}\Lambda^{(s)-}_{\bar{a}b}(w))\partial^t\right)\gamma^{j\bar{c}}(w),\\
\delta_{\Lambda^-_{\bar{a}b}}c^{j\bar{c}}(w) &= -\left(\delta_{b\bar{c}}\frac{(s-1+2\lambda)}{(2s-1)}\sum_{i=0}^{s-1}a^i\left(s,\lambda+\frac{1}{2}\right)(-1)^s i!\sum_{t=0}^{i+1}(i+1-t)_{s-1-i}\frac{1}{t!}(\partial^{s-1-t}\Lambda^{(s)-}_{\bar{a}b}(w))\partial^t\right)c^{j\bar{c}}(w),\\
\delta_{\Theta^-_{\bar{a}b}}\beta^{\bar{m}c}(w) &= \left(\delta_{c\bar{a}}\sum_{i=0}^{s-2}\beta^i(s,\lambda)(-1)^{s-2-i}(s-2-i)!(\partial^{s-2-i}\Theta^{(s)-}_{\bar{a}b}(w))\partial^i\right)b^{\bar{m}c}(w),
\end{aligned}$$

$$\delta_{\Theta^{-}_{\bar{a}b}} b^{\bar{j}c}(w) = \left(\delta_{c\bar{a}} \sum_{i=0}^{s-1} \alpha^i(s,\lambda)(-1)^{s-1-i}(s-1-i)!(\partial^{s-1-i}\Theta^{(s)-}_{\bar{a}b}(w))\partial^i\right)\beta^{\bar{j}c}(w),$$
$$\delta_{\Theta^{-}_{\bar{a}b}} \gamma^{j\bar{c}}(w) = \left(\delta_{b\bar{c}} \sum_{i=0}^{s-1} \alpha^i(s,\lambda)(-1)^{s} i! \sum_{t=0}^{i+1} (i+1-t)_{s-1-i} \frac{1}{t!}(\partial^{s-1-t}\Theta^{(s)-}_{\bar{a}b}(w))\partial^t\right) c^{j\bar{c}}(w),$$
$$\delta_{\Theta^{-}_{\bar{a}b}} c^{j\bar{c}}(w) = -\left(\delta_{b\bar{c}} \sum_{i=0}^{s-2} \beta^i(s,\lambda)(-1)^{s-1} i! \sum_{t=0}^{i+1} (i+1-t)_{s-2-i} \frac{1}{t!}(\partial^{s-2-t}\Theta^{(s)-}_{\bar{a}b}(w))\partial^t\right)\gamma^{j\bar{c}}(w). \quad (4.11)$$

It is straightforward to obtain the corresponding transformations of the $\beta\gamma$ and bc ghost systems for the $\mathcal{N}=4$ currents studied in this paper, by taking the linear combinations between the above results summarized in (4.3), (4.5), (4.7), and (4.9)–(4.11).

ACKNOWLEDGMENTS

We thank M. H. Kim for the intensive discussions. This work was supported by a National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (No. 2020R1F1A1066893).

APPENDIX A: THE OPES BETWEEN THE $\mathcal{N}=4$ STRESS ENERGY TENSOR AND ITSELF IN THE COMPONENT APPROACH

For the previous result (3.14), the component results can be summarized by

$$L(z)L(w) = \frac{1}{(z-w)^4}\left[\frac{1}{2}c\right] + \frac{1}{(z-w)^2}[2L](w) + \frac{1}{(z-w)}[\partial L](w) + \cdots,$$
$$L(z)G^i(w) = \frac{1}{(z-w)^2}\left[\frac{3}{2}G^i\right](w) + \frac{1}{(z-w)}[\partial G^i](w) + \cdots,$$
$$L(z)T^{ij}(w) = \frac{1}{(z-w)^2}[T^{ij}](w) + \frac{1}{(z-w)}[\partial T^{ij}](w) + \cdots,$$
$$L(z)\Gamma^i(w) = \frac{1}{(z-w)^2}\left[\frac{1}{2}\Gamma^i\right](w) + \frac{1}{(z-w)}[\partial \Gamma^i](w) + \cdots,$$
$$L(z)U(w) = -\frac{1}{(z-w)^3}[N] + \frac{1}{(z-w)^2}[U](w) + \frac{1}{(z-w)}[\partial U](w) + \cdots,$$
$$G^i(z)G^j(w) = \frac{1}{(z-w)^3}\left[\frac{2}{3}c\delta^{ij}\right] + \frac{1}{(z-w)^2}[-2iT^{ij} - i(1-4\lambda)\varepsilon^{ijkl}T^{kl}](w)$$
$$+ \frac{1}{(z-w)}\left[2\delta^{ij}L - i\partial T^{ij} - i\frac{1}{2}(1-4\lambda)\varepsilon^{ijkl}\partial T^{kl}\right](w) + \cdots,$$
$$G^i(z)T^{jk}(w) = -\frac{1}{(z-w)^2}[\varepsilon^{ijkl}\Gamma^l + (1-4\lambda)(\delta^{ik}\Gamma^j - \delta^{ij}\Gamma^k)](w)$$
$$- \frac{1}{(z-w)}[\varepsilon^{ijkl}\partial\Gamma^l + (1-4\lambda)(\delta^{ik}\partial\Gamma^j - \delta^{ij}\partial\Gamma^k) + i\delta^{ik}G^j - i\delta^{ij}G^k](w) + \cdots,$$
$$G^i(z)\Gamma^j(w) = -\frac{1}{(z-w)^2}[iN\delta^{ij}] + \frac{1}{(z-w)}\left[-\frac{1}{2}\varepsilon^{ijkl}T^{kl} + i\,\delta^{ij}U\right](w) + \cdots,$$
$$G^i(z)U(w) = -\frac{1}{(z-w)^2}[i\Gamma^i](w) - \frac{1}{(z-w)}[i\partial\Gamma^i](w) + \cdots,$$
$$T^{ij}(z)T^{kl}(w) = \frac{1}{(z-w)^2}[\varepsilon^{ijkl}N] - \frac{1}{(z-w)}i[\delta^{ik}T^{jl} - \delta^{il}T^{jk} - \delta^{jk}T^{il} + \delta^{jl}T^{ik}](w) + \cdots,$$
$$T^{ij}(z)\Gamma^k(w) = -\frac{1}{(z-w)}i[\delta^{ik}\Gamma^j - \delta^{jk}\Gamma^i](w) + \cdots. \quad (A1)$$

Compared to the large $\mathcal{N}=4$ superconformal algebra [11,15], there are two additional central terms in the fifth and eighth of (A1).[17] Moreover, there are trivial OPEs $\Gamma^i(z)\Gamma^j(w)$ and $U(z)U(w)$. Finally, the central term of the OPE $T^{ij}(z)T^{kl}(w)$ which is proportional to $(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk})$ does not appear.

APPENDIX B: THE OPES BETWEEN THE $\mathcal{N}=4$ STRESS ENERGY TENSOR AND THE FIRST $\mathcal{N}=4$ MULTIPLET IN THE COMPONENT APPROACH

We present the complete OPEs corresponding to (3.34) as follows:

$$\begin{aligned}
L(z)\Phi_2^{(1)}(w) &= -\frac{1}{(\mathtt{z}-\mathtt{w})^5}[\mathtt{96N}\lambda(\mathtt{1}-\mathtt{2}\lambda)] + \frac{1}{(z-w)^4}[-6(1-4\lambda)\Phi_0^{(1)} - \mathtt{96}\lambda(\mathtt{1}-\mathtt{2}\lambda)\mathtt{U}](w) \\
&\quad + \frac{1}{(z-w)^3}[2(1-4\lambda)\partial\Phi_0^{(1)}](w) + \frac{1}{(z-w)^2}[3\Phi_2^{(1)}](w) + \frac{1}{(z-w)}[\partial\Phi_2^{(1)}](w) + \cdots, \\
L(z)\Phi_{\frac{3}{2}}^{(1),i}(w) &= -\frac{1}{(\mathtt{z}-\mathtt{w})^4}[\mathtt{24i}\lambda(\mathtt{1}-\mathtt{2}\lambda)\mathit{\Gamma}^{\mathtt{i}}](\mathtt{w}) + \frac{1}{(z-w)^3}[(1-4\lambda)\Phi_{\frac{1}{2}}^{(1),i}](w) + \frac{1}{(z-w)^2}\left[\frac{5}{2}\Phi_{\frac{3}{2}}^{(1),i}\right](w) \\
&\quad + \frac{1}{(z-w)}[\partial\Phi_{\frac{3}{2}}^{(1),i}](w) + \cdots, \\
L(z)\Phi_1^{(1),ij}(w) &= \frac{1}{(z-w)^2}[2\Phi_1^{(1),ij}](w) + \frac{1}{(z-w)}[\partial\Phi_1^{(1),ij}](w) + \cdots, \\
L(z)\Phi_{\frac{1}{2}}^{(1),i}(w) &= \frac{1}{(z-w)^2}\left[\frac{3}{2}\Phi_{\frac{1}{2}}^{(1),i}\right](w) + \frac{1}{(z-w)}[\partial\Phi_{\frac{1}{2}}^{(1),i}](w) + \cdots, \\
L(z)\Phi_0^{(1)}(w) &= \frac{1}{(z-w)^2}[\Phi_0^{(1)}](w) + \frac{1}{(z-w)^1}[\partial\Phi_0^{(1)}](w) + \cdots, \\
G^i(z)\Phi_2^{(1)}(w) &= \frac{1}{(\mathtt{z}-\mathtt{w})^4}[\mathtt{48i}\lambda(\mathtt{1}-\mathtt{2}\lambda)\mathit{\Gamma}^{\mathtt{i}}](\mathtt{w}) + \frac{1}{(z-w)^3}[-\mathtt{48i}\lambda(\mathtt{1}-\mathtt{2}\lambda)\partial\mathit{\Gamma}^{\mathtt{i}} - 6(1-4\lambda)\Phi_{\frac{1}{2}}^{(1),i}](w) \\
&\quad + \frac{1}{(z-w)^2}[-5\Phi_{\frac{3}{2}}^{(1),i} + (1-4\lambda)\partial\Phi_{\frac{1}{2}}^{(1),i}](w) - \frac{1}{(z-w)}[\partial\Phi_{\frac{3}{2}}^{(1),i}](w) + \cdots, \\
G^i(z)\Phi_{\frac{3}{2}}^{(1),j}(w) &= \frac{1}{(\mathtt{z}-\mathtt{w})^4}[\mathtt{48N}\lambda(\mathtt{1}-\mathtt{2}\lambda)\delta^{\mathtt{ij}}] \\
&\quad + \frac{1}{(z-w)^3}[\mathtt{48}\lambda(\mathtt{1}-\mathtt{2}\lambda)\delta^{\mathtt{ij}}\mathtt{U} - \mathtt{8i}\lambda(\mathtt{1}-\mathtt{2}\lambda)\varepsilon^{\mathtt{ijkl}}\mathtt{T}^{\mathtt{kl}} + 4(1-4\lambda)\delta^{ij}\Phi_0^{(1)}](w) \\
&\quad - \frac{1}{(z-w)^2}\left[4\Phi_1^{(1),ij} + \frac{1}{2}(1-4\lambda)\varepsilon^{ijkl}\Phi_1^{(1),kl} + (1-4\lambda)\delta^{ij}\partial\Phi_0^{(1)}\right](w) \\
&\quad - \frac{1}{(z-w)}[\partial\Phi_1^{(1),ij} + \delta^{ij}\Phi_2^{(1)}](w) + \cdots, \\
G^i(z)\Phi_1^{(1),jk}(w) &= \frac{1}{(\mathtt{z}-\mathtt{w})^3}[\mathtt{16i}\lambda(\mathtt{1}-\mathtt{2}\lambda)(\delta^{\mathtt{ij}}\mathit{\Gamma}^{\mathtt{k}} - \delta^{\mathtt{ik}}\mathit{\Gamma}^{\mathtt{j}})](\mathtt{w}) \\
&\quad + \frac{1}{(z-w)^2}[-(1-4\lambda)(\delta^{ij}\Phi_{\frac{1}{2}}^{(1),k} - \delta^{ik}\Phi_{\frac{1}{2}}^{(1),j}) + 3\varepsilon^{ijkl}\Phi_{\frac{1}{2}}^{(1),l}](w) \\
&\quad + \frac{1}{(z-w)}[-(\delta^{ij}\Phi_{\frac{3}{2}}^{(1),k} - \delta^{ik}\Phi_{\frac{3}{2}}^{(1),j}) + \varepsilon^{ijkl}\partial\Phi_{\frac{1}{2}}^{(1),l}](w) + \cdots, \\
G^i(z)\Phi_{\frac{1}{2}}^{(1),j}(w) &= -\frac{1}{(z-w)^2}[2\delta^{ij}\Phi_0^{(1)}](w) - \frac{1}{(z-w)}\left[\delta^{ij}\partial\Phi_0^{(1)} - \frac{1}{2}\varepsilon^{ijkl}\Phi_1^{(1),kl}\right](w) + \cdots, \\
G^i(z)\Phi_0^{(1)}(w) &= -\frac{1}{(z-w)}[\Phi_{\frac{1}{2}}^{(1),i}](w) + \cdots,
\end{aligned}$$

[17]They are denoted by the typewriter fonts.

$$
\begin{aligned}
T^{ij}(z)\Phi_2^{(1)}(w) &= \frac{1}{(z-w)^3}[4(1-4\lambda)T^{ij}-2\varepsilon^{ijkl}T^{kl}](w)\\
&+\frac{1}{(z-w)^2}[-2(1-4\lambda)\partial T^{ij}+\varepsilon^{ijkl}\partial T^{kl}+4i\Phi_1^{(1),ij}](w)+\cdots,\\
T^{ij}(z)\Phi_{\frac{3}{2}}^{(1),k}(w) &= \frac{1}{(z-w)^3}[-2(\delta^{ik}\Gamma^j-\delta^{jk}\Gamma^i)+2(1-4\lambda)\varepsilon^{ijkl}\Gamma^l](w)\\
&+\frac{1}{(z-w)^2}[(3-8\lambda+16\lambda^2)(\delta^{ik}\partial\Gamma^j-\delta^{jk}\partial\Gamma^i)-3(1-4\lambda)\varepsilon^{ijkl}\partial\Gamma^l\\
&-3i\varepsilon^{ijkl}\Phi_{\frac{1}{2}}^{(1),l}+i(1-4\lambda)(\delta^{ik}G^j-\delta^{jk}G^i)-i\varepsilon^{ijkl}G^l](w)-\frac{1}{(z-w)}[i\delta^{ik}\Phi_{\frac{3}{2}}^{(1),j}-i\delta^{jk}\Phi_{\frac{3}{2}}^{(1),i}](w)+\cdots,\\
T^{ij}(z)\Phi_1^{(1),kl}(w) &= \frac{1}{(z-w)^3}[2Ni\delta^{ik}\delta^{jl}-2Ni\delta^{il}\delta^{jk}-2Ni(1-4\lambda)\varepsilon^{ijkl}]\\
&+\frac{1}{(z-w)^2}\Big[-2i(1-4\lambda)\varepsilon^{ijkl}U-2i(\delta^{il}\delta^{jk}-\delta^{ik}\delta^{jl})U\\
&-(1-4\lambda)(\delta^{ik}T^{jl}-\delta^{il}T^{jk}-\delta^{jk}T^{il}+\delta^{jl}T^{ik})\\
&+\frac{1}{2}(\delta^{ik}\varepsilon^{jlmn}-\delta^{il}\varepsilon^{jkmn}-\delta^{jk}\varepsilon^{ilmn}+\delta^{jl}\varepsilon^{ikmn})T^{mn}+2i\varepsilon^{ijkl}\Phi_0^{(1)}\Big](w)\\
&-\frac{1}{(z-w)}[i\delta^{ik}\Phi_1^{(1),jl}-i\delta^{il}\Phi_1^{(1),jk}-i\delta^{jk}\Phi_1^{(1),il}+i\delta^{jl}\Phi_1^{(1),ik}](w)+\cdots,\\
T^{ij}(z)\Phi_{\frac{1}{2}}^{(1),k}(w) &= \frac{1}{(z-w)^2}[(1-4\lambda)(\delta^{ik}\Gamma^j-\delta^{jk}\Gamma^i)-\varepsilon^{ijkl}\Gamma^l](w)+\frac{1}{(z-w)}[-i\delta^{ik}\Phi_{\frac{1}{2}}^{(1),j}+i\delta^{jk}\Phi_{\frac{1}{2}}^{(1),i}](w)+\cdots,\\
U(z)\Phi_2^{(1)}(w) &= \frac{1}{(z-w)^4}[6N(1-4\lambda)]+\frac{1}{(z-w)^3}[-4\Phi_0^{(1)}+8(1-4\lambda)U](w)\\
&+\frac{1}{(z-w)^2}[2\partial\Phi_0^{(1)}-4(1-4\lambda)\partial U-8L](w)+\cdots,\\
U(z)\Phi_{\frac{3}{2}}^{(1),i}(w) &= \frac{1}{(z-w)^3}[2i(1-4\lambda)\Gamma^i](w)+\frac{1}{(z-w)^2}[-3i(1-4\lambda)\partial\Gamma^i+\Phi_{\frac{1}{2}}^{(1),i}+3G^i](w)+\cdots,\\
U(z)\Phi_1^{(1),ij}(w) &= \frac{1}{(z-w)^2}[2iT^{ij}](w)+\cdots,\\
U(z)\Phi_{\frac{1}{2}}^{(1),i}(w) &= -\frac{1}{(z-w)^2}[i\Gamma^i](w)+\cdots,\\
U(z)\Phi_0^{(1)}(w) &= \frac{1}{(z-w)^2}[N]+\cdots,\\
\Gamma^i(z)\Phi_2^{(1)}(w) &= \frac{1}{(z-w)^3}[2(1-4\lambda)\Gamma^i](w)+\frac{1}{(z-w)^2}[-6(1-4\lambda)\partial\Gamma^i-3i\Phi_{\frac{1}{2}}^{(1),i}-3iG^i](w)\\
&+\frac{1}{(z-w)}[(1-4\lambda)\partial^2\Gamma^i+i\partial\Phi_{\frac{1}{2}}^{(1),i}+i\partial G^i](w)+\cdots,\\
\Gamma^i(z)\Phi_{\frac{3}{2}}^{(1),j}(w) &= -\frac{1}{(z-w)^3}[2Ni(1-4\lambda)\delta^{ij}]\\
&+\frac{1}{(z-w)^2}\Big[2T^{ij}+2i\delta^{ij}\Phi_0^{(1)}-3i(1-4\lambda)\delta^{ij}U-\frac{1}{2}(1-4\lambda)\varepsilon^{ijkl}T^{kl}\Big](w)\\
&+\frac{1}{(z-w)}\Big[-\partial T^{ij}-i\delta^{ij}\partial\Phi_0^{(1)}+i(1-4\lambda)\delta^{ij}\partial U-\frac{1}{2}i\varepsilon^{ijkl}\Phi_1^{(1),kl}+2i\delta^{ij}L\Big](w)+\cdots,
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^i(z)\Phi_1^{(1),jk}(w) &= \frac{1}{(\mathtt{z}-\mathtt{w})^2}[-(1-4\lambda)(\delta^{\mathtt{ik}}\mathit{\Gamma}^{\mathtt{j}}-\delta^{\mathtt{ij}}\mathit{\Gamma}^{\mathtt{k}})+\varepsilon^{\mathtt{ijkl}}\mathit{\Gamma}^{\mathtt{l}}](\mathtt{w}) \\
&+\frac{1}{(z-w)}[(1-4\lambda)(\delta^{\mathtt{ik}}\partial\mathit{\Gamma}^{\mathtt{j}}-\delta^{\mathtt{ij}}\partial\mathit{\Gamma}^{\mathtt{k}})-\varepsilon^{\mathtt{ijkl}}\partial\mathit{\Gamma}^{\mathtt{l}}-i\delta^{ij}\Phi_{\frac{1}{2}}^{(1),k} \\
&+i\delta^{ik}\Phi_{\frac{1}{2}}^{(1),j}+\mathtt{i}(\delta^{\mathtt{ik}}\mathtt{G}^{\mathtt{j}}-\delta^{\mathtt{ij}}\mathtt{G}^{\mathtt{k}})](w)+\cdots, \\
\Gamma^i(z)\Phi_{\frac{1}{2}}^{(1),j}(w) &= -\frac{1}{(\mathtt{z}-\mathtt{w})^2}[\mathtt{Ni}\delta^{\mathtt{ij}}]+\frac{1}{(\mathtt{z}-\mathtt{w})}\left[-\mathtt{i}\delta^{\mathtt{ij}}\mathtt{U}-\frac{1}{2}\varepsilon^{\mathtt{ijkl}}\mathtt{T}^{\mathtt{kl}}\right](\mathtt{w})+\cdots, \\
\Gamma^i(z)\Phi_0^{(1)}(w) &= -\frac{1}{(\mathtt{z}-\mathtt{w})}[\mathit{\Gamma}^{\mathtt{i}}](\mathtt{w})+\cdots. \qquad \text{(B1)}
\end{aligned}
$$

Compared to the $\mathcal{N}=4$ primary condition [11] of the first $\mathcal{N}=4$ multiplet, there are additional terms in (B1): either central terms or the field contents of the $\mathcal{N}=4$ stress energy tensor.[18] In other words, the first $\mathcal{N}=4$ multiplet is not $\mathcal{N}=4$ primary.

APPENDIX C: THE OPES BETWEEN THE FIRST $\mathcal{N}=4$ MULTIPLET AND ITSELF IN THE COMPONENT APPROACH

As before, we list all the component results corresponding to (3.43) as follows, the details are given in the Ref. [27]:

$$
\begin{aligned}
\Phi_0^{(1)}(z)\Phi_0^{(1)}(w) &= \frac{1}{(z-w)^2}[2N(1-4\lambda)]+\cdots, \\
\Phi_0^{(1)}(z)\Phi_{\frac{1}{2}}^{(1),i}(w) &= \frac{1}{(z-w)}[G^i](w)+\cdots, \\
\Phi_0^{(1)}(z)\Phi_1^{(1),ij}(w) &= \frac{1}{(z-w)^2}[2i(1-4\lambda)T^{ij}+i\varepsilon^{ijkl}T^{kl}](w)+\cdots, \\
\Phi_0^{(1)}(z)\Phi_{\frac{3}{2}}^{(1),i}(w) &= -\frac{1}{(z-w)^3}[16i\lambda(1-2\lambda)\Gamma^i](w)+\frac{1}{(z-w)^2}[3(1-4\lambda)G^i+16i\lambda(1-2\lambda)\partial\Gamma^i](w) \\
&+\frac{1}{(z-w)}\left[\frac{1}{3}(1-4\lambda)\partial G^i-\frac{1}{2}\Phi_{\frac{1}{2}}^{(2),i}\right](w)+\cdots, \\
\Phi_0^{(1)}(z)\Phi_2^{(1)}(w) &= \frac{1}{(z-w)^4}[4N(1-12\lambda+24\lambda^2)]-\frac{1}{(z-w)^3}[32\lambda(1-2\lambda)U](w) \\
&+\frac{1}{(z-w)^2}\left[2\left(\Phi_0^{(2)}-\frac{8}{3}(1-4\lambda)L\right)+16\lambda(1-2\lambda)\partial U\right](w)+\cdots, \\
\Phi_{\frac{1}{2}}^{(1),i}(z)\Phi_{\frac{1}{2}}^{(1),j}(w) &= -\frac{1}{(z-w)^3}[4N(1-4\lambda)\delta^{ij}]+\frac{1}{(z-w)^2}[2iT^{ij}+i(1-4\lambda)\varepsilon^{ijkl}T^{kl}](w) \\
&+\frac{1}{(z-w)}\left[-2\delta^{ij}L+i\partial T^{ij}+\frac{1}{2}i(1-4\lambda)\varepsilon^{ijkl}\partial T^{kl}\right](w)+\cdots, \\
\Phi_{\frac{1}{2}}^{(1),i}(z)\Phi_1^{(1),jk}(w) &= \frac{1}{(z-w)^3}[16i\lambda(1-2\lambda)(\delta^{ij}\Gamma^k-\delta^{ik}\Gamma^j)](w)+\frac{1}{(z-w)^2}[-(1-4\lambda)(\delta^{ij}G^k-\delta^{ik}G^j)+3\varepsilon^{ijkl}G^l](w) \\
&+\frac{1}{(z-w)}\left[\delta^{ij}\left(\frac{1}{2}\Phi_{\frac{1}{2}}^{(2),k}-\frac{1}{3}(1-4\lambda)\partial G^k\right)-\delta^{ik}\left(\frac{1}{2}\Phi_{\frac{1}{2}}^{(2),j}-\frac{1}{3}(1-4\lambda)\partial G^j\right)+\varepsilon^{ijkl}\partial G^l\right](w)+\cdots,
\end{aligned}
$$

[18] They have the typewriter fonts.

$$\begin{aligned}
\Phi_{\frac{1}{2}}^{(1),i}(z)\Phi_{\frac{3}{2}}^{(1),j}(w) &= -\frac{1}{(z-w)^4}[4N(1-12\lambda+24\lambda^2)\delta^{ij}] \\
&+\frac{1}{(z-w)^3}[16\delta^{ij}\lambda(1-2\lambda)U-4i(1-4\lambda)T^{ij}-2i(1+4\lambda-8\lambda^2)\varepsilon^{ijkl}T^{kl}](w) \\
&+\frac{1}{(z-w)^2}\left[i(1-4\lambda)\partial T^{ij}+\frac{1}{2}i(1-4\lambda)^2\varepsilon^{ijkl}\partial T^{kl}-2\delta^{ij}\Phi_0^{(2)}-\frac{2}{3}(1-4\lambda)L\right](w) \\
&+\frac{1}{(z-w)}\left[\frac{1}{3}i(1-4\lambda)\partial^2T^{ij}+\frac{1}{6}i(1-4\lambda)^2\varepsilon^{ijkl}\partial^2T^{kl}\right. \\
&\left.-\frac{2}{3}(1-4\lambda)\delta^{ij}\partial L-\frac{1}{2}\delta^{ij}\partial\Phi_0^{(2)}+\frac{1}{4}\varepsilon^{ijkl}\Phi_1^{(2),kl}\right](w)+\cdots, \\
\Phi_{\frac{1}{2}}^{(1),i}(z)\Phi_2^{(1)}(w) &= \frac{1}{(z-w)^4}[48i\lambda(1-2\lambda)\Gamma^i](w)+\frac{1}{(z-w)^3}[-6(1-4\lambda)G^i-48i\lambda(1-2\lambda)\partial\Gamma^i](w) \\
&+\frac{1}{(z-w)^2}\left[-\frac{2}{3}(1-4\lambda)\partial G^i+\frac{5}{2}\Phi_{\frac{1}{2}}^{(2),i}\right](w)+\frac{1}{(z-w)}\left[-\frac{1}{3}(1-4\lambda)\partial^2G^i+\frac{1}{2}\partial\Phi_{\frac{1}{2}}^{(2),i}\right](w)+\cdots, \\
\Phi_1^{(1),ij}(z)\Phi_1^{(1),kl}(w) &= \frac{1}{(z-w)^4}[-12N(1-4\lambda)(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk})+4N(1-12\lambda+24\lambda^2)\varepsilon^{ijkl}] \\
&+\frac{1}{(z-w)^3}[-4i(1+4\lambda-8\lambda^2)\times(-\delta^{ik}T^{jl}+\delta^{il}T^{jk}+\delta^{jk}T^{il}-\delta^{jl}T^{ik})-2i(1-4\lambda) \\
&\times(-\delta^{ik}\varepsilon^{jlmn}+\delta^{il}\varepsilon^{jkmn}+\delta^{jk}\varepsilon^{ilmn}-\delta^{jl}\varepsilon^{ikmn})T^{mn}](w)+\frac{1}{(z-w)^2}\left[-2i(1+4\lambda-8\lambda^2)\right. \\
&\times(-\delta^{ik}\partial T^{jl}+\delta^{il}\partial T^{jk}+\delta^{jk}\partial T^{il}-\delta^{jl}\partial T^{ik})-i(1-4\lambda) \\
&\times(-\delta^{ik}\varepsilon^{jlmn}+\delta^{il}\varepsilon^{jkmn}+\delta^{jk}\varepsilon^{ilmn}-\delta^{jl}\varepsilon^{ikmn})\partial T^{mn}-8(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk})L+2\varepsilon^{ijkl}\Phi_0^{(2)} \\
&\left.+\frac{8}{3}(1-4\lambda)\varepsilon^{ijkl}L\right](w)+\frac{1}{(z-w)}\left[-\frac{2}{3}i(1+4\lambda-8\lambda^2)\right. \\
&\times(-\delta^{ik}\partial^2T^{jl}+\delta^{il}\partial^2T^{jk}+\delta^{jk}\partial^2T^{il}-\delta^{jl}\partial^2T^{ik})-i\frac{1}{3}(1-4\lambda) \\
&\times(-\delta^{ik}\varepsilon^{jlmn}+\delta^{il}\varepsilon^{jkmn}+\delta^{jk}\varepsilon^{ilmn}-\delta^{jl}\varepsilon^{ikmn})\partial^2T^{mn}-4(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk})\partial L+\varepsilon^{ijkl}\partial\Phi_0^{(2)} \\
&\left.+\frac{4}{3}(1-4\lambda)\varepsilon^{ijkl}\partial L+\frac{1}{2}(-\delta^{ik}\Phi_1^{(2),jl}+\delta^{il}\Phi_1^{(2),jk}+\delta^{jk}\Phi_1^{(2),il}-\delta^{jl}\Phi_1^{(2),ik})\right](w)+\cdots, \\
\Phi_1^{(1),ij}(z)\Phi_{\frac{3}{2}}^{(1),k}(w) &= -\frac{1}{(z-w)^4}[48i\lambda(1-2\lambda)\varepsilon^{ijkl}\Gamma^l](w)+\frac{1}{(z-w)^3}[2(5+8\lambda-16\lambda^2)(\delta^{ik}G^j-\delta^{jk}G^i) \\
&+2(1-4\lambda)\varepsilon^{ijkl}G^l-16i\lambda(1-2\lambda)(1-4\lambda)(\delta^{ik}\partial\Gamma^j-\delta^{jk}\partial\Gamma^i)](w) \\
&+\frac{1}{(z-w)^2}\left[(3-4\lambda)(1+4\lambda)(\delta^{ik}\partial G^j-\delta^{jk}\partial G^i)+\frac{5}{3}(1-4\lambda)\varepsilon^{ijkl}\partial G^l\right. \\
&\left.-8i\lambda(1-2\lambda)(1-4\lambda)(\delta^{ik}\partial^2\Gamma^j-\delta^{jk}\partial^2\Gamma^i)-\frac{5}{2}\varepsilon^{ijkl}\Phi_{\frac{1}{2}}^{(2),l}\right](w) \\
&+\frac{1}{(z-w)}\left[\frac{2}{3}(1+4\lambda-8\lambda^2)(\delta^{ik}\partial^2G^j-\delta^{jk}\partial^2G^i)+\frac{2}{3}(1-4\lambda)\varepsilon^{ijkl}\partial^2G^l\right. \\
&\left.-\frac{8}{3}i\lambda(1-2\lambda)(1-4\lambda)(\delta^{ik}\partial^3\Gamma^j-\delta^{jk}\partial^3\Gamma^i)-\varepsilon^{ijkl}\partial\Phi_{\frac{1}{2}}^{(2),l}-\frac{1}{2}(\delta^{ik}\Phi_{\frac{3}{2}}^{(2),j}-\delta^{jk}\Phi_{\frac{3}{2}}^{(2),i})\right](w)+\cdots,
\end{aligned}$$

$$\begin{aligned}
\Phi_1^{(1),ij}(z)\Phi_2^{(1)}(w) &= \frac{1}{(z-w)^4}[-12i(1+8\lambda-16\lambda^2)T^{ij}-6i(1-4\lambda)\varepsilon^{ijkl}T^{kl}](w) \\
&+\frac{1}{(z-w)^3}[4i(1-4\lambda)^2\partial T^{ij}+2i(1-4\lambda)\varepsilon^{ijkl}\partial T^{kl}](w) \\
&+\frac{1}{(z-w)^2}[2i(1-4\lambda)^2\partial^2 T^{ij}+i(1-4\lambda)\varepsilon^{ijkl}\partial^2 T^{kl}+3\Phi_1^{(2),ij}](w) \\
&+\frac{1}{(z-w)}\left[\frac{2}{3}i(1-4\lambda)^2\partial^3 T^{ij}+\frac{1}{3}i(1-4\lambda)\varepsilon^{ijkl}\partial^3 T^{kl}+\partial\Phi_1^{(2),ij}\right](w)+\cdots, \\
\Phi_{\frac{3}{2}}^{(1),i}(z)\Phi_{\frac{3}{2}}^{(1),j}(w) &= \frac{1}{(z-w)^5}[48N(1-4\lambda)\delta^{ij}]+\frac{1}{(z-w)^4}[-12i(1+8\lambda-16\lambda^2)T^{ij}-6i(1-4\lambda)\varepsilon^{ijkl}T^{kl}](w) \\
&+\frac{1}{(z-w)^3}[-6i(1+8\lambda-16\lambda^2)\partial T^{ij}-3i(1-4\lambda)\varepsilon^{ijkl}\partial T^{kl}+4(9+8\lambda-16\lambda^2)\delta^{ij}L \\
&+16\lambda(1-2\lambda)(1-4\lambda)\delta^{ij}\partial U](w)+\frac{1}{(z-w)^2}\Big[-i(1+24\lambda-48\lambda^2)\partial^2 T^{ij} \\
&-\frac{1}{2}i(1-4\lambda)\varepsilon^{ijkl}\partial^2 T^{kl}+2(9+8\lambda-16\lambda^2)\partial L+8\lambda(1-2\lambda)(1-4\lambda)\delta^{ij}\partial^2 U+3\Phi_1^{(2),ij}\Big](w) \\
&+\frac{1}{(z-w)}\Big[-8i\lambda(1-2\lambda)\partial^3 T^{ij}+\frac{8}{3}(2-2\lambda)(1+2\lambda)\partial^2 L \\
&+\frac{8}{3}\lambda(1-2\lambda)(1-4\lambda)\delta^{ij}\partial^3 U+\frac{3}{2}\partial\Phi_1^{(2),ij}+\frac{1}{2}\delta^{ij}\Phi_2^{(2)}\Big](w)+\cdots, \\
\Phi_{\frac{3}{2}}^{(1),i}(z)\Phi_2^{(1)}(w) &= \frac{1}{(z-w)^4}[-6(7+24\lambda-48\lambda^2)G^i+144i\lambda(1-2\lambda)(1-4\lambda)\partial\Gamma^i](w) \\
&+\frac{1}{(z-w)^3}[-4(3+16\lambda-32\lambda^2)\partial G^i+64i\lambda(1-2\lambda)(1-4\lambda)\partial^2\Gamma^i](w) \\
&+\frac{1}{(z-w)^2}\left[-\frac{5}{3}(1+16\lambda-32\lambda^2)\partial^2 G^i+\frac{80}{3}i\lambda(1-2\lambda)(1-4\lambda)\partial^3\Gamma^i+\frac{7}{2}\Phi_{\frac{3}{2}}^{(2),i}\right](w) \\
&+\frac{1}{(z-w)}\left[-8\lambda(1-2\lambda)\partial^3 G^i+8i\lambda(1-2\lambda)(1-4\lambda)\partial^4\Gamma^i+\frac{3}{2}\partial\Phi_{\frac{3}{2}}^{(2),i}\right](w)+\cdots, \\
\Phi_2^{(1)}(z)\Phi_2^{(1)}(w) &= \frac{1}{(z-w)^6}[240N(1-4\lambda)]+\frac{1}{(z-w)^4}[192(1+2\lambda-4\lambda^2)L+192\lambda(1-2\lambda)(1-4\lambda)\partial U](w) \\
&+\frac{1}{(z-w)^3}[96(1+2\lambda-4\lambda^2)\partial L+96\lambda(1-2\lambda)(1-4\lambda)\partial^2 U](w) \\
&+\frac{1}{(z-w)^2}\left[\frac{16}{3}(5+14\lambda-28\lambda^2)\partial^2 L+\frac{112}{3}\lambda(1-2\lambda)(1-4\lambda)\partial^3 U+4\Phi_2^{(2)}\right](w) \\
&+\frac{1}{(z-w)}\left[\frac{16}{3}(1+4\lambda-8\lambda^2)\partial^3 L+\frac{32}{3}\lambda(1-2\lambda)(1-4\lambda)\partial^4 U+2\partial\Phi_2^{(2)}\right](w)+\cdots. \qquad \text{(C1)}
\end{aligned}$$

Note that on the right-hand sides of (C1), the field contents of the $\mathcal{N}=4$ stress energy tensor and the second $\mathcal{N}=4$ multiplet appear which is manifest in (3.43).

APPENDIX D: THE $\mathcal{N}=4$ COSET MODEL RESULTS UNDER THE LARGE (N,k) LIMIT

We rewrite the previous results under the large (N,k) limit in [10] as

$$
\begin{aligned}
\Phi_0^{(1)}(z)\Phi_0^{(1)}(w) &= \frac{1}{(z-w)^2}2N(1-\lambda_{co}) + \cdots,\\
\Phi_{\frac{1}{2}}^{(1),i}(z)\Phi_0^{(1)}(w) &= -\frac{1}{(z-w)}G^i(w) + \cdots,\\
\Phi_1^{(1),ij}(z)\Phi_0^{(1)}(w) &= \frac{1}{(z-w)^2}[2i(1-2\lambda_{co})T^{ij} + i\varepsilon^{ijkl}T^{kl}](w) + \frac{1}{(z-w)}[2i(1-2\lambda_{co})\partial T^{ij} + i\varepsilon^{ijkl}\partial T^{kl}](w) + \cdots,\\
\Phi_{\frac{3}{2}}^{(1),i}(z)\Phi_0^{(1)}(w) &= \frac{1}{(z-w)^3}[8i\lambda_{co}(1-\lambda_{co})\Gamma^i](w) + \frac{1}{(z-w)^2}[16i\lambda_{co}(1-\lambda_{co})\partial\Gamma^i + 3(1-2\lambda_{co})G^i](w)\\
&\quad + \frac{1}{(z-w)}\left[12i\lambda_{co}(1-\lambda_{co})\partial^2\Gamma^i + \frac{8}{3}(1-2\lambda_{co})\partial G^i + \frac{1}{2}\Phi_{\frac{1}{2}}^{(2),i}\right](w) + \cdots,\\
\Phi_2^{(1)}(z)\Phi_0^{(1)}(w) &= \frac{1}{(z-w)^4}[4N(1-\lambda_{co})(1-2\lambda_{co})] + \frac{1}{(z-w)^3}[16\lambda_{co}(1-\lambda_{co})U](w)\\
&\quad + \frac{1}{(z-w)^2}\left[24\lambda_{co}(1-\lambda_{co})\partial U + 2\left(\Phi_0^{(2)} - \frac{8}{3}(1-2\lambda_{co})L\right)\right](w)\\
&\quad + \frac{1}{(z-w)}\left[16\lambda_{co}(1-\lambda_{co})\partial^2 U + 2\left(\partial\Phi_0^{(2)} - \frac{8}{3}(1-2\lambda_{co})\partial L\right)\right](w) + \cdots. \qquad \text{(D1)}
\end{aligned}
$$

It is straightforward to express the OPE as in (3.43) in $\mathcal{N}=4$ superspace.

APPENDIX E: THE OPES BETWEEN THE $\mathcal{N}=4$ STRESS ENERGY TENSOR AND THE SECOND $\mathcal{N}=4$ MULTIPLET

We present the super OPE between the $\mathcal{N}=4$ stress energy tensor and the second $\mathcal{N}=4$ multiplet as follows:

$$
\begin{aligned}
\mathbf{J}(Z_1)\mathbf{\Phi}^{(2)}(Z_2) &= -\frac{\theta_{12}^{4-0}}{z_{12}^4}8N(1-4\lambda+8\lambda^2) - \frac{1}{z_{12}^2}\frac{4}{3}N(1-4\lambda) + \frac{\theta_{12}^{4-i}}{z_{12}^3}\frac{16}{3}(1-2\lambda+4\lambda^2)D^i\mathbf{J}(Z_2) - \frac{\theta_{12}^i}{z_{12}^2}\frac{8}{3}(1-4\lambda)D^i\mathbf{J}(Z_2)\\
&\quad + \frac{\theta_{12}^{4-0}}{z_{12}^3}\left[8(1-4\lambda)\mathbf{\Phi}^{(1)} + \frac{16}{3}(1-4\lambda)^2\partial\mathbf{J}\right](Z_2) + \frac{1}{z_{12}}\left[4\mathbf{\Phi}^{(1)} - \frac{8}{3}(1-4\lambda)\partial\mathbf{J}\right](Z_2)\\
&\quad + \frac{\theta_{12}^{4-ij}}{z_{12}^2}\left[-\frac{4}{3}(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}D^kD^l\mathbf{J} - 2\varepsilon^{ijkl}\frac{1}{2}\varepsilon^{klmn}D^mD^n\mathbf{J}\right](Z_2)\\
&\quad + \frac{\theta_{12}^{4-i}}{z_{12}^2}\left[-\frac{8}{3}(1-4\lambda)^2\partial D^i\mathbf{J} + 2(1-4\lambda)D^i\mathbf{\Phi}^{(1)} + 2(1-4\lambda)\left(-\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{J} - (1-4\lambda)\partial D^i\mathbf{J}\right)\right](Z_2)\\
&\quad + \frac{\theta_{12}^i}{z_{12}}\left[\frac{8}{3}(1-4\lambda)\partial D^i\mathbf{J} + 2D^i\mathbf{\Phi}^{(1)} + 2\left(-\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{J} - (1-4\lambda)\partial D^i\mathbf{J}\right)\right](Z_2)\\
&\quad + \frac{\theta_{12}^{4-0}}{z_{12}^2}4\mathbf{\Phi}^{(2)}(Z_2) + \frac{\theta_{12}^{4-i}}{z_{12}}D^i\mathbf{\Phi}^{(2)}(Z_2) + \frac{\theta_{12}^{4-0}}{z_{12}}2\partial\mathbf{\Phi}^{(2)}(Z_2) + \cdots. \qquad \text{(E1)}
\end{aligned}
$$

Compared to the $\mathcal{N}=4$ primary condition [11] for the second $\mathcal{N}=4$ multiplet, there are additional terms except the last line of (E1). The $\mathcal{N}=4$ stress energy tensor and the first $\mathcal{N}=4$ multiplet including their descendants appear in these extra terms.

APPENDIX F: THE OPES BETWEEN THE FIRST $\mathcal{N}=4$ MULTIPLET AND THE SECOND $\mathcal{N}=4$ MULTIPLET

We describe the super OPE between the first $\mathcal{N}=4$ multiplet and the second $\mathcal{N}=4$ multiplet as follows:

$$
\begin{aligned}
\mathbf{\Phi}^{(1)}(Z_1)\mathbf{\Phi}^{(2)}(Z_2) = & -\frac{\theta_{12}^{4-0}}{z_{12}^5}[128N\lambda(1-2\lambda)(1-4\lambda)] + \frac{\theta_{12}^{4-0}}{z_{12}^4}[-32(-1-4\lambda+8\lambda^2)\mathbf{\Phi}^{(1)} - 64\lambda(1-2\lambda)(1-4\lambda)\partial\mathbf{J}] \\
& + \frac{\theta_{12}^{4-i}}{z_{12}^4}[32\lambda(1-2\lambda)(1-4\lambda)D^i\mathbf{J}](Z_2) + \frac{\theta_{12}^{4-i}}{z_{12}^3}\left[\frac{64}{3}(1-\lambda)(1+2\lambda)D^i\mathbf{\Phi}^{(1)}\right](Z_2) \\
& + \frac{\theta_{12}^{i}}{z_{12}^3}[-32\lambda(1-2\lambda)D^i\mathbf{J}](Z_2) + \frac{\theta_{12}^{4-0}}{z_{12}^3}\left[\frac{128}{3}(1-\lambda)(1+2\lambda)\partial\mathbf{\Phi}^{(1)}\right](Z_2) + \frac{1}{z_{12}^3}[-32N\lambda(1-2\lambda)] \\
& + \frac{1}{z_{12}^2}\left[-\frac{16}{3}(1-4\lambda)\mathbf{\Phi}^{(1)} - 32\lambda(1-2\lambda)\partial\mathbf{J}\right](Z_2) \\
& + \frac{\theta_{12}^{4-ij}}{z_{12}^2}\left[-\frac{8}{3}(1-4\lambda)\frac{1}{2}\varepsilon^{ijkl}D^kD^l\mathbf{\Phi}^{(1)} - 4\varepsilon^{ijkl}\frac{1}{2}\varepsilon^{klmn}D^mD^n\mathbf{\Phi}^{(1)}\right](Z_2) \\
& + \frac{\theta_{12}^{4-i}}{z_{12}^2}\left[-\frac{2}{3}(-23-8\lambda+16\lambda^2)\partial D^i\mathbf{\Phi}^{(1)} - \frac{10}{3}(1-4\lambda)\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{\Phi}^{(1)}\right](Z_2) \\
& + \frac{\theta_{12}^{4-0}}{z_{12}^2}\left[-\frac{8}{15}(-41-32\lambda+64\lambda^2)\partial^2\mathbf{\Phi}^{(1)} - \frac{8}{5}(1-4\lambda)\frac{1}{4!}\varepsilon^{ijkl}D^iD^jD^kD^l\mathbf{\Phi}^{(1)} + \mathbf{\Phi}^{(3)}\right](Z_2) \\
& + \frac{\theta_{12}^{i}}{z_{12}}\left[-\frac{2}{3}(1-4\lambda)\partial D^i\mathbf{\Phi}^{(1)} + 2\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{\Phi}^{(1)}\right](Z_2) \\
& + \frac{\theta_{12}^{4-ij}}{z_{12}}\left[-\frac{4}{3}(1-4\lambda)\frac{1}{2}\varepsilon^{ijkl}\partial D^kD^l\mathbf{\Phi}^{(1)} - 2\varepsilon^{ijkl}\frac{1}{2}\varepsilon^{klmn}\partial D^mD^n\mathbf{\Phi}^{(1)}\right](Z_2) \\
& + \frac{\theta_{12}^{4-i}}{z_{12}}\left[\frac{8}{15}(-11-2\lambda+4\lambda^2)\partial^2D^i\mathbf{\Phi}^{(1)} - \frac{8}{5}(1-4\lambda)\frac{1}{3!}\varepsilon^{ijkl}\partial D^jD^kD^l\mathbf{\Phi}^{(1)} + \frac{1}{6}D^i\mathbf{\Phi}^{(3)}\right](Z_2) \\
& + \frac{\theta_{12}^{4-0}}{z_{12}}\left[-\frac{16}{15}(-7-4\lambda+8\lambda^2)\partial^3\mathbf{\Phi}^{(1)} - \frac{16}{5}(1-4\lambda)\frac{1}{4!}\varepsilon^{ijkl}\partial D^iD^jD^kD^l\mathbf{\Phi}^{(1)} + \frac{2}{3}\partial\mathbf{\Phi}^{(3)}\right](Z_2) + \cdots. \qquad \text{(F1)}
\end{aligned}
$$

On the right-hand sides of (F1), there are the $\mathcal{N}=4$ stress energy tensor, the first $\mathcal{N}=4$ multiplet, the third $\mathcal{N}=4$ multiplet, as well as their descendants.

The third $\mathcal{N}=4$ multiplet can be summarized by

$$
\begin{aligned}
\Phi_0^{(3)} &= -\frac{48}{5}(-3+2\lambda)(W_{\mathrm{F},3}^{\lambda,11} + W_{\mathrm{F},3}^{\lambda,22}) - \frac{96}{5}(1+\lambda)(W_{\mathrm{B},3}^{\lambda,11} + W_{\mathrm{B},3}^{\lambda,22}), \\
\Phi_{\frac{1}{2}}^{(3),1} &= (-6)(-2)\times\left[\frac{1}{2}(Q_{\frac{7}{2}}^{\lambda,11} + i\sqrt{2}Q_{\frac{7}{2}}^{\lambda,12} + 2i\sqrt{2}Q_{\frac{7}{2}}^{\lambda,21} - 2Q_{\frac{7}{2}}^{\lambda,22} + 2\bar{Q}_{\frac{7}{2}}^{\lambda,11} + 2i\sqrt{2}\bar{Q}_{\frac{7}{2}}^{\lambda,12} + i\sqrt{2}\bar{Q}_{\frac{7}{2}}^{\lambda,21} - \bar{Q}_{\frac{7}{2}}^{\lambda,22})\right], \\
\Phi_{\frac{1}{2}}^{(3),2} &= (-6)(-2)\times\left[-\frac{i}{2}(Q_{\frac{7}{2}}^{\lambda,11} + 2i\sqrt{2}Q_{\frac{7}{2}}^{\lambda,21} - 2Q_{\frac{7}{2}}^{\lambda,22} + 2\bar{Q}_{\frac{7}{2}}^{\lambda,11} + 2i\sqrt{2}\bar{Q}_{\frac{7}{2}}^{\lambda,12} - \bar{Q}_{\frac{7}{2}}^{\lambda,22})\right], \\
\Phi_{\frac{1}{2}}^{(3),3} &= (-6)(-2)\times\left[-\frac{i}{2}(Q_{\frac{7}{2}}^{\lambda,11} + i\sqrt{2}Q_{\frac{7}{2}}^{\lambda,12} - 2Q_{\frac{7}{2}}^{\lambda,22} + 2\bar{Q}_{\frac{7}{2}}^{\lambda,11} + i\sqrt{2}\bar{Q}_{\frac{7}{2}}^{\lambda,21} - \bar{Q}_{\frac{7}{2}}^{\lambda,22})\right], \\
\Phi_{\frac{1}{2}}^{(3),4} &= (-6)(-2)\times\left[-\frac{1}{2}Q_{\frac{7}{2}}^{\lambda,11} - Q_{\frac{7}{2}}^{\lambda,22} + \bar{Q}_{\frac{7}{2}}^{\lambda,11} + \frac{1}{2}\bar{Q}_{\frac{7}{2}}^{\lambda,22}\right], \\
\Phi_1^{(3),12} &= (-6)(-2)\times[2iW_{\mathrm{B},4}^{\lambda,11} - \sqrt{2}W_{\mathrm{B},4}^{\lambda,12} - 2iW_{\mathrm{B},4}^{\lambda,22} + 2iW_{\mathrm{F},4}^{\lambda,11} - 2\sqrt{2}W_{\mathrm{F},4}^{\lambda,12} - 2iW_{\mathrm{F},4}^{\lambda,22}], \\
\Phi_1^{(3),13} &= (-6)(-2)\times[-2iW_{\mathrm{B},4}^{\lambda,11} + 4\sqrt{2}W_{\mathrm{B},4}^{\lambda,21} + 2iW_{\mathrm{B},4}^{\lambda,22} - 2iW_{\mathrm{F},4}^{\lambda,11} + 2\sqrt{2}W_{\mathrm{F},4}^{\lambda,21} + 2iW_{\mathrm{F},4}^{\lambda,22}], \\
\Phi_1^{(3),14} &= (-6)(-2)\times[2W_{\mathrm{B},4}^{\lambda,11} + i\sqrt{2}W_{\mathrm{B},4}^{\lambda,12} + 4i\sqrt{2}W_{\mathrm{B},4}^{\lambda,21} - 2W_{\mathrm{B},4}^{\lambda,22} - 2W_{\mathrm{F},3}^{\lambda,11} - 2i\sqrt{2}W_{\mathrm{F},4}^{\lambda,12} - 2i\sqrt{2}W_{\mathrm{F},4}^{\lambda,21} + 2W_{\mathrm{F},4}^{\lambda,22}],
\end{aligned}
$$

$$\Phi_1^{(3),23} = (-6)(-2)\times[-2W_{\mathrm{B},4}^{\lambda,11} - i\sqrt{2}W_{\mathrm{B},4}^{\lambda,12} - 4i\sqrt{2}W_{\mathrm{B},4}^{\lambda,21} + 2W_{\mathrm{B},4}^{\lambda,22} - 2W_{\mathrm{F},4}^{\lambda,11} - 2i\sqrt{2}W_{\mathrm{F},4}^{\lambda,12} - 2i\sqrt{2}W_{\mathrm{F},4}^{\lambda,21} + 2W_{\mathrm{F},4}^{\lambda,22}],$$
$$\Phi_1^{(3),24} = (-6)(-2)\times[-2iW_{\mathrm{B},4}^{\lambda,11} + 4\sqrt{2}W_{\mathrm{B},4}^{\lambda,21} + 2iW_{\mathrm{B},4}^{\lambda,22} + 2iW_{\mathrm{F},4}^{\lambda,11} - 2\sqrt{2}W_{\mathrm{F},4}^{\lambda,21} - 2iW_{\mathrm{F},4}^{\lambda,22}],$$
$$\Phi_1^{(3),34} = (-6)(-2)\times[-2iW_{\mathrm{B},4}^{\lambda,11} + \sqrt{2}W_{\mathrm{B},4}^{\lambda,12} + 2iW_{\mathrm{B},4}^{\lambda,22} + 2iW_{\mathrm{F},4}^{\lambda,11} - 2\sqrt{2}W_{\mathrm{F},4}^{\lambda,12} - 2iW_{\mathrm{F},4}^{\lambda,22}],$$
$$\tilde{\Phi}_{\frac{3}{2}}^{(3),1} \equiv \Phi_{\frac{3}{2}}^{(3),1} - \frac{1}{7}(1-4\lambda)\partial\Phi_{\frac{1}{2}}^{(3),1}$$
$$= (-6)(-2)\times\left[-\frac{1}{2}(Q_{\frac{9}{2}}^{\lambda,11} + i\sqrt{2}Q_{\frac{9}{2}}^{\lambda,12} + 2i\sqrt{2}Q_{\frac{9}{2}}^{\lambda,21} - 2Q_{\frac{9}{2}}^{\lambda,22} - 2\bar{Q}_{\frac{9}{2}}^{\lambda,11} - 2i\sqrt{2}\bar{Q}_{\frac{9}{2}}^{\lambda,12} - i\sqrt{2}\bar{Q}_{\frac{9}{2}}^{\lambda,21} + \bar{Q}_{\frac{9}{2}}^{\lambda,22})\right],$$
$$\tilde{\Phi}_{\frac{3}{2}}^{(3),2} \equiv \Phi_{\frac{3}{2}}^{(3),2} - \frac{1}{7}(1-4\lambda)\partial\Phi_{\frac{1}{2}}^{(3),2}$$
$$= (-6)(-2)\times\left[\frac{i}{2}(Q_{\frac{9}{2}}^{\lambda,11} + 2i\sqrt{2}Q_{\frac{9}{2}}^{\lambda,21} - 2Q_{\frac{9}{2}}^{\lambda,22} - 2\bar{Q}_{\frac{9}{2}}^{\lambda,11} - 2i\sqrt{2}\bar{Q}_{\frac{9}{2}}^{\lambda,12} + \bar{Q}_{\frac{9}{2}}^{\lambda,22})\right],$$
$$\tilde{\Phi}_{\frac{3}{2}}^{(3),3} \equiv \Phi_{\frac{3}{2}}^{(3),3} - \frac{1}{7}(1-4\lambda)\partial\Phi_{\frac{1}{2}}^{(3),3}$$
$$= (-6)(-2)\times\left[\frac{i}{2}(Q_{\frac{9}{2}}^{\lambda,11} + i\sqrt{2}Q_{\frac{9}{2}}^{\lambda,12} - 2Q_{\frac{9}{2}}^{\lambda,22} - 2\bar{Q}_{\frac{9}{2}}^{\lambda,11} - i\sqrt{2}\bar{Q}_{\frac{9}{2}}^{\lambda,21} + \bar{Q}_{\frac{9}{2}}^{\lambda,22})\right],$$
$$\tilde{\Phi}_{\frac{3}{2}}^{(3),4} \equiv \Phi_{\frac{3}{2}}^{(3),4} - \frac{1}{7}(1-4\lambda)\partial\Phi_{\frac{1}{2}}^{(3),4}$$
$$= (-6)(-2)\times\left[\frac{1}{2}(Q_{\frac{9}{2}}^{\lambda,11} + 2Q_{\frac{9}{2}}^{\lambda,22} + 2\bar{Q}_{\frac{9}{2}}^{\lambda,11} + \bar{Q}_{\frac{9}{2}}^{\lambda,22})\right],$$
$$\tilde{\Phi}_2^{(3)} \equiv \Phi_2^{(3)} - \frac{1}{7}(1-4\lambda)\partial^2\Phi_0^{(3)}$$
$$= (-6)(-2)\times[-2(W_{\mathrm{B},5}^{\lambda,11} + W_{\mathrm{B},5}^{\lambda,22} + W_{\mathrm{F},5}^{\lambda,11} + W_{\mathrm{F},5}^{\lambda,22})]. \qquad \text{(F2)}$$

All of these in (F2) are quasiprimary under the stress energy tensor (3.1).

APPENDIX G: THE OPES BETWEEN THE SECOND $\mathcal{N}=4$ MULTIPLET AND ITSELF

We summarize the super OPE between the second $\mathcal{N}=4$ multiplet and itself as follows:

$$\begin{aligned}
\mathbf{\Phi}^{(2)}(Z_1)\mathbf{\Phi}^{(2)}(Z_2) = {}& \frac{1}{z_{12}^4}\left[\frac{128}{3}N(1-4\lambda)(1+2\lambda-4\lambda^2)\right] + \frac{\theta_{12}^{4-0}}{z_{12}^6}\left[-\frac{512}{3}N(-1+10\lambda-80\lambda^3+80\lambda^4)\right] \\
&+ \frac{\theta_{12}^{4-i}}{z_{12}^5}\left[\frac{2048}{3}\lambda(1-\lambda)(1-2\lambda)(1+2\lambda)D^i\mathbf{J}\right](Z_2) + \frac{\theta_{12}^{4-0}}{z_{12}^5}\left[\frac{4096}{3}\lambda(1-\lambda)(1-2\lambda)(1+2\lambda)\partial\mathbf{J}\right](Z_2) \\
&+ \frac{\theta_{12}^{4-ij}}{z_{12}^4}\left[-\frac{256}{3}(1-\lambda)(1+2\lambda)(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}D^kD^l\mathbf{J} - \frac{128}{3}(1-\lambda)(1+2\lambda)\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}D^mD^n\mathbf{J}\right](Z_2) \\
&+ \frac{\theta_{12}^{4-i}}{z_{12}^4}\left[\frac{4096}{3}\lambda(1-\lambda)(1-2\lambda)(1+2\lambda)\partial D^i\mathbf{J}\right. \\
&\left. + 128(1-\lambda)(1+2\lambda)(1-4\lambda)\left(-\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{J} - (1-4\lambda)\partial D^i\mathbf{J}\right)\right](Z_2)
\end{aligned}$$

$$+\frac{\theta_{12}^{4-0}}{z_{12}^4}\left[2048\lambda(1-\lambda)(1-2\lambda)(1+2\lambda)\partial^2\mathbf{J}-\frac{512}{3}(1-\lambda)(1+2\lambda)(1-4\lambda)\left(\frac{1}{2\cdot 4!}\varepsilon^{ijkl}D^iD^jD^kD^l\mathbf{J}\right.\right.$$
$$\left.\left.-\frac{1}{2}(1-4\lambda)\partial^2\mathbf{J}\right)-32(-7-4\lambda+8\lambda^2)\mathbf{\Phi}^{(2)}\right](Z_2)$$
$$+\frac{\theta_{12}^{i}}{z_{12}^3}\left[-\frac{128}{3}(1-\lambda)(1+2\lambda)\left(-\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{J}-(1-4\lambda)\partial D^i\mathbf{J}\right)\right](Z_2)$$
$$+\frac{\theta_{12}^{4-ij}}{z_{12}^3}\left[-\frac{256}{3}(1-\lambda)(1+2\lambda)(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}\partial D^kD^l\mathbf{J}-\frac{128}{3}(1-\lambda)(1+2\lambda)\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}\partial D^mD^n\mathbf{J}\right](Z_2)$$
$$+\frac{\theta_{12}^{4-i}}{z_{12}^3}\left[1024\lambda(1-\lambda)(1-2\lambda)(1+2\lambda)\partial^2D^i\mathbf{J}\right.$$
$$+\frac{1024}{9}(1-\lambda)(1+2\lambda)(1-4\lambda)\left(-\frac{1}{3!}\varepsilon^{ijkl}\partial D^jD^kD^l\mathbf{J}-(1-4\lambda)\partial^2D^i\mathbf{J}\right)$$
$$\left.-\frac{16}{3}(-11-2\lambda+4\lambda^2)D^i\mathbf{\Phi}^{(2)}\right](Z_2)+\frac{\theta_{12}^{4-0}}{z_{12}^3}\left[\frac{4096}{3}\lambda(1-\lambda)(1-2\lambda)(1+2\lambda)\partial^3\mathbf{J}\right.$$
$$+\frac{1792}{9}(1-\lambda)(1+2\lambda)(1-4\lambda)\left(\frac{1}{2\cdot 4!}\varepsilon^{ijkl}\partial D^iD^jD^kD^l\mathbf{J}-\frac{1}{2}(1-4\lambda)\partial^3\mathbf{J}\right)$$
$$\left.-\frac{16}{3}(-43-16\lambda+32\lambda^2)\partial\mathbf{\Phi}^{(2)}\right](Z_2)+\frac{1}{z_{12}^2}\left[-\frac{16}{3}(1-4\lambda)\mathbf{\Phi}^{(2)}\right.$$
$$\left.+\frac{256}{9}(1-\lambda)(1+2\lambda)\left(\frac{1}{2\cdot 4!}\varepsilon^{ijkl}D^iD^jD^kD^l\mathbf{J}-\frac{1}{2}(1-4\lambda)\partial^2\mathbf{J}\right)\right](Z_2)$$
$$+\frac{\theta_{12}^{i}}{z_{12}^2}\left[-\frac{256}{9}(1-\lambda)(1+2\lambda)\left(-\frac{1}{3!}\varepsilon^{ijkl}\partial D^jD^kD^l\mathbf{J}-(1-4\lambda)\partial^2D^i\mathbf{J}\right)-\frac{8}{3}(1-4\lambda)D^i\mathbf{\Phi}^{(2)}\right](Z_2)$$
$$+\frac{\theta_{12}^{4-ij}}{z_{12}^2}\left[\frac{128}{3}(1-\lambda)(1+2\lambda)(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}\partial^2D^kD^l\mathbf{J}\right.$$
$$-\frac{64}{3}(1-\lambda)(1+2\lambda)\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}\partial^2D^mD^n\mathbf{J}-4(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}D^kD^l\mathbf{\Phi}^{(2)}$$
$$\left.-6\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}D^mD^n\mathbf{\Phi}^{(2)}\right](Z_2)+\frac{\theta_{12}^{4-i}}{z_{12}^2}\left[\frac{4096}{9}\lambda(1-\lambda)(1-2\lambda)(1+2\lambda)\partial^3D^i\mathbf{J}\right.$$
$$+\frac{160}{3}(1-\lambda)(1+2\lambda)(1-4\lambda)\left(-\frac{1}{3!}\varepsilon^{ijkl}\partial^2D^jD^kD^l\mathbf{J}-(1-4\lambda)\partial^3D^i\mathbf{J}\right)$$
$$\left.-\frac{2}{3}(-71-8\lambda+16\lambda^2)\partial D^i\mathbf{\Phi}^{(2)}-\frac{14}{3}(1-4\lambda)\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{\Phi}^{(2)}\right](Z_2)$$
$$+\frac{\theta_{12}^{4-0}}{z_{12}^2}\left[\frac{5120}{9}\lambda(1-\lambda)(1-2\lambda)(1+2\lambda)\partial^4\mathbf{J}\right.$$
$$-\frac{512}{5}(1-\lambda)(1+2\lambda)(1-4\lambda)\left(\frac{1}{2\cdot 4!}\varepsilon^{ijkl}\partial^2D^iD^jD^kD^l\mathbf{J}-\frac{1}{2}(1-4\lambda)\partial^4\mathbf{J}\right)$$
$$\left.-\frac{64}{21}(-38-11\lambda+22\lambda^2)\partial^2\mathbf{\Phi}^{(2)}-\frac{64}{21}(1-4\lambda)\frac{1}{4!}\varepsilon^{ijkl}D^iD^jD^kD^l\mathbf{\Phi}^{(2)}+\frac{1}{3}\mathbf{\Phi}^{(4)}\right](Z_2)$$
$$+\frac{1}{z_{12}}\left[-\frac{8}{3}(1-4\lambda)\partial\mathbf{\Phi}^{(2)}+\frac{128}{9}(1-\lambda)(1+2\lambda)\left(\frac{1}{2\cdot 4!}\varepsilon^{ijkl}\partial D^iD^jD^kD^l\mathbf{J}-\frac{1}{2}(1-4\lambda)\partial^3\mathbf{J}\right)\right](Z_2)$$

$$+\frac{\theta_{12}^{i}}{z_{12}}\left[-\frac{32}{3}(1-\lambda)(1+2\lambda)\left(-\frac{1}{3!}\varepsilon^{ijkl}\partial^2 D^j D^k D^l \mathbf{J}-(1-4\lambda)\partial^3 D^i \mathbf{J}\right)\right.$$
$$\left.-2(1-4\lambda)\partial D^i \mathbf{\Phi}^{(2)}+2\frac{1}{3!}\varepsilon^{ijkl}D^j D^k D^l \mathbf{\Phi}^{(2)}\right](Z_2)$$
$$+\frac{\theta_{12}^{4-ij}}{z_{12}}\left[-\frac{128}{9}(1-\lambda)(1+2\lambda)(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}\partial^3 D^k D^l \mathbf{J}\right.$$
$$-\frac{64}{9}(1-\lambda)(1+2\lambda)\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}\partial^3 D^m D^n \mathbf{J}+\frac{8}{3}(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}\partial D^k D^l \mathbf{\Phi}^{(2)}$$
$$\left.+4\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}\partial D^m D^n \mathbf{\Phi}^{(2)}\right](Z_2)+\frac{\theta_{12}^{4-i}}{z_{12}}\left[\frac{1280}{9}\lambda(1-\lambda)(1-2\lambda)(1+2\lambda)\partial^4 D^i \mathbf{J}\right.$$
$$\frac{256}{15}(1-\lambda)(1+2\lambda)(1-4\lambda)\left(-\frac{1}{3!}\varepsilon^{ijkl}\partial^3 D^j D^k D^l \mathbf{J}-(1-4\lambda)\partial^4 D^i \mathbf{J}\right)$$
$$\left.-\frac{32}{21}(-13-\lambda+2\lambda^2)\partial^2 D^i \mathbf{\Phi}^{(2)}-\frac{64}{21}(1-4\lambda)\frac{1}{3!}\varepsilon^{ijkl}\partial D^j D^k D^l \mathbf{\Phi}^{(2)}+\frac{1}{24}D^i \mathbf{\Phi}^{(4)}\right](Z_2)$$
$$+\frac{\theta_{12}^{4-0}}{z_{12}}\left[\frac{512}{3}\lambda(1-\lambda)(1-2\lambda)(1+2\lambda)\partial^5 \mathbf{J}\right.$$
$$-\frac{512}{15}(1-\lambda)(1+2\lambda)(1-4\lambda)\left(\frac{1}{2\cdot 4!}\varepsilon^{ijkl}\partial^3 D^i D^j D^k D^l \mathbf{J}-\frac{1}{2}(1-4\lambda)\partial^5 \mathbf{J}\right)$$
$$\left.-\frac{67}{7}(-17-4\lambda+8\lambda^2)\partial^3 \mathbf{\Phi}^{(2)}-\frac{16}{7}(1-4\lambda)\frac{1}{4!}\varepsilon^{ijkl}\partial D^i D^j D^k D^l \mathbf{\Phi}^{(2)}+\frac{1}{4}\partial \mathbf{\Phi}^{(4)}\right](Z_2)+\cdots. \qquad \text{(G1)}$$

There exist the $\mathcal{N}=4$ stress energy tensor, the second $\mathcal{N}=4$ multiplet, the fourth $\mathcal{N}=4$ multiplet, as well as their descendants on the right-hand sides of (G1).

The fourth $\mathcal{N}=4$ multiplet can be summarized by

$$\Phi_0^{(4)}=\frac{768}{7}(-2+\lambda)(W_{\mathrm{F},4}^{\lambda,11}+W_{\mathrm{F},4}^{\lambda,22})+\frac{384}{7}(3+2\lambda)(W_{\mathrm{B},4}^{\lambda,11}+W_{\mathrm{B},4}^{\lambda,22}),$$
$$\Phi_{\frac{1}{2}}^{(4),1}=(-8)(-6)(-2)\times\left[\frac{1}{2}(Q_{\frac{9}{2}}^{\lambda,11}+i\sqrt{2}Q_{\frac{9}{2}}^{\lambda,12}+2i\sqrt{2}Q_{\frac{9}{2}}^{\lambda,21}-2Q_{\frac{9}{2}}^{\lambda,22}+2\bar{Q}_{\frac{9}{2}}^{\lambda,11}+2i\sqrt{2}\bar{Q}_{\frac{9}{2}}^{\lambda,12}+i\sqrt{2}\bar{Q}_{\frac{9}{2}}^{\lambda,21}-\bar{Q}_{\frac{9}{2}}^{\lambda,22})\right],$$
$$\Phi_{\frac{1}{2}}^{(4),2}=(-8)(-6)(-2)\times\left[-\frac{i}{2}(Q_{\frac{9}{2}}^{\lambda,11}+2i\sqrt{2}Q_{\frac{9}{2}}^{\lambda,21}-2Q_{\frac{9}{2}}^{\lambda,22}+2\bar{Q}_{\frac{9}{2}}^{\lambda,11}+2i\sqrt{2}\bar{Q}_{\frac{9}{2}}^{\lambda,12}-\bar{Q}_{\frac{9}{2}}^{\lambda,22})\right],$$
$$\Phi_{\frac{1}{2}}^{(4),3}=(-8)(-6)(-2)\times\left[-\frac{i}{2}(Q_{\frac{9}{2}}^{\lambda,11}+i\sqrt{2}Q_{\frac{9}{2}}^{\lambda,12}-2Q_{\frac{9}{2}}^{\lambda,22}+2\bar{Q}_{\frac{9}{2}}^{\lambda,11}+i\sqrt{2}\bar{Q}_{\frac{9}{2}}^{\lambda,21}-\bar{Q}_{\frac{9}{2}}^{\lambda,22})\right],$$
$$\Phi_{\frac{1}{2}}^{(4),4}=(-8)(-6)(-2)\times\left[-\frac{1}{2}Q_{\frac{9}{2}}^{\lambda,11}-Q_{\frac{9}{2}}^{\lambda,22}+\bar{Q}_{\frac{9}{2}}^{\lambda,11}+\frac{1}{2}\bar{Q}_{\frac{9}{2}}^{\lambda,22}\right],$$
$$\Phi_1^{(4),12}=(-8)(-6)(-2)\times[2iW_{\mathrm{B},5}^{\lambda,11}-\sqrt{2}W_{\mathrm{B},5}^{\lambda,12}-2iW_{\mathrm{B},5}^{\lambda,22}+2iW_{\mathrm{F},5}^{\lambda,11}-2\sqrt{2}W_{\mathrm{F},5}^{\lambda,12}-2iW_{\mathrm{F},5}^{\lambda,22}],$$
$$\Phi_1^{(4),13}=(-8)(-6)(-2)\times[-2iW_{\mathrm{B},5}^{\lambda,11}+4\sqrt{2}W_{\mathrm{B},5}^{\lambda,21}+2iW_{\mathrm{B},5}^{\lambda,22}-2iW_{\mathrm{F},5}^{\lambda,11}+2\sqrt{2}W_{\mathrm{F},5}^{\lambda,21}+2iW_{\mathrm{F},5}^{\lambda,22}],$$
$$\Phi_1^{(4),14}=(-8)(-6)(-2)\times[2W_{\mathrm{B},5}^{\lambda,11}+i\sqrt{2}W_{\mathrm{B},5}^{\lambda,12}+4i\sqrt{2}W_{\mathrm{B},5}^{\lambda,21}-2W_{\mathrm{B},5}^{\lambda,22}-2W_{\mathrm{F},5}^{\lambda,11}-2i\sqrt{2}W_{\mathrm{F},5}^{\lambda,12}-2i\sqrt{2}W_{\mathrm{F},5}^{\lambda,21}+2W_{\mathrm{F},5}^{\lambda,22}],$$
$$\Phi_1^{(4),23}=(-8)(-6)(-2)\times[-2W_{\mathrm{B},5}^{\lambda,11}-i\sqrt{2}W_{\mathrm{B},5}^{\lambda,12}-4i\sqrt{2}W_{\mathrm{B},5}^{\lambda,21}+2W_{\mathrm{B},5}^{\lambda,22}-2W_{\mathrm{F},5}^{\lambda,11}-2i\sqrt{2}W_{\mathrm{F},5}^{\lambda,12}-2i\sqrt{2}W_{\mathrm{F},5}^{\lambda,21}+2W_{\mathrm{F},5}^{\lambda,22}],$$
$$\Phi_1^{(4),24}=(-8)(-6)(-2)\times[-2iW_{\mathrm{B},5}^{\lambda,11}+4\sqrt{2}W_{\mathrm{B},5}^{\lambda,21}+2iW_{\mathrm{B},5}^{\lambda,22}+2iW_{\mathrm{F},5}^{\lambda,11}-2\sqrt{2}W_{\mathrm{F},5}^{\lambda,21}-2iW_{\mathrm{F},5}^{\lambda,22}],$$
$$\Phi_1^{(4),34}=(-8)(-6)(-2)\times[-2iW_{\mathrm{B},5}^{\lambda,11}+\sqrt{2}W_{\mathrm{B},5}^{\lambda,12}+2iW_{\mathrm{B},5}^{\lambda,22}+2iW_{\mathrm{F},5}^{\lambda,11}-2\sqrt{2}W_{\mathrm{F},5}^{\lambda,12}-2iW_{\mathrm{F},5}^{\lambda,22}],$$

$$
\begin{aligned}
\tilde{\Phi}^{(4),1}_{\frac{3}{2}} &\equiv \Phi^{(4),1}_{\frac{3}{2}} - \frac{1}{9}(1-4\lambda)\partial\Phi^{(4),1}_{\frac{1}{2}} \\
&= (-8)(-6)(-2) \times \left[-\frac{1}{2}(Q^{\lambda,11}_{\frac{11}{2}} + i\sqrt{2}Q^{\lambda,12}_{\frac{11}{2}} + 2i\sqrt{2}Q^{\lambda,21}_{\frac{11}{2}} - 2Q^{\lambda,22}_{\frac{11}{2}} - 2\bar{Q}^{\lambda,11}_{\frac{11}{2}} - 2i\sqrt{2}\bar{Q}^{\lambda,12}_{\frac{11}{2}} - i\sqrt{2}\bar{Q}^{\lambda,21}_{\frac{11}{2}} + \bar{Q}^{\lambda,22}_{\frac{11}{2}})\right], \\
\tilde{\Phi}^{(4),2}_{\frac{3}{2}} &\equiv \Phi^{(4),2}_{\frac{3}{2}} - \frac{1}{9}(1-4\lambda)\partial\Phi^{(4),2}_{\frac{1}{2}} \\
&= (-8)(-6)(-2) \times \left[\frac{i}{2}(Q^{\lambda,11}_{\frac{11}{2}} + 2i\sqrt{2}Q^{\lambda,21}_{\frac{11}{2}} - 2Q^{\lambda,22}_{\frac{11}{2}} - 2\bar{Q}^{\lambda,11}_{\frac{11}{2}} - 2i\sqrt{2}\bar{Q}^{\lambda,12}_{\frac{11}{2}} + \bar{Q}^{\lambda,22}_{\frac{11}{2}})\right], \\
\tilde{\Phi}^{(4),3}_{\frac{3}{2}} &\equiv \Phi^{(4),3}_{\frac{3}{2}} - \frac{1}{9}(1-4\lambda)\partial\Phi^{(4),3}_{\frac{1}{2}} \\
&= (-8)(-6)(-2) \times \left[\frac{i}{2}(Q^{\lambda,11}_{\frac{11}{2}} + i\sqrt{2}Q^{\lambda,12}_{\frac{11}{2}} - 2Q^{\lambda,22}_{\frac{11}{2}} - 2\bar{Q}^{\lambda,11}_{\frac{11}{2}} - i\sqrt{2}\bar{Q}^{\lambda,21}_{\frac{11}{2}} + \bar{Q}^{\lambda,22}_{\frac{11}{2}})\right], \\
\tilde{\Phi}^{(4),4}_{\frac{3}{2}} &\equiv \Phi^{(4),4}_{\frac{3}{2}} - \frac{1}{9}(1-4\lambda)\partial\Phi^{(4),4}_{\frac{1}{2}} \\
&= (-8)(-6)(-2) \times \left[\frac{1}{2}(Q^{\lambda,11}_{\frac{11}{2}} + 2Q^{\lambda,22}_{\frac{11}{2}} + 2\bar{Q}^{\lambda,11}_{\frac{11}{2}} + \bar{Q}^{\lambda,22}_{\frac{11}{2}})\right], \\
\tilde{\Phi}^{(4)}_{2} &\equiv \Phi^{(4)}_{2} - \frac{1}{9}(1-4\lambda)\partial^2\Phi^{(4)}_{0} \\
&= (-8)(-6)(-2) \times [-2(W^{\lambda,11}_{\mathrm{B},6} + W^{\lambda,22}_{\mathrm{B},6} + W^{\lambda,11}_{\mathrm{F},6} + W^{\lambda,22}_{\mathrm{F},6})]. \qquad \text{(G2)}
\end{aligned}
$$

All of these are quasiprimary under the stress energy tensor (3.1). The λ dependence in the weight-4 operator in (G2) can be obtained from the factor $(4-2\lambda)$ appearing in $W^{\lambda,\bar{a}a}_{B,4}$ and the factor $(3+2\lambda)$ appearing in $W^{\lambda,\bar{a}a}_{F,4}$, respectively.

This implies that for the hth $\mathcal{N}=4$ multiplet, the λ dependence in the weight-h operator can be obtained from the factor $(h-2\lambda)$ appearing in $W^{\lambda,\bar{a}a}_{F,h}$ and the factor $(h-1+2\lambda)$ appearing in $W^{\lambda,\bar{a}a}_{B,h}$, respectively. For the four weight-$(h+\frac{1}{2})$ operators, we simply take $(-2h)\cdots(-8)(-6)(-2)$ multiplied by the quantities inside the brackets after we replace $\frac{9}{2}$ with $(h+\frac{1}{2})$. For the six weight-$(h+1)$ operators, we simply take $(-2h)\cdots(-8)(-6)(-2)$ multiplied by the quantities inside the brackets after we replace 5 with $(h+1)$. Similarly, for the four weight-$(h+\frac{3}{2})$ operators, we simply take $(-2h)\cdots(-8)(-6)(-2)$ multiplied by the quantities inside the brackets after we replace $\frac{11}{2}$ with $(h+\frac{3}{2})$. For the weight-$(h+2)$ operator, we simply take $(-2h)\cdots(-8)(-6)(-2)$ multiplied by the quantity inside the bracket after we replace 6 with $(h+2)$. The $W^{\lambda,\bar{a}b}_{F,h}$ and the $W^{\lambda,\bar{a}b}_{B,h}$ can be written in terms of $\Phi^{(h)}_0$, $\Phi^{(h-1),ij}_1$, and $\Phi^{(h-2)}_2$. Similarly, the $Q^{\lambda,\bar{a}b}_{h+\frac{1}{2}}$ and the $\bar{Q}^{\lambda,a\bar{b}}_{h+\frac{1}{2}}$ can be written in terms of $\Phi^{(h),i}_{\frac{1}{2}}$ and $\Phi^{(h-1),i}_{\frac{3}{2}}$.

APPENDIX H: THE OPES BETWEEN THE $\mathcal{N}=4$ STRESS ENERGY TENSOR AND THE THIRD $\mathcal{N}=4$ MULTIPLET

The OPEs between the $\mathcal{N}=4$ stress energy tensor and the third $\mathcal{N}=4$ multiplet can be described by

$$
\begin{aligned}
\mathbf{J}(Z_1)\mathbf{\Phi}^{(3)}(Z_2) = &-\frac{\theta^{4-0}_{12}}{z^5_{12}}\left[-\frac{1536}{5}N\lambda(1-2\lambda)(1-4\lambda)\right] + \frac{1}{z^3_{12}}\left[-\frac{192}{15}N(1-3\lambda+6\lambda^2)\right] \\
&+ \frac{\theta^{4-i}_{12}}{z^4_{12}}\left[\frac{1536}{5}\lambda(1-2\lambda)(1-4\lambda)D^i\mathbf{J}\right](Z_2) + \frac{\theta^{i}_{12}}{z^3_{12}}\left[-\frac{192}{5}(1-3\lambda+6\lambda^2)D^i\mathbf{J}\right](Z_2) \\
&+ \frac{\theta^{4-ij}_{12}}{z^3_{12}}\left[\frac{192}{5}(1+2\lambda-4\lambda^2)\frac{1}{2!}\varepsilon^{ijkl}D^kD^l\mathbf{J} - \frac{96}{5}(1-4\lambda)\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}D^mD^n\mathbf{J}\right](Z_2) \\
&+ \frac{\theta^{4-0}_{12}}{z^4_{12}}\left[\frac{1152}{5}(1+2\lambda-4\lambda^2)\mathbf{\Phi}^{(1)} - \frac{1536}{5}\lambda(1-2\lambda)(1-4\lambda)\partial\mathbf{J}\right](Z_2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{z_{12}^2}\left[\frac{96}{5}(1-4\lambda)\mathbf{\Phi}^{(1)}-\frac{192}{5}(1-3\lambda+6\lambda^2)\partial\mathbf{J}\right](Z_2)\\
&+\frac{\theta_{12}^{4-i}}{z_{12}^3}\left[-\frac{1536}{5}\lambda(1-2\lambda)(1-4\lambda)\partial D^i\mathbf{J}+\frac{576}{5}(1+2\lambda-4\lambda^2)D^i\mathbf{\Phi}^{(1)}\right.\\
&\left.+192\lambda(1-2\lambda)\left(-\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{J}-(1-4\lambda)\partial D^i\mathbf{J}\right)\right](Z_2)\\
&+\frac{\theta_{12}^{i}}{z_{12}^2}\left[\frac{384}{5}(1-3\lambda+6\lambda^2)\partial D^i\mathbf{J}-\frac{96}{5}(1-4\lambda)D^i\mathbf{\Phi}^{(1)}\right.\\
&\left.-\frac{96}{5}(1-4\lambda)\left(-\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{J}-(1-4\lambda)\partial D^i\mathbf{J}\right)\right](Z_2)\\
&+\frac{1}{z_{12}}\left[-\frac{96}{5}(1-4\lambda)\partial\mathbf{\Phi}^{(1)}+\frac{192}{5}(1-3\lambda+6\lambda^2)\partial^2\mathbf{J}\right.\\
&\left.+\frac{64}{5}(1-4\lambda)\left(\frac{1}{2\cdot 4!}\varepsilon^{ijkl}D^iD^jD^kD^l\mathbf{J}-\frac{1}{2}(1-4\lambda)\partial^2\mathbf{J}\right)+24\mathbf{\Phi}^{(2)}\right](Z_2)\\
&+\frac{\theta_{12}^{4-ij}}{z_{12}^2}\left[-\frac{96}{5}(1+2\lambda-4\lambda^2)\frac{1}{2!}\varepsilon^{ijkl}\partial D^kD^l\mathbf{J}+\frac{48}{5}(1-4\lambda)\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}\partial D^mD^n\mathbf{J}\right.\\
&\left.-\frac{48}{5}(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}D^kD^l\mathbf{\Phi}^{(1)}-24\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}D^mD^n\mathbf{\Phi}^{(1)}\right](Z_2)\\
&+\frac{\theta_{12}^{i}}{z_{12}}\left[-\frac{96}{5}(1-3\lambda+6\lambda^2)\partial^2D^i\mathbf{J}+\frac{12}{5}(1-4\lambda)\partial D^i\mathbf{\Phi}^{(1)}\right.\\
&\left.+\frac{32}{5}(1-4\lambda)\left(-\frac{1}{3!}\varepsilon^{ijkl}\partial D^jD^kD^l\mathbf{J}-(1-4\lambda)\partial^2D^i\mathbf{J}\right)+12\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{\Phi}^{(1)}-6D^i\mathbf{\Phi}^{(2)}\right](Z_2)\\
&+\frac{\theta_{12}^{4-0}}{z_{12}^2}[6\mathbf{\Phi}^{(3)}](Z_2)+\frac{\theta_{12}^{4-i}}{z_{12}}[D^i\mathbf{\Phi}^{(3)}](Z_2)+\frac{\theta_{12}^{4-0}}{z_{12}}[2\partial\mathbf{\Phi}^{(3)}](Z_2)+\cdots. \qquad \text{(H1)}
\end{aligned}
$$

Except for the last three terms for the $\mathcal{N}=4$ primary condition [11], the additional terms consisting of the $\mathcal{N}=4$ stress energy tensor, the first and the second $\mathcal{N}=4$ multiplets (and their descendants) appear in (H1).

APPENDIX I: THE OPES BETWEEN THE $\mathcal{N}=4$ STRESS ENERGY TENSOR AND THE FOURTH $\mathcal{N}=4$ MULTIPLET

The OPEs between the operators in (3.12) and the operators in (G2) can be summarized by

$$
\begin{aligned}
\mathbf{J}(Z_1)\mathbf{\Phi}^{(4)}(Z_2)&=\frac{1}{z_{12}^4}\left[-\frac{3072}{35}N(1-4\lambda)(3-2\lambda+4\lambda^2)\right]+\frac{\theta_{12}^{4-0}}{z_{12}^6}\left[-\frac{147456}{7}N\lambda(1-2\lambda)(1-\lambda+2\lambda^2)\right]\\
&+\frac{\theta_{12}^{4-i}}{z_{12}^5}\left[\frac{147456}{7}\lambda(1-2\lambda)(1-\lambda+2\lambda^2)D^i\mathbf{J}\right](Z_2)+\frac{\theta_{12}^{i}}{z_{12}^4}\left[-\frac{12288}{35}(1-4\lambda)(3-2\lambda+4\lambda^2)D^i\mathbf{J}\right](Z_2)\\
&+\frac{\theta_{12}^{4-0}}{z_{12}^5}\left[\frac{49152}{7}\lambda(1-2\lambda)(1-4\lambda)\mathbf{\Phi}^{(1)}-\frac{294912}{7}\lambda(1-2\lambda)(1-\lambda+2\lambda^2)\partial\mathbf{J}\right](Z_2)\\
&+\frac{\theta_{12}^{4-ij}}{z_{12}^4}\left[\frac{18432}{35}(1-\lambda)(1+2\lambda)(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}D^kD^l\mathbf{J}-\frac{9216}{7}(1-\lambda+2\lambda^2)\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}D^mD^n\mathbf{J}\right](Z_2)\\
&+\frac{1}{z_{12}^3}\left[\frac{6144}{7}(1-\lambda+2\lambda^2)\mathbf{\Phi}^{(1)}-\frac{12288}{35}(1-4\lambda)(3-2\lambda+4\lambda^2)\partial\mathbf{J}\right](Z_2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\theta_{12}^{4-i}}{z_{12}^4}\left[-\frac{294912}{7}\lambda(1-2\lambda)(1-\lambda+2\lambda^2)\partial D^i\mathbf{J}+\frac{24576}{7}\lambda(1-2\lambda)(1-4\lambda)D^i\mathbf{\Phi}^{(1)}\right.\\
&\left.-\frac{36864}{35}(1-4\lambda)(2-3\lambda+6\lambda^2)\left(-\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{J}-(1-4\lambda)\partial D^i\mathbf{J}\right)\right](Z_2)\\
&+\frac{\theta_{12}^{i}}{z_{12}^3}\left[\frac{36864}{35}(1-4\lambda)(3-2\lambda+4\lambda^2)\partial D^i\mathbf{J}-\frac{9216}{7}(1-\lambda+2\lambda^2)D^i\mathbf{\Phi}^{(1)}\right.\\
&\left.-\frac{9216}{7}(1-\lambda+2\lambda^2)\left(-\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{J}-(1-4\lambda)D^i\mathbf{J}\right)\right](Z_2)\\
&+\frac{\theta_{12}^{4-0}}{z_{12}^4}\left[-\frac{24576}{7}\lambda(1-2\lambda)(1-4\lambda)\partial\mathbf{\Phi}^{(1)}+\frac{147456}{7}\lambda(1-2\lambda)(1-\lambda+2\lambda^2)\partial^2\mathbf{J}\right.\\
&-\frac{9216}{7}(-4-3\lambda+6\lambda^2)\mathbf{\Phi}^{(2)}-\frac{24576}{35}(1-4\lambda)(2-3\lambda+6\lambda^2)\\
&\left.\times\left(\frac{1}{2\cdot 4!}\varepsilon^{ijkl}D^iD^jD^kD^l\mathbf{J}-\frac{1}{2}(1-4\lambda)\partial^2\mathbf{J}\right)\right](Z_2\\
&+\frac{\theta_{12}^{4-ij}}{z_{12}^3}\left[-\frac{18432}{35}(1-\lambda)(1+2\lambda)(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}\partial D^kD^l\mathbf{J}+\frac{9216}{7}(1-\lambda+2\lambda^2)\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}\partial D^mD^n\mathbf{J}\right.\\
&\left.-\frac{1536}{7}(-3-4\lambda+8\lambda^2)\frac{1}{2!}\varepsilon^{ijkl}D^kD^l\mathbf{\Phi}^{(1)}-\frac{2304}{7}(1-4\lambda)\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}D^mD^n\mathbf{\Phi}^{(1)}\right](Z_2)\\
&+\frac{1}{z_{12}^2}\left[-\frac{9216}{7}(1-\lambda+2\lambda^2)\partial\mathbf{\Phi}^{(1)}+\frac{18432}{35}(1-4\lambda)(3-2\lambda+4\lambda^2)\partial^2\mathbf{J}\right.\\
&\left.+\frac{6144}{7}(1-\lambda+2\lambda^2)\left(\frac{1}{2\cdot 4!}\varepsilon^{ijkl}D^iD^jD^kD^l\mathbf{J}-\frac{1}{2}(1-4\lambda)\partial^2\mathbf{J}\right)+\frac{1152}{7}(1-4\lambda)\mathbf{\Phi}^{(2)}\right](Z_2)\\
&+\frac{\theta_{12}^{4-i}}{z_{12}^3}\left[\frac{73728}{7}\lambda(1-2\lambda)(1-\lambda+2\lambda^2)\partial^2D^i\mathbf{J}-\frac{4608}{7}\lambda(1-2\lambda)(1-4\lambda)\partial D^i\mathbf{\Phi}^{(1)}\right.\\
&+\frac{12288}{35}(1-4\lambda)(2-3\lambda+6\lambda^2)\left(-\frac{1}{3!}\varepsilon^{ijkl}\partial D^jD^kD^l\mathbf{J}-(1-4\lambda)\partial^2D^i\mathbf{J}\right)\\
&\left.-1536\lambda(1-2\lambda)\frac{1}{3!}\varepsilon^{ijkl}\partial D^jD^kD^l\mathbf{\Phi}^{(1)}-\frac{2304}{7}(-4-3\lambda+6\lambda^2)D^i\mathbf{\Phi}^{(2)}\right](Z_2)\\
&+\frac{\theta_{12}^{i}}{z_{12}^2}\left[-\frac{18432}{35}(1-4\lambda)(3-2\lambda+4\lambda^2)\partial^2D^i\mathbf{J}+\frac{384}{7}(15-8\lambda+16\lambda^2)\partial D^i\mathbf{\Phi}^{(1)}\right.\\
&+\frac{6144}{7}(1-\lambda+2\lambda^2)\left(-\frac{1}{3!}\varepsilon^{ijkl}\partial D^jD^kD^l\mathbf{J}-(1-4\lambda)\partial^2D^i\mathbf{J}\right)+\frac{1152}{7}(1-4\lambda)\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{\Phi}^{(1)}\\
&\left.-\frac{576}{7}(1-4\lambda)D^i\mathbf{\Phi}^{(2)}\right](Z_2)+\frac{\theta_{12}^{4-ij}}{z_{12}^2}\left[\frac{3072}{35}(1-\lambda)(1+2\lambda)(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}\partial^2D^kD^l\mathbf{J}\right.\\
&-\frac{1536}{7}(1-\lambda+2\lambda^2)\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}\partial^2D^mD^n\mathbf{J}+\frac{384}{7}(-3-4\lambda+8\lambda^2)\frac{1}{2!}\varepsilon^{ijkl}\partial D^kD^l\mathbf{\Phi}^{(1)}\\
&\left.+\frac{576}{7}(1-4\lambda)\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}\partial D^mD^n\mathbf{\Phi}^{(1)}-\frac{288}{7}(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}D^kD^l\mathbf{\Phi}^{(2)}-144\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}D^mD^n\mathbf{\Phi}^{(2)}\right](Z_2)\\
&+\frac{1}{z_{12}}\left[\frac{384}{35}(39-32\lambda+64\lambda^2)\partial^2\mathbf{\Phi}^{(1)}-\frac{6144}{35}(1-4\lambda)(3-2\lambda+4\lambda^2)\partial^3\mathbf{J}\right.\\
&-\frac{3072}{7}(1-\lambda+2\lambda^2)\left(\frac{1}{2\cdot 4!}\varepsilon^{ijkl}\partial D^iD^jD^kD^l\mathbf{J}-\frac{1}{2}(1-4\lambda)\partial^3\mathbf{J}\right)-\frac{576}{7}(1-4\lambda)\partial\mathbf{\Phi}^{(2)}+48\mathbf{\Phi}^{(3)}\\
&\left.+\frac{1152}{35}(1-4\lambda)\frac{1}{4!}\varepsilon^{ijkl}D^iD^jD^kD^l\mathbf{\Phi}^{(1)}\right](Z_2)
\end{aligned}
$$

$$+\frac{\theta_{12}^i}{z_{12}}\left[\frac{2048}{35}(1-4\lambda)(3-2\lambda+4\lambda^2)\partial^3 D^i\mathbf{J}-\frac{384}{35}(9-2\lambda+4\lambda^2)\partial^2 D^i\mathbf{\Phi}^{(1)}\right.$$
$$-\frac{768}{7}(1-\lambda+2\lambda^2)\left(-\frac{1}{3!}\varepsilon^{ijkl}\partial^2 D^j D^k D^l\mathbf{J}-(1-4\lambda)\partial^3 D^i\mathbf{J}\right)-\frac{1152}{35}(1-4\lambda)\frac{1}{3!}\varepsilon^{ijkl}\partial D^j D^k D^l\mathbf{\Phi}^{(1)}$$
$$\left.+\frac{48}{7}(1-4\lambda)\partial D^i\mathbf{\Phi}^{(2)}-8D^i\mathbf{\Phi}^{(3)}+48\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{\Phi}^{(2)}\right](Z_2)+\frac{\theta_{12}^{4-0}}{z_{12}^2}[8\mathbf{\Phi}^{(4)}](Z_2)$$
$$+\frac{\theta_{12}^{4-i}}{z_{12}}[D^i\mathbf{\Phi}^{(4)}](Z_2)+\frac{\theta_{12}^{4-0}}{z_{12}}[2\partial\mathbf{\Phi}^{(4)}](Z_2)+\cdots. \qquad \text{(I1)}$$

Except for the last three terms for the $\mathcal{N}=4$ primary condition [11], the additional terms consisting of the $\mathcal{N}=4$ stress energy tensor, the first, the second, and the third $\mathcal{N}=4$ multiplets (and their descendants) appear in (I1). We expect that the OPEs between the $\mathcal{N}=4$ stress energy tensor and the hth $\mathcal{N}=4$ multiplet contain the first, the second, through the hth $\mathcal{N}=4$ multiplets.

APPENDIX J: THE OPES BETWEEN THE FIRST $\mathcal{N}=4$ MULTIPLET AND THE THIRD $\mathcal{N}=4$ MULTIPLET

The OPEs between the operators in (3.32) and the operators in (F2) can be summarized by

$$\mathbf{\Phi}^{(1)}(Z_1)\mathbf{\Phi}^{(3)}(Z_2)=\frac{1}{z_{12}^4}\left[-\frac{768}{5}N\lambda(1-2\lambda)(1-4\lambda)\right]+\frac{\theta_{12}^{4-0}}{z_{12}^6}[3072N\lambda(-1+4\lambda-8\lambda^2+8\lambda^3)]$$
$$+\frac{\theta_{12}^{4-i}}{z_{12}^5}\left[\frac{6144}{5}\lambda(1-2\lambda)(2-\lambda+2\lambda^2)D^i\mathbf{J}\right](Z_2)+\frac{\theta_{12}^{i}}{z_{12}^4}\left[-\frac{1536}{5}\lambda(1-2\lambda)(1-4\lambda)D^i\mathbf{J}\right](Z_2)$$
$$+\frac{\theta_{12}^{4-0}}{z_{12}^5}\left[1536\lambda(1-2\lambda)(1-4\lambda)\mathbf{\Phi}^{(1)}+\frac{6144}{5}\lambda(1-2\lambda)(1-4\lambda)^2\partial\mathbf{J}\right](Z_2)$$
$$+\frac{1}{z_{12}^3}\left[384\lambda(1-2\lambda)\mathbf{\Phi}^{(1)}-\frac{1536}{5}\lambda(1-2\lambda)(1-4\lambda)\partial\mathbf{J}\right](Z_2)$$
$$+\frac{\theta_{12}^{4-ij}}{z_{12}^4}\left[-\frac{384}{5}(1-4\lambda)(3+2\lambda-4\lambda^2)\frac{1}{2!}\varepsilon^{ijkl}D^kD^l\mathbf{J}\right.$$
$$\left.-\frac{576}{5}(1+2\lambda-4\lambda^2)\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}D^mD^n\mathbf{J}\right](Z_2)$$
$$+\frac{\theta_{12}^{4-i}}{z_{12}^4}\left[-\frac{1536}{5}\lambda(1-2\lambda)(1-4\lambda)^2\partial D^i\mathbf{J}+192\lambda(1-2\lambda)(1-4\lambda)D^i\mathbf{\Phi}^{(1)}\right.$$
$$\left.+\frac{576}{5}(1-4\lambda)(1+2\lambda-4\lambda^2)\left(-\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{J}-(1-4\lambda)\partial D^i\mathbf{J}\right)\right](Z_2)$$
$$+\frac{\theta_{12}^{i}}{z_{12}^3}\left[\frac{1536}{5}\lambda(1-2\lambda)(1-4\lambda)\partial D^i\mathbf{J}-192\lambda(1-2\lambda)D^i\mathbf{\Phi}^{(1)}\right.$$
$$\left.-\frac{576}{5}(1+2\lambda-4\lambda^2)\left(-\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{J}-(1-4\lambda)\partial D^i\mathbf{J}\right)\right](Z_2)$$
$$+\frac{\theta_{12}^{4-0}}{z_{12}^4}\left[-384\lambda(1-2\lambda)(1-4\lambda)\partial\mathbf{\Phi}^{(1)}+\frac{1536}{5}\lambda(1-2\lambda)(1-4\lambda)^2\partial^2\mathbf{J}\right.$$
$$+\frac{768}{5}(1-4\lambda)(1+2\lambda-4\lambda^2)\left(\frac{1}{2\cdot 4!}\varepsilon^{ijkl}D^iD^jD^kD^l\mathbf{J}-\frac{1}{2}(1-4\lambda)\partial^2\mathbf{J}\right)$$
$$\left.+\frac{288}{5}(11+12\lambda-24\lambda^2)\mathbf{\Phi}^{(2)}\right](Z_2)$$

$$
\begin{aligned}
&+\frac{1}{z_{12}^2}\left[-192\lambda(1-2\lambda)\partial\mathbf{\Phi}^{(1)}+\frac{768}{5}\lambda(1-2\lambda)(1-4\lambda)\partial^2\mathbf{J}-\frac{144}{5}(1-4\lambda)\mathbf{\Phi}^{(2)}\right.\\
&\left.-\frac{384}{5}(-1-2\lambda+4\lambda^2)\left(\frac{1}{2\cdot 4!}\varepsilon^{ijkl}D^iD^jD^kD^l\mathbf{J}-\frac{1}{2}(1-4\lambda)\partial^2\mathbf{J}\right)\right](Z_2)\\
&+\frac{\theta_{12}^{4-i}}{z_{12}^3}\left[\frac{288}{5}(1+\lambda)(3-2\lambda)D^i\mathbf{\Phi}^{(2)}\right](Z_2)\\
&+\frac{\theta_{12}^{4-ij}}{z_{12}^2}\left[-\frac{36}{5}(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}D^kD^l\mathbf{\Phi}^{(2)}-18\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}D^mD^n\mathbf{\Phi}^{(2)}\right](Z_2)\\
&+\frac{\theta_{12}^{4-0}}{z_{12}^3}\left[\frac{576}{5}(1+\lambda)(3-2\lambda)\partial\mathbf{\Phi}^{(2)}\right](Z_2)\\
&+\frac{\theta_{12}^{4-i}}{z_{12}^2}\left[\frac{6}{5}(59+8\lambda-16\lambda^2)\partial D^i\mathbf{\Phi}^{(2)}-\frac{42}{5}(1-4\lambda)\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{\Phi}^{(2)}\right](Z_2)\\
&+\frac{\theta_{12}^{i}}{z_{12}}\left[-\frac{6}{5}(1-4\lambda)\partial D^i\mathbf{\Phi}^{(2)}+6\frac{1}{3!}\varepsilon^{ijkl}D^jD^kD^l\mathbf{\Phi}^{(2)}\right](Z_2)\\
&+\frac{\theta_{12}^{4-0}}{z_{12}^2}\left[\frac{96}{35}(38+11\lambda-22\lambda^2)\partial^2\mathbf{\Phi}^{(2)}-\frac{96}{35}(1-4\lambda)\frac{1}{4!}\varepsilon^{ijkl}D^iD^jD^kD^l\mathbf{\Phi}^{(2)}+\mathbf{\Phi}^{(4)}\right](Z_2)\\
&+\frac{\theta_{12}^{4-ij}}{z_{12}}\left[-\frac{12}{5}(1-4\lambda)\frac{1}{2!}\varepsilon^{ijkl}\partial D^kD^l\mathbf{\Phi}^{(2)}-6\varepsilon^{ijkl}\frac{1}{2!}\varepsilon^{klmn}\partial D^mD^n\mathbf{\Phi}^{(2)}\right](Z_2)\\
&+\frac{\theta_{12}^{4-i}}{z_{12}}\left[\frac{48}{35}(13+\lambda-2\lambda^2)\partial^2D^i\mathbf{\Phi}^{(2)}-\frac{96}{35}(1-4\lambda)\frac{1}{3!}\varepsilon^{ijkl}\partial D^jD^kD^l\mathbf{\Phi}^{(2)}+\frac{1}{8}D^i\mathbf{\Phi}^{(4)}\right](Z_2)\\
&+\frac{\theta_{12}^{4-0}}{z_{12}}\left[\frac{48}{35}(17+4\lambda-8\lambda^2)\partial^3\mathbf{\Phi}^{(2)}-\frac{48}{35}(1-4\lambda)\frac{1}{4!}\varepsilon^{ijkl}\partial D^iD^jD^kD^l\mathbf{\Phi}^{(2)}+\frac{1}{2}\partial\mathbf{\Phi}^{(4)}\right](Z_2)+\cdots. \qquad \text{(J1)}
\end{aligned}
$$

There are the $\mathcal{N}=4$ stress energy tensor, the second and the fourth $\mathcal{N}=4$ multiplets (and their descendants) in (J1).

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[1] D. Friedan, E. J. Martinec, and S. H. Shenker, Conformal invariance, supersymmetry and string theory, Nucl. Phys. **B271**, 93 (1986).

[2] E. Bergshoeff, M. A. Vasiliev, and B. de Wit, The super-W$_\infty(\lambda)$ algebra, Phys. Lett. B **256**, 199 (1991).

[3] E. Bergshoeff, B. de Wit, and M. A. Vasiliev, The structure of the super-W$_\infty(\lambda)$ algebra, Nucl. Phys. **B366**, 315 (1991).

[4] I. Bakas and E. Kiritsis, Grassmannian coset models and unitary representations of W$_\infty$, Mod. Phys. Lett. A **05**, 2039 (1990).

[5] S. Odake and T. Sano, W$_{(1)+\infty}$ and super-W$_\infty$ algebras with SU(N) symmetry, Phys. Lett. B **258**, 369 (1991).

[6] S. Odake, Unitary representations of W$_\infty$ algebras, Int. J. Mod. Phys. A **07**, 6339 (1992).

[7] C. Ahn, The $N=2$ supersymmetric $w_{1+\infty}$ symmetry in the two-dimensional SYK models, J. High Energy Phys. 05 (2022) 115.

[8] T. Creutzig, Y. Hikida, and P. B. Ronne, Extended higher spin holography and Grassmannian models, J. High Energy Phys. 11 (2013) 038.

[9] L. Eberhardt, M. R. Gaberdiel, and I. Rienacker, Higher spin algebras and large $\mathcal{N}=4$ holography, J. High Energy Phys. 03 (2018) 097.

[10] C. Ahn, D. g. Kim, and M. H. Kim, The $\mathcal{N}=4$ coset model and the higher spin algebra, Int. J. Mod. Phys. A **35**, 2050046 (2020).

[11] C. Ahn and M. H. Kim, The operator product expansion between the 16 lowest higher spin currents in the $\mathcal{N}=4$ superspace, Eur. Phys. J. C **76**, 389 (2016).

[12] K. Thielemans, A Mathematica package for computing operator product expansions, Int. J. Mod. Phys. C **02**, 787 (1991).

[13] Wolfram Research, Inc., Mathematica, Version 13.0.0, Champaign, IL (2021).

[14] E. Bergshoeff, C. N. Pope, L. J. Romans, E. Sezgin, and X. Shen, The super W$_\infty$ algebra, Phys. Lett. B **245**, 447 (1990).

[15] K. Schoutens, O(n) extended superconformal field theory in superspace, Nucl. Phys. **B295**, 634 (1988).

[16] M. R. Gaberdiel and R. Gopakumar, Large $N=4$ holography, J. High Energy Phys. 09 (2013) 036.

[17] C. Ahn, Adding complex fermions to the Grassmannian-like coset model, Eur. Phys. J. C **81**, 1125 (2021).

[18] M. Beccaria, C. Candu, and M. R. Gaberdiel, The large $N = 4$ superconformal W_∞ algebra, J. High Energy Phys. 06 (2014) 117.

[19] C. Ahn and M. H. Kim, The $\mathcal{N} = 4$ higher spin algebra for generic μ parameter, J. High Energy Phys. 02 (2021) 123.

[20] S. Pasterski, M. Pate, and A. M. Raclariu, Celestial holography, arXiv:2111.11392.

[21] I. Bakas, The large N limit of extended conformal symmetries, Phys. Lett. B **228**, 57 (1989).

[22] A. Strominger. $\mathrm{W}_{(1+\infty)}$ and the celestial sphere, arXiv:2105.14346.

[23] A. Guevara, E. Himwich, M. Pate, and A. Strominger, Holographic symmetry algebras for gauge theory and gravity, J. High Energy Phys. 11 (2021) 152.

[24] C. Ahn, Towards a supersymmetric $w_{1+\infty}$ symmetry in the celestial conformal field theory, Phys. Rev. D **105**, 086028 (2022).

[25] C. Ahn, A deformed supersymmetric $w_{1+\infty}$ symmetry in the celestial conformal field theory, arXiv:2202.02949.

[26] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, Nucl. Phys. **B241**, 333 (1984).

[27] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.106.026008 for the calculation of Appendix C.