String theory corrections to holographic black hole chemistry

Suvankar Dutta* and Gurmeet Singh Punia†
Department of Physics, Indian Institute of Science Education and Research Bhopal, Bhopal 462066, India

(Received 9 June 2022; accepted 2 July 2022; published 13 July 2022)

The connection between the bulk and the boundary first law of thermodynamics in anti–de Sitter space has been discussed in generic higher derivative gravity. String theory corrections to supergravity render higher derivative terms in the bulk action, proportional to different powers of string theory parameter α'. A variation in the cosmological constant induces a variation in the 't Hooft coupling in the boundary theory. We show that in order to match the bulk first law and Smarr relation with the boundary side we need to include the variation of α' in the bulk thermodynamics as a bookkeeping device. Accordingly, the boundary first law and Euler relation are modified with the inclusion of two central charges (a, c) and/or other chemical potentials as thermodynamic variables. We consider four- and six-derivative terms as well as the Weyl^4 terms (in type IIB) in bulk in support of our generic result.

DOI: 10.1103/PhysRevD.106.026003

I. INTRODUCTION

The thermodynamics of anti–de Sitter (AdS) black holes entered into a new paradigm after considering the cosmological constant as the thermodynamic pressure and inclusion of its variation in the first law [1–9]. This new paradigm is dubbed as *black hole chemistry* [8,9]. In the context of the AdS/CFT correspondence the black hole provides a dual description of the field theory living on the boundary, and hence it is expected that the thermodynamic variables and the laws on both sides match. To make the bulk first law and Smarr relation consistent with the boundary thermodynamics it was shown by [10–12] (following the earlier work [13]) that the inclusion of variation of Newton's constant along with the cosmological constant in the bulk first law is required. Before we elaborate further let us summarize the current status of the first law and the Smarr relation for uncharged AdS-black holes in two-derivative gravity.[1]

The first law in the bulk is given by

$$dM = \frac{T}{4G}dA + \frac{\Theta}{8\pi G}d\Lambda - M\frac{dG}{G}, \tag{1.1}$$

where

$$\begin{aligned} M &= \text{ADM mass of the black hole},\\ T &= \text{Hawking temperature},\\ A &= \text{Area of the event horizon},\\ \Lambda &= \text{Cosmological constant}, \quad G = \text{Newton's constant}. \end{aligned} \tag{1.2}$$

The parameter Θ has a geometrical interpretation in terms of proper volume weighted locally by the norm of the Killing vector ξ [1,14,15],

$$\Theta = \int_{\text{BH}} |\xi| dV - \int_{\text{AdS}} |\xi| dV, \tag{1.3}$$

where the integrations are taken over the constant time hypersurfaces in black hole and AdS spacetimes. The generalized Smarr relation for $(d+1)$-dimensional bulk is given by [16,17]

$$M = \frac{d-1}{d-2}\frac{TA}{4G} - \frac{1}{d-2}\frac{\Theta\Lambda}{4\pi G}. \tag{1.4}$$

The inclusion of variation of Newton's constant in the first law is needed in order to make the black hole thermodynamics consistent with the boundary one. Without including the variation of G and considering the bulk pressure $P = -\Lambda/8\pi G$, the first law (1.1) takes the form

$$dM = TdS + \mathcal{V}dP \tag{1.5}$$

with $\mathcal{V} = -\Theta$, and the Smarr relation becomes $M = (d-1)/(d-2)TS - 2/(d-2)P\mathcal{V}$. There are two

*suvankar@iiserb.ac.in
†gurmeet17@iiserb.ac.in

[1]The formulas can be written for electrically charged black holes as well.

main objections with (1.5), in the context of the AdS/CFT correspondence. From a simple computation of the asymptotic stress energy tensor [18] one can find both the energy and the pressure of the boundary theory. The boundary pressure evaluated in this way does not match with the bulk pressure defined above. Also the energy of the boundary theory turns out to be equal to the Arnowitt-Deser-Misner (ADM) mass M of the black hole. Whereas M, which appears in the first law (1.5), is identified with the enthalpy of the black hole system and not energy, hence we see that the bulk first law (1.5) cannot be interpreted as the boundary first law.

In $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence the AdS radius L, the effective $(d+1)$-dimensional Newton's constant G, and the number of colors N (number of coincident D branes) are related by

$$\text{dictionary 1:}\ \frac{L^{d-1}}{G} \sim N^2. \tag{1.6}$$

The exact relation depends on the particular string theory and its reduction over the compact manifold. Since the bulk cosmological constant is given by

$$\Lambda = -\frac{d(d-1)}{2L^2}, \tag{1.7}$$

a variation of Λ in the bulk, therefore, induces a variation in the number of color N (degrees of freedom or the central charge) in the boundary. Also, from the asymptotic structure of the AdS metric it follows that the spatial volume of the boundary theory goes as $V \sim L^{d-1}$. As a result Λ variation also induces a variation in the spatial volume of the boundary theory. To disentangle the variations of N and V on the boundary (as a result of Λ variation) the variation of bulk Newton's constant was included in the bulk thermodynamics as a "bookkeeping" device [10], such that a variation in N at fixed V in the boundary corresponds to a variation of G^{-1} at fixed L in the bulk and a variation of V at fixed N corresponds to a variation of L keeping L^{d-1}/G fixed in the bulk. After the inclusion of a G variation, the bulk first law takes the form (1.1).

Inclusion of a G variation in the first law has an advantage that the first law (1.1) can be rewritten in the following way:

$$dM = TdS - \frac{M}{d-1}\frac{dL^{d-1}}{L^{d-1}} + (M - TS)\frac{d(L^{d-1}/G)}{L^{d-1}/G} \tag{1.8}$$

and hence can immediately be mapped to the boundary first law [11,12,19]. L^{d-1} is proportional to the thermodynamic volume of the boundary theory. The coefficient of the dL^{d-1} term is therefore identified with the pressure of the boundary theory which satisfies the equation of state: $E = M = (d-1)pV$. Finally, the last term in the first law $d(L^{d-1}/G)/(L^{d-1}/G)$ is identified with the variation of the central charge c, and its coefficient is a new chemical potential μ_c. This new chemical potential satisfies the boundary Euler relation

$$E = M = TS + \mu_c c. \tag{1.9}$$

In this paper we establish the connection between the bulk and the boundary thermodynamics in AdS space in generic higher derivative gravity.[2] String theory correction to supergravity renders higher derivative terms in the bulk action, proportional to different powers of string theory parameter α'. *A priori* it may seem trivial to extend the connection beyond the supergravity limit by including higher derivative corrections to all the thermodynamic variables. But one has to be careful because the second holographic dictionary (2.2) implies that the variation of the cosmological constant in the bulk also induces a variation of the 't Hooft coupling λ of the boundary theory. To disentangle the variation of λ from the variation of N and volume V, one needs to include the variation of α' (along with G and L) in the bulk as a bookkeeping device. We show that under a suitable change of thermodynamic variables the bulk first law can be interpreted as the boundary first law and the Smarr relation renders the generic Euler relation of the boundary theory.

The summary of our results is as follows. We find that the variation of G and α' in the bulk first law can be traded with the variations of two boundary central charges, and thus the boundary first law can be written as

$$dM_c = T_c dS_c - \frac{M_c}{d-1}\frac{dL_c^{d-1}}{L_c^{d-1}} + \mu_+ dc_+ + \mu_- dc_-. \tag{1.10}$$

The subscript c denotes the higher derivative corrected thermodynamic variables. $c_\pm$ are related to two boundary central charges c and a as $c_\pm = (c \pm a)/2$, $\mu_\pm$ are the corresponding chemical potentials (associated with $c_\pm$, respectively) satisfying the generic Euler relation

$$E_c = M_c = T_c S_c + \mu_+ c_+ + \mu_- c_-. \tag{1.11}$$

The organization of this paper is as follows: In Sec. II we discuss how bulk first law and the Smarr relation are modified in the presence of generic higher derivative terms. In the next section (Sec. III) we show how the bulk first law can be interpreted as the boundary first law in generic higher derivative gravity. In Sec. IV we discuss a few examples in support of our generic statement. In particular, we consider six-derivative corrections as well as the Weyl[4] correction to supergravity action.

[2]Large N corrections to the holographic Smarr relation in the presence of Lovelock gravity was considered in [20].

II. THE BULK FIRST LAW AND SMARR RELATION IN HIGHER DERIVATIVE GRAVITY

In the throat limit [21] the effective $(d+1)$-dimensional action has the following qualitative form[3] (in the Einstein frame):

$$\mathcal{I} = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{-g}\left(R - 2\Lambda + \sum_{n\geq 2}(\alpha')^{n-1}\mathcal{R}^{(2n)}\right), \tag{2.1}$$

where α' is proportional to the square of the string length. $\mathcal{R}^{(2n)}$ is the $2n$-derivative term in the action.

Such higher derivative terms appear in low energy effective action of different closed string theories. For example, the appearance of the curvature square term (Riemann$^2 \sim \mathcal{R}^{(4)}$) in heterotic string theory is well known. $\mathcal{R}^{(8)}$ terms appear in superstring theories, whereas $\mathcal{R}^{(6)}$ appears in bosonic string theory. In general, such lower dimensional effective actions are also endowed with other fields, for example $U(1)$ gauge fields, higher form fields, and dilaton. However, in this section we shall consider the effect of pure curvature higher derivative terms on the boundary thermodynamics. In Sec. IV B we include the effect of dilaton coupled to higher derivative terms in the bulk and boundary thermodynamics.

The supergravity limit corresponds to $\alpha'/L^2 \ll 1$. In this limit all the higher derivative terms drop out. The structures of these higher derivative terms are completely fixed for a specific string theory. The only parameter that appears in front of these terms is different powers of α' as this is the only dimensional full parameter in the theory. In the context of gauge/gravity duality such higher derivative terms correspond to the large 't Hooft coupling correction in the strongly correlated field theory. The AdS/CFT correspondence provides another relation between the parameters in the string theory and gauge theory

$$\text{dictionary 2: } \alpha' = \frac{L^2}{\sqrt{\lambda}}, \tag{2.2}$$

where $\lambda = N g_s$ is the 't Hooft coupling of the boundary theory and g_s is the string coupling. This means that the two-derivative gravity (supergravity) is dual to strongly coupled gauge theory on the boundary.

From the relation (2.2) we see that a variation of Λ induces a variation in the 't Hooft coupling constant λ. Therefore, as before, to disentangle the λ variation from N and V variations on the boundary side we allow the parameter α' to vary in the bulk along with L and G as a bookkeeping device [10,19]. A variation with respect to λ on the boundary corresponds to the variation of α' keeping other combinations fixed.

The Smarr relation for Lovelock gravity in AdS spacetime was considered in [13,20]. Considering the variations of the coefficients of Lovelock terms they derived the Smarr relation and showed that it gives the mass of the black hole in terms of geometrical quantities together with the parameters of the Lovelock theory. Following a similar argument the Smarr relation in a generic higher derivative gravity (2.1) takes the following form:

$$M_c = \frac{d-1}{d-2}\frac{T_c A_c}{4G} - \frac{1}{d-2}\frac{\Theta_c \Lambda_c}{4\pi G} + \frac{2}{d-2}\frac{U_{\alpha'}}{G}\alpha', \tag{2.3}$$

and the first law turns out to be

$$dM_c = \frac{T_c}{4G}dA_c + \frac{\Theta_c}{8\pi G}d\Lambda_c - M_c\frac{dG}{G} + \frac{U_{\alpha'}}{G}d\alpha'. \tag{2.4}$$

In the presence of higher derivative terms the mass, entropy, temperature, and cosmological constant receive corrections. Here we denote the higher derivative corrected thermodynamic quantities with the same variables with a subscript c. The new thermodynamic variable $U_{\alpha'}$ is conjugate to the coupling constant α', and the $U_{\alpha'}d\alpha'$ term in the first law disappears in the supergravity limit: $\alpha' = 0$. The quantity A_c appearing in the first law is given by

$$A_c = 4GS_c. \tag{2.5}$$

We call this quantity the "Wald area." Later we shall see that in our parametrization the horizon area remains unchanged under higher derivative corrections. The Wald area is equal to the horizon area in the $\alpha' \to 0$ limit. The Wald area will appear in the first law of black hole thermodynamics. In the presence of higher derivative terms the effective radius of the AdS spacetime changes. Denoting the effective radius by L_c, the corrected cosmological constant Λ_c is given by $\Lambda_c = -d(d-1)/2L_c^2$. The value of L_c depends on the nature of higher derivative terms.

The first law and Smarr relation are consistent with those given in [13]. However, unlike [13] we have only one coupling constant α'. Another important point to note here is that unlike two-derivative gravity the variable Θ_c does not have the geometrical meaning (1.3) any more. In Secs. IV A and IV B we explicitly calculate all the higher derivative corrected thermodynamic variables up to order $(\alpha')^2$ and $(\alpha')^3$, respectively, and show that the first law and Smarr relation are satisfied.

Our next goal is to use the AdS/CFT dictionary (1.6) and (2.2) to obtain the boundary first law from the bulk first law by suitably choosing boundary thermodynamic variables. We also show that the bulk Smarr relation boils down to the generic Euler relation under that choice.

[3] In writing (2.1) we have made another simplification. In principle, there could be other matter terms also, but we have ignored those terms.

III. HOLOGRAPHIC FIRST LAW AND EULER EQUATION

The first law written in terms of $(A_c, \Lambda_c, G, \alpha')$ cannot immediately be identified with the boundary first law. To do so we need to express the first law in terms of boundary thermodynamic variables. In two-derivative gravity it was shown that instead of (A, Λ, G) one can write the first law in the (S, L^{d-1}, c) basis, where L^{d-1} is (proportional to) spatial volume and c is the central charge of the boundary theory [11,12,19]. In higher derivative gravity we have an extra variable α' in the bulk. Therefore the natural question is what is the correct thermodynamic basis in the boundary theory in this case.

CFTs in higher dimensions are endowed with two central charges c and a. These two central charges are an artifact of the breaking of conformal symmetry at quantum level. The expectation value of the trace of the CFT stress tensor is given by $\langle T^\mu_\nu \rangle = -aE_4 - cI_4$ where E_4 and I_4 are two invariants made of the Riemann tensor. The holographic computation of these central charges [22] shows that $c = a \sim L^{d-1}/G$ in the supergravity limit. However, in the presence of string theory corrections they are not the same anymore; they differ by the inverse powers of the 't Hooft coupling [23–25]. Motivated by [11,12,19] we identify that the extra bulk parameter α' can be replaced in terms of the second central charge a. However, instead of writing the first law in terms of (c, a) we define a new set

$$c_\pm = \frac{c \pm a}{2} \tag{3.1}$$

and write down the first law in the (c_+, c_-) basis such that in the $\alpha' \to 0$ limit we readily get back the two-derivative results.

The holographic dictionary (2.2) relates two dimension full parameters α' and L in the bulk with a dimensionless parameter λ on the boundary. The effective (corrected) length of AdS spacetime therefore can be written as

$$L_c = L\tilde{b}(\lambda), \tag{3.2}$$

where $\tilde{b}(\lambda)$ depends on the form of the higher derivative terms and $\tilde{b}(\lambda) = 1$ as $\lambda \to \infty$. Therefore the string theory parameter α' can be written in terms of the effective radius of AdS spacetime as

$$\alpha' = \frac{L_c^2}{\sqrt{\lambda}\tilde{b}^2(\lambda)} = L_c^2 b(\lambda), \quad \text{where} \quad b(\lambda) = \frac{1}{\sqrt{\lambda}\tilde{b}^2(\lambda)}. \tag{3.3}$$

From the similar dimensional analysis the generic form of $c_\pm$ in higher derivative gravity can be written as

$$c_+ = \frac{L_c^{d-1}}{G} h_+(\lambda) \quad \text{and} \quad c_- = \frac{L_c^{d-1}}{G} h_-(\lambda), \tag{3.4}$$

where h_+ and h_- are functions of dimensionless parameter λ and depend on the nature of the higher derivative terms added. In holographic theory they also satisfy

$$h_+(\lambda) \sim 1 \quad \text{and} \quad h_-(\lambda) \sim \frac{1}{\sqrt{\lambda}} \quad \lambda \to \infty. \tag{3.5}$$

Varying Eqs. (3.4) we find

$$dc_\pm = c_\pm \frac{dL_c^3}{L_c^3} - c_\pm \frac{dG}{G} + \frac{c_\pm h'_\pm}{h_\pm} d\lambda. \tag{3.6}$$

Taking the variation of Eq. (3.3),

$$d\alpha' = \frac{2bL_c^2}{3} \frac{dL_c^3}{L_c^3} + L_c^2 b' d\lambda, \tag{3.7}$$

we replace $d\alpha'$ in terms of $\frac{dL_c^3}{L_c^3}$ and $d\lambda$ in the bulk first law. We then solve (3.6) to replace dG and $d\lambda$ in the first law in terms of $dc_\pm$. We also use the Smarr relation (2.3) to replace Θ_c in the first law. After simplification the final result is given by

$$dM_c = T_c dS_c - \frac{M_c}{(d-1)L_c^{d-1}} dL_c^{d-1} + \left(\frac{h'_+(M_c - S_cT_c) - c_- U b' L_c^{3-d}}{(c_+h'_- - c_-h'_+)}\right) dc_+ + \left(\frac{c_+ U b' L_c^{3-d} - h'_+(M_c - S_cT_c)}{(c_+h'_- - c_-h'_+)}\right) dc_-. \tag{3.8}$$

Identifying the coefficient of $dc_\pm$ as the chemical potentials $\mu_\pm$ associated with $c_\pm$ we see that $\mu_\pm$ satisfy the generic Euler relation (1.11).

Thus we see that the bulk first law (2.4) can immediately be identified with the extended first law of the boundary CFT (1.10), and the generic Smarr relation renders the Euler relation (1.11). As a consistency check we see that the chemical potentials $c_- = 0$, $c_+ = c = a$ in the limit $\lambda \to \infty$, and we get back (1.8) and (1.9). We also note that though individual thermodynamic quantities explicitly depend on specific combinations of higher derivative terms, the first law and the Euler relation are independent of any specific combinations.

In the next section we consider a few examples of higher derivative gravity and compute different thermodynamic quantities and the chemical potential in order to check our generic statement.

IV. HIGHER DERIVATIVE THERMODYNAMICS: EXAMPLES

The higher derivative terms that appear in the effective $(d+1)$-dimensional Lagrangian under a consistent truncation of string theory have a very specific form. For example, if we consider type IIB string theory on $\mathrm{AdS}_5 \times S^5$ background and truncate the string theory action over S^5, the resulting effective theory in AdS_5 has the first nontrivial higher derivative correction term at the order of $(\alpha')^3$ [26–30]. We shall consider the effect of such terms on bulk and boundary thermodynamics in Sec. IV B. Before that, we consider a bulk action with four- and six-derivative terms. The curvature square terms appear in heterotic string theory, whereas the curvature cube terms appear in bosonic string theory. The four-derivative terms in the action are proportional to α' and the six-derivative terms appear at the order $(\alpha')^2$. Our goal is to compute all the thermodynamic quantities up to order $(\alpha')^2$ and check the generic first law, the Smarr relation, and the Euler relation. In this section we work in $4+1$ dimensions.

A. Six-derivative theory

We start with the most general four- and six-derivative terms. Before we find the corrections to bulk metric and thermodynamic quantities, we briefly discuss the field redefinition ambiguity with these terms. There are five possible dimension-six invariants that do not involve Ricci tensors or curvature scalars,

$$\begin{aligned} I_1 &= R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\rho}R^{\lambda\rho}{}_{\mu\nu}, \qquad I_2 = R^{\mu\nu}{}_{\rho\sigma}R^{\rho\tau}{}_{\lambda\mu}R^{\sigma}{}_{\tau}{}^{\lambda}{}_{\nu}, \qquad I_3 = R^{\alpha\nu}{}_{\mu\beta}R^{\beta\gamma}{}_{\nu\lambda}R^{\lambda\mu}{}_{\gamma\alpha}, \\ I_4 &= R_{\mu\nu\alpha\beta}R^{\mu\alpha}{}_{\gamma\delta}R^{\nu\beta\gamma\delta}, \qquad I_5 = R_{\mu\nu\alpha\beta}\mathcal{D}^2R^{\mu\nu\alpha\beta}. \end{aligned} \tag{4.1}$$

These five invariants satisfy the following relations:

$$I_3 = I_2 - \frac{1}{4}I_1, \qquad I_4 = \frac{1}{2}I_1, \qquad I_5 = -I_1 - 4I_2. \tag{4.2}$$

Hence only two of them are independent. We choose these two invariants to be I_1 and I_2.

Including the invariants made out of Ricci tensor and scalar the most general Lagrangian (density) containing all possible independent curvature invariants are given by

$$\begin{aligned} \mathcal{L} = a_0R - 2\Lambda &+ \alpha'(\beta_1R^2 + \beta_2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \beta_3R_{\mu\nu}R^{\mu\nu}) \\ &+ \alpha'^2(\alpha_1I_1 + \alpha_2I_2 + \alpha_3R_{\mu\alpha\beta\gamma}R_{\nu}^{\alpha\beta\gamma}R^{\mu\nu} + \alpha_4RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \alpha_5R_{\mu\nu\rho\lambda}R^{\nu\lambda}R^{\mu\rho} \\ &+ \alpha_6R_{\mu\nu}R^{\nu\lambda}R^{\mu}{}_{\lambda} + \alpha_7R_{\mu\nu}\mathcal{D}^2R^{\mu\nu} + \alpha_8RR_{\mu\nu}R^{\mu\nu} + \alpha_9R^3 + \alpha_{10}R\mathcal{D}^2R) + \mathcal{O}(\alpha'^3). \end{aligned} \tag{4.3}$$

However, many terms in this Lagrangian are ambiguous up to a field redefinition [23,31]. Under the following field redefinition:

$$\begin{aligned} g_{\mu\nu} \to \tilde{g}_{\mu\nu} = g_{\mu\nu} &+ \alpha'(d_1g_{\mu\nu}R + d_2R_{\mu\nu}) + \alpha'^2(d_3R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} + d_4g_{\mu\nu}R_{\alpha\beta\gamma\sigma}R^{\alpha\beta\gamma\sigma} + d_5R_{\mu\alpha\beta\nu}R^{\alpha\beta} + d_6R_{\mu\lambda}R^{\lambda}{}_{\nu} + d_7\mathcal{D}^2R_{\mu\nu} \\ &+ d_8g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + d_9g_{\mu\nu}R^2 + d_{10}g_{\mu\nu}\mathcal{D}^2R) + \mathcal{O}(\alpha'^3), \end{aligned} \tag{4.4}$$

only the coefficients a_0, β_2, α_1, and α_2 remain invariant as it is not possible to generate any higher rank tensor from a lower rank tensor in (4.4). Therefore the coefficients β_2, α_1, and α_2 are unambiguous. By proper choice of field redefinition one can set all other ambiguous coefficients to zero

$$\mathcal{L} \to a_0R - 2\Lambda + \alpha'\beta_2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \alpha'^2(\alpha_1I_1 + \alpha_2I_2). \tag{4.5}$$

In the Euclidean approach the thermodynamic quantities are obtained by evaluating the Euclidean on-shell action. Since the action is not invariant under such field redefinition, the thermodynamic quantities depend on unambiguous as well as different ambiguous coefficients. However, in this section we turn on only the unambiguous higher derivative terms, i.e., β_2, α_1, and α_2 to maintain the simplicity of different thermodynamic variables. In Appendix A we present the results for other ambiguous coefficients also.

1. The solution

We start with the following six-derivative action:

$$\begin{aligned} \mathcal{I} = \frac{1}{16\pi G}\int d^5x\sqrt{-g}[R - 2\Lambda &+ \alpha'\beta_2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ &+ \alpha'^2(\alpha_1I_1 + \alpha_2I_2)]. \end{aligned} \tag{4.6}$$

The coefficients β_2, α_1, and α_2 are fixed, so they do not vary.

The equations of motion for the metric obtained from the action (4.6) are given by

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} - \frac{6}{L^2} g_{\alpha\beta} = \alpha' T^{(4)}_{\alpha\beta} + \alpha'^2 T^{(6)}_{\alpha\beta}, \quad (4.7)$$

where $T^{(4)}_{\alpha\beta}$ and $T^{(6)}_{\alpha\beta}$ are given by (A1) and (A2), respectively. To solve these equations we consider the following metric ansatz (for a static spherically symmetric solution):

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)} dr^2 + r^2 d\Omega_3^2, \quad (4.8)$$

where $d\Omega_3^2$ is a metric on a 3-sphere of unit radius. We solve Einstein's equations perturbatively to obtain $f(r)$ and $g(r)$. In the absence of any higher derivative terms the equations of motion admit an asymptotically AdS black hole solution given by[4]

$$f_0(r) = g_0(r) = \frac{(r^2 - r_+^2)(L^2 + r^2 + r_+^2)}{L^2 r^2}, \quad (4.9)$$

where r_+ is the horizon radius that is related to the ADM mass by [32]

$$M = \frac{3\pi r_+^2 (L^2 + r_+^2)}{8GL^2}. \quad (4.10)$$

Treating the higher derivative terms perturbatively one can systematically find the corrections to the leading solutions. The equations of motion (4.7) are ordinary second order differential equations (while treated perturbatively). Therefore at every order in α' we have two integration constants. We fix these constants by demanding that the horizon radius r_+ remains unaffected under perturbations and the spacetime remains AdS in the limit $r \to \infty$. With these two conditions the final form of $f(r)$ and $g(r)$ are given in Appendix B. Asymptotically the metric takes the following form:

$$ds^2 \sim -\left(1 + \frac{r^2}{L_c^2}\right) dt^2 + \left(1 + \frac{r^2}{L_c^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2, \quad (4.11)$$

where

$$L_c = L\left[1 - \frac{\beta_2}{3}\frac{\alpha'}{L^2} - \frac{1}{6}\left(4\alpha_1 + 3\alpha_2 + \frac{5}{3}\beta_2^2\right)\left(\frac{\alpha'}{L^2}\right)^2\right] \quad (4.12)$$

is the corrected AdS radius. For a consistency, one can also compute the Ricci scalar for the corrected solution (B2), (B3) and check that

$$R = -\frac{20}{L_c^2}. \quad (4.13)$$

2. The thermodynamic variables

Once we have the black hole metric corrected up to $(\alpha')^2$, one can compute different thermodynamic variables associated with the corrected geometry.

The correction to the black hole temperature can be computed in different ways. One simple method to compute is the Euclidean method. In this method we first Euclideanize the time direction by replacing $t \to i\tau$. The Euclidean metric will show a conical singularity at $r = r_+$ unless the Euclidean time τ is periodic. One can compute the periodicity β of τ, and the black hole temperature is inversely proportional to the periodicity

$$T = \frac{1}{\beta} = \frac{1}{4\pi}\sqrt{g'(r) f'(r)}|_{r_+}. \quad (4.14)$$

After simplification we find that the corrected temperature is given by

$$T_c = \left(\frac{r_+}{\pi L^2} + \frac{1}{2\pi r_+}\right) - \left(\frac{2\beta_2(3L^4 + 6L^2 r_+^2 + 2r_+^4)}{3\pi L^2 r_+^3}\right)\frac{\alpha'}{L^2} + \left(\frac{2(-21L^6 + 18L^4 r_+^2 + 99L^2 r_+^4 + 62 r_+^6)\alpha_1}{3\pi L^2 r_+^5} + \frac{(L^6 - 3r_+^2(2L^2 + r_+^2)(L^2 + 2r_+^2))\alpha_2}{2\pi L^2 r_+^5} + \frac{8(36L^6 + 36L^4 r_+^2 - 39L^2 r_+^4 - 38 r_+^6)\beta_2^2}{9\pi L^2 r_+^5}\right)\left(\frac{\alpha'}{L^2}\right)^2. \quad (4.15)$$

The entropy of the black hole can also be computed using either Wald's formula or the Euclidean method. In either method the corrected entropy turns out to be

$$S_c = \frac{\pi^2 r_+^3}{2G}\left[1 + \beta_2\left(\frac{12L^2}{r_+^2} + 8\right)\frac{\alpha'}{L^2} + \left(\frac{12L^2(12\alpha_1 + 3\alpha_2 - 24\beta_2^2)}{r_+^2} + \frac{3L^4(36\alpha_1 + 3\alpha_2 - 48\beta_2^2)}{r_+^4} + \frac{36(4\alpha_1 + 3\alpha_2) - 416\beta_2^2}{3}\right)\left(\frac{\alpha'}{L^2}\right)^2\right]. \quad (4.16)$$

[4]The subscript 0 means the leading solution.

The ADM mass of the black hole can also be obtained by either computing the asymptotic stress tensor [18,33] or on-shell Euclidean action [32,34]. The result is given by

$$M_c = \frac{3\pi}{8\pi G}\left[r_+^2 + \frac{r_+^4}{L^2} + \beta_2\left(\frac{6L^4 + 20r_+^4 + 24r_+^2L^2}{3L^2}\right)\frac{\alpha'}{L^2} + \left(\frac{2L^4(32\alpha_1 + \alpha_2 - 48\beta_2^2)}{r_+^2} + 3(84\alpha_1 + 5\alpha_2 - 128\beta_2^2)L^2 \right.\right.$$
$$\left.\left. + \frac{r_+^2}{3}(3(276\alpha_1 + 45\alpha_2) - 1424\beta_2^2) + \frac{r_+^4}{9L^2}(3(268\alpha_1 + 99\alpha_2) - 1648\beta_2^2)\right)\left(\frac{\alpha'}{L^2}\right)^2\right]. \tag{4.17}$$

3. The first law and chemical potentials

To write down the higher derivative corrected first law of thermodynamics we first note that the effective radius of AdS spacetime has been modified (4.12). As a result the cosmological constant Λ will also be corrected

$$\Lambda \to \Lambda_c = -\frac{6}{L_c^2}. \tag{4.18}$$

Allowing the variations of G, L, and α' we find that the thermodynamic variables M_c, T_c, A_c, and Λ_c satisfy the first law (2.4). The thermodynamic potentials Θ_c and $U_{\alpha'}$ are given by

$$\Theta_c = -\frac{1}{2}\pi^2 r_+^4 - \frac{2\pi^2\beta_2 r_+^2\alpha'(2L^2 + r_+^2)}{3L^2} + \frac{\pi^2}{18}(-36\alpha_1(39L^4 + 74L^2r_+^2 + 35r_+^4)$$
$$+ 27\alpha_2(3L^4 + 2L^2r_+^2 - r_+^4) + 16\beta_2^2(96L^4 + 190L^2r_+^2 + 93r_+^4))\left(\frac{\alpha'}{L^2}\right)^2 \tag{4.19}$$

and

$$U_{\alpha'} = -\frac{\pi(9L^4 + 20L^2r_+^2 + 6r_+^4)\beta_2}{4L^4} + \frac{\pi}{12r_+^2L^4}(12\alpha_1(-6L^6 + 9L^4r_+^2 + 39L^2r_+^4 + 19r_+^6)$$
$$- 9\alpha_2(4L^6 + 21L^4r_+^2 + 27L^2r_+^4 + 15r_+^6) + 16\beta_2^2(9L^6 + 24L^4r_+^2 + 19L^2r_+^4 + 5r_+^6))\frac{\alpha'}{L^2}. \tag{4.20}$$

As we mentioned in the Introduction, the thermodynamic potential Θ_c associated with the Λ_c variation does not have the geometrical meaning (1.3) in the presence of higher derivative terms. For the spacetime metric (B2), (B3) the correction to the geometric volume Θ_c is different from (4.19). With these corrected thermodynamic potentials it is easy to check that the corrected Smarr relation (2.3) is satisfied.

To cast the bulk first law in terms of boundary variables we compute the anomaly coefficients c and a in the presence of higher derivative terms in the action (4.6). The answers are given by [23]

$$c = \frac{L_c^3}{128\pi G}\left[1 + \frac{4\beta_2}{\sqrt{\lambda}} + \frac{(-36\alpha_1 + 21\alpha_2)}{\lambda}\right] \tag{4.21}$$

and

$$a = \frac{L_c^3}{128\pi G}\left[1 - \frac{4\beta_2}{\sqrt{\lambda}} + \frac{12\alpha_1 + 9\alpha_2}{\lambda}\right]. \tag{4.22}$$

Replacing the variations dG and $d\alpha'$ as discussed in Sec. III, in terms of $dc_\pm$, we can write the first law in the form given in (1.10) with

$$\mu_+ = \frac{16\pi^2 r_+^2(r_+^2 - L^2)}{L^5} - \frac{128\pi^2\beta_2(3L^4 + 6L^2r_+^2 + 2r_+^4)}{3\sqrt{\lambda}L^5} - \frac{8\pi^2}{9\lambda L^5 r_+^2}(36L^6(40\alpha_1 + 5\alpha_2 - 64\beta_2^2) + 36L^4r_+^2(12\alpha_1 + 24\alpha_2 - 61\beta_2^2)$$
$$+ 3L^2r_+^4(-1200\alpha_1 + 396\alpha_2 + 875\beta_2^2) + r_+^6(-2688\alpha_1 + 432\alpha_2 + 2423\beta_2^2)) \tag{4.23}$$

and[5]

[5]Note that β_2 is a nonzero coefficient. It comes in the denominator because $c_- \sim \beta_2$.

$$\mu_- = \frac{8\pi^2(9L^4 + 20L^2 r_+^2 + 7r_+^4)}{L^5} + \frac{8\pi^2}{3\beta_2\sqrt{\lambda}L^5 r_+^2}(24(L^2 + r_+^2)(3L^4(2\alpha_1 + \alpha_2) + 6L^2 r_+^2(2\alpha_1 + \alpha_2) + r_+^4(4\alpha_1 + 3\alpha_2)) - \beta_2^2(288L^6 + 756L^4 r_+^2 + 591L^2 r_+^4 + 147r_+^6)), \tag{4.24}$$

and they satisfy Euler relation (1.11) up to order $\frac{1}{\lambda}$.

B. $\mathcal{W}^4$ term

Finally, we discuss a string theory example in the context of the AdS/CFT. Since the conjecture is valid for the complete string theory, one should consider the stringy corrections to the ten-dimensional (10D) supergravity action. In particular, we consider the string theory correction to type IIB supergravity. The first corrections occur at order $(\alpha')^3$ [26–28]. The bosonic part of the action in the Einstein frame is given by

$$I = \frac{1}{16\pi G_{10}}\int d^{10}x\sqrt{-g}\left\{R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4.5!}\frac{1}{N^2}F_5^2 + \gamma e^{-\frac{3}{2}\phi}\mathcal{W}^4\right\}, \tag{4.25}$$

where F_5 is a self-dual five-form field strength, ϕ is the dilaton, and $\mathcal{W}^4$ denotes the *eight-derivative* term in action, which can be expressed as a contraction of four Weyl tensors,

$$\mathcal{W}^4 = C_{abcd}C^{ebcf}C^a{}_{ghe}C_f^{ghd} + \frac{1}{2}C_{adbc}C^{efbc}C^a{}_{ghe}C_f^{ghd}. \tag{4.26}$$

The coupling constant γ is given by

$$\gamma = \frac{1}{8}\zeta(3)(\alpha')^3. \tag{4.27}$$

These higher derivative terms do not alter the extremal $AdS_5 \times S^5$ geometry [29,35]. However, this observation is not true in the nonextremal case [30,36]. Moreover, the higher derivative action (4.25) is not supersymmetrically complete. Supersymmetric completion of eight-derivative terms in type IIB string theory can be found in [37–43]. Tentative corrections to black hole free energy and other thermodynamic quantities in the presence of these supersymmetrically complete terms were considered in [43]. The answer is tentative in a sense that the author did not compute the corrections to the full black hole geometry in the presence of these terms since those computations are extremely cumbersome. In this paper we shall consider only the $\mathcal{W}^4$ term (4.26) in the action and see how the bulk and boundary first laws are modified.

In the supergravity limit ($\gamma \to 0$) the type IIB vacuum admits a solution of the form $AdS_5 \times S^5$ with constant five-form field strength over AdS_5 and S^5 with a constant dilaton. As a result, one can truncate the ten-dimensional action over S^5 and the effective five-dimensional action takes the form of Einstein-Hilbert action in AdS_5. Fo the $\mathcal{W}^4$ term apparently it appears that one can proceed exactly in the similar way that we discussed in the previous sections, i.e., replacing the dG and $d\alpha'$ terms in the bulk first law in terms of the variations of two central charges $c_\pm$. However, it turns out that the central charges a and c do not receive any corrections in the presence of $\mathcal{W}^4$ terms and hence $c = a + \mathcal{O}(\gamma^2)$. Therefore the question is what is the relevant boundary parameter with which we trade the $d\alpha'$ term in the bulk first law.

The dilaton field plays an important role here. The massless dilaton field in $4+1$ dimensions corresponds to a dimension-four scalar operator $\hat{\mathcal{O}}_4 \sim \frac{1}{g_{YM}^2}\int \mathrm{Tr}F^2$ in the boundary theory. The dilaton has the asymptotic falloff

$$\phi(r) = \varphi_\infty + \frac{\varphi_1}{r^4} + \mathcal{O}(r^{-6}), \tag{4.28}$$

where φ_∞ plays the role of source for $\hat{\mathcal{O}}_4$ and φ_1 is the expectation value of $\hat{\mathcal{O}}_4$ [44,45]. At the leading order ($\gamma = 0$) the dilaton is constant and the expectation value of $\hat{\mathcal{O}}_4$ is zero. However, at subleading order φ_∞ induces an expectation value for $\mathcal{O}_4$ proportional to γ. Hence the boundary first law and Euler relation can be written in terms of $\langle\mathcal{O}_4\rangle$ and other standard thermodynamic variables.

1. $\mathcal{W}^4$ corrected geometry and the boundary first law

Following [30,36] we consider the following ansatz for the ten-dimensional metric and the five-form field strength:

$$ds^2 = \frac{r^2}{L^2}e^{-\frac{10}{3}C(r)}(e^{2A(r)+8B(r)}d\tau^2 + e^{2B(r)}dr^2 + L^2 d\Omega_3^2) + e^{2C(r)}L^2 d\Omega_5^2, \tag{4.29}$$

$$F_5 = \mathcal{F}_5 + \star\mathcal{F}_5, \qquad \mathcal{F}_5 = dB_4 \tag{4.30}$$

where

$$B_4 = f(r)dt \wedge d\mathrm{vol}_3. \tag{4.31}$$

Here $d\mathrm{vol}_3$ is the volume element of S^3 with radius L. The leading order metric solution is given by (subscript 0 stands for leading solution)

$$A_0(r) = -2\log\left(\frac{r}{L}\right) + \frac{5}{2}\log\left(\frac{r^2}{L^2} + \frac{r^4}{L^4} - \frac{r_0^2}{L^4}\right);$$
$$B_0(r) = -\frac{1}{2}\log\left(\frac{r^2}{L^2} + \frac{r^4}{L^4} - \frac{r_0^2}{L^4}\right);$$
$$C_0(r) = 0. \tag{4.32}$$

The leading dilaton field is constant and denoted as φ_∞, and the solution of the four-form field is $B_4 = \sqrt{2}\frac{r^4}{L^4} dt \wedge d\mathrm{vol}_3$.

Here r_0 is the nonextremality parameter; i.e., in the limit $r_0 \to 0$ the ten-dimensional geometry corresponds to the near horizon limit of an extremal $D3$ branes. We consider the higher derivative terms perturbatively, such that $A(r) = A_0(r) + \gamma A_1(r)$, $B(r) = B_0(r) + \gamma B_1(r)$, $f(r) = f_0(r) + \gamma f_1(r)$, and $\varphi = \varphi_\infty + \gamma\phi_1$. Solving the ten-dimensional equations of motion up to $\mathcal{O}(\gamma)$ we find that the corrections are given by

$$A_1(r) = e^{-\frac{3\varphi_\infty}{2}}(r^4 + L^2(r^2 - r_0^2))^{-1}\left(\frac{5r_0^6(312r^4 + L^2(272r^2 - 237r_0^2))}{2r^{12}} - \frac{5r_0^6(312r_+^4 + L^2(272r_+^2 - 237r_0^2))}{2r_+^{12}}\right), \tag{4.33}$$

$$B_1(r) = e^{-\frac{3\varphi_\infty}{2}}(r^4 + L^2(r^2 - r_0^2))^{-1}\left(\frac{r_0^6(312r_+^4 + L^2(272r_+^2 - 237r_0^2))}{2r_+^{12}} - \frac{5r_0^6(64L^2r^2 - 57L^2r_0^2 + 72r^4)}{2r^{12}}\right), \tag{4.34}$$

$$f_1(r) = 60\sqrt{2}e^{-\frac{3}{2}\varphi_\infty}\frac{r_0^6}{L^4r^8}. \tag{4.35}$$

Here r_+ is the corrected horizon radius. Correction to the dilaton solution is given by

$$\begin{aligned}\phi_1(r) = {} & 45e^{-\frac{3}{2}\varphi_\infty}\frac{(L^8 + 4L^6r_+^2 + 7L^4r_+^4 + 6L^2r_+^6 + 3r_+^8)}{4L^4r_+^4(L^2 + r_+^2)^3}\log\left(\frac{L^2 + r^2 + r_+^2}{r^2}\right) \\ & - \frac{e^{-\frac{3}{2}\varphi_\infty}}{16L^4r^{12}r_0^4}(36L^4r_0^8r^2 + 30L^4r_0^{10} + 60L^2r_0^4r^6(L^2 + 2r_0^2) + 45L^4r_0^6r^4 \\ & + 45L^4r_0^8r^4 + 180r^{10}(L^4 + 4L^2r_0^2 + 3r_0^4) + 90r_0^2r^8(L^4 + 3L^2r_0^2 + r_0^4)).\end{aligned} \tag{4.36}$$

The asymptotic expansion of the dilaton field near the AdS boundary is given by

$$\phi(r) = \varphi_\infty - \frac{45\gamma e^{-\frac{3}{2}\varphi_\infty}(L^2 + r_+^2)^4}{8L^6r_+^4}\frac{1}{r^4} + \mathcal{O}\left(\frac{1}{r^6}\right). \tag{4.37}$$

Following [44,45] we find the expectation value of the corresponding boundary operator $\mathcal{O}_4$ is given by

$$\langle\hat{\mathcal{O}}_4\rangle = 4\varphi_1 = -\frac{45\gamma e^{-\frac{3}{2}\varphi_\infty}(L^2 + r_+^2)^4}{2L^6r_+^4}. \tag{4.38}$$

From the asymptotic expansion of the metric one can easily show that unlike four- and six-derivative cases, the AdS radius L does not receive any higher derivative correction.

2. Bulk thermodynamics and the first law

Using the Euclidean technique (4.14) one can correct the higher derivative correction to the black hole temperature, and it is given by [also using the relation (4.27)]

$$T_c = \frac{L^2 + 2r_+^2}{2\pi L^2 r_+}\left(1 - \frac{5\zeta(3)e^{-\frac{3}{2}\varphi_\infty}(L^2 - 3r_+^2)(L^2 + r_+^2)^3(\alpha')^3}{4L^6r_+^6(L^2 + 2r_+^2)}\right). \tag{4.39}$$

Correction to the entropy can be computed using the Wald formula or Euclidean method. The corrected entropy is given by

$$S_c = \frac{\pi^2 {r_+}^3}{2G}\left(1 + \frac{15\zeta(3)e^{-\frac{3}{2}\varphi_\infty}(L^2 + r_+^2)^3(\alpha')^3}{2L^6r_+^6}\right). \tag{4.40}$$

The ADM mass of the black hole is given by

$$M_c = \frac{3\pi r_+^2(L^2 + r_+^2)}{8GL^2}\left(1 + \frac{5\zeta(3)e^{-\frac{3}{2}\varphi_\infty}(L^2 + r_+^2)^2(7L^2 + 15r_+^2)(\alpha')^3}{8L^6r_+^6}\right). \tag{4.41}$$

In the presence of moduli the first law of the black hole changes [46]. Including the variation of φ_∞ in the first law we find that

$$dM_c = \frac{T_c}{4G} dA_c + \frac{\Theta_c}{8\pi G} d\Lambda_c - \frac{M_c}{G} dG + U_{\alpha'} d(\alpha')^3 + \mu_\varphi d\varphi_\infty, \tag{4.42}$$

where

$$\Theta_c = -\frac{\pi^2 r_+^4}{2} - \frac{15\pi^2 \zeta(3) e^{-\frac{3}{2}\varphi_\infty} (L^2 + r_+^2)^3 (\alpha')^3}{4L^6 r_+^2}, \tag{4.43}$$

$$U_{\alpha'} = -\frac{15\pi \zeta(3) e^{-\frac{3}{2}\varphi_\infty} (L^2 + r_+^2)^4}{64 G L^8 r_+^4}, \tag{4.44}$$

$$\mu_\varphi = \left(\frac{\partial M_c}{\partial \varphi_\infty}\right) = -\frac{\pi}{8GL^2} \langle \hat{\mathcal{O}}_4 \rangle, \tag{4.45}$$

and these thermodynamics quantities satisfy the Smarr relation

$$M_c = \frac{3}{2} T_c S_c + \frac{\Theta_c}{8\pi G} \Lambda_c + 3 \frac{U_{\alpha'}}{G} (\alpha')^3. \tag{4.46}$$

3. Boundary thermodynamics

The dilaton couples to the bulk operator $\frac{1}{g_{\rm YM}^2} {\rm Tr} F^2$. We define a variable

$$\psi = e^{-\varphi_\infty} \tag{4.47}$$

and use the holographic dictionary

$$\alpha' \sim \frac{L^2}{\sqrt{\lambda}}, \qquad \lambda \sim \frac{L^{3/2} e^{\varphi_\infty}}{\sqrt{G}}, \quad \text{and} \quad c = c_+ \sim \frac{L^3}{G} \tag{4.48}$$

to replace variations of α', φ_∞, and G in terms of the variation of c_+ and ψ in the bulk first law (4.42) to obtain the same in terms of boundary parameters

$$dE_c = dM_c = T_c dS_c - \frac{M_c}{3} \frac{dL_c^3}{L_c^3} + \mu_c dc - \mu_\psi d\psi, \tag{4.49}$$

where $L_c = L$ and

$$\mu_c = \frac{16\pi^2 r_+^2 (L^2 - r_+^2)}{L^5} + \frac{5\pi^2 \zeta(3) e^{-3\varphi/2} (13L^2 - 51 r_+^2)(L^2 + r_+^2)^3 (\alpha')^3}{2L^{11} r_+^4},$$

$$\mu_\psi = -\frac{2\mu_\varphi}{\psi}. \tag{4.50}$$

Here we see that the pressure satisfies the equation of state $E_c = 3pV$. The chemical potentials μ_c and μ_ψ satisfy the Euler equation

$$E_c = T_c S_c + \mu_c c + \frac{1}{4} \mu_\psi \psi. \tag{4.51}$$

V. SUMMARY AND DISCUSSION

In this paper we consider string theory corrections to black hole thermodynamics in AdS space and its consistency with the thermodynamics of the boundary theory in the context of the AdS/CFT correspondence. Since the higher derivative terms are low energy effects of some bona fide string theories, their couplings are also fixed. The only parameter that appears in the bulk action with these higher derivative terms is α'. From the AdS/CFT dictionary we see that a variation of Λ induces a variation in the 't Hooft coupling λ, apart from variations in color N and boundary volume V. Therefore, to disentangle the λ variation from that of N and V we allow the parameter α' to vary in the bulk along with L and G as a bookkeeping device. This allows us to establish the equivalence between the bulk and boundary thermodynamics. We consider two types of examples. In the first type we added pure metric higher derivative terms in the action (for example, four- and six-derivative terms). In the presence of such terms we include the variation of α' in the bulk first law and show that trading the variations of G and α' with the variations of c_+ and c_-, where $c_\pm = (c \pm a)/2$, the bulk first law can be beautifully interpreted as the boundary first law which is written in terms of variations of $c_\pm$. As a result, the boundary theory is endowed with two chemical potentials $\mu_\pm$ (corresponding to $c_\pm$), and they satisfy the generalized Euler relation (1.11). In the second example we considered an eight-derivative term in the bulk Lagrangian coming from the superstring theory. In this case the term is coupled with the dilaton. In the leading case the dilaton solution is constant, and we see that the effective five-dimensional bulk first law is the same as before. In the presence of the higher derivative term the dilaton solution is modified, and it turns out that the dilaton sources an expectation value of a dimension-four operator, namely ${\rm Tr}F^2$, and the expectation value is $\sim\alpha'^3$. In this case we trade the G and α' variations in the bulk with the variations of c and the asymptotic value of the dilaton, which acts as a source for the ${\rm Tr}F^2$ operator, and we write the boundary first law in terms of their variations. Again the boundary theory is endowed with two chemical potential μ_c, which corresponds to c, and μ_φ, proportional to $\langle {\rm Tr}F^2 \rangle$, which corresponds to φ_∞. These two chemical potentials satisfy the generalized Euler relation (4.51).

The phase structure of AdS black holes in higher derivative gravity is endowed with an extra chemical potential, and hence the dimension of the thermodynamic phase will increase. In this paper we study the thermodynamics perturbatively; however, it would be interesting to study the black hole phase structure in the presence of the extra parameter, even perturbatively. It would also be interesting to find an effective van der Waals–type description (following [47]) of higher derivative black holes and understand the effect of the central charges on the mean-field potential.

ACKNOWLEDGMENTS

We are thankful to Arnab Rudra, Muktajyoti Saha, and Archit Vidyarthi for their helpful discussion. The work of S. D. is supported by the Mathematical Research Impact Centric Support (MATRICS) (Grant No. MTR/2019/000390, the Department of Science and Technology, Government of India). Finally, we are indebted to the people of India for their unconditional support toward the research in basic science.

APPENDIX A: EOM's FOR FOUR-DERIVATIVE AND SIX-DERIVATIVE TERMS

Here we present the expressions for $T^{(4)}_{ab}$ and $T^{(6)}_{ab}$ that appear in the equations of motion (4.7) in the presence of four- and six-derivative terms.

The contribution from the four-derivative term, i.e., $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$, is given by

$$T^{(4)}_{\alpha\beta} = \frac{\beta_2}{16\pi G}\Big(4R^{\gamma\delta}R_{\alpha\gamma\beta\delta} + 2R_\alpha{}^{\gamma\delta\zeta}R_{\beta\gamma\delta\zeta} - 4R_\alpha{}^\gamma R_{\beta\gamma} - \frac{1}{2}g_{\alpha\beta}R_{\gamma\delta\zeta\lambda}R^{\gamma\delta\zeta\lambda} - 2\nabla_\beta\nabla_\alpha R + 4\Box R_{\alpha\beta}\Big), \tag{A1}$$

and the same for the six-derivative terms is given by

$$T^{(6)}_{\alpha\beta} = \frac{1}{16\pi G}(T^{(a)}{}_{\alpha\beta} + T^{(b)}{}_{\alpha\beta}), \tag{A2}$$

where

$$T^{(a)}{}_{\alpha\beta} = \alpha_1\Big(6R^{\gamma\delta}(2R_\alpha{}^\zeta{}_\gamma{}^\lambda R_{\beta\delta\zeta\lambda} + R_{\alpha\gamma}{}^{\zeta\lambda}(R_{\beta\delta\zeta\lambda} + 2R_{\beta\zeta\delta\lambda})) - 6R_{(\alpha|}{}^\gamma R_{|\beta)}{}^{\delta\zeta\lambda}R_{\gamma\zeta\delta\lambda} - 3R_\alpha{}^{\gamma\delta\zeta}(R_\beta{}^\mu{}_\gamma{}^\nu R_{\delta\zeta\mu\nu} + 4R_\beta{}^\mu{}_\delta{}^\nu R_{\gamma\mu\zeta\nu}) + \frac{3}{2}R_\alpha{}^{\gamma\delta\zeta}R_{\beta[\gamma|}{}^{\mu\nu}R_{|\delta]\zeta\mu\nu} - \frac{1}{2}g_{\alpha\beta}I_1 + 6\nabla^\delta R_\alpha{}^\gamma\nabla_{[\delta|}R_{\beta|\gamma]} + 6R_{(\alpha|\delta\gamma\zeta}\nabla^\zeta\nabla^\delta R_{|\beta)}{}^\gamma + 6\nabla_\gamma R_{\beta\lambda\delta\zeta}\nabla^\lambda R_\alpha{}^{\gamma\delta\zeta}\Big) \tag{A3}$$

and

$$T^{(b)}_{\alpha\beta} = \frac{\alpha_2}{2}(6R^{\gamma\delta}(-R_\gamma{}^\zeta R_{\alpha\delta\beta\zeta} - R_\alpha{}^\zeta{}_\gamma{}^\lambda R_{\beta\lambda\delta\zeta} + R_\alpha{}^\zeta{}_\beta{}^\lambda R_{\gamma\zeta\delta\lambda}) + 9R_\alpha{}^{\gamma\delta\zeta}R_{\beta\delta}{}^{\mu\nu}R_{\gamma\mu\zeta\nu} + 3R_\alpha{}^{\gamma\delta\zeta}R_\beta{}^\mu{}_\delta{}^\nu R_{\gamma(\mu|\zeta|\nu)} + 3R_\alpha{}^{\gamma\delta\zeta}R_\beta{}^\lambda{}_\gamma{}^{\lambda 1}R_{\delta\lambda\zeta\lambda 1} - g_{\alpha\beta}I_2 - 3\nabla_\alpha R^{\gamma\delta}\nabla_{[\beta|}R_{\gamma|\delta]} + 3\nabla^\delta R_\alpha{}^\gamma\nabla_{(\beta|}R_{\gamma|\delta)} - 3R_{\alpha\gamma\beta\delta}\nabla^\delta\nabla^\gamma R + 3R_{(\alpha|\gamma\delta\zeta}\nabla^\zeta\nabla_{|\beta)}R^{\gamma\delta} + 6R_{\alpha\gamma\beta\delta}\Box R^{\gamma\delta} - 3\nabla_\delta R_{\alpha(\gamma|\beta|\zeta)}\nabla^\zeta R^{\gamma\delta} + 12\nabla_\zeta R_{\alpha\gamma\beta\delta}\nabla^\zeta R^{\gamma\delta} - 3R_{(\alpha|\gamma\delta\zeta}\nabla^\zeta\nabla^\delta R_{|\beta)}{}^\gamma + 6R^{\gamma\delta\zeta\lambda}\nabla_\lambda\nabla_\delta R_{\alpha\gamma\beta\zeta} - 6\nabla_\zeta R_{\beta\delta\gamma\lambda}\nabla^\lambda R_\alpha{}^{\gamma\delta\zeta}). \tag{A4}$$

APPENDIX B: α' CORRECTED METRIC

In this appendix we present the higher derivative corrected metric in the presence of generic four- and six-derivative terms. Although in our calculations in the main text only three unambiguous terms are turned on, in general one can study black hole thermodynamics in the presence of these generic terms. Therefore, it will be helpful to find the perturbative metric up to $\mathcal{O}(\alpha'^2)$ in the presence of all these terms.

Einstein-Hilbert action with negative cosmological constant and generic four-derivative and six-derivative terms is given by

$$I = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\{R - 2\Lambda + \alpha'(\beta_1 R^2 + \beta_2 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \beta_3 R_{\mu\nu}R^{\mu\nu}) + \alpha'^2(\alpha_1 R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\rho}R^{\lambda\rho}{}_{\mu\nu} + \alpha_2 R^{\mu\nu}{}_{\rho\sigma}R^{\rho\tau}{}_{\lambda\mu}R^\sigma{}_\tau{}^\lambda{}_\nu + \alpha_3 R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma}R^{\mu\nu} + \alpha_4 RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \alpha_5 R_{\mu\nu\rho\lambda}R^{\nu\lambda}R^{\mu\rho} + \alpha_6 R_{\mu\nu}R^{\nu\lambda}R^\mu{}_\lambda + \alpha_7 R_{\mu\nu}\mathcal{D}^2R^{\mu\nu} + \alpha_8 RR_{\mu\nu}R^{\mu\nu} + \alpha_9 R^3 + \alpha_{10}R\mathcal{D}^2R)\}. \tag{B1}$$

With the metric ansatz (4.8), the higher derivative corrected metric solution for action (B1) is given by

$$f(r)=1+\frac{r^2}{L^2}-\frac{r_0^2}{r^2}+\frac{(\alpha'/L^2)}{3L^2r^6}(6\beta_2L^4r_0^4+2(10\beta_1+\beta_2+2\beta_3)r^8)-\frac{(\alpha'/L^2)^2}{9L^2r^{10}}[72L^6r_0^6(\alpha_1+2\alpha_2-2\alpha_3-12\alpha_4+\beta_2(8\beta_1-7\beta_2)) \\ -18L^6r_0^4r^2(36\alpha_1+9\alpha_2-24\alpha_4-16\beta_2(-\beta_1+5\beta_2+\beta_3))-9L^4r_0^4r^4(132\alpha_1+27\alpha_2+8\alpha_3-40\alpha_4 \\ +8\beta_2(10\beta_1-33\beta_2-6\beta_3))-r^{12}(3(4\alpha_1+3\alpha_2+8\alpha_3+40\alpha_4+16\alpha_5+16\alpha_6+80\alpha_8+400\alpha_9)+8(10\beta_1+\beta_2+2\beta_3)^2)] \tag{B2}$$

and

$$g(r)=1+\frac{r^2}{L^2}-\frac{r_0^2}{r^2}+\frac{(\alpha'/L^2)}{3L^2r^6}(6\beta_2L^6r_0^4+2(10\beta_1+\beta_2+2\beta_3)L^2r^8) \\ +\frac{(\alpha'/L^2)^2}{9L^2r^{10}}[144L^6r_0^4r^2(24\alpha_1+7\alpha_3+24\alpha_4-4\beta_2(4\beta_1+8\beta_2+3\beta_3)) \\ +18L^6r_0^6(-160\alpha_1+\alpha_2-48\alpha_3-168\alpha_4+4\beta_2(28\beta_1+51\beta_2+20\beta_3)) \\ +9L^4r_0^4r^4(444\alpha_1+9\alpha_2+120\alpha_3+392\alpha_4-8\beta_2(26\beta_1+77\beta_2+26\beta_3)) \\ +r^{12}(12\alpha_1+9\alpha_2+8(3\alpha_3+15\alpha_4+6\alpha_5+6\alpha_6+30\alpha_8+150\alpha_9+(10\beta_1+\beta_2+2\beta_3)^2))]. \tag{B3}$$

Here r_0 is the integration constant and is related to the mass of the black hole, and r_+ is the event horizon $f(r_+)=0$. $r_0=0$ corresponds to the pure AdS spacetime solution.

APPENDIX C: EUCLIDEAN FORMULATION

In this appendix we give a quick review of the Euclidean approach to calculating the total energy, entropy, and other thermodynamic variables of the AdS black holes. We start with the Lorentzian metric that describes pure AdS spacetime; after the Wick rotation, $\tau=it$, the metric becomes Euclidean, i.e., positive definite, where we can construct a thermal state in AdS space where the imaginary time coordinate is periodic. The period of the τ direction is mapped to the inverse temperature of thermal AdS gas. The Euclidean metric of AdS_5 is given by

$$ds^2=\left(1+\frac{r^2}{L^2}\right)d\tau^2+\frac{dr^2}{(1+\frac{r^2}{L^2})}+r^2d\Omega_3^2, \tag{C1}$$

where L is the radius of AdS space. Similarly, the Euclidean AdS_5 Schwarzschild metric is given by

$$ds^2=\left(1+\frac{r^2}{L^2}-\frac{r_0^2}{r^2}\right)d\tau^2+\left(1+\frac{r^2}{L^2}-\frac{r_0^2}{r^2}\right)^{-1}dr^2+r^2d\Omega_3^2, \tag{C2}$$

where r_0 is the black hole parameter related with the ADM mass of the black hole given as $M=\frac{3\Omega_3}{16\pi G}r_0^2$. This spacetime has a horizon at $r=r_+$, given by the relation $1+\frac{r_+^2}{L^2}-\frac{r_0^2}{r_+^2}=0$. The Euclidean black hole metric has a conical singularity at $r=r_+$ unless we consider the imaginary time direction to be periodic. This fixes the temperature of the AdS Schwarzschild spacetime.

To discuss the thermodynamics of the black hole in the Euclidean framework [32] we define the canonical partition function as the functional integration of Euclidean action I_E, $Z=\int[\mathcal{D}g]e^{-I_E}$. In the semiclassical limit $G\to 0$, the predominant contributions to the path integral come from the classical solution. So $\ln\mathcal{Z}=-I_E^{\mathrm{OS}}$, where I_E^{OS} is the on-shell action. Thus, the free energy of the system is

$$\log Z=-I_E=-\beta F, \tag{C3}$$

and the thermodynamical energy (or ADM mass) of the black hole is

$$E=-\frac{\partial(\log Z(\beta))}{\partial\beta}=\frac{\partial I_E}{\partial\beta}, \tag{C4}$$

and the entropy is given by

$$S=\beta\frac{\partial I_E}{\partial\beta}-I_E. \tag{C5}$$

However, the above thermodynamic quantities receive divergence contributions since the black hole spacetime has infinite volume (the integration over the radial direction ranges to ∞). To read off the finite values of the thermodynamic quantities, we subtract the contribution of thermal AdS spacetime from the black hole as a regularization prescription. First, we evaluate the action integral for the black hole putting a cutoff on the radial integration:

$r_+ \leq r \leq R_c$ where R_c is an IR cutoff on the spacetime and r_+ is the outermost horizon. In the pure AdS spacetime the region of integration is $0 \leq r \leq R_c$. The important point here is that the temperature of the pure AdS spacetime cannot be fixed from the conical singularity of the metric as the AdS metric is well defined everywhere between $0 \leq r < \infty$. Rather we demand that both the AdS and black hole spacetimes have the same geometry asymptotically. This fixes the temperature of the AdS spacetime

$$\beta_{\rm BH}\sqrt{g^{\rm BH}_{\tau\tau}(r=R_c)} = \beta_{\rm AdS}\sqrt{g^{\rm AdS}_{\tau\tau}(r=R_c)}. \quad \text{(C6)}$$

Via this consideration, the periodicity of the reference AdS spacetime depends on the black hole parameters [such as the mass and temperature $(1/\beta_{\rm BH})$].

1. Six-derivative gravity

Here we present the higher derivative corrected [for the action (B1)] thermodynamic quantities using the Euclidean method.

The temperature of the black hole can be computed from the Euclidean metric using (4.14) where $\beta_{\rm BH}$ is the inverse Hawking temperature in corrected geometry,

$$\begin{aligned}\beta_{\rm BH}^{-1} = T = &\frac{1}{2\pi r_+} + \frac{r_+}{\pi L^2} + \frac{\alpha'}{L^2}\left(\frac{1}{3\pi r_+^3 L^2}\right)(4(5\beta_1-\beta_2+\beta_3)r_+^4 - 6\beta_2 L^2(L^2+2r_+^2))\\ &+\left(\frac{\alpha'}{L^2}\right)^2\left(\frac{1}{18\pi L^2 r_+^5}\right)[12\alpha_1(-21L^6+18L^4r_+^2+99L^2r_+^4+62r_+^6)+9\alpha_2(L^6-6L^4r_+^2-15L^2r_+^4-6r_+^6)\\ &+24\alpha_3(-3L^6+6L^4r_+^2+21L^2r_+^4+14r_+^6)-24\alpha_4(9L^6-30L^4r_+^2-87L^2r_+^4-58r_+^6)\\ &+32r_+^6(3\alpha_5+3\alpha_6+15\alpha_8+75\alpha_9)+16\beta_2^2(36L^6+36L^4r_+^2-39L^2r_+^4-38r_+^6)+64r_+^6(5\beta_1+\beta_3)^2\\ &+16\beta_2(9\beta_1L^6+9\beta_3L^6-18(5\beta_1+\beta_3)L^4r_+^2-3(79\beta_1+23\beta_3)L^2r_+^4-2(59\beta_1+19\beta_3)r_+^6)]. \end{aligned}\quad \text{(C7)}$$

The temperature of the thermal AdS can be fixed using Eq. (C6). For a six-derivative it becomes

$$\begin{aligned}\beta_{\rm AdS} = \beta_{\rm BH}\Bigg[&1-\frac{L^2r_0}{2R_c^4}-\frac{\alpha'}{L^2}\left(\frac{(10\beta_1+\beta_2+2\beta_3)L^2r_0}{3R_c^4}\right)+\frac{(\alpha'/L^2)^2}{18R_c^4}(L^2r_0(3(4\alpha_1+3\alpha_2+8\alpha_3+40\alpha_4+16\alpha_5+16\alpha_6\\ &+80\alpha_8+400\alpha_9)+4(10\beta_1+\beta_2+2\beta_3)^2))\Bigg]. \end{aligned}\quad \text{(C8)}$$

Following the background subtraction method the regularized on-shall Euclidean action is given by

$$\begin{aligned}I_E{}^{\rm OS} = -\frac{\beta_{\rm BH}\Omega_3}{16\pi G}\Bigg[&r_+^2\left(1-\frac{r_+^2}{L^2}\right)+\frac{\alpha'}{L^2}\left(-10\beta_2L^2-8(5\beta_1+3\beta_2+\beta_3)r_+^2+\frac{20}{3L^2}(5\beta_1-\beta_2+\beta_3)r_+^4\right)\\ &+\left(\frac{\alpha'}{L^2}\right)^2\Bigg(\frac{2L^4}{r_+^2}(16\alpha_1-7\alpha_2-8\alpha_3+24\alpha_4)-\frac{32L^4}{r_+^2}\beta_2(\beta_1+\beta_2+\beta_3)-L^2(8(5\alpha_3-25\alpha_4-16\beta_2^2)+12\alpha_1+51\alpha_2)\\ &-\frac{r_+^4}{3L^2}(268\alpha_1+99\alpha_2-56\alpha_3+88\alpha_4)+\frac{16r_+^4}{9L^2}(250\beta_1^2+11\beta_2\beta_1+100\beta_3\beta_1+103\beta_2^2+10\beta_3^2+31\beta_2\beta_3)\\ &-r_+^2(108\alpha_1+51\alpha_2+56\alpha_3-376\alpha_4)-\frac{16r_+^2}{3}(50\beta_1^2+17\beta_2\beta_1+20\beta_3\beta_1-63\beta_2^2+2\beta_3^2-11\beta_2\beta_3)\Bigg)\Bigg]. \end{aligned}\quad \text{(C9)}$$

The ADM mass of a black hole can be computed from the free energy (C3) using the definition (C4)

$$\begin{aligned} M = \frac{3\Omega_3}{16\pi G}\bigg[& r_+^2 + \frac{r_+^4}{L^2} + \frac{(\alpha'/L^2)}{3L^2}(6\beta_2 L^4 - 4(5\beta_1 - \beta_2 + \beta_3) r_+^2 (6L^2 + 5r_+^2)) \\ & + \frac{(\alpha'/L^2)^2}{9r_+^2 L^2}(12\alpha_1(48L^6 + 189L^4 r_+^2 + 207L^2 r_+^4 + 67r_+^6) + 9\alpha_2(2L^6 + 15L^4 r_+^2 \\ & + 45L^2 r_+^4 + 33r_+^6) + 24\alpha_3(6L^6 + 21L^4 r_+^2 + 21L^2 r_+^4 + 7r_+^6) + 24\alpha_4(18L^6 \\ & + 57L^4 r_+^2 + 45L^2 r_+^4 + 11r_+^6) + 48\alpha_5 r_+^4(9L^2 + 10r_+^2) + 48\alpha_6 r_+^4(9L^2 + 10r_+^2) \\ & + 240\alpha_8(9L^2 r_+^4 + 10r_+^6) + 1200\alpha_9(9L^2 r_+^4 + 10r_+^6) - 288\beta_2(\beta_1 + \beta_3)L^6 \\ & - 32(5\beta_1 + \beta_3)^2 r_+^4(3L^2 + 5r_+^2) - 16\beta_2^2(54L^6 + 216L^4 r_+^2 + 267L^2 r_+^4 + 103r_+^6) \\ & - 16\beta_2(+72(\beta_1 + \beta_3)L^4 r_+^2 + 3(25\beta_1 + 29\beta_3)L^2 r_+^4 + (11\beta_1 + 31\beta_3)r_+^6))\bigg]. \end{aligned} \tag{C10}$$

The entropy of the black hole can be computed using (C5)

$$\begin{aligned} S = \frac{\Omega_3 r_+^3}{4G}\bigg[& 1 + \frac{(\alpha'/L^2)}{r_+^2}(12\beta_2 L^2 - 8(5\beta_1 - \beta_2 + \beta_3)r_+^2) + \frac{(\alpha'/L^2)^2}{3r_+^4}(27\alpha_2(L^2 + 2r_+^2)^2 + \alpha_1(3L^2 + 2r_+^2)^2 + 72\alpha_3 L^4 \\ & + 72\alpha_4(3L^4 - 4L^2 r_+^2 - 2r_+^4) + 16(9\alpha_5 + 9\alpha_6 + 45\alpha_8 + 225\alpha_9)r_+^4 - 16\beta_2^2(27L^4 + 54L^2 r_+^2 + 26r_+^4) - 16\beta_2(9(\beta_1 + \beta_3)L^4 \\ & + 18(\beta_1 + \beta_3)L^2 r_+^2 + 4(\beta_1 + 2\beta_3)r_+^4) - 32(5\beta_1 + \beta_3)^2 r_+^4)\bigg], \end{aligned} \tag{C11}$$

and this expression is identical to what we computed from Wald's approach. In the main text we use these expressions by setting all the coefficients to zero except β_2, α_1, and α_2.

2. $\mathcal{W}^4$ gravity

Here we present the Euclidean computation for the AdS_5 black hole in type IIB string theory with the $\mathcal{W}^4$ term. The on-shell action can be depicted by

$$I_E{}^{\mathrm{OS}} = \beta_{\mathrm{BH}} \frac{\pi r_+^2 (L^2 - r_+^2)}{8GL^2}\left(1 + \gamma e^{-\frac{3}{2}\varphi_\infty}\frac{5(L^2 - 15r_+^2)(L^2 + r_+^2)^3}{L^6 r_+^6 (L^2 - r_+^2)}\right). \tag{C12}$$

Thus (C3) implies the free energy of the black hole is

$$F = \frac{\pi r_+^2 (L^2 - r_+^2)}{8GL^2}\left(1 + \gamma e^{-\frac{3}{2}\varphi_\infty}\frac{5(L^2 - 15r_+^2)(L^2 + r_+^2)^3}{L^6 r_+^6 (L^2 - r_+^2)}\right), \tag{C13}$$

and the internal energy or ADM mass and the entropy of the black hole is $S = \frac{\partial F}{\partial T}$. The final expression is given by

$$\frac{\partial I_E{}^{\mathrm{OS}}}{\partial \beta} \Rightarrow M = \frac{3\pi r_+^2 (L^2 + r_+^2)}{8GL^2}\left(1 + \gamma e^{-\frac{3}{2}\varphi_\infty}\frac{5(L^2 + r_+^2)^2 (7L^2 + 15r_+^2)}{L^6 r_+^6}\right), \tag{C14}$$

$$\beta\frac{\partial I_E}{\partial \beta} - I_E \Rightarrow S = \frac{\pi^2 r_+{}^3}{2G}\left(1 + \gamma e^{-\frac{3}{2}\varphi_\infty}\frac{60(L^2 + r_+^2)^3}{L^6 r_+^6}\right). \tag{C15}$$

[1] David Kastor, Sourya Ray, and Jennie Traschen, Enthalpy and the mechanics of AdS black holes, Classical Quantum Gravity **26**, 195011 (2009).

[2] David Kubiznak and Robert B. Mann, P-V criticality of charged AdS black holes, J. High Energy Phys. 07 (2012) 033.

[3] Rong-Gen Cai, Li-Ming Cao, Li Li, and Run-Qiu Yang, P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space, J. High Energy Phys. 09 (2013) 005.

[4] Natacha Altamirano, David Kubizňák, Robert B. Mann, and Zeinab Sherkatghanad, Kerr-AdS analogue of triple point and solid/liquid/gas phase transition, Classical Quantum Gravity **31**, 042001 (2014).

[5] Natacha Altamirano, David Kubiznak, and Robert B. Mann, Reentrant phase transitions in rotating anti–de Sitter black holes, Phys. Rev. D **88**, 101502 (2013).

[6] Suvankar Dutta, Akash Jain, and Rahul Soni, Dyonic black hole and holography, J. High Energy Phys. 12 (2013) 060.

[7] Clifford V. Johnson, Holographic heat engines, Classical Quantum Gravity **31**, 205002 (2014).

[8] David Kubiznak and Robert B. Mann, Black hole chemistry, Can. J. Phys. **93**, 999 (2015).

[9] David Kubiznak, Robert B. Mann, and Mae Teo, Black hole chemistry: Thermodynamics with Lambda, Classical Quantum Gravity **34**, 063001 (2017).

[10] Andreas Karch and Brandon Robinson, Holographic black hole chemistry, J. High Energy Phys. 12 (2015) 073.

[11] Manus R. Visser, Holographic thermodynamics requires a chemical potential for color, Phys. Rev. D **105**, 106014 (2022).

[12] Wan Cong, David Kubiznak, Robert Mann, and Manus Visser, Holographic CFT phase transitions and criticality for charged AdS black holes, 2021, arXiv:2112.14848.

[13] David Kastor, Sourya Ray, and Jennie Traschen, Smarr formula and an extended first law for Lovelock gravity, Classical Quantum Gravity **27**, 235014 (2010).

[14] M. Cvetic, G. W. Gibbons, D. Kubiznak, and C. N. Pope, Black hole enthalpy and an entropy inequality for the thermodynamic volume, Phys. Rev. D **84**, 024037 (2011).

[15] Ted Jacobson and Manus Visser, Gravitational thermodynamics of causal diamonds in (A)dS, SciPost Phys. **7**, 079 (2019).

[16] Larry Smarr, Mass Formula for Kerr Black Holes, Phys. Rev. Lett. **30**, 71 (1973); **30**, 521(E) (1973).

[17] Rabin Banerjee, Bibhas Ranjan Majhi, Sujoy Kumar Modak, and Saurav Samanta, Killing symmetries and smarr formula for black holes in arbitrary dimensions, Phys. Rev. D **82**, 124002 (2010).

[18] Vijay Balasubramanian and Per Kraus, A stress tensor for Anti-de Sitter gravity, Commun. Math. Phys. **208**, 413 (1999).

[19] Wan Cong, David Kubiznak, and Robert B. Mann, Thermodynamics of AdS Black Holes: Critical Behavior of the Central Charge, Phys. Rev. Lett. **127**, 091301 (2021).

[20] Musema Sinamuli and Robert B. Mann, Higher order corrections to holographic black hole chemistry, Phys. Rev. D **96**, 086008 (2017).

[21] Juan Martin Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. **2**, 231 (1998).

[22] M. Henningson and K. Skenderis, The Holographic Weyl anomaly, J. High Energy Phys. 07 (1998) 023.

[23] Nabamita Banerjee and Suvankar Dutta, Shear viscosity to entropy density ratio in six derivative gravity, J. High Energy Phys. 07 (2009) 024.

[24] Shin'ichi Nojiri and Sergei D. Odintsov, On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence, Int. J. Mod. Phys. A **15**, 413 (2000).

[25] Matthias Blau, K. S. Narain, and Edi Gava, On subleading contributions to the AdS/CFT trace anomaly, J. High Energy Phys. 09 (1999) 018.

[26] Marcus T. Grisaru and D. Zanon, σ model superstring corrections to the Einstein-hilbert action, Phys. Lett. B **177**, 347 (1986).

[27] M. D. Freeman, C. N. Pope, M. F. Sohnius, and K. S. Stelle, Higher order σ model counterterms and the effective action for superstrings, Phys. Lett. B **178**, 199 (1986).

[28] David J. Gross and Edward Witten, Superstring modifications of Einstein's equations, Nucl. Phys. **B277**, 1 (1986).

[29] Tom Banks and Michael B. Green, Non-perturbative effects in $AdS_5 \times S^5$ string theory and $d = 4$ SUSY Yang-Mills, J. High Energy Phys. 05 (1998) 002.

[30] Jacek Pawelczyk and Stefan Theisen, $AdS_5 \times S^5$ black hole metric at $O(\alpha'^3)$, J. High Energy Phys. 09 (1998) 010.

[31] R. R. Metsaev and Arkady A. Tseytlin, Curvature cubed terms in string theory effective actions, Phys. Lett. B **185**, 52 (1987).

[32] S. W. Hawking and Don N. Page, Thermodynamics of black holes in anti-De Sitter space, Commun. Math. Phys. **87**, 577 (1983).

[33] R. Arnowitt, S. Deser, and C. W. Misner, Dynamical structure and definition of energy in general relativity, Phys. Rev. **116**, 1322 (1959).

[34] Suvankar Dutta and Rajesh Gopakumar, On Euclidean and Noetherian entropies in AdS space, Phys. Rev. D **74**, 044007 (2006).

[35] Renata Kallosh and Arvind Rajaraman, Vacua of M theory and string theory, Phys. Rev. D **58**, 125003 (1998).

[36] Karl Landsteiner, String corrections to the Hawking-Page phase transition, Mod. Phys. Lett. A **14**, 379 (1999).

[37] Sebastian de Haro, Annamaria Sinkovics, and Kostas Skenderis, On a supersymmetric completion of the R4 term in 2B supergravity, Phys. Rev. D **67**, 084010 (2003).

[38] Michael B. Green and Christian Stahn, D3-branes on the Coulomb branch and instantons, J. High Energy Phys. 09 (2003) 052.

[39] Kasper Peeters, Pierre Vanhove, and Anders Westerberg, Towards complete string effective actions beyond leading order, Fortschr. Phys. **52**, 630 (2004).

[40] Kasper Peeters and Anders Westerberg, The Ramond-Ramond sector of string theory beyond leading order, Classical Quantum Gravity **21**, 1643 (2004).

[41] Arvind Rajaraman, On a supersymmetric completion of the R^4 term in type IIB supergravity, Phys. Rev. D **72**, 125008 (2005).

[42] Giuseppe Policastro and Dimitrios Tsimpis, R^4, purified, Classical Quantum Gravity **23**, 4753 (2006).
[43] Miguel F. Paulos, Higher derivative terms including the Ramond-Ramond five-form, J. High Energy Phys. 10 (2008) 047.
[44] Vijay Balasubramanian, Per Kraus, and Albion E. Lawrence, Bulk versus boundary dynamics in anti-de Sitter space-time, Phys. Rev. D **59**, 046003 (1999).
[45] Igor R. Klebanov and Edward Witten, AdS/CFT correspondence and symmetry breaking, Nucl. Phys. **B556**, 89 (1999).
[46] Gary W. Gibbons, Renata Kallosh, and Barak Kol. Moduli, Scalar Charges, and the First Law of Black Hole Thermodynamics, Phys. Rev. Lett. **77**, 4992 (1996).
[47] Suvankar Dutta and Gurmeet Singh Punia, Interactions between AdS black hole molecules, Phys. Rev. D **104**, 126009 (2021).