Localization and observers in ϱ-Minkowski spacetime

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We consider the ϱ-Minkowski spacetime, a model with linear noncommutativity involving the time and azimuthal angle. We study its quantum symmetries, the ϱ-Poincaré quantum group, and analyze the concepts of localizability and quantum observers.

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I. INTRODUCTION

In this work, we investigate the basic features of localizability in a specific model of noncommutative spacetime, ϱ-Minkowski, based on a previous analysis [1,2] which has been carried out for the more famous κ-Minkowski spacetime.

The interest for quantum, or noncommutative, spacetimes is deeply motivated by the search for a consistent theory of quantum gravity, valid at Planck scale, of which quantum spacetimes should represent a signature at lower energies. Noncommutative spacetimes, whose symmetry groups are deformations of the Poincaré group, are therefore the natural candidates to be investigated.

The space we investigate, ϱ-Minkowski, has a noncommutativity of the angular type, involving time and the azimuthal angle. Time has a quantized discrete spectrum. Noncommutativity in the coordinates implies the presence of uncertainty relations among time and angle. A perfectly localized state in time is totally spread in φ, and vice versa, angle localization increases the time measurement uncertainty.

The presence of a noncommutative spacetime, invariant under a quantum group, implies that also observers, reference frames, have to be quantized, and this is the important feature of this activity. Observers are quantum objects as well. We have collected observations and definitions about states, observers, and observables in Appendix A.

We start in Sec. II by defining the commutation relations of the ϱ-Minkowski model and showing explicitly its angular nature. In Sec. III, we obtain the κ-Poincaré quantum group in order to illustrate the differences between the ϱ and κ deformations and explain the approach carried on for the ϱ case by means of comparison. Once having obtained the ϱ-Poincaré quantum group as the symmetry group of our model, we tackle in Sec. IV the problem of localizability in the ϱ-Minkowski spacetime, comparing the results with the known ones for the κ case. To this, we realize both the spacetime observables and the quantum group generators as operators and represent them on a suitable Hilbert space. Section V, dealing with conclusions and perspectives, closes the work. In Appendix A, as we mentioned, the notions of states, observables, and observers employed throughout the paper are formally stated, while Appendix B is devoted to recalling the classical r-matrix deformation method to obtain the Poincaré quantum groups.

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II. THE ϱ-MINKOWSKI SPACETIME

The ϱ-Minkowski spacetime is characterized by the following commutation relations of the angular type:

$$[x^0, x^1] = \mathrm{i}\varrho x^2,$$
$$[x^0, x^2] = -\mathrm{i}\varrho x^1, \tag{2.1}$$

all other commutators being zero. In particular, x^3 is central; it commutes with all coordinates. The ϱ parameter has the dimension of a length, and it is often identified with the Planck length, the scale at which quantum gravity effects are expected to be manifest.

These relations are a part of a larger family of "Lie-algebra-type" commutation relations, of which the most famous case is κ-Minkowski spacetime defined by[1]

[1] We use the standard convention for which the latin indices $i, j\ldots$ go from 1 to 3, while the greek ones $\mu, \nu\ldots$ go from 0 to 3.

$$[x^0, x^i] = \mathrm{i}\lambda x^i, \tag{2.2}$$

again all other commutators vanishing. The λ parameter with the dimension of a length is sometimes expressed as $\frac{1}{\kappa}$, hence, the name of the model.

The Lie algebra (2.1) has the structure of the Euclidean algebra in $2+1$ dimensions; it goes back to at least [3] (also see [4,5]). In the context of twisted symmetries, it was analyzed by Lukierski and Woronowicz in [6]. In [7] it was discussed in relation to the principle of relative locality [8]. This kind of noncommutative spacetime might have concrete physical relevance [9–11] and phenomenological/observational consequences [12]. In [13], a field theory on this space has been built; in the same paper, a different physical identification of the noncommuting variables has been also considered, with time a commuting coordinate. The latter has been studied in [14] in the context of Poisson gauge models and in [15] in the context of double quantization.[2]

The commutation relations (2.1) give rise to two nontrivial uncertainty relations:

$$\Delta x^0 \Delta x^1 \geq \frac{\varrho}{2}|\langle x^2\rangle|,$$
$$\Delta x^0 \Delta x^2 \geq \frac{\varrho}{2}|\langle x^1\rangle|. \tag{2.3}$$

This implies that in this kind of noncommutative spacetime, sharp localization of event operators is not always possible. Note that, by the centrality of x^3, this coordinate can be determined with absolute precision.

A realization of ϱ-Minkowski spacetime is given by [16]

$$x^i\psi(x) = x^i\psi(x),$$
$$x^0\psi(x) = -i\varrho(x^1\partial_2 - x^2\partial_1)\psi(x), \tag{2.4}$$

with x^i a complete set of observables on the Hilbert space $L^2(\mathbb{R}^3)$, x^0 a self-adjoint operator on $L^2(\mathbb{R}^3)$ acting like an angular momentum along the 3-axis, and $\psi(x)$ a state in the Hilbert space.

We can choose a more convenient way of writing commutators and uncertainty relations, given by the fact that the ϱ deformation is of angular nature. We therefore use cylindrical coordinates defining

$$r = \sqrt{(x^1)^2 + (x^2)^2},$$
$$z = x^3,$$
$$\varphi = \arctan\frac{x^2}{x^1}. \tag{2.5}$$

[2]In order to distinguish the two spacetimes, the one with commutative time has been named λ-Minkowski spacetime.

We take $e^{i\varphi}$ instead of φ, for the latter is a multivalued function, and it cannot be promoted to a self-adjoint operator, so that the commutation relations (2.1) become

$$[x^0, r] = 0,$$
$$[x^0, z] = 0,$$
$$[x^0, e^{i\varphi}] = \varrho e^{i\varphi}. \tag{2.6}$$

In this way, we have two complete sets of commuting observables given by (r, z, φ) and (r, z, x^0). On the Hilbert space of L^2 functions of the first set, the operators r, z, φ act as multiplication operators, while the action of x^0 is that of the angular momentum along the 3-axis

$$x^0\psi(r, z, x^0) = -i\varrho\partial_\varphi\psi(r, z, x^0). \tag{2.7}$$

Expressing the functions of $\vec{x}$ in cylindrical coordinates as

$$\psi(\vec{x}) = \psi(r, z, \varphi) = \sum_{n=-\infty}^{\infty}\psi_n(r, z)e^{in\varphi}, \tag{2.8}$$

we have that[3]

$$[x^0, \psi] = \sum_n n\varrho\psi_n(r, z)e^{in\varphi} = -i\varrho\partial_\varphi\psi(r, z, \varphi) = n\varrho\psi. \tag{2.9}$$

Therefore, in this case the spectrum of time is discrete, being the whole of $\mathbb{Z}$ [16]. The eigenstates of φ are given by a Fourier superposition

$$\delta(\varphi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\varphi}. \tag{2.10}$$

III. ϱ VS κ: THE QUANTUM GROUPS

Central in the discussion on localizability are the symmetry groups of noncommutative spacetimes. In this section, we will present the quantum group ϱ-Poincaré by means of comparison with the well-known κ-Poincaré quantum group starting by a review of the latter.

A. The κ-Poincaré quantum group $\mathcal{C}_\kappa(P)$

Although historically, the κ-Minkowski spacetime was found starting from the κ-Poincaré Hopf algebra as a quotient by the Lorentz subgroup, here we will follow the opposite path; i.e., we will find the algebra and the group as

[3]Equation (2.9) is valid (convergent) only for a class of functions. However, since this class is sufficiently large, we will ignore this subtlety.

symmetries of κ-Minkowski spacetime. The content of this subsection is not new; we present it as a prelude to the ϱ case.

We start by defining the κ-Poincaré $\mathcal{C}_\kappa(P)$[4] as the deformation of the algebra of continuous functions on the Poincaré group that preserves the κ-Minkowski commutation relations, i.e., the algebra generated by $\{\Lambda^\mu_\nu, a^\mu\}$ that leaves (2.2) invariant under the transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1, \tag{3.1}$$

from $\mathcal{M}_\kappa$ to $\mathcal{C}_\kappa(P) \otimes \mathcal{M}_\kappa$. Note that (3.1) has the form of a left coaction of $\mathcal{C}_\kappa(P)$ on $\mathcal{M}_\kappa \subset \mathfrak{p}^*$, the latter being the dual of the Poincaré algebra. Let us recall that, given an algebra $(\mathcal{A}, \mu, \eta)$ and a coalgebra $(\mathcal{C}, \Delta, \varepsilon)$, a left coaction $\beta_L\colon \mathcal{A} \to \mathcal{C} \otimes \mathcal{A}$ is a linear mapping satisfying

$$(id \otimes \beta_L) \circ \beta_L = (\Delta \otimes id) \circ \beta_L \qquad \text{(coassociativity)}, \tag{3.2a}$$

$$(\varepsilon \otimes id) \circ \beta_L = id \qquad \text{(counitality)}. \tag{3.2b}$$

The coaction is said to be covariant if it is a homomorphism:

$$\beta_L(ab) = \beta_L(a)\beta_L(b), \quad a, b \in \mathcal{A}, \tag{3.3a}$$

$$\beta_L(1) = 1 \otimes 1; \tag{3.3b}$$

in this case, it preserves the algebra structure on which it coacts.

We require (3.1) to be a covariant left coaction. In other words, recalling (2.2), and since from (3.3a) $\beta_L([x^\mu, x^\nu]) = [\beta_L(x^\mu), \beta_L(x^\nu)]$, we ask that

$$[x'^\mu, x'^\nu] = i\lambda(\delta^\mu{}_0 x'^\nu - \delta^\nu{}_0 x'^\mu). \tag{3.4}$$

By imposing Eq. (3.4), it is possible to recover part of the full algebra structure of $\mathcal{C}_\kappa(P)$. Indeed, the left-hand side (lhs) of (3.4) yields

$$\begin{aligned}[x'^\mu, x'^\nu] &= [\Lambda^\mu{}_\alpha \otimes x^\alpha + a^\mu \otimes 1, \Lambda^\nu{}_\beta \otimes x^\beta + a^\nu \otimes 1] \\ &= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \otimes x^\alpha x^\beta - \Lambda^\nu{}_\beta \Lambda^\mu{}_\alpha \otimes x^\beta x^\alpha \\ &\quad + \Lambda^\mu{}_\alpha a^\nu \otimes x^\alpha - a^\nu \Lambda^\mu{}_\alpha \otimes x^\alpha + a^\mu \Lambda^\nu{}_\beta \otimes x^\beta \\ &\quad - \Lambda^\nu{}_\beta a^\mu \otimes x^\beta + [a^\mu, a^\nu] \otimes 1, \end{aligned} \tag{3.5}$$

while the right-hand side (rhs) of (3.4) assumes the form

$$\begin{aligned} i\lambda(\delta^\mu{}_0 x'^\nu - \delta^\nu{}_0 x'^\mu) = i\lambda(&\delta^\mu{}_0(\Lambda^\nu{}_\sigma \otimes x^\sigma + a^\nu \otimes 1) \\ &- \delta^\nu{}_0(\Lambda^\mu{}_\rho \otimes x^\rho + a^\mu \otimes 1)). \end{aligned} \tag{3.6}$$

Thus, equating terms at order 0 in x, it follows straightforwardly

$$[a^\mu, a^\nu] = i\lambda(\delta^\mu{}_0 a^\nu - \delta^\nu{}_0 a^\mu), \tag{3.7}$$

and the translational parameters, unlike the classical Poincaré group case, do not commute. This poses problems in localizability of κ-Poincaré transformed observables [1]. These commutation relations are isomorphic to the κ-Minkowski ones, a feature connected to the bicrossproduct structure of the quantum group [17].

Consider the terms quadratic in Λ:

$$\begin{aligned} &\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \otimes x^\alpha x^\beta - \Lambda^\nu{}_\beta \Lambda^\mu{}_\alpha \otimes x^\beta x^\alpha \\ &= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \otimes x^\alpha x^\beta - \Lambda^\nu{}_\beta \Lambda^\mu{}_\alpha \otimes x^\beta x^\alpha \\ &\quad + \Lambda^\nu{}_\beta \Lambda^\mu{}_\alpha \otimes x^\alpha x^\beta - \Lambda^\nu{}_\beta \Lambda^\mu{}_\alpha \otimes x^\alpha x^\beta \\ &= [\Lambda^\mu{}_\alpha, \Lambda^\nu{}_\beta] \otimes x^\alpha x^\beta + \Lambda^\nu{}_\beta \Lambda^\mu{}_\alpha \otimes i\lambda(\delta^\alpha{}_0 x^\beta - \delta^\beta{}_0 x^\alpha), \end{aligned} \tag{3.8}$$

from which it follows, since (3.4) has no second order terms in x on the right-hand side,

$$[\Lambda^\mu{}_\alpha, \Lambda^\nu{}_\beta] = 0. \tag{3.9}$$

Therefore, the Lorentz sector remains undeformed, having trivial commutators.

We remark that in this discussion we are considering a single particle. One could consider also multiparticle systems described by set of coordinates $x^{\mu(m)}$ and consider commutators $[x^{\mu(m)}, x^{\nu(n)}]$. The situation is then more complicated, and the coaction is not covariant for ordinary κ-Minkowski spacetime. Nevertheless, it becomes covariant for a lightlike version of it [18]. In this paper, we will remain in the usual one-particle case, but it would be interesting to consider in the ϱ-Minkowski setting also the two-particle case.

Let us equate the remaining terms on the left- and right-hand sides:

$$\begin{aligned} &\Lambda^\nu{}_\beta \Lambda^\mu{}_\alpha \otimes i\lambda(\delta^\alpha{}_0 x^\beta - \delta^\beta{}_0 x^\alpha) + \Lambda^\mu{}_\alpha a^\nu \otimes x^\alpha - a^\nu \Lambda^\mu{}_\alpha \otimes x^\alpha \\ &\quad + a^\mu \Lambda^\nu{}_\beta \otimes x^\beta - \Lambda^\nu{}_\beta a^\mu \otimes x^\beta \\ &= i\lambda(\delta^\mu{}_0 \Lambda^\nu{}_\sigma \otimes x^\sigma - \delta^\nu{}_0 \Lambda^\mu{}_\rho \otimes x^\rho). \end{aligned} \tag{3.10}$$

It is easy to see that this last condition imposes that the remaining commutators $[\Lambda^\mu_\nu, a^\rho]$ satisfy a compatibility condition:

$$[\Lambda^\mu{}_\alpha, a^\nu] + [a^\mu, \Lambda^\nu{}_\alpha] = i\lambda(\Lambda^\mu{}_\alpha(\Lambda^\nu{}_0 - \delta^\nu{}_0) - \Lambda^\nu{}_\alpha(\Lambda^\mu{}_0 - \delta^\mu{}_0)), \tag{3.11}$$

which should be settled by further requests. This is a consequence of the fact that relations (2.2) admit more than one single covariance group.

[4]To be distinguished from the quantum group $U_\kappa(\mathfrak{p})$ obtained deforming the Hopf algebra of the universal enveloping algebra $\mathfrak{p}$.

From (3.2a) and (3.2b), we can find the coproducts and the counits. Acting with the lhs of (3.2a) on x^μ, and recalling (3.3b), we find

$$(id\otimes\beta_L)\circ(\Lambda^\mu{}_\nu\otimes x^\nu+a^\mu\otimes 1)$$
$$=\Lambda^\mu{}_\nu\otimes\Lambda^\nu{}_\alpha\otimes x^\alpha+\Lambda^\mu{}_\nu\otimes a^\nu\otimes 1+a^\mu\otimes 1\otimes 1, \tag{3.12}$$

while from the rhs we have

$$(\Delta\otimes id)\circ(\Lambda^\mu{}_\nu\otimes x^\nu+a^\mu\otimes 1)$$
$$=\Delta(\Lambda^\mu{}_\nu)\otimes x^\nu+\Delta(a^\mu)\otimes 1. \tag{3.13}$$

Comparing the results, we have that

$$\Delta(a^\mu)=\Lambda^\mu{}_\nu\otimes a^\nu+a^\mu\otimes 1, \tag{3.14a}$$

$$\Delta(\Lambda^\mu{}_\nu)=\Lambda^\mu{}_\alpha\otimes\Lambda^\alpha{}_\nu. \tag{3.14b}$$

Turning to (3.2b), and acting on x^μ, we have that

$$(\varepsilon\otimes id)\circ(\Lambda^\mu{}_\nu\otimes x^\nu+a^\mu\otimes 1)$$
$$=\varepsilon(\Lambda^\mu{}_\nu)\otimes x^\nu+\varepsilon(a^\mu)\otimes 1=id(x^\mu)=x^\mu, \tag{3.15}$$

and therefore,

$$\varepsilon(a^\mu)=0, \tag{3.16a}$$

$$\varepsilon(\Lambda^\mu{}_\nu)=\delta^\mu{}_\nu. \tag{3.16b}$$

As far the antipodes are concerned, by the Hopf algebra axioms, it can be shown that they remain undeformed:

$$S(a^\mu)=-a^\nu(\Lambda^{-1})^\mu{}_\nu, \tag{3.17a}$$

$$S(\Lambda^\mu{}_\nu)=(\Lambda^{-1})^\mu{}_\nu. \tag{3.17b}$$

B. $\mathcal{C}_\kappa(P)$ structure from the r matrix

To fully compute the commutators between coordinate functions of $\mathcal{C}_\kappa(P)$, we may follow a different approach based on the introduction of the classical r matrix (see Appendix B for details), which will turn out to be useful for the ϱ-Minkowski case.

A classical r matrix for $\mathcal{C}_\kappa(P)$ is found to be [19]

$$r=\mathrm{i}\lambda M_{0\nu}\wedge P^\nu \tag{3.18}$$

with $M_{\mu\nu}$ and P_ν the generators of the Poincaré algebra. It can be checked to satisfy the modified Yang-Baxter equation

$$[\![r,r]\!]=\mathrm{i}\lambda^2\left(\frac{1}{2}g_{00}M_{\mu\nu}\wedge P^\mu\wedge P^\nu-M_{\nu0}\wedge P^\nu\wedge P_0\right), \tag{3.19}$$

where $[\![\cdot,\cdot]\!]$ denotes the bracket (B1) described in Appendix B, while the rhs is invariant under the group action. In order to compute the Sklyanin brackets (B2) of the group parameters, we need the Poincaré left- and right-invariant vector fields. These are obtained starting from the five-dimensional representation[5] of ISO(1, 3):

$$g=\begin{pmatrix}\Lambda & \vec{a}\\ \vec{0}^T & 1\end{pmatrix} \tag{3.20}$$

through the left- and right-invariant Maurer-Cartan 1-forms

$$\Theta_L=g^{-1}dg=\Theta_L^{\alpha\beta}M_{\alpha\beta}+\Theta_L^\alpha P_\alpha, \tag{3.21}$$

$$\Theta_R=dgg^{-1}=\Theta_R^{\alpha\beta}M_{\alpha\beta}+\Theta_R^\alpha P_\alpha. \tag{3.22}$$

By duality, the left- and right-invariant vector fields result to be

$$X^L_{\alpha\beta}=\Lambda^\mu{}_\alpha\frac{\partial}{\partial\Lambda^{\mu\beta}}-\Lambda^\mu{}_\beta\frac{\partial}{\partial\Lambda^{\mu\alpha}},\qquad X^L_\alpha=\Lambda^\mu{}_\alpha\frac{\partial}{\partial a^\mu},$$
$$X^R_{\alpha\beta}=\Lambda_{\beta\nu}\frac{\partial}{\partial\Lambda^\alpha{}_\nu}-\Lambda_{\alpha\nu}\frac{\partial}{\partial\Lambda^\beta{}_\nu}+a_\beta\frac{\partial}{\partial a^\alpha}-a_\alpha\frac{\partial}{\partial a^\beta},$$
$$X^R_\alpha=\frac{\partial}{\partial a^\alpha}, \tag{3.23}$$

which enable us to rewrite (B2) as

$$\{f,g\}=-\lambda(X^R_{0\nu}\wedge X^{R\nu}-X^L_{0\nu}\wedge X^{L\nu})(\mathrm{d}f,\mathrm{d}g). \tag{3.24}$$

Here we have rescaled the vector fields $X_{\alpha\beta}$ by a factor of i. Performing the calculation for a^ρ and a^σ:

$$\{a^\rho,a^\sigma\}=-\lambda\left(a_\nu\frac{\partial}{\partial a^0}-a_0\frac{\partial}{\partial a^\nu}\right)\wedge\frac{\partial}{\partial a_\nu}(\mathrm{d}a^\rho,\mathrm{d}a^\sigma)$$
$$=-\lambda(a^\sigma\delta^\rho{}_0-a^\rho\delta^\sigma{}_0). \tag{3.25}$$

The commutators are then obtained via the canonical prescription $\{,\}\to\frac{1}{i}[,]$, and we find the previously stated result (3.7) quantizing the Poisson-Hopf algebra to a deformed one.

A calculation of $\{\Lambda^\alpha{}_\beta,\Lambda^\mu{}_\nu\}$ gives identically 0, since P^μ does not contain derivatives in Λ in the left or right bases, so the result (3.9) comes straightforwardly.

Unlike what we found employing the covariance method, we can now fix the mixed brackets:

[5]The arrow indicates four-dimensional vectors.

$$\begin{aligned}\{\Lambda^\alpha{}_\beta, a^\rho\} &= -\lambda\left(\Lambda_{\nu\mu}\frac{\partial}{\partial\Lambda^0{}_\mu} - \Lambda_{0\mu}\frac{\partial}{\partial\Lambda^\nu{}_\mu} + a_\nu\frac{\partial}{\partial a^0} - a_0\frac{\partial}{\partial a^\nu}\right)\\ &\wedge\frac{\partial}{\partial a_\nu}(\mathrm{d}\Lambda^\alpha{}_\beta, \mathrm{d}a^\rho) + \lambda\left(\Lambda^\mu{}_0\frac{\partial}{\partial\Lambda^{\mu\nu}} - \Lambda^\mu{}_\nu\frac{\partial}{\partial\Lambda^{\mu0}}\right)\\ &\wedge\Lambda^{\kappa\nu}\frac{\partial}{\partial a^\kappa}(\mathrm{d}\Lambda^\alpha{}_\beta, \mathrm{d}a^\rho)\\ &= \lambda((\Lambda^\alpha{}_0 - \delta^\alpha{}_0)\Lambda^\rho{}_\beta + (\Lambda_{0\beta} - g_{0\beta})g^{\alpha\rho}). \qquad (3.26)\end{aligned}$$

Considering the commutators,[6] we obtain

$$[\Lambda^\alpha{}_\beta, a^\rho] = -\mathrm{i}\lambda((\Lambda^\alpha{}_0 - \delta^\alpha{}_0)\Lambda^\rho{}_\beta + (\Lambda_{0\beta} - g_{0\beta})g^{\alpha\rho}). \qquad (3.27)$$

Having completed the algebra structure of $\mathcal{C}_\kappa(P)$, we note that in this formulation the Lorentz sector is undeformed, while the translational one and the cross-relations are noncommutative, giving intuitively an increase in uncertainty of transformed observables.

C. The ϱ-Poincaré quantum group $\mathcal{C}_\varrho(P)$

Following the discussion in Sec. III B, we will derive the commutation relations (i.e., the algebra sector) of the $\mathcal{C}_\varrho(P)$ quantum group starting from the classical r matrix of ϱ-Minkowski spacetime.

First, note that left- and right-invariant vector fields retain the same expressions (3.23). The only difference with the κ-Poincaré quantum group is in the r matrix, which in this case assumes the form [6,16]

$$r = -i\varrho(P_0 \wedge M_{12}). \qquad (3.28)$$

Note that, unlike the case of the classical r matrix of κ-Minkowski spacetime which satisfies a modified Yang-Baxter equation (MYBE), (3.28) satisfies the classical Yang-Baxter equation (CYBE)—in fact, computing the brackets

$$\begin{aligned}[r_{12}, r_{13}] &= -\varrho^2[M_{12}, M_{12}] \otimes P_0 \otimes P_0 = 0,\\ [r_{12}, r_{23}] &= \varrho^2 P_0 \otimes [M_{12}, M_{12}] \otimes P_0 = 0,\\ [r_{13}, r_{23}] &= -\varrho^2 P_0 \otimes P_0 \otimes [M_{12}, M_{12}] = 0, \qquad (3.29)\end{aligned}$$

and thus, $[\![r, r]\!] = 0$.

The Sklyanin bracket (B2) assumes the form

$$\{f, g\} = -\varrho(X^R_{12} \wedge X^R_0 - X^L_{12} \wedge X^L_0)(df, dg), \qquad (3.30)$$

so that we can compute the brackets between Poincaré coordinates as done earlier:

$$\begin{aligned}\{a^\mu, a^\nu\} &= -\varrho[\delta^\nu{}_0(a_2\delta^\mu{}_1 - a_1\delta^\mu{}_2) - \delta^\mu{}_0(a_2\delta^\nu{}_1 - a_1\delta^\nu{}_2)],\\ \{\Lambda^\mu{}_\nu, \Lambda^\varrho{}_\sigma\} &= 0,\\ \{\Lambda^\mu{}_\nu, a^\varrho\} &= -\varrho[\delta^\varrho{}_0(\Lambda_{2\nu}\delta^\mu{}_1 - \Lambda_{1\nu}\delta^\mu{}_2)\\ &\quad - \Lambda^\varrho{}_0(\Lambda^\mu{}_1 g_{2\nu} - \Lambda^\mu{}_2 g_{1\nu})]. \qquad (3.31)\end{aligned}$$

Therefore, the commutators are

$$[a^\mu, a^\nu] = -i\varrho[\delta^\nu{}_0(a_2\delta^\mu{}_1 - a_1\delta^\mu{}_2) - \delta^\mu{}_0(a_2\delta^\nu{}_1 - a_1\delta^\nu{}_2)], \qquad (3.32a)$$

$$[\Lambda^\mu{}_\nu, \Lambda^\varrho{}_\sigma] = 0, \qquad (3.32b)$$

$$\begin{aligned}[\Lambda^\mu{}_\nu, a^\varrho] &= -i\varrho[\delta^\varrho{}_0(\Lambda_{2\nu}\delta^\mu{}_1 - \Lambda_{1\nu}\delta^\mu{}_2)\\ &\quad - \Lambda^\varrho{}_0(\Lambda^\mu{}_1 g_{2\nu} - \Lambda^\mu{}_2 g_{1\nu})]. \qquad (3.32c)\end{aligned}$$

Again, it is easy to see that the commutation relations between a^μ and a^ν reproduce Eqs. (2.1), and ϱ-Minkowski spacetime can therefore be recovered from the momenta sector of $\mathcal{C}_\varrho(P)$. Moreover, it can be checked that the commutation relations of the ϱ-Minkowski spacetime (2.1) are covariant under the left coaction (3.1) if the commutation relations (3.32a)–(3.32c) are implemented.

For the coalgebra sector and the antipode, since the left coaction is the same as that of the κ-Poincaré quantum group, they retain the forms (3.14a), (3.14b), (3.16a), (3.16b), (3.17a), and (3.17b). It is then trivial to see that taking the limit $\varrho \to 0$, the classical commutative case is recovered.

The fundamental result of this analysis is that the algebra sector of the translational parameters and the cross-relations between translational and Lorentz parameters are noncommutative [16]. This will lead to an increase in uncertainty in ϱ-Poincaré transformations, as we will show in the following.

IV. LOCALIZABILITY IN ϱ-MINKOWSKI SPACE

We now analyze localizability in the ϱ-Minkowski space, following what has been done in [1] for the κ case. We first consider coordinate localizability features coming from (2.3), then we realize the elements of the quantum group on a suitable Hilbert space, we derive uncertainty relations for them, and we discuss localizability in ϱ-Minkowski space in relation to observers and observables.

A. Localized states in $\mathcal{M}_\varrho$

Let us suppose we sharply measure an eigenvalue $\varrho\bar{n}$ of the time operator. The system would be in an eigenstate of time $\bar{\chi}(\varphi) = e^{i\bar{n}\varphi}$, so that we would have complete delocalization in φ. If the measure has instead some degree of uncertainty in time, we would have a finite sum over the available elements of the basis, and this would give, in turn,

[6]Note that the canonical substitution prescription is ordering unambiguous due to the commutativity of the Λ's.

a degree of uncertainty in φ, as in the ordinary quantum mechanical angular momentum theory.

From (2.3), we expect, however, that sharp spacetime localization is possible in the case $\langle x^1\rangle = \langle x^2\rangle = 0$. In our cylindrical coordinates, this corresponds to perfect localization in $\langle r\rangle = 0$. Since r commutes with z and x^0, we can find a state that localizes in r as well as in z and x^0. As usual, the state will not be a proper square integrable vector, but a δ-like distribution reachable via a limiting process. A state of this kind can be constructed as

$$\psi_{n_0}(r,z,x^0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\varphi \mathrm{e}^{-i(n-n_0)\varphi}\xi(r,z), \qquad (4.1)$$

where the integral yields a $\delta(n-n_0)$ that gives a state localized in time at n_0, and $\xi(r,z)$ is a function of r and z localized around (r_0, z_0). This can be taken to be a factorized product of two states in the Hilbert space (e.g., Gaussian distributions) that tend to delta distributions in the limit of their amplitudes going to 0 (e.g., the Gaussian variances $\to 0$).

From (2.5), $x^1 = r\cos\varphi$, $x^2 = r\sin\varphi$, but φ is completely undetermined since we are in an eigenstate of x^0. Computing the mean values on the state, we have $\langle x^1\rangle = r_0\cos\varphi$ and $\langle x^2\rangle = r_0\sin\varphi$; hence, perfect localization in x^μ is possible only if $r_0 = 0$. We obtain then a two-parameter localized family of states $|o_{n,z}\rangle$. In the particular case of $n_0 = z_0 = 0$, we can define a localized origin state $|o\rangle$. This result is analogous with the case of κ-Minkowski spacetime [1,2], for which it was found that a one-parameter family of localized states $|o_\tau\rangle$ does exist, allowing for the definition of a localized origin state $|o\rangle$.

Let us note here an important fact. The following function also gives a localized state at time $n_0 + \alpha$:

$$\psi_{n_0+\alpha}(r,z,x^0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\varphi \mathrm{e}^{-i(n-n_0+\alpha)\varphi}\xi(r,z), \qquad (4.2)$$

which is only periodic in φ up to the phase $\mathrm{e}^{i2\pi\alpha}$. This means that these two states belong to different domains of self-adjointness of the operator x^0. This aspect will be discussed elsewhere.

B. ϱ-Poincaré realization

Since later on we will deal with the localization properties of the quantum group parameters, we now present a realization for the ϱ-Poincaré group, following the approach carried on for the κ-Poincaré group in [1].

We start noting that, as in the κ-Poincaré case, the Λ's commute with each other, and so they can be realized classically. In terms of the infinitesimal generators of the Lorentz group $\omega^\mu{}_\nu$, we have that

$$\Lambda^\mu{}_\nu = (\exp\omega)^\mu{}_\nu \qquad (4.3)$$

with the auxiliary antisymmetry condition $\omega^\mu{}_\varrho g^{\varrho\nu} = -\omega^\nu{}_\varrho g^{\varrho\mu}$.

For the a's, by considering the commutation relation (3.32c), we formulate the ansatz

$$a^\varrho = i\varrho[\delta^\varrho{}_0(\Lambda_{2\nu}\delta^\mu{}_1 - \Lambda_{1\nu}\delta^\mu{}_2) - \Lambda^\varrho{}_0(\Lambda^\mu{}_1 g_{2\nu} - \Lambda^\mu{}_2 g_{1\nu})]\frac{\partial}{\partial\Lambda^\mu{}_\nu}. \qquad (4.4)$$

To have a realization of the group, we must show that this is coherent with (3.32a). From (3.32a), it has to be

$$[a^\mu, a^\nu] = -\varrho^2(-\delta^\mu{}_0\Lambda_{20}\delta^\nu{}_1 + \delta^\mu{}_0\Lambda_{10}\delta^\nu{}_2 + \delta^\nu{}_0\Lambda_{20}\delta^\mu{}_1 - \delta^\nu{}_0\Lambda_{10}\delta^\mu{}_2) \times (\Lambda^\alpha{}_1 g_{2\beta} - \Lambda^\alpha{}_2 g_{1\beta})\frac{\partial}{\partial\Lambda^\alpha{}_\beta}. \qquad (4.5)$$

On computing the lhs, we find

$$[a^\varrho, a^\sigma] = \varrho^2[\delta^\varrho{}_0(\Lambda_{20}\delta^\sigma{}_1 - \Lambda_{10}\delta^\sigma{}_2)(\Lambda^\delta{}_2 g_{1\lambda} - \Lambda^\delta{}_1 g_{2\lambda}) + -\delta^\sigma{}_0(\Lambda_{20}\delta^\varrho{}_1 - \Lambda_{10}\delta^\varrho{}_2)(\Lambda^\delta{}_2 g_{1\lambda} - \Lambda^\delta{}_1 g_{2\lambda})] \times \frac{\partial}{\partial\Lambda^\delta{}_\lambda}, \qquad (4.6)$$

which is in agreement with Eq. (4.5); therefore, Eqs. (4.3) and (4.4) give a true realization of the ϱ-Poincaré quantum group.

Finally, in analogy with the κ case (cf. [1]), we add to Eq. (4.4) the realization of ϱ-Minkowski Eq. (2.4):

$$a^\varrho = i\frac{\varrho}{2}[\delta^\varrho{}_0(\Lambda_{2\nu}\delta^\mu{}_1 - \Lambda_{1\nu}\delta^\mu{}_2) - \Lambda^\varrho{}_0(\Lambda^\mu{}_1 g_{2\nu} - \Lambda^\mu{}_2 g_{1\nu})] \times \frac{\partial}{\partial\Lambda^\mu{}_\nu} + i\frac{\varrho}{2}[\delta^\varrho{}_i q^i - \delta^\varrho{}_0(q^1\partial_2 - q^2\partial_1)] + \text{H.c.} \qquad (4.7)$$

defined on the Hilbert space $L^2(SO(1,3)\times\mathbb{R}^3)$.

C. ϱ-Poincaré parameters, localization, and constraints on transformations

Since the symmetry group of ϱ-Minkowski spacetime is deformed according to Eqs. (3.32a)–(3.32c), we expect localization problems to arise also in observer transformations. Indeed, we obtain uncertainty relations in the form

$$\Delta a^\mu \Delta a^\nu \geq \frac{\varrho}{2}|\delta^\nu{}_0(\langle a_2\rangle\delta^\mu{}_1 - \langle a_1\rangle\delta^\mu{}_2) - \delta^\mu{}_0(\langle a_2\rangle\delta^\nu{}_1 - \langle a_1\rangle\delta^\nu{}_2)|, \qquad (4.8a)$$

$$\Delta\Lambda^\mu{}_\alpha \Delta\Lambda^\nu{}_\beta \geq 0, \qquad (4.8b)$$

$$\Delta\Lambda^\mu{}_\nu \Delta a^\rho \geq \frac{\varrho}{2}|\delta^\varrho{}_0(\langle\Lambda_{2\nu}\rangle\delta^\mu{}_1 - \langle\Lambda_{1\nu}\rangle\delta^\mu{}_2) - \langle\Lambda^\varrho{}_0\Lambda^\mu{}_1\rangle g_{2\nu} + \langle\Lambda^\varrho{}_0\Lambda^\mu{}_2\rangle g_{1\nu}|. \qquad (4.8c)$$

Let us analyze the localization properties of this algebra structure. We start with the case of pure ϱ-Lorentz transformations, i.e., transformations for which translational parameters are sharply localized in 0 ($\langle a^\mu \rangle = 0$, $\Delta a^\mu = 0$). The relevant constraint on localizability comes from (4.8c):

$$\delta^\varrho{}_0(\langle \Lambda_{2\nu} \rangle \delta^\mu{}_1 - \langle \Lambda_{1\nu} \rangle \delta^\mu{}_2) - \langle \Lambda^\varrho{}_0 \Lambda^\mu{}_1 \rangle g_{2\nu} + \langle \Lambda^\varrho{}_0 \Lambda^\mu{}_2 \rangle g_{1\nu} = 0. \tag{4.9}$$

This, like the case of κ-Poincaré quantum group [20], admits a solution[7] for $\langle \Lambda^\varrho{}_0 \rangle = \delta^\varrho{}_0$, $\langle \Lambda^3{}_1 \rangle = \Lambda^3{}_2 = 0$, $\langle \Lambda^1{}_1 \rangle = \langle \Lambda^2{}_2 \rangle$, and so the only admitted pure ϱ-Lorentz transformations are rotations around the 3-axis and the identical transformation, and they can be sharply localized. For the κ-Poincaré quantum group, a slightly different result was found in [20], namely, that just pure boosts are not admitted, in accord with [21].

For the case of pure translations, i.e., $\langle \Lambda^\mu{}_\nu \rangle = \delta^\mu{}_\nu$ and $\Delta \Lambda^\mu{}_\nu = 0$, substituting in (4.8c) we see that the relation is automatically satisfied, and the only relevant condition is (4.8a). Since a^3 is central in the algebra, pure translations along the 3-axis do exist without issues and can be sharply localized.

Considering a pure time translation, the conditions to impose on (4.8a) are that $\langle a^i \rangle = 0$ and $\Delta a^i = 0$, and the equation is trivially satisfied, meaning that pure time translations do exist and can be localized. For pure translations along the 1- and 2-axes the result is different: If we consider, for example, the first case, one would have $\langle a^2 \rangle = 0$ that is compatible with $\Delta x^0 = 0$, but this last condition imposes also that $\langle a^1 \rangle = 0$, the same being true switching a^1 and a^2. This means that the ϱ-Poincaré quantum group admits only pure time translations and pure space translations along the 3-axis. For comparison, in the κ case, it was found that the only possible pure translation is the temporal one.

Summarizing the localization features of the quantum group, the only transformations that can be sharply localized are translations along x^0, translations along x^3, rotations around x^3, and their combinations.

As a special case, we turn our attention to the identical transformation $\langle a^\mu \rangle = 0$, $\langle \Lambda^\mu{}_\nu \rangle = \delta^\mu{}_\nu$, $\Delta a^\mu = 0$, $\Delta \Lambda^\mu{}_\nu = 0$; as we expect, the uncertainty relations are satisfied, and therefore, the identity in the ϱ-Poincaré quantum group is a well-defined sharp state.

D. Observers, observables, and uncertainties on ϱ-Poincaré quantum group

Let us analyze the uncertainties in Poincaré transformations (3.1) coming from the deformation features of the quantum group. Since our transformation is a left coaction from $x \in \mathcal{M}_\varrho$ to $x' \in \mathcal{C}_\varrho(P) \otimes \mathcal{M}_\varrho$, we want to find a realization of the tensor product $\mathcal{C}_\varrho(P) \otimes \mathcal{M}_\varrho$. It is convenient to lift $x \in \mathcal{M}_\varrho$ to $1 \otimes x \in \mathcal{C}_\varrho(P) \otimes \mathcal{M}_\varrho$. We can find the action of elements $x'^\mu \in \mathcal{C}_\varrho(P) \otimes \mathcal{M}_\varrho$ on functions $f(\omega, q, x) \in L^2(SO(1,3) \times \mathbb{R}^3_q) \times L^2(\mathbb{R}^3_x) \sim L^2(SO(1,3) \times \mathbb{R}^3_q \times \mathbb{R}^3_x)$ by means of the direct sum of realizations (2.4) and (4.7):

$$\begin{aligned} x'^\varrho f(\omega, q, x) &= i\varrho \Lambda^\varrho{}_\sigma [\delta^\sigma{}_i x^i - \delta^\sigma{}_0 (x^1 \partial_{x_2} - x^2 \partial_{x_1})] f(\omega, q, x) \\ &\quad + i\frac{\varrho}{2} [\delta^\varrho{}_0 (\Lambda_{2\nu} \delta^\mu{}_1 - \Lambda_{1\nu} \delta^\mu{}_2) - \Lambda^\varrho{}_0 (\Lambda^\mu{}_1 g_{2\nu} - \Lambda^\mu{}_2 g_{1\nu})] \\ &\quad \times \frac{\partial}{\partial \Lambda^\mu{}_\nu} f(\omega, q, x) + i\frac{\varrho}{2} [\delta^\varrho{}_i q^i - \delta^\varrho{}_0 (q^1 \partial_{q_2} - q^2 \partial_{q_1})] \\ &\quad \times f(\omega, q, x) + \frac{1}{2} \mathrm{H.c.} \end{aligned} \tag{4.10}$$

The Hilbert space admits separable states of the kind

$$|\phi, \psi\rangle = |\phi\rangle \otimes |\psi\rangle, \tag{4.11}$$

with $|\phi\rangle \in L^2(SO(1,3) \times \mathbb{R}^3_q)$ and $|\psi\rangle \in L^2(\mathbb{R}^3_x)$ normalized according to $\langle \phi | \phi \rangle = 1$, $\langle \psi | \psi \rangle = 1$.

We are ready to give an interpretation of the realization constructed here. We define $L^2(SO(1,3) \times \mathbb{R}^3_q)$ as the space of states of an observer (i.e., the space of ϱ-Poincaré states) and $L^2(\mathbb{R}^3_x)$ as the space of observables (i.e., the space of states of ϱ-Minkowski spacetime); furthermore, we assume that a generic state can be realized as a separable element $|\phi, \psi\rangle = |\phi\rangle \otimes |\psi\rangle$, a reasonable assumption since it reflects the fact that the relation between two inertial observers does not depend on the observed state.

The point here is that we have at the same time a noncommutative spacetime on which observables are defined and a noncommutative observer state space, meaning that in general, a ϱ-Poincaré transformation between different observers could decrease localizability of states.

Taking into account (3.1), and interpreting x^μ as the coordinates of an inertial observer $\mathcal{O}$, while x'^μ as those of a transformed observer $\mathcal{O}'$, the mean value of the coordinates of a transformed observer would be

$$\begin{aligned} \langle x'^\mu \rangle &= \langle \phi | \otimes \langle \psi | (\Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1) | \phi \rangle \otimes | \psi \rangle \\ &= \langle \phi | \Lambda^\mu{}_\nu | \phi \rangle \langle \psi | x^\nu | \psi \rangle + \langle \phi | a^\mu | \phi \rangle, \end{aligned} \tag{4.12}$$

while for the uncertainties of transformed states in relation to those of the starting ones, we can write

$$\begin{aligned} \Delta(x'^\mu)^2 = \langle (x'^\mu)^2 \rangle - \langle x'^\mu \rangle^2 &= \Delta(\Lambda^\mu{}_\nu \otimes x^\nu)^2 + \Delta(a^\mu)^2 \\ &\quad + 2\mathrm{cov}(\Lambda^\mu{}_\nu, a^\mu) \langle x^\nu \rangle, \end{aligned} \tag{4.13}$$

[7] We consider a state $|\phi\rangle$ on which $\Delta a^\mu = 0$. Then, if we take an eigenstate $|\phi_\lambda\rangle$ of $\Lambda^\mu{}_\nu$, we have that, since the Λ's commute, their eigenvalues on $|\phi_\lambda\rangle$ are classical Lorentz parameters $\lambda^\mu{}_\nu$. It is possible to show that the only solution of (4.9) is $\lambda^\mu{}_0 = \delta^\mu{}_0$, $\lambda^1{}_3 = \lambda^2{}_3 = 0$, $\lambda^1{}_1 = \lambda^2{}_2$ for every eigenstate such that $\langle \phi | \phi_\lambda \rangle \neq 0$.

since $\langle a\otimes b\rangle = \langle a\rangle \otimes \langle b\rangle$ and the covariance between elements on different sides of the tensor product is 0. In the following, we will specialize the construction to three notable cases, that is, identity transformations, origin states transformations, and translations.

1. Identity transformation state

Since we know from our analysis that a sharp identity state does exist in the ϱ-Poincaré quantum group, we can consider identity transformations. We define the identity state $|i\rangle$ for our realization of $\mathcal{C}_\varrho(P)$ as follows:

$$\langle i|f(a,\Lambda)|i\rangle = \varepsilon(f) \tag{4.14}$$

with $f(a,\Lambda)\in\mathcal{C}_\varrho(P)$. Then the state

$$|i,\psi\rangle = |i\rangle\otimes|\psi\rangle \tag{4.15}$$

can be linked to the ϱ-Poincaré transformation between two coincident observers, as one can see working the following calculation:

$$\begin{aligned}\langle x'^\mu\rangle &= \langle i|\otimes\langle\psi|(\Lambda^\mu{}_\nu\otimes x^\nu + a^\mu\otimes 1)|i\rangle\otimes|\psi\rangle\\ &= \langle i|\Lambda^\mu{}_\nu|i\rangle\langle\psi|x^\nu|\psi\rangle + \langle i|a^\mu|i\rangle,\end{aligned} \tag{4.16}$$

but recalling the counits (3.16a) and (3.16b)

$$\langle x'^\mu\rangle = \langle\psi|x^\mu|\psi\rangle. \tag{4.17}$$

The same result is achieved for a generic monomial in coordinates $x'^{\mu_1}\cdots x'^{\mu_n}$:

$$\begin{aligned}\langle x'^{\mu_1}\cdots x'^{\mu_n}\rangle &= \langle i|\otimes\langle\psi|x'^{\mu_1}\cdots x'^{\mu_n}|i\rangle\otimes|\psi\rangle\\ &= \langle i|a^{\mu_1}\cdots a^{\mu_n}|i\rangle + \langle i|\mathcal{O}^{\mu_1\cdots\mu_n}_\nu(a,\Lambda)|i\rangle\langle\psi|x^\nu|\psi\rangle\\ &\quad+\cdots+\langle i|\mathcal{O}^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}(a,\Lambda)|i\rangle\langle\psi|x^{\nu_1}\cdots x^{\nu_n}|\psi\rangle\end{aligned} \tag{4.18}$$

with $\mathcal{O}(a,\Lambda)$ generic monomials in a's and Λ's. Since the counit map is a homomorphism, every monomial that contains at least one a vanishes $[\varepsilon(a^\mu)=0]$, and the only surviving term is the one with an equal number of upper and lower indices that is a product of Λ's only. Again from the homomorphism property, one obtains that $\varepsilon(\mathcal{O}^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}(a,\Lambda)) = \delta^{\mu_1}{}_{\nu_1}\cdots\delta^{\mu_n}{}_{\nu_n}$, and

$$\langle x'^{\mu_1}\cdots x'^{\mu_n}\rangle = \langle\psi|x^{\mu_1}\cdots x^{\mu_n}|\psi\rangle. \tag{4.19}$$

Then one easily sees that uncertainties between the two events coincide:

$$\Delta(x'^\mu)^2 = \langle(x'^\mu)^2\rangle - \langle x'^\mu\rangle^2 = \langle(\delta^\mu{}_\nu x^\nu)^2\rangle - \langle x^\mu\rangle^2 = \Delta(x^\mu)^2. \tag{4.20}$$

Coincident observers are well defined in ϱ-Minkowski spacetime and they agree on every measurement they make. These results are identical to those found in [1] for κ-Minkowski spacetime.

2. Origin state transformations

We ask what the observer $\mathcal{O}'$ measures after the ϱ-Poincaré quantum group transforming the origin state; the starting state is

$$|\phi,o\rangle = |\phi\rangle\otimes|o\rangle, \tag{4.21}$$

therefore,

$$\begin{aligned}\langle x'^\mu\rangle &= \langle\phi|\otimes\langle o|(\Lambda^\mu{}_\nu\otimes x^\nu + a^\mu\otimes 1)|\phi\rangle\otimes|o\rangle\\ &= \langle\phi|\Lambda^\mu{}_\nu|\phi\rangle\langle o|x^\nu|o\rangle + \langle\phi|a^\mu|\phi\rangle.\end{aligned} \tag{4.22}$$

Recalling that $\langle o|x^\mu|o\rangle = 0$, we have

$$\langle x'^\mu\rangle = \langle\phi|a^\mu|\phi\rangle. \tag{4.23}$$

This result entails the fact that the two observers $\mathcal{O}$ and $\mathcal{O}'$ are comparing positions and not directions, so the expectation value is determined only by the mean value of translation operators.

It can be shown by an analogous computation that the result remains true also for a generic monomial in coordinates $x'^{\mu_1}\cdots x'^{\mu_n}$; in fact, since $\langle o|x^{\mu_1}\cdots x^{\mu_n}|o\rangle = 0\ \forall\ n$,

$$\begin{aligned}\langle x'^{\mu_1}\cdots x'^{\mu_n}\rangle &= \langle\phi|\otimes\langle o|x'^{\mu_1}\cdots x'^{\mu_n}|\phi\rangle\otimes|o\rangle\\ &= \langle\phi|a^{\mu_1}\cdots a^{\mu_n}|\phi\rangle.\end{aligned} \tag{4.24}$$

In this case, the uncertainty of the transformed event coincides with that of the translation operator:

$$\Delta(x'^\mu)^2 = \langle(x'^\mu)^2\rangle - \langle x'^\mu\rangle^2 = \langle(a^\mu)^2\rangle - \langle a^\mu\rangle^2 = \Delta(a^\mu)^2. \tag{4.25}$$

Comparing with the κ case [1], we notice that when the translational parameter can be localized, in both cases the uncertainty on the final state is zero. For the ϱ-Poincaré quantum group recalling (4.8), this occurs when $\langle a^1\rangle = \langle a^2\rangle = 0$, namely, for pure translations along a^0, a^3 or even mixed translations in a^0, a^3, while for the κ-Poincaré quantum group, this occurs only for pure temporal translations.

3. Translations

Another interesting case is that of a pure translation $x'^\mu = 1\otimes x^\mu + a^\mu\otimes 1$ of a generic state. To demonstrate that states $|\phi_T\rangle$ corresponding to translations do exist in $L^2(SO(1,3)\times\mathbb{R}^3_q)$, it is necessary to take a sequence of functions which converge to a δ for the diagonal elements of Λ and to 0 for off-diagonal ones, so that $\langle\phi_T|\Lambda^\mu{}_\nu|\phi_T\rangle = \delta^\mu{}_\nu$. We observe that taking such states and (co)acting with the usual coaction (3.1), it is the same thing as (co)acting on a generic state of $L^2(SO(1,3)\times\mathbb{R}^3_q)\times L^2(\mathbb{R}^3_x)$ with $x'^\mu = 1\otimes x^\mu + a^\mu\otimes 1$. The expectation value is then

$$\begin{aligned}\langle x'^\mu\rangle &= \langle\phi|\otimes\langle\psi|(1\otimes x^\mu + a^\mu\otimes 1)|\phi\rangle\otimes|\psi\rangle\\ &= \langle\psi|x^\mu|\psi\rangle + \langle\phi|a^\mu|\phi\rangle,\end{aligned} \tag{4.26}$$

while the variance

$$\begin{aligned}\Delta(x'^\mu)^2 &= \langle (x^\mu)^2 + (a^\mu)^2 + x^\mu a^\mu + a^\mu x^\mu\rangle\\ &\quad - \langle x^\mu\rangle^2 - \langle a^\mu\rangle^2 - 2\langle x^\mu\rangle\langle a^\mu\rangle\\ &= \Delta(x^\mu)^2 + \Delta(a^\mu)^2 \geq \Delta(x^\mu)^2.\end{aligned} \tag{4.27}$$

Therefore, one sees that acting with a pure translation leads, in general, to an increase in the state uncertainty. As for the comparison with the κ case, the same considerations apply as those at the end of Sec. IV D 2.

V. CONCLUSIONS AND OUTLOOK

We have analyzed the localization features of spacetime states of $\mathcal{M}_\varrho$ as well as those of the quantum group $\mathcal{C}_\varrho(P)$ and their consequences on Poincaré-deformed transformations.

The main difference between the κ- and ϱ-Minkowski spacetimes is in the nature of the commutation relations. While for the former, these are clearly of radial nature; for the latter, they are explicitly of an angular one. In the first case, there are no central Cartesian coordinates, while in the second case, x^3 commutes with every other one. It is therefore legitimate to think that this coordinate can be determined without any uncertainty and will not pose problems for its localizability. We have shown that perfect localization of observable states can be achieved in the "special position" $x^1 = x^2 = 0$, in accord with the angular nature of the only nonmultiplicative operator x^0 that acts as an angular momentum along the 3-axis.

Turning our attention to the issue of symmetries, we have shown that the deformed nature of the Poincaré quantum groups leads to the interesting feature of having uncertainties arising from deformed-Poincaré transformations. This implies that two different observers will, in general, not agree on the localizability properties of the same state. The localizability properties of the quantum groups can be analyzed by writing uncertainty relations between the noncommutative group parameters. These relations, surprisingly, pose constraints on the admissible deformed-Poincaré transformations; for example, we have seen that for the ϱ-Poincaré quantum group, pure space translations along the 1- and 2-axes and pure Lorentz transformations are not allowed except for the rotations around the 3-axis. These features were previously discussed for the κ case, leading to the so-called "no-pure" features of the quantum group [20,21].

It is worth noticing that in [1], a particular mixed transformation in the $(1+1)$-dimensional κ case leading to a decrease in uncertainty was found. It would be interesting to see if also the ϱ-Poincaré quantum group admits some transformation of this kind and to give some physical interpretation to it.

We have not considered the dual picture of the quantum universal enveloping algebra $U_\varrho(\mathfrak{p})$ that was obtained in [9,13] within the twist approach. In that framework, the Lie algebra sector is naturally underformed, whereas the cosector is modified. There is, however, the possibility of finding a nonlinear change of basis which could lead to a different quantum universal enveloping algebra $\tilde{U}_\varrho(\mathfrak{p})$ with a bi-cross-product structure, in analogy with the κ-Poincaré case. This could have interesting physical implications (such as consequences on deformed infinitesimal symmetries and deformed dispersion relations) and it is presently under investigation.

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APPENDIX A: STATES, OBSERVABLES, AND OBSERVERS

We review in this appendix the notions of states, observables, and observers, which hold true both in the commutative and the noncommutative cases.

States: A state ϕ is a linear functional from a C^*-algebra $\mathcal{C}$ to the complex field (see, for example, [22]):

$$\phi\colon\ \mathcal{C}\to\mathbb{C}, \tag{A1}$$

positive defined

$$\phi(a^*a)\geq 0, \quad \forall\ a\in\mathcal{C}, \tag{A2}$$

and normalized

$$\|\phi\| = \sup_{\|a\|\leq 1}\{\phi(a)\} = 1. \tag{A3}$$

The space of states can be shown to be convex. Any state that can be expressed as a convex combination is said to be a mixed state, while states that cannot are called pure states.

From a commutative algebra and its set of pure states, it is possible to define a topology and thus obtain the associated topological space through the so-called Connes construction (see [22] for details). Furthermore, we can associate the notion of (functional) states to that of vector states on a Hilbert space via a Gelfand-Naimark-Segal (GNS) construction. Given, in fact, an algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, any normalized vector $|\xi\rangle$ defines a state with expectation value $\phi_\xi(a) = \langle\xi|\hat{a}|\xi\rangle$, $\hat{a} \in \mathcal{B}(\mathcal{H})$. On the contrary, to any state ϕ it corresponds a vector state $\xi_\phi \in \mathcal{H}$ such that $\langle\xi_\phi|\hat{a}|\xi_\phi\rangle = \phi(a)$. If the variance $\Delta(a) = \sqrt{\phi(a^2) - \phi(a)^2} = \sqrt{\langle\xi_\phi|\hat{a}^2|\xi_\phi\rangle - (\langle\xi_\phi|\hat{a}|\xi_\phi\rangle)^2}$ is equal to zero, the state is said to be localized.

Observables. An observable $\mathcal{A}$ is, heuristically, a physical quantity that can be measured. Formally, in classical mechanics it is defined as a real-valued function on the phase space, while in quantum mechanics as a self-adjoint operator defined on a Hilbert space. Therefore, in the present context, an observable $\mathcal{A}$ is a self-adjoint element of the C^*-algebra $\mathcal{C}$. In this way, we can say that a state is a mapping from physical observables to their measured value.

Observers. The notion of an observer is a more subtle one. Loosely speaking, an observer in classical mechanics is something that performs a measure on a physical system and associates a real numerical value to the corresponding observable function; in quantum mechanics instead, it is a filter procedure that sends, after having performed a measure on a quantum object, a quantum state to a classical one associating numerical eigenvalues to observable operators with discrete spectra, or continuous density eigenvalues to operators with continuous spectra. In this work, we avoid the problem of giving a rigorous definition by relating an observer to its reference frame.

An observer $\mathcal{O}$ is a reference frame with respect to which the ordinary theory of measurement (i.e., the possibility of finding mean values, variances, and other higher moments of one or more observables in a state) can be applied. As a final remark, let us notice that, since we are dealing with special-relativistic theories, not taking into account general relativity (GR) features, we always mean inertial observers.

APPENDIX B: DEFORMATION OF HOPF ALGEBRAS

In this appendix, we recall the basic facts about the classical r matrix [23], as well as an approach [24,25] to quantize solvable Lie algebras employing them, which is used to obtain the quantum Poincaré groups associated with κ- and ϱ-Minkowski spacetimes.

Given a Lie algebra $\mathfrak{g}$ the classical r matrix is a tensor $r \in \bigwedge^2 \mathfrak{g}$, satisfying the MYBE, namely,

$$[\![r, r]\!] = t \tag{B1}$$

with $t \in \otimes^3 \mathfrak{g}$ a $\mathfrak{g}$-invariant element and $[\![r, r]\!] = [r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}]$. In the case $[\![r, r]\!] = 0$, this is the CYBE. Here, $r_{\alpha\beta} \in \otimes^3 \mathfrak{g}$, $\alpha, \beta = 1, 2, 3$ so that

$$r_{12} = c_{ij} a_i \otimes a_j \otimes 1, \qquad r_{23} = c_{ij} 1 \otimes a_i \otimes a_j,$$
$$r_{13} = c_{ij} a_i \otimes 1 \otimes a_j$$

with $a_i \in \mathfrak{g}$.

The classical r matrix has the important property of defining a Lie bialgebra structure on the Lie algebra $\mathfrak{g}$. Moreover, it allows for the definition of a Poisson bracket on the group manifold, which is compatible with the group structure, yielding to the notion of the Poisson-Lie group, whose quantum counterpart is a quantum group. Hence, a Poisson-Lie group G is a Lie group with group operations being Poisson maps [23]. The algebra of smooth functions $\mathcal{C}^\infty(G)$ is a Hopf algebra (with trivial cosector and antipode), which is referred to as a Poisson-Hopf algebra. The classical r matrix provides the Poisson-Lie structure through the following Sklyanin bracket:

$$\{f, g\} = r^{\alpha\beta}(X^R_\alpha f X^R_\beta g - X^L_\alpha f X^L_\beta g), \qquad f, g \in \mathcal{C}^\infty(G), \tag{B2}$$

where X^L, X^R are the left- and right-invariant vector fields.

If there are no ordering issues, one can quantize the Poisson-Hopf algebra by means of the usual canonical quantization $\{,\} \to \frac{1}{i}[,]$ to obtain the corresponding quantum Hopf algebra, namely, the quantum group. This is the case for both κ-Poincaré and ϱ-Poincaré deformations considered in Secs. III B and III C.

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