Transmutation operators and expansions for one-loop Feynman integrands

Kang Zhou*

Center for Gravitation and Cosmology, College of Physical Science and Technology, Yangzhou University, Yangzhou 225009, China

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In this paper, the connections among one-loop Feynman integrands of a wide range of theories are further investigated. The work includes two parts. First, we construct a new class of differential operators which transmute the one-loop gravitational Feynman integrands to Einstein-Yang-Mills and Yang-Mills Feynman integrands, thus linking these theories together. Second, by using one-loop level transmutation relations, together with some general conditions such as gauge and Lorentz invariance, we derive the expansions of Feynman integrands of certain theories to those of other theories. In particular, we find the Feynman integrands of all theories under consideration can be expanded to integrands of bi-adjoint scalar theory. The unified web for expansions is established, including gravity, Einstein-Yang-Mills theory, Einstein-Maxwell theory, Born-Infeld theory, pure Yang-Mills theory, Yang-Mills-scalar theory, special Yang-Mills-scalar theory, Dirac-Born-Infeld theory, extended Dirac-Born-Infeld theory, special Galileon theory, and nonlinear sigma model. The systematic rules for evaluating coefficients in the expansions are provided, and the duality between transmutation relations and expansions is shown.

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I. INTRODUCTION

Modern research on the S-matrix has exposed amazing relations among amplitudes of different theories, such as the Kawai-Lewellen-Tye relations [1], Bern-Carrasco-Johansson (BCJ) color-kinematics duality [2–7], Cachazo-He-Yuan (CHY) formulas [8–12], transmutation relations proposed by Cheung, Shen and Wen [13], which are invisible through the traditional Feynman rules. The transmutation relations, which based on constructing some Lorentz and gauge invariant differential operators, reveal the marvelous unity for tree amplitudes. By acting these transmutation operators on gravitational tree amplitudes, one can generate the tree amplitudes of a variety of theories. These unifying relations were verified and further studied in [14–16], by acting transmutation operators on CHY formulas of different theories.

At the level of tree amplitudes, another important reflection of connections among different theories is the amplitudes of certain theories can be expanded as the combination of amplitudes of other theories. Such expansions have been studied in various literatures, especially for expanding the tree Einstein-Yang-Mills amplitudes to Yang-Mills ones [17–26]. To evaluate the coefficients in expansions, one of the methods is solving differential equations indicated by transmutation operators [26]. This approach manifests the underlying connection between transmutation operators and expansions, allowing us to understand the unified web for expansions as the dual version of the web for transmutation relations [27,28].

It is natural to search the similar unities at the loop levels. This interesting question was first considered in [15], which exposed the strong evidence for the existence of transmutation operators at one-loop level. Recently, we studied this issue in [29]. In [29], the one-loop level differential operators which link one-loop Feynman integrands of different theories together were found, by employing the tree level operators, as well as the forward limit operation. Based on these one-loop level transmutation relations, the complete unified web for one-loop Feynman integrands was established. On the other hand, the expansions of one-loop Einstein-Yang-Mills and gravitational Feynman integrands to Yang-Mills ones, and the related one-loop BCJ numerators, were studied in [30,31].

In this paper, we further investigate the unifying relations among one-loop Feynman integrands of different theories. We first continue the study of the one-loop level transmutation relations. We construct a new kind of transmutation operators which link the Feynman integrands of gravity, Einstein-Yang-Mills and Yang-Mills theories together, in a manner different from those in [29]. The basic idea is similar as in [29], which can be summarized as follows. Suppose the tree amplitudes of theories A and B

*zhoukang@yzu.edu.cn

are linked by the operator $\mathcal{O}$ as $\mathcal{A}_B = \mathcal{O}\mathcal{A}_A$, we seek the one-loop level operator $\mathcal{O}_\circ$ satisfying $\mathcal{O}_\circ\mathcal{F}\mathcal{A}_A = \mathcal{F}\mathcal{O}\mathcal{A}_A$, where the operator $\mathcal{F}$ denotes taking the forward limit. Since the one-loop Feynman integrands can be obtained by taking the forward limit of tree amplitudes, one can conclude that the operator $\mathcal{O}_\circ$ transmutes the Feynman integrands as $\mathbf{I}_B = \mathcal{O}_\circ\mathbf{I}_A$. In this paper, we make new choices of tree level operators, which lead to new one-loop level operators. These new operators play the crucial role when solving the expansions of one-loop gravitational Feynman integrands.

The expansions of one-loop Feynman integrands of various theories are also investigated. We first solve the expansions of Einstein-Yang-Mills and gravitational integrands to Yang-Mills integrands, by employing the one-loop level differential operators, together with some general requirements such as Lorentz and gauge invariance. The idea is, we regard the transmutation relation $\mathbf{I}_B = \mathcal{O}\mathbf{I}_A$ as a differential equation, which allows us to solve $\mathbf{I}_A$ from it. A general difficulty is that the full Feynman integrands always contain terms which are annihilated by the differential operators under consideration, thus cannot be determined by solving differential equations. We fix these terms by imposing some general principles/assumptions, in particular the gauge invariance condition, and obtain the expansion

$$\mathbf{I}_{\text{GR}} = \sum_i C_i \mathbf{I}_{\text{YM};i}, \tag{1}$$

where $\mathbf{I}_{\text{GR}}$ stands for the gravitational Feynman integrands while $\mathbf{I}_{\text{YM};i}$ are Yang-Mills integrands.

Then, by applying the one-loop level transmutation relations further, we obtain expansions of Feynman integrands of other theories, and find that all the integrands under consideration can be double expanded to integrands of bi-adjoint scalar (BAS) theory, and provide the rules for constructing coefficients in the expansions. The resulting unified web includes a wide range of theories, which are gravitational (GR) theory, Einstein-Yang-Mills (EYM) theory, Einstein-Maxwell (EM) theory, Born-Infeld (BI) theory, pure Yang-Mills (YM) theory, Yang-Mills-scalar (YMS) theory, special Yang-Mills-scalar (SYMS) theory, Dirac-Born-Infeld (DBI) theory, extended Dirac-Born-Infeld (EDBI) theory, special Galileon (SG) theory, non-linear sigma model (NLSM). The whole process only depends on the knowledge of transmutation relations, as well as some general principles/assumptions, without knowing any detail of Feynman integrands, and without applying any tool for evaluating Feynman integrands such as Feynman rules, CHY formulas, and so on. We also show the tree level duality between transmutation relations and expansions can be generalized to the one-loop level, and give the map between differential operators and coefficients in expansions.

The remainder of this paper is organized as follows. In Sec. II, we give a brief introduction to the forward limit approach, and the differential operators at tree and one-loop levels, which are crucial for subsequent discussions. In Sec. III, we construct the new operators which transmute GR Feynman integrands to EYM ones. Then, in Sec. IV, we solve the expansions of EYM and GR integrands to YM ones, by using the one-loop level operators together with some general principles/assumptions. Section V is devoted to providing the full unified web for expansions, and showing the duality between transmutation relations and expansions. Finally, we end with a summary and discussions in Sec. VI.

II. BACKGROUND

For the readers' convenience, in this section we rapidly review the background for later sections. In Sec. II A, we give a brief introduction to the forward limit which generates the one-loop Feynman integrands from the tree amplitudes, as well as the color ordered Feynman integrands which will be discussed frequently in this paper. In Sec. II B, we review the tree level differential operators which link the tree amplitudes of a wide range of theories together, as well as the one-loop level generalization of these tree level operators. Most of the notations and conventions which will be used in later sections are also introduced in this section.

A. Forward limit and one-loop Feynman integrand

As is well known, the one-loop Feynman integrands can be generated from the tree amplitudes, via the so-called forward limit procedure. For instance, the one-loop CHY formulas can be obtained by applying this method, as studied in [32–35]. In this subsection, we review the general idea and characters of the forward limit.

Diagrammatically, the forward limit can be understood as gluing two external legs of the tree together to generate the loop. Here we give the strict definition, especially the manipulations $\mathcal{L}$ and $\mathcal{E}$ which will be used in later sections. The forward limit is reached as follows:

(i) Consider an $(n+2)$-point tree amplitude $\mathcal{A}_{n+2}(k_+, k_-)$ including n on-shell legs with momenta in $\{k_1, \ldots, k_n\}$ and two off-shell legs with $k_+^2 = k_-^2$.

(ii) Take the limit $k_\pm \to \pm\ell$. We denote this manipulation as $\mathcal{L}$.

(iii) Glue two external legs with ℓ and $-\ell$ together by summing over all allowed internal states, such as polarization vectors, colors, flavors, and so on. We denote this manipulation as $\mathcal{E}$.

Roughly speaking, the obtained object times the factor $1/\ell^2$,

$$\frac{1}{\ell^2}\mathcal{F}\mathcal{A}_{n+2}(k_+^{h_+},k_-^{h_-}), \tag{2}$$

contributes to the n-point one-loop Feynman integrand $\mathbf{I}_n$. Here we introduced the operator

$$\mathcal{F}\equiv\mathcal{E}\mathcal{L}, \tag{3}$$

to denote the operation of taking the forward limit.

For the individual Feynman diagram, the manipulation in (2) obviously turns the tree to the loop. However, it does not mean the forward limit of tree amplitude gives rise to the one-loop Feynman integrand directly, since the full tree amplitude and full one-loop Feynman integrand are obtained by summing over all appropriate diagrams, and the map between tree diagrams and one-loop diagrams is not one to one. For example, after gluing legs $+$ and $-$ together, three different tree diagrams at the rhs of Fig. 1 are turned to the same one-loop diagram at the lhs. This difficulty is solved by the decomposition based on the so-called partial fraction identity [32,36]. Figure 1 is an example of such decomposition. We will not discuss the details of the partial fraction identity; the introduction of it can be found in [32,36]. We only point out an important feature related to such decomposition: when considering the color ordered Feynman integrands, one needs to distinguish the full ones and partial ones, as can be seen in (6).

Let us give a rapid introduction to the color ordered Feynman integrands. We start with the tree Yang-Mills amplitudes. Consider a theory that external particles are in the adjoint representation of the $U(N)$ group; the full tree amplitude can be expanded using the standard color decomposition as a sum over $(n+1)!$ terms [37–39]

$$\mathcal{A}_{n+2}=\sum_{\sigma\in S_{n+2}/\mathbb{Z}_{n+2}}\mathrm{Tr}(T^{a_{\sigma_+}}T^{a_{\sigma_1}}\cdots T^{a_{\sigma_n}}T^{a_{\sigma_-}})\times\mathcal{A}(\sigma_+,\sigma_1,\ldots,\sigma_n,\sigma_-), \tag{4}$$

where σ denotes the permutations of external particles. Each $\mathcal{A}(\sigma_+,\sigma_1,\ldots,\sigma_n,\sigma_-)$, which has the fixed ordering among external legs, is called the color ordered amplitude. The color ordered amplitudes $\mathcal{A}(\sigma_+,\sigma_1,\ldots,\sigma_n,\sigma_-)$, as well as the color ordered Feynman integrands which will be introduced soon, are independent of the choices of gauge groups, thus are unique for all Yang-Mills theories. Notice that each color ordering is invariant under the cyclic permutation of external legs, as can be observed from the factor $\mathrm{Tr}(T^{a_{\sigma_+}}T^{a_{\sigma_1}}\cdots T^{a_{\sigma_n}}T^{a_{\sigma_-}})$, thus the summation is over $\sigma\in S_{n+2}/\mathbb{Z}_{n+2}$ rather than $\sigma\in S_{n+2}$.

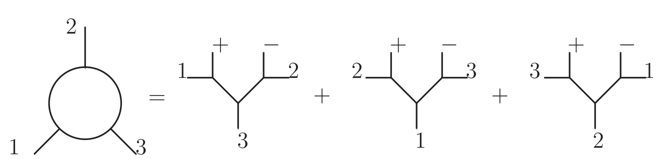

FIG. 1. Decomposition of one-loop Feynman integrand.

Now we glue the external legs $+$ and $-$ of $\mathcal{A}(\sigma_+,\sigma_1,\ldots,\sigma_n,\sigma_-)$ together. Taking the forward limit requires summing over the $U(N)$ degrees of freedom of two internal particles. This gives rise to two kinds of terms. The first comes from permutations such that legs $+$ and $-$ are adjacent, the corresponding color factors are given as

$$\sum_{a_+=a_-=1}^{N^2}\delta_{a_+a_-}\mathrm{Tr}(T^{a_+}T^{a_{\sigma_1}}\cdots T^{a_{\sigma_n}}T^{a_-})=N\mathrm{Tr}(T^{a_{\sigma_1}}\cdots T^{a_{\sigma_n}}), \tag{5}$$

thus contribute to the n-point color ordered Feynman integrand $\mathbf{I}_\circ(\sigma_1,\ldots,\sigma_n)$. The second case that $+$ and $-$ are not adjacent gives rise to double-trace terms. In this paper, we only consider the single-trace terms, since the double-trace ones are determined by the single-trace ones [40], as can be proved via the tree level Kleiss-Kuijf relation together with the forward limit operation [41]. For the single trace case, the above discussion shows that the partial integrand obtained from taking the forward limit for $\mathcal{A}(+,\sigma_1,\ldots,\sigma_n,-)$ contributes to $\mathbf{I}_\circ(\sigma_1,\ldots,\sigma_n)$. There are several original color orderings giving rise to the same trace factor after summing over a_+ and a_-, due to the cyclic symmetry of the trace factors. For instance, the object obtained by taking the forward limit for $\mathcal{A}(+,\sigma_2,\ldots,\sigma_n,\sigma_1,-)$ also contributes to $\mathbf{I}_\circ(\sigma_1,\ldots,\sigma_n)$. Collecting these equivalent color orderings together, one finds that the full color ordered Feynman integrand can be expanded as the following cyclic summation:

$$\mathbf{I}_\circ(\sigma_1,\ldots,\sigma_n)=\sum_{j=0}^{n-1}\mathbf{I}(+,\sigma_{1+j},\ldots,\sigma_{n+j},-), \tag{6}$$

where the partial color ordered integrands $\mathbf{I}(+,\sigma_{1+j},\ldots,\sigma_{n+j},-)$ are obtained from the color ordered tree amplitudes via the standard forward limit procedure in (2), namely,

$$\mathbf{I}(+,\sigma_{1+j},\ldots,\sigma_{n+j},-)=\frac{1}{\ell^2}\mathcal{F}\mathcal{A}(+,\sigma_{1+j},\ldots,\sigma_{n+j},-). \tag{7}$$

This equality (6) is supported by the partial fraction identity. Notice that throughout this paper we denote the full color ordered Feynman integrands by $\mathbf{I}_\circ$, and the partial ones by $\mathbf{I}$.

The forward limit is well defined for the $\mathcal{N}=4$ SYM theory. For other theories, a quite general feature is the obtained Feynman integrand suffers from divergence in the forward limit. Fortunately, the singular parts are found to be physically irrelevant, at least for theories under consideration in this paper. From the Feynman diagrams point of view, the singular parts correspond to tadpoles, as well as babbles carried by external legs, which do not contribute to the S matrix. From the CHY point of view, the singular parts can be ignored due to the following observation [33]: as long as the CHY integrand is homogeneous in ℓ^μ, the singular solutions of scattering equations contribute to the scaleless integrals which vanish under the dimensional regularization. The homogeneity in ℓ^μ is satisfied by all theories under consideration in this paper. Thus, we just assume that the singular parts are excluded.

B. Transmutation operators at tree and one-loop levels

The differential operators at tree level, proposed by Cheung, Shen and Wen, link a wide range of theories together by transmuting tree amplitudes of one theory to those of another theory [13–15]. Three kinds of basic operators are defined as follows:

(i) Trace operator:

$$\mathcal{T}^\epsilon_{ij} \equiv \partial_{\epsilon_i\cdot\epsilon_j}, \tag{8}$$

where ϵ_i is the polarization vector of the ith external leg. The up index ϵ means the operators are defined through polarization vectors in $\{\epsilon_i\}$. Since the graviton carries the polarization tensor $\epsilon^{\mu\nu} = \epsilon^u\tilde{\epsilon}^\nu$, the operators can always be defined via $\{\epsilon_i\}$ or $\{\tilde{\epsilon}_i\}$.[1]

(ii) Insertion operator:

$$\mathcal{I}^\epsilon_{ikj} \equiv \partial_{\epsilon_k\cdot k_i} - \partial_{\epsilon_k\cdot k_j}, \tag{9}$$

where k_i denotes the momentum carried by the ith external leg. When applying to physical amplitudes, the insertion operator $\mathcal{I}^\epsilon_{ik(i+1)}$ inserts the external leg k between external legs i and $i+1$, thus turns the color ordering $\ldots, i, i+1, \ldots$ to $\ldots, i, k, i+1, \ldots$. For general $\mathcal{I}^\epsilon_{ikj}$ with $i < j$, one can use definition (9) to decompose $\mathcal{I}^\epsilon_{ikj}$ as

$$\mathcal{I}^\epsilon_{ikj} = \sum_{a=i}^{j-1} \mathcal{I}^\epsilon_{ak(a+1)}. \tag{10}$$

Each $\mathcal{I}^\epsilon_{ak(a+1)}$ on the rhs can be interpreted as inserting the leg k between a and $(a+1)$. Thus, the effect of applying $\mathcal{I}^\epsilon_{ikj}$ can be understood as inserting k between i and j in the color ordering $\ldots, i, \ldots, j, \ldots$, and summing over all possible positions.

(iii) Longitudinal operator:

$$\mathcal{L}^\epsilon_i \equiv \sum_{j\neq i} (k_i\cdot k_j)\partial_{\epsilon_i\cdot k_j}, \quad \mathcal{L}^\epsilon_{ij} \equiv -(k_i\cdot k_j)\partial_{\epsilon_i\cdot\epsilon_j}. \tag{11}$$

With the basic operators given above, three combinatory operators are defined as follows:

(i) For a length-m ordered set $\vec{\boldsymbol{a}}_m = \langle a_1, \ldots, a_m\rangle$ of external particles, the operator $\mathcal{T}^\epsilon_{\vec{\boldsymbol{a}}_m}$ is defined as[2]

$$\mathcal{T}^\epsilon_{\vec{\boldsymbol{a}}_m} \equiv \left(\prod_{i=2}^{m-1} \mathcal{I}^\epsilon_{a_1a_ia_{i+1}}\right)\mathcal{T}^\epsilon_{a_1a_m}. \tag{12}$$

In this paper, we use $\boldsymbol{a}_m = \{a_1, \ldots, a_m\}$ to denote an unordered set with length m, and $\vec{\boldsymbol{a}}_m = \langle a_1, \ldots, a_m\rangle$ for an ordered set. For amplitudes/Feynman integrands carry color orderings, sometimes we write down the orderings explicitly if necessary, and sometimes we use $\vec{\boldsymbol{a}}_m$ to denote orderings. The combinatory operator $\mathcal{T}^\epsilon_{\vec{\boldsymbol{a}}_m}$ turns the spin of a_ith external particle with $a_i \in \boldsymbol{a}_m$ from s^{a_i} to $s^{a_i}-1$ by removing the polarization vector ϵ^{a_i}, and generates the color ordering $a_1, a_2, \ldots, a_m$ as follows: fixes two reference legs a_1 and a_m at two ends in the color ordering via the operator $\mathcal{T}^\epsilon_{a_1a_m}$, then inserts other elements between them by using insertion operators $\mathcal{I}^\epsilon_{a_1a_ia_{i+1}}$. The interpretation of insertion operators indicates that $\mathcal{T}^\epsilon_{\vec{\boldsymbol{a}}_m}$ has various equivalent choices, for example

$$\begin{aligned} \mathcal{T}^\epsilon_{\vec{\boldsymbol{a}}_m} &= \left(\prod_{i=m-1}^{2} \mathcal{I}^\epsilon_{a_{i-1}a_ia_m}\right)\mathcal{T}^\epsilon_{a_1a_m}, \\ \mathcal{T}^\epsilon_{\vec{\boldsymbol{a}}_m} &= \left(\prod_{i=3}^{m-3} \mathcal{I}^\epsilon_{a_2a_ia_{i+1}}\right)\mathcal{I}^\epsilon_{a_2a_{m-2}a_{m-1}} \\ &\quad\times \mathcal{I}^\epsilon_{a_1a_2a_{m-1}}\mathcal{I}^\epsilon_{a_{m-1}a_ma_1}\mathcal{T}^\epsilon_{a_1a_{m-1}}, \end{aligned} \tag{13}$$

and so on. The second example shows that it is not necessary to choose the first operator to be $\mathcal{T}^\epsilon_{a_1a_m}$. In other words, two reference legs in the color ordering can be chosen arbitrary.

(ii) For n-point amplitudes, the operator $\mathcal{L}^\epsilon$ is defined as

$$\mathcal{L}^\epsilon \equiv \prod_i \mathcal{L}^\epsilon_i, \qquad \bar{\mathcal{L}}^\epsilon \equiv \sum_{\rho\in\text{pair}} \prod_{i,j\in\rho} \mathcal{L}^\epsilon_{ij}. \tag{14}$$

[1]Here the gravity is understood as a generalized version, i.e., gravitons coupled to dilatons and B fields.

[2]In this paper, we adopt the convention that the operator at the lhs acts after the operator at the rhs. From the mathematical point of view, the order of operators is irrelevant, since all operators are commutable with each other. We choose the order of operators in the definition to emphasize the interpretation of each one.

Two definitions $\mathcal{L}^\epsilon$ and $\bar{\mathcal{L}}^\epsilon$ are not equivalent to each other at the algebraic level. However, when acting on proper on-shell physical amplitudes, two combinations $\mathcal{L}^\epsilon \cdot \mathcal{T}^\epsilon_{ab}$ and $\bar{\mathcal{L}}^\epsilon \cdot \mathcal{T}^\epsilon_{ab}$, with subscripts of $\mathcal{L}^\epsilon_i$ and $\mathcal{L}^\epsilon_{ij}$ run through all nodes in $\{1,2,\ldots,n\}\backslash\{a,b\}$, giving the same effect which can be interpreted physically.

(iii) For a length-$2m$ set, the operator $\mathcal{T}^\epsilon_{\mathcal{X}_{2m}}$ is defined as

$$\mathcal{T}^\epsilon_{\mathcal{X}_{2m}} \equiv \sum_{\rho\in\text{pair}} \prod_{i_k,j_k\in\rho} \delta_{I_{i_k}I_{j_k}} \mathcal{T}^\epsilon_{i_kj_k}, \tag{15}$$

where I_a denotes the flavor carried by the ath particle. For the special case $2m$ particles do not carry any flavor, the operator $\mathcal{T}^\epsilon_{X_{2m}}$ is defined by removing $\delta_{I_{i_k}I_{j_k}}$,

$$\mathcal{T}^\epsilon_{X_{2m}} \equiv \sum_{\rho\in\text{pair}} \prod_{i_k,j_k\in\rho} \mathcal{T}^\epsilon_{i_kj_k}. \tag{16}$$

Here is the explanation for the notation $\sum_{\rho\in\text{pair}}\prod_{i_k,j_k\in\rho}$. Let Γ be the set of all partitions of the set $\{1,2,\ldots,2m\}$ into pairs without regard to the order. An element in Γ can be written as

$$\rho = \{(i_1,j_1),(i_2,j_2),\ldots,(i_m,j_m)\}, \tag{17}$$

with conditions $i_1 < i_2 < \cdots < i_m$ and $i_t < j_t,\ \ \forall\, t$. Then, $\prod_{i_k,j_k\in\rho}$ stands for the product of $\mathcal{T}^\epsilon_{i_kj_k}$ for all pairs (i_k,j_k) in ρ, and $\sum_{\rho\in\text{pair}}$ denotes the summation over all partitions.

The combinatory operators defined above link tree amplitudes of a wide range of theories together, by transmuting the GR amplitudes to amplitudes of other theories, formally expressed as

$$\mathcal{A} = \mathcal{O}^\epsilon \mathcal{O}^{\tilde\epsilon} \mathcal{A}^{\epsilon,\tilde\epsilon}_{\text{GR}}. \tag{18}$$

Operators $\mathcal{O}^\epsilon$ and $\mathcal{O}^{\tilde\epsilon}$ for different theories are listed in Table I.

The EYM theory appearing in Table I is the single-trace Einstein-Yang-Mills theory, denoted by sEYM.[3] Let us explain other notations in turn. The symbol $\mathbb{I}$ stands for the identical operator. Up indexes h, p, g and s denote gravitons, photons, gluons and scalars. For instance, $\boldsymbol{a}^h_n$ is the unordered set of gravitons with length n, $\vec{\boldsymbol{a}}^g_m$ is the ordered set of gluons with length m. The total number of external legs is denoted by n, each set with length m is a subset of external legs. We use $\mathcal{A}^{\tilde\epsilon}_{\text{SYMS}}(\boldsymbol{a}^s_{2m};\boldsymbol{a}^g_{n-m}\|\vec{\boldsymbol{a}}^A_n)$ as the example to explain notations $\|\vec{\boldsymbol{a}}^A_n$ and ;. For the amplitude including more than one kind of particles, such as scalars and gluons in the example, $\|\vec{\boldsymbol{a}}^A_n$ stands for the color ordering among all external legs, without distinguishing the kinds of them. Notation ; is used to separate different kinds of external particles, with the convention that particles at the lhs of ; carry lower spin. In our example, the lhs of ; are scalars, while the rhs are gluons. The up index of $\mathcal{A}$ denotes the polarization vectors carried by external particles. When the amplitude includes external gravitons, the rule is: the previous polarization vectors are only carried by gravitons, while the later ones are carried by all particles. For instance, in the notation $\mathcal{A}^{\epsilon,\tilde\epsilon}_{\text{EMf}}(\boldsymbol{a}^p_{2m};\boldsymbol{a}^h_{n-2m})$, polarization vectors in $\{\epsilon_i\}$ are only carried by gravitons, while those in $\{\tilde\epsilon_i\}$ are carried by both photons and gravitons. For the BAS amplitude, we have used $\vec{\boldsymbol{a}}_n$ and $\vec{\boldsymbol{s}}_n$ to distinguish two color orderings among external legs. In later sections, when considering more than one color orderings simultaneously, we frequently use $\vec{\boldsymbol{s}}$ in addition to $\vec{\boldsymbol{a}}$, to avoid the ambiguity.

[3]In [13–15], the operators which generate the general multiple-trace tree EYM amplitudes are also considered. These operators are not included in Table I, since we will not consider them throughout this paper.

TABLE I. Unifying relations for differential operators at tree level.

Amplitude	$\mathcal{O}^\epsilon$	$\mathcal{O}^{\tilde\epsilon}$
$\mathcal{A}^{\epsilon,\tilde\epsilon}_{\text{GR}}(\boldsymbol{a}^h_n)$	$\mathbb{I}$	$\mathbb{I}$
$\mathcal{A}^{\epsilon,\tilde\epsilon}_{\text{sEYM}}(\vec{\boldsymbol{a}}_m;\boldsymbol{a}^h_{n-m})$	$\mathcal{T}^\epsilon_{\vec{\boldsymbol{a}}_m}$	$\mathbb{I}$
$\mathcal{A}^{\epsilon,\tilde\epsilon}_{\text{EMf}}(\boldsymbol{a}^p_{2m};\boldsymbol{a}^h_{n-2m})$	$\mathcal{T}^\epsilon_{\mathcal{X}_{2m}}$	$\mathbb{I}$
$\mathcal{A}^{\epsilon,\tilde\epsilon}_{\text{EM}}(\boldsymbol{a}^p_{2m};\boldsymbol{a}^h_{n-2m})$	$\mathcal{T}^\epsilon_{X_{2m}}$	$\mathbb{I}$
$\mathcal{A}^{\tilde\epsilon}_{\text{BI}}(\boldsymbol{a}^p_n)$	$\mathcal{L}^\epsilon\mathcal{T}^\epsilon_{ab}$	$\mathbb{I}$
$\mathcal{A}^{\tilde\epsilon}_{\text{YM}}(\vec{\boldsymbol{a}}^g_n)$	$\mathcal{T}^\epsilon_{\vec{\boldsymbol{a}}_n}$	$\mathbb{I}$
$\mathcal{A}^{\tilde\epsilon}_{\text{sYMS}}(\vec{\boldsymbol{a}}^s_m;\boldsymbol{a}^g_{n-m}\|\vec{\boldsymbol{a}}^A_n)$	$\mathcal{T}^\epsilon_{\vec{\boldsymbol{a}}_n}$	$\mathcal{T}^{\tilde\epsilon}_{\vec{\boldsymbol{a}}_m}$
$\mathcal{A}^{\tilde\epsilon}_{\text{SYMS}}(\boldsymbol{a}^s_{2m};\boldsymbol{a}^g_{n-m}\|\vec{\boldsymbol{a}}^A_n)$	$\mathcal{T}^\epsilon_{\vec{\boldsymbol{a}}_n}$	$\mathcal{T}^{\tilde\epsilon}_{\mathcal{X}_{2m}}$
$\mathcal{A}_{\text{NLSM}}(\vec{\boldsymbol{a}}^s_n)$	$\mathcal{T}^\epsilon_{\vec{\boldsymbol{a}}_n}$	$\mathcal{L}^{\tilde\epsilon}\mathcal{T}^{\tilde\epsilon}_{a'b'}$
$\mathcal{A}_{\text{BAS}}(\vec{\boldsymbol{a}}^s_n\|\vec{\boldsymbol{s}}^s_n)$	$\mathcal{T}^\epsilon_{\vec{\boldsymbol{a}}_n}$	$\mathcal{T}^{\tilde\epsilon}_{\vec{\boldsymbol{s}}_n}$
$\mathcal{A}^{\tilde\epsilon}_{\text{DBI}}(\boldsymbol{a}^s_{2m};\boldsymbol{a}^p_{n-2m})$	$\mathcal{L}^\epsilon\mathcal{T}^\epsilon_{ab}$	$\mathcal{T}^{\tilde\epsilon}_{\mathcal{X}_{2m}}$
$\mathcal{A}^{\tilde\epsilon}_{\text{EDBI}}(\vec{\boldsymbol{a}}^s_m;\boldsymbol{a}^p_{n-2m})$	$\mathcal{L}^\epsilon\mathcal{T}^\epsilon_{ab}$	$\mathcal{T}^{\tilde\epsilon}_{\vec{\boldsymbol{a}}_m}$
$\mathcal{A}_{\text{SG}}(\boldsymbol{a}^s_n)$	$\mathcal{L}^\epsilon\mathcal{T}^\epsilon_{ab}$	$\mathcal{L}^{\tilde\epsilon}\mathcal{T}^{\tilde\epsilon}_{a'b'}$

The above tree level unity can be generalized to one-loop Feynman integrands via the following idea, as studied in [29]. Suppose the tree amplitudes of theories A and B are connected by the operator $\mathcal{O}$ as $\mathcal{A}_B = \mathcal{O}\mathcal{A}_A$; we seek the one-loop level operator $\mathcal{O}_\circ$ satisfying $\mathcal{O}_\circ\mathcal{F}\mathcal{A}_A = \mathcal{F}\mathcal{O}\mathcal{A}_A$. Since the one-loop Feynman integrands are obtained through the forward limit as $\mathbf{I}_A = (1/\ell^2)\mathcal{F}\mathcal{A}_A$ and $\mathbf{I}_B = (1/\ell^2)\mathcal{F}\mathcal{A}_B$, one can conclude that the operator $\mathcal{O}_\circ$ transmutes the Feynman integrand as $\mathbf{I}_B = \mathcal{O}_\circ\mathbf{I}_A$.

Using the above idea, the one-loop generalization of the tree level unifying relation (18) is found to be

$$\mathbf{I} = \mathcal{O}^\epsilon_\circ \mathcal{O}^{\tilde\epsilon}_\circ \mathbf{I}^{\epsilon,\tilde\epsilon}_{\text{GR}}. \tag{19}$$

The one-loop level operators $\mathcal{O}^\epsilon_\circ$ and $\mathcal{O}^{\tilde\epsilon}_\circ$ for different theories are listed in Table II.

TABLE II. $\mathcal{O}^{\epsilon}_{\circ}$ and $\mathcal{O}^{\tilde{\epsilon}}_{\circ}$ for various theories.

Feynman integrand	$\mathcal{O}^{\epsilon}_{\circ}$	$\mathcal{O}^{\tilde{\epsilon}}_{\circ}$
$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n)$	$\mathbb{I}$	$\mathbb{I}$
$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_m,-^g;\boldsymbol{a}^h_{n-m})$	$\mathbb{I}$	$\mathcal{T}^{\tilde{\epsilon}}_{+\vec{\boldsymbol{a}}_m-}$
$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{EMf};\circ}(\boldsymbol{a}^p_{2m};\boldsymbol{a}^h_{n-2m})$	$\mathbb{I}$	$\mathcal{T}^{\tilde{\epsilon}}_{\mathcal{X}_{2m}}(N\tilde{\mathcal{D}}+1)$
$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{EM};\circ}(\boldsymbol{a}^p_{2m};\boldsymbol{a}^h_{n-2m})$	$\mathbb{I}$	$\mathcal{T}^{\tilde{\epsilon}}_{\mathcal{X}_{2m}}(\tilde{\mathcal{D}}+1)$
$\mathbf{I}^{\epsilon}_{\mathrm{BI};\circ}(\boldsymbol{a}^p_n)$	$\mathbb{I}$	$\mathcal{L}^{\tilde{\epsilon}}\tilde{\mathcal{D}}$
$\mathbf{I}^{\epsilon}_{\mathrm{YM}}(+\vec{\boldsymbol{a}}^g_n-)$	$\mathbb{I}$	$\mathcal{T}^{\tilde{\epsilon}}_{+\vec{\boldsymbol{a}}_n-}$
$\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{ssYMS}}(+^s,\vec{\boldsymbol{a}}^s_m,-^s;\boldsymbol{a}^g_{n-m}\Vert+^A,\vec{\boldsymbol{a}}^A_n,-^A)$	$\mathcal{T}^{\epsilon}_{+\vec{\boldsymbol{a}}_n-}$	$\mathcal{T}^{\tilde{\epsilon}}_{+\vec{\boldsymbol{a}}_m-}$
$\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{SYMS}}(\boldsymbol{a}^s_{2m};\boldsymbol{a}^g_{n-m}\Vert+^A,\vec{\boldsymbol{a}}^A_n,-^A)$	$\mathcal{T}^{\epsilon}_{+\vec{\boldsymbol{a}}_n-}$	$\mathcal{T}^{\tilde{\epsilon}}_{\mathcal{X}_{2m}}(N\tilde{\mathcal{D}}+1)$
$\mathbf{I}_{\mathrm{NLSM};\circ}(\vec{\boldsymbol{a}}^s_n)$	$\mathcal{T}^{\epsilon}_{+\vec{\boldsymbol{a}}_n-}$	$\mathcal{L}^{\tilde{\epsilon}}\tilde{\mathcal{D}}$
$\mathbf{I}_{\mathrm{BAS}}(+^s,\vec{\boldsymbol{a}}^s_n,-^s\Vert+^s,\vec{\boldsymbol{s}}^s_n,-^s)$	$\mathcal{T}^{\epsilon}_{+\vec{\boldsymbol{a}}_n-}$	$\mathcal{T}^{\tilde{\epsilon}}_{+\vec{\boldsymbol{s}}_n-}$
$\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{DBI};\circ}(\boldsymbol{a}^s_{2m};\boldsymbol{a}^p_{n-2m})$	$\mathcal{L}^{\epsilon}\mathcal{D}$	$\mathcal{T}^{\tilde{\epsilon}}_{\mathcal{X}_{2m}}(N\tilde{\mathcal{D}}+1)$
$\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{ssEDBI}}(+^s,\vec{\boldsymbol{a}}^s_m,-^s;\boldsymbol{a}^p_{n-2m})$	$\mathcal{L}^{\epsilon}\mathcal{D}$	$\mathcal{T}^{\tilde{\epsilon}}_{+\vec{\boldsymbol{a}}_m-}$
$\mathbf{I}_{\mathrm{SG};\circ}(\boldsymbol{a}^s_n)$	$\mathcal{L}^{\epsilon}\mathcal{D}$	$\mathcal{L}^{\tilde{\epsilon}}\tilde{\mathcal{D}}$

In Table II, ssEYM denotes the special part of the sEYM Feynman integrand that the virtual particle propagating in the loop is only a gluon. Similarly, ssYMS denotes the special sYMS integrand with a virtual scalar in the loop, and ssEDBI is the special EDBI integrand with a virtual scalar in the loop. Integrands $\mathbf{I}_\circ$ with the subscript $\circ$ are full one-loop Feynman integrands, while $\mathbf{I}$ without $\circ$ are partial Feynman integrands, as introduced in the previous subsection. We used $+,\vec{\boldsymbol{a}}_m,-$ and $+,\vec{\boldsymbol{a}}_n,-$ to denote the color orderings of partial integrands, where $+,-$ are external legs carrying k_+ and k_- respectively before taking the forward limit. After doing the cyclic summation over the one-loop level equivalent color orderings, we use $\vec{\boldsymbol{a}}_m$ or $\vec{\boldsymbol{a}}_n$ instead of $+,\vec{\boldsymbol{a}}_m,-$ or $+,\vec{\boldsymbol{a}}_n,-$. For example,

$$\mathbf{I}_{\mathrm{BAS}}(\vec{\boldsymbol{a}}^s_n\Vert+^s,\vec{\boldsymbol{s}}^s_n,-^s)=\sum_{\pi_c}\mathbf{I}_{\mathrm{BAS}}(+^s,\pi_c(\vec{\boldsymbol{a}}^s_n),-^s\Vert+^s,\vec{\boldsymbol{s}}^s_n,-^s),$$
$$\mathbf{I}_{\mathrm{BAS};\circ}(\vec{\boldsymbol{a}}^s_n\Vert\vec{\boldsymbol{s}}^s_n)=\sum_{\pi_c}\mathbf{I}_{\mathrm{BAS}}(\pi_c(\vec{\boldsymbol{a}}^s_n)\Vert+^s,\pi_c(\vec{\boldsymbol{s}}^s_n),-^s), \tag{20}$$

where π_c denotes the cyclic permutation.

In Table II, the one-loop level differential operators are defined as follows. The operator $\mathcal{T}^{\epsilon}_{+\vec{\boldsymbol{a}}_m-}$ is given as

$$\mathcal{T}^{\epsilon}_{+\vec{\boldsymbol{a}}_m-}\equiv\left(\prod_{i=1}^{m-1}\mathcal{I}^{\epsilon}_{+a_ia_{i+1}}\right)\mathcal{I}^{\epsilon}_{+a_m-}\mathcal{D}. \tag{21}$$

The operator $\mathcal{D}$ in (21) is defined in the following way. We think the Lorentz vectors before taking the forward limit as, the momenta in $\{k_1,\ldots,k_n,\ell\}$ and polarization vectors in $\{\epsilon_1,\ldots,\epsilon_n\}$ lie in the d dimensional space where d is regarded as a constant, while the polarization vectors ϵ_+ and ϵ_- are in the D dimensional space where D is regarded as a variable. We can set $D=d$ finally to obtain a physically acceptable object. Then we define

$$\mathcal{D}\equiv\partial_D. \tag{22}$$

For gravitons, we regard $D=\sum_r\epsilon^r_+\cdot\epsilon^r_-+2$ and $\tilde{D}=\sum_r\tilde{\epsilon}^r_+\cdot\tilde{\epsilon}^r_-+2$ as two independent variables, namely, $\partial_D\tilde{D}=0$, $\partial_{\tilde{D}}D=0$. The insertion operators are defined by

$$\mathcal{I}^{\epsilon}_{+a_i-}\equiv\partial_{\epsilon_i\cdot\ell},\qquad\mathcal{I}^{\epsilon}_{+a_ia_{i+1}}\equiv\partial_{\epsilon_{a_i}\cdot\ell}-\partial_{\epsilon_{a_i}\cdot k_{a_{i+1}}}. \tag{23}$$

Here ℓ is understood as k_+, since one can always let $k_-=-\ell$ to be removed, due to the momentum conservation law. The operators $\mathcal{T}^{\epsilon}_{\mathcal{X}_{2m}}$, $\mathcal{T}^{\epsilon}_{\mathcal{X}_{2m}}$ and $\mathcal{L}^{\epsilon}$ are the same as the tree level ones. The number N in the combinatory operator $\mathcal{T}^{\tilde{\epsilon}}_{\mathcal{X}_{2m}}(N\tilde{\mathcal{D}}+1)$ stands for the number of different flavors. When applying $\mathcal{L}^{\epsilon}$ at the one-loop level, the operator $\mathcal{L}^{\epsilon}_i$ should include $\partial_{\epsilon_i\cdot k_+}=\partial_{\epsilon_i\cdot\ell}$.

III. NEW OPERATORS TRANSMUTE GR TO YM AND YM TO BAS

In this section, we construct a new class of differential operators which transmute the GR Feynman integrands to EYM and YM integrands.

In [29], the differential operators transmuting the one-loop GR Feynman integrand to the YM partial ones are constructed as follows. At tree level, the transmutation operator is chosen as

$$\mathcal{T}^{\epsilon}_{+\vec{\boldsymbol{a}}_m-}=\left(\prod_{i=2}^{m}\mathcal{I}^{\epsilon}_{a_{i-1}a_i-}\right)\mathcal{I}^{\epsilon}_{+a_1-}\mathcal{T}^{\epsilon}_{+-}. \tag{24}$$

The trace operator $\mathcal{T}^{\epsilon}_{+-}$ turns the external gravitons $+^h$ and $-^h$ to gluons, and fixes legs $+$ and $-$ at two ends in the color ordering. Then, the insertion operators turn other gravitons to gluons, and insert them between $+$ and $-$ to generate the full color ordering. The obtained color ordered tree amplitude is $\mathcal{A}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\vec{\boldsymbol{a}}^g_n,-^g)$. Based on the above tree level manipulation, by seeking the operator $\mathcal{O}^{\epsilon}_{\circ}$ satisfying $\mathcal{O}^{\epsilon}_{\circ}\mathcal{F}\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}=\mathcal{F}\mathcal{T}^{\epsilon}_{+\vec{\boldsymbol{a}}_m-}\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}$, one can construct the corresponding one-loop level operator $\mathcal{O}^{\epsilon}_{\circ}$ which transmutes the GR integrand $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n)$ to the partial YM one $\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\vec{\boldsymbol{a}}^g_n,-^g)$. However, to generate the tree color ordered YM amplitude $\mathcal{A}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\vec{\boldsymbol{a}}^g_n,-^g)$, the operator (24) is not the only choice. Actually, one can turn arbitrary two gravitons to gluons at the first step, and insert other legs between them to get the desired result. Thus it is natural to ask, if we make the different choices of operators at tree level, what level operators can be constructed at one-loop level? What physical effects will these new operators have when acting on $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n)$?

In this section, we show that choosing two reference legs as one in $\{+,-\}$ and another one in $\boldsymbol{a}_n$ leads to new operators at the one-loop level which transmute $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n)$ to ssEYM and YM integrands. These new operators also link the YM, ssYMS and BAS integrands together, as indicated by the tree level transmutation relations.

A. Construction of new operators

The goal of this subsection is to construct the operator (35) which transmutes the GR Feynman integrands to ssEYM ones as in (39). The YM integrands serve as the special case of ssEYM, as can be reached by taking $\boldsymbol{a}_m = \boldsymbol{a}_n$ in (39). As pointed out in Sec. II B, to transmute the tree GR amplitude $\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n \cup \{+^h,-^h\})$ to tree single-trace EYM (sEYM) amplitude $\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{EYM}}(+,\vec{\boldsymbol{a}}^g_m,-;\boldsymbol{a}^h_n\backslash\boldsymbol{a}^h_m)$, the choices for transmutation operator $\mathcal{T}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ are not unique. Here we still restrict ourselves to the choices that the color ordering is created by generating two reference legs at two ends, and inserting other legs between them. However, for $\boldsymbol{a}^h_m \neq \emptyset$, we use the cyclic symmetry of color orderings to rewrite $\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{EYM}}(+^g,\vec{\boldsymbol{a}}^g_m,-^g;\boldsymbol{a}^h_n\backslash\boldsymbol{a}^h_m)$ as $\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{EYM}}(\vec{\boldsymbol{a}}^g_m,-^g,+^g;\boldsymbol{a}^h_n\backslash\boldsymbol{a}^h_m)$. The new representation of color ordering indicates the new choice of the tree level operator,

$$\mathcal{T}^{\epsilon}_{\vec{\boldsymbol{a}}_m-+} = \mathcal{I}^{\epsilon}_{a_m-+}\left(\prod_{i=2}^{m}\mathcal{I}^{\epsilon}_{a_{i-1}a_i+}\right)\mathcal{T}^{\epsilon}_{a_1+}. \tag{25}$$

Then we seek the operator $\mathcal{O}^{\epsilon}_{\circ}$ satisfying

$$\mathcal{O}^{\epsilon}_{\circ}\mathcal{F}\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}} = \mathcal{F}\mathcal{T}^{\epsilon}_{\vec{\boldsymbol{a}}_m-+}\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}. \tag{26}$$

The difference between the two choices (24) and (25) is quite trivial at the tree level. However, since the forward limit glues legs $+$ and $-$ together to create the loop, the new choice (25) leads to totally new operators at the one-loop level. For tree sEYM amplitudes with at least four external gluons, the above method leads to well defined and physically meaningful operators. For tree sEYM amplitudes containing only three external gluons, this method does not make sense, as explained in Appendix B.

Thus we start with the four-gluon tree amplitude $\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{sEYM}}(+^g,a^g,b^g,-^g;\boldsymbol{a}^h_n\backslash\{a^h,b^h\})$. Using the cyclic symmetry of color orderings, as well as the tree level differential operators, we have

$$\begin{aligned}&\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{sEYM}}(a^g,b^g,-^g,+^g;\boldsymbol{a}^h_n\backslash\{a^h,b^h\})\\&=\mathcal{I}^{\epsilon}_{b-+}\mathcal{I}^{\epsilon}_{ab+}\mathcal{T}^{\epsilon}_{a+}\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n\cup\{+^h,-^h\})\\&=(\partial_{\epsilon_-\cdot k_b}-\partial_{\epsilon_-\cdot\ell})(\partial_{\epsilon_b\cdot k_a}-\partial_{\epsilon_b\cdot\ell})\partial_{\epsilon_a\cdot\epsilon_+}\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n\cup\{+^h,-^h\}).\end{aligned} \tag{27}$$

The above manipulation is understood as turning gravitons a^h and $+^h$ to gluons and regarding them as the reference legs, then turning the graviton b^h to a gluon and inserting it between a and $+$, and turning the graviton $-^h$ to a gluon and inserting it between b and $+$ finally. The object $\epsilon_-\cdot\ell$ vanishes under the action of $\mathcal{L}$, since $\epsilon_-\cdot k_-=0$ and $k_-=-k_+=-\ell$, thus we focus on the $\partial_{\epsilon_-\cdot k_b}$ part in $\mathcal{I}^{\epsilon}_{b-+}$. At the tree level, the combinatory operator $\partial_{\epsilon_-\cdot k_b}\partial_{\epsilon_a\cdot\epsilon_+}$ turns $(\epsilon_-\cdot k_b)(\epsilon_a\cdot\epsilon_+)$ to 1, and annihilates all terms which do not contain $(\epsilon_-\cdot k_b)(\epsilon_a\cdot\epsilon_+)$. Under the action of $\mathcal{E}$, the object $(\epsilon_-\cdot k_b)(\epsilon_a\cdot\epsilon_+)$ behaves as

$$\sum_r(\epsilon_a\cdot\epsilon^r_+)(\epsilon^r_-\cdot k_b)=\epsilon_a\cdot k_b, \tag{28}$$

thus the effect of the operator $\partial_{\epsilon_-\cdot k_b}\partial_{\epsilon_a\cdot\epsilon_+}$ at the tree level is equivalent to $\partial_{\epsilon_a\rhd k_b}$ at the one-loop level. In other words, we find

$$\begin{aligned}&\partial_{\epsilon_a\rhd k_b}\mathcal{F}\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n\cup\{+^h,-^h\})\\&\quad=\mathcal{F}\mathcal{I}^{\epsilon}_{b-+}\mathcal{T}^{\epsilon}_{a+}\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n\cup\{+^h,-^h\}).\end{aligned} \tag{29}$$

From now on, we use $A\rhd B$ to denote $A\cdot B$ arises from $\sum_r(A\cdot\epsilon^r_+)(\epsilon^r_-\cdot B)$, and $A\lhd B$ to denote $A\cdot B$ from $\sum_r(A\cdot\epsilon^r_-)(\epsilon^r_+\cdot B)$. Notice that in general the summation over $\epsilon^r_+\epsilon^r_-$ should be

$$\sum_r(\epsilon^r_+)^\mu(\epsilon^r_-)^\nu=\eta^{\mu\nu}-\frac{\ell^\mu q^\nu+\ell^\nu q^\mu}{\ell\cdot q}, \tag{30}$$

where the null q satisfies $\epsilon^r_+\cdot q=\epsilon^r_-\cdot q=0$. Here we are allowed to drop the q-dependent terms, since their contributions vanish on the solution to the scattering equations, see in [42]. When the contribution from summing over $\epsilon^r_+\epsilon^r_-$ is included in $\epsilon_a\rhd k_b$, the operator $\mathcal{I}^{\epsilon}_{ab+}$ cannot act on $\epsilon_b\rhd k_a$, $\epsilon_b\rhd k_l$, or $\epsilon_b\lhd k_a$, $\epsilon_b\lhd k_l$, therefore is commutable with $\mathcal{F}$. Then we arrive at the relation

$$\begin{aligned}&\mathcal{I}^{\epsilon}_{ab+}\partial_{\epsilon_a\rhd k_b}\mathcal{F}\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n\cup\{+^h,-^h\})\\&\quad=\mathcal{F}\mathcal{I}^{\epsilon}_{b-+}\mathcal{I}^{\epsilon}_{ab+}\mathcal{T}^{\epsilon}_{a+}\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n\cup\{+^h,-^h\}),\end{aligned} \tag{31}$$

which implies

$$\mathcal{I}^{\epsilon}_{ab+}\partial_{\epsilon_a\rhd k_b}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n)=\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM}}(+^g,a^g,b^g,-^g;\boldsymbol{a}^h_n\backslash\{a^h,b^h\}). \tag{32}$$

Here $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM}}$ denotes the special single-trace EYM partial Feynman integrand with a virtual gluon running in the loop. The reason for interpreting the rhs of (32) as such a special sEYM Feynman integrand is as follows. The EYM theory includes three kinds of interaction vertices in Fig. 2, which indicate that for the tree sEYM amplitude including external gluons $+^g$ and $-^g$, one can always start from one of them, go along the gluon lines, and arrive at another one. It means, after gluing legs $+^g$ and $-^g$ together, the

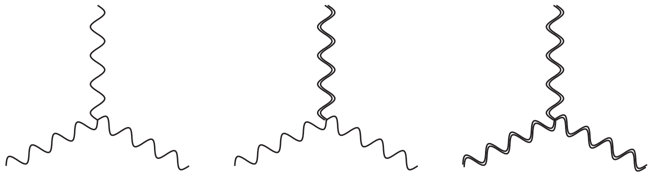

FIG. 2. Three vertices of EYM theory, the single wavy lines denote gluons while the double wavy lines denote gravitons.

obtained loop contains only gluon lines. This observation fixes the virtual particle in the loop to be a gluon.

The operator $\mathcal{I}^{\epsilon}_{ab+}\partial_{\epsilon_a \rhd k_b}$ can be simplified to $\partial_{\epsilon_b\cdot k_a}\partial_{\epsilon_a \rhd k_b}$, since $\partial_{\epsilon_b\cdot \ell}\partial_{\epsilon_a \rhd k_b}$ gives no contribution at the one-loop level, as proved in Appendix B by using the CHY formulas introduced in Appendix A. This observation simplifies the relation (32) to

$$\partial_{\epsilon_b\cdot k_a}\partial_{\epsilon_a \rhd k_b}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n) = \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM}}(+^g, a^g, b^g, -^g; \boldsymbol{a}^h_n\setminus\{a^h, b^h\}). \tag{33}$$

The generalization to the cases with more external gluons is straightforward. Along the similar line, one arrives at the relation

$$\left(\prod_{i=1}^{m-1}\partial_{\epsilon_{a_{i+1}}\cdot k_{a_i}}\right)\partial_{\epsilon_{a_1}\rhd k_{a_m}}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n) = \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM}}(+^g, a^g_1, \ldots, a^g_m, -^g; \boldsymbol{a}^h_n\setminus\{a^h_1, \ldots, a^h_m\}), \tag{34}$$

with $m \geq 2$.

However, in the one-loop integrand $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n)$, the tree level information associated to ϵ^r_+ and ϵ^r_- is lost, thus $\epsilon_a \rhd k_b$ or $\epsilon_a \lhd k_b$ cannot be distinguished from the original $\epsilon_a\cdot k_b$ included in the tree amplitude. It means the operator $\partial_{\epsilon_a\rhd k_b}$ is not well defined at the one-loop level, and we should replace it by $\partial_{\epsilon_a\cdot k_b}$. Motivated by the relation (34), it is natural to act the operator $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ on $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n)$, where the cyclical operator $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ for the ordered set $\vec{\boldsymbol{a}}_m = \langle a_1, \ldots, a_m\rangle$ with $m \geq 2$ is defined as

$$\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m} \equiv \prod_{i=1}^{m}\partial_{k_{a_i}\cdot\epsilon_{a_{i+1}}} = \partial_{C^{\epsilon}_{\vec{\boldsymbol{a}}_m}}, \tag{35}$$

where

$$C^{\epsilon}_{\vec{\boldsymbol{a}}_m} = \prod_{i=1}^{m} k_{a_i}\cdot\epsilon_{a_{i+1}}. \tag{36}$$

The second equality in (35) holds as long as the Feynman integrand is linear in each polarization vector. When saying this is a cyclical operator, we mean $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ is invariant under the arbitrary cyclic permutation of the ordered set $\vec{\boldsymbol{a}}_m$.

Thus we need to figure out the effect of acting $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ on $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n)$. The operator $\partial_{\epsilon_{a_1}\cdot k_{a_m}}$ acts on both $\epsilon_{a_1}\rhd k_{a_m}$ and $\epsilon_{a_1}\lhd k_{a_m}$, as well as ordinary $\epsilon_{a_1}\cdot k_{a_m}$ in $\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n\cup\{+^h, -^h\})$. The effect of acting $\partial_{\epsilon_{a_1}\cdot k_{a_m}}$ on $\epsilon_{a_1}\lhd k_{a_m}$ in $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n)$ is equivalent to acting $\partial_{\epsilon_{a_1}\lhd k_{a_m}}$ on $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n)$. Following the similar method for obtaining (34), we find

$$\left(\prod_{i=1}^{m-1}\partial_{\epsilon_{a_{i+1}}\cdot k_{a_i}}\right)\partial_{\epsilon_{a_1}\lhd k_{a_m}}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n) = (-)^m\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM}}(+^g, a^g_m, \ldots, a^g_1, -^g; \boldsymbol{a}^h_n\setminus\boldsymbol{a}^h_m), \tag{37}$$

with $m \geq 2$. The derivation can be seen in Appendix B. On the other hand, the cyclical operator $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ annihilates $\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n\cup\{+^h, -^h\})$, i.e.,

$$\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n\cup\{+^h, -^h\}) = 0, \tag{38}$$

which means one need not consider the case $\partial_{\epsilon_{a_1}\cdot k_{a_m}}$ acts on ordinary $\epsilon_{a_1}\cdot k_{a_m}$ in $\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n\cup\{+^h, -^h\})$. The proof of the equality (38) is also provided in Appendix B.

With the results (34), (37) and (38), now we can determine the resulting object of acting $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ on $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n)$. Since there is only one ϵ_+ and one ϵ_- at the tree level, among m operators $\partial_{\epsilon_{a_i}\cdot k_{a_{i-1}}}$, at most one of them can act on $\epsilon_{a_i}\rhd k_{a_{i-1}}$ or $\epsilon_{a_i}\lhd k_{a_{i-1}}$, the remaining operators must act on the original $\epsilon_{a_i}\cdot k_{a_{i-1}}$ in $\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n\cup\{+^h, -^h\})$. On the other hand, if none of them acts on $\epsilon_{a_i}\rhd k_{a_{i-1}}$ or $\epsilon_{a_i}\lhd k_{a_{i-1}}$, the equality (38) indicates the vanishing of the result. Thus, the nonvanishing contributions arise from acting $(\prod_{j=i}^{i-2}\partial_{\epsilon_{a_{j+1}}\cdot k_{a_j}})\partial_{\epsilon_{a_i}\rhd k_{a_{i-1}}}$, as well as $(\prod_{j=i}^{i-2}\partial_{\epsilon_{a_{j+1}}\cdot k_{a_j}})\partial_{\epsilon_{a_i}\lhd k_{a_{i-1}}}$, for all $i\in\{1, \ldots, m\}$. Applying (34) and (37), we finally find

$$\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n) = \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\vec{\boldsymbol{a}}^g_m; \boldsymbol{a}^h_n\setminus\boldsymbol{a}^h_m) + (-)^m\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\overleftarrow{\boldsymbol{a}}^g_m; \boldsymbol{a}^h_n\setminus\boldsymbol{a}^h_m), \quad \text{with } m\geq 2, \tag{39}$$

where $\overleftarrow{\boldsymbol{a}}_m$ is the reversed set of $\vec{\boldsymbol{a}}_m$, i.e., $\overleftarrow{\boldsymbol{a}}_m = \langle a_m, \ldots, a_1\rangle$, and

$$\begin{aligned}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\vec{\boldsymbol{a}}^g_m; \boldsymbol{a}^h_n\setminus\boldsymbol{a}^h_m) &= \sum_{\pi_c}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM}}(+, \pi_c(\vec{\boldsymbol{a}}^g_m), -; \boldsymbol{a}^h_n\setminus\boldsymbol{a}^h_m),\\ \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\overleftarrow{\boldsymbol{a}}^g_m; \boldsymbol{a}^h_n\setminus\boldsymbol{a}^h_m) &= \sum_{\pi_c}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM}}(+, \pi_c(\overleftarrow{\boldsymbol{a}}^g_m), -; \boldsymbol{a}^h_n\setminus\boldsymbol{a}^h_m),\end{aligned} \tag{40}$$

where π_c are the cyclic permutations. Now we have found the one-loop level operators $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$, which transmute the GR

Feynman integrands to the ssEYM ones, formally expressed in (39). The rhs of (39) is invariant under the cyclic permutation of $\vec{\boldsymbol{a}}_m$, as indicated by the cyclic symmetry of the operator $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$. We emphasize that $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM};\circ}(\vec{\boldsymbol{a}}^g_m;\boldsymbol{a}^h_n\backslash\boldsymbol{a}^h_m)$ and $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM};\circ}(\overleftarrow{\boldsymbol{a}}^g_m;\boldsymbol{a}^h_n\backslash\boldsymbol{a}^h_m)$ appearing in (39) are full one-loop color ordered ssEYM integrands, rather than partial ones without the cyclic summation. The transmutation relation (39) can be verified by using the CHY formulas, as shown in Appendix C.

Some discussions are in order. First, the operators $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ do not act on any Lorentz invariants that include the loop momentum ℓ, thus are commutable with the integration of ℓ. This observation implies that the relation (39) holds at not only the integrand level, but also the integral level.

Second, the operator $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ preserves the gauge invariance. To see this, we consider the following operator for the external leg a_i, defined based on the Ward's identity:

$$\mathcal{W}^{\epsilon}_{a_i} \equiv \sum_V k_{a_i}\cdot V \partial_{\epsilon_{a_i}\cdot V}, \tag{41}$$

where V denotes either momenta or polarization vectors contract with ϵ_{a_i}. Any gauge invariant object should be annihilated by this operator. Suppose the object A is gauge invariant for the leg a_i, i.e., $\mathcal{W}^{\epsilon}_{a_i}A=0$, the commutator $[\mathcal{W}^{\epsilon}_{a_i},\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}]=0$ indicates

$$\mathcal{W}^{\epsilon}_{a_i}\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}A=0. \tag{42}$$

Third, at the tree level, the operators, which transmute the tree GR amplitudes to the tree sEYM amplitudes, also transmute the color ordered YM ones to the sYMS ones, as can be seen in Table I. Thus, replacing $\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\text{GR}}$ by $\mathcal{A}^{\epsilon}_{\text{YM}}$ in (26), we see the operators $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ also transmute the one-loop YM partial Feynman integrands to the ssYMS ones,

$$\begin{aligned}\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}\mathbf{I}^{\epsilon}_{\text{YM}}(+^g,\vec{\boldsymbol{a}}^g_n,-^g)=&\,\mathbf{I}^{\epsilon}_{\text{ssYMS}}(\vec{\boldsymbol{a}}^s_m;\boldsymbol{a}^g_n\backslash\boldsymbol{a}^g_m\|+^A,\vec{\boldsymbol{a}}^A_n,-^A)\\&+(-)^m\mathbf{I}^{\epsilon}_{\text{ssYMS}}(\overleftarrow{\boldsymbol{a}}^s_m;\boldsymbol{a}^g_n\backslash\boldsymbol{a}^g_m\|+^A,\vec{\boldsymbol{a}}^A_n,-^A).\end{aligned} \tag{43}$$

Doing the cyclic summation for the color orderings $+,\vec{\boldsymbol{a}}_n,-$, one obtains

$$\begin{aligned}\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}\mathbf{I}^{\epsilon}_{\text{YM};\circ}(\vec{\boldsymbol{a}}^g_n)=&\,\mathbf{I}^{\epsilon}_{\text{ssYMS};\circ}(\vec{\boldsymbol{a}}^s_m;\boldsymbol{a}^g_n\backslash\boldsymbol{a}^g_m\|\vec{\boldsymbol{a}}^A_n)\\&+(-)^m\mathbf{I}^{\epsilon}_{\text{ssYMS};\circ}(\overleftarrow{\boldsymbol{a}}^s_m;\boldsymbol{a}^g_n\backslash\boldsymbol{a}^g_m\|\vec{\boldsymbol{a}}^A_n),\end{aligned} \tag{44}$$

which links the full color ordered YM and ssYMS integrands together.

IV. EXPANDING GR AND EYM TO YM

In this section, we demonstrate that the one-loop level transmutation relations naturally lead to the expansions of one-loop EYM and GR Feynman integrands to YM ones. The main goal of this section is the expansions (75) and (76), as well as the rules for evaluating coefficients $\mathbf{C}^{\epsilon}_1(\sigma)$ and $\mathbf{C}^{\epsilon}_2(\sigma,\vec{\boldsymbol{a}}_m)$ in them.

At the tree level, the unifying relations among different theories can also be represented by expansions, i.e., the amplitude of one theory can be expanded to amplitudes of another theory [17–26,38]. In particular, all theories in Table I can be expanded to BAS amplitudes, with double copied coefficients [28]. The unified web for expansions serves as the dual version of the web for transmutation relations [28]. Inspired by the tree level story, an interesting question is, can the unified web for expansions at one-loop level be constructed from the one-loop transmutation relations, together with other appropriate general principles and assumptions? The answer is positive, as will be shown in this and the next sections.

Here we list the principles and assumptions beside differential operators, which will be used to solve expansions:

(i) Lorentz invariance
(ii) Gauge invariance
(iii) Property of GR Feynman integrands: We assume each external graviton i^h carries the polarization tensor $\epsilon^{\mu\nu}_i=\epsilon^{\mu}_i\tilde{\epsilon}^{\nu}_i$, and the GR integrands carry no color ordering.
(iv) Double-copy structure: We assume each polarization vector in the set $\{\epsilon_i\}$ cannot contract with another polarization vector in the set $\{\tilde{\epsilon}_i\}$, and vice versa.
(v) On-shell condition: We assume $\epsilon_i\cdot k_i=0$ for each external leg i.
(vi) Linearity in ϵ_i: We assume the Feynman integrand is linear in each polarization vector ϵ_i.
(vii) Forward limit: We assume the one-loop integrands can be generated from the tree amplitudes via the forward limit.

The first six principles/assumptions are also used for deriving the expansions of tree amplitudes, while the last one only makes sense at the one-loop level. Here we give a brief discussion about the third assumption. It seems that one should make similar assumptions for other theories, but indeed it is not necessary, since the assumption for the GR integrands, together with differential operators, uniquely fix the information about polarization vectors and color orderings for the Feynman integrands of other theories. For example, the relation

$$\mathcal{T}^{\epsilon}_{+\vec{\boldsymbol{a}}^g_n-}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{GR}}(\boldsymbol{a}^h_n)=\mathbf{I}^{\tilde{\epsilon}}_{\text{YM}}(+^g,\vec{\boldsymbol{a}}^g_n,-^g) \tag{45}$$

indicates each external gluon i^g carries the polarization vector $\tilde{\epsilon}_i$, and the YM partial integrand carries the color ordering $+,\vec{\boldsymbol{a}}_n,-$, as long as each graviton carries $\epsilon_i\tilde{\epsilon}_i$ and the GR integrand carries no color ordering. As will be seen soon, the information carried by transmutation operators, together with the seven general principles/assumptions mentioned above, fully determine the expansions of ssEYM and GR integrands to YM ones, with polynomial coefficients.

It is worth classifying the basis for expansions. At the tree level, for YM amplitudes with $n+2$ external legs, one can take the basis as $n!$ color ordered YM amplitudes with two legs fixed at two ends in the color orderings, and expand the EYM and GR amplitudes to these YM amplitudes, with polynomial coefficients. Such basis is called the KK basis, since its completeness is ensured by the Kleiss-Kuijf relation [41]. Suppose we fix legs $+$ and $-$ at two ends in the color orderings to obtain the tree level KK basis; taking the forward limit naturally leads to the one-loop KK basis including YM partial Feynman integrands $\mathbf{I}^{\tilde{\epsilon}}_{\rm YM}(+^g,\sigma(\vec{\boldsymbol{a}}^g_n),-^g)$, where σ denotes the permutations. This is the choice of basis in the current section. In the next section, we will generalize the one-loop KK basis to color ordered partial integrands of other theories.

The main idea in this section is as follows. Suppose a one-loop level operator $\mathcal{O}_\circ$ transmutes the Feynman integrand of theory A to that of theory B, i.e.,

$$\mathcal{O}_\circ \mathbf{I}_A = \mathbf{I}_B. \tag{46}$$

We regard (46) as a differential equation, rather than a transmutation relation. Then, one can solve this equation to get $\mathbf{I}_A$. The general feature is $\mathbf{I}_A$ contains terms which are annihilated by the operator $\mathcal{O}_\circ$; these terms cannot be fixed by solving the differential equation. Terms vanishing under the action of $\mathcal{O}_\circ$ are called undetectable terms for the operator $\mathcal{O}_\circ$. The undetectable terms should be determined via other conditions, such as imposing the gauge invariance requirement, and so on. By applying the method described above, in Sec. IV A, we solve the recursive expansions of ssEYM and GR Feynman integrands to ssEYM partial integrands with less external gravitons. In Sec. IV B, we use the results in Sec. IV A to get the expansions of GR and ssEYM integrands to YM ones, and give the rules for constructing the coefficients.

A. Recursive expansions of EYM and GR

This subsection focuses on recursive expansions of ssEYM and GR Feynman integrands implied by the one-loop level differential operators. Since the technique for treating ssEYM bears strong similarity with the approach for solving the expansions of the tree sEYM amplitudes in [26], we only give the resulting expanded formula. For the readers' convenience, the details are provided in Appendix D. On the other hand, the expansions of GR integrands to EYM ones will be discussed in detail, since the process has no analog at tree level.

The recursive expansion, which decomposes the ssEYM Feynman integrands to ssEYM ones with less external gravitons, is found to be

$$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^h_m)=\sum_{\vec{\boldsymbol{s}}:\boldsymbol{s}\subseteq\boldsymbol{a}^h_m\setminus h_m}\sum_{\shuffle}K^{\epsilon}_{\vec{\boldsymbol{s}}}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM}(+^g,\langle\vec{\boldsymbol{s}}^g,h^g_m\rangle\shuffle\vec{\boldsymbol{a}}^g_{n-m},-;\boldsymbol{a}^h_m\setminus\{h^h_m,\boldsymbol{s}^h\}). \tag{47}$$

The lhs is the partial ssEYM integrand with $n-m$ external gluons and m external gravitons. The color ordered set of gluons is labeled as $\vec{\boldsymbol{a}}_{n-m}=\langle 1,\ldots,n-m\rangle$, while the set of gravitons is labeled as $\boldsymbol{a}^h_m=\{h_1,\ldots,h_m\}$. The first summation is over all ordered sets $\vec{\boldsymbol{s}}$ with $\boldsymbol{s}\subseteq\boldsymbol{a}^h_m\setminus h_m$; here $\boldsymbol{s}$ is allowed to be empty. The second summation over possible shuffles $\shuffle$ of two ordered sets $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{s}}$ is the summation over all permutations of $\vec{\boldsymbol{a}}\cup\vec{\boldsymbol{s}}$ those preserving the orderings of $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{s}}$. For example, $\langle 1,2\rangle\shuffle\langle 3,4\rangle$ includes the following ordered sets: $\langle 2,3,4,5\rangle$, $\langle 2,4,3,5\rangle$, $\langle 2,4,5,3\rangle$, $\langle 4,2,3,5\rangle$, $\langle 4,2,5,3\rangle$, $\langle 4,5,2,3\rangle$. The kinematic factor $K^{\epsilon}_{\vec{\boldsymbol{s}}}$ is defined as

$$K^{\epsilon}_{\vec{\boldsymbol{s}}}=\epsilon_{h_m}\cdot f_{s_{|\boldsymbol{s}|}}\cdots f_{s_1}\cdot Y_{s_1}, \tag{48}$$

for any $\vec{\boldsymbol{s}}=\langle s_1,\ldots,s_{|\boldsymbol{s}|}\rangle$, where the antisymmetric strength tensors are defined as

$$f^{\mu\nu}_i\equiv k^\mu_i\epsilon^\nu_i-\epsilon^\mu_i k^\nu_i,\qquad \tilde{f}^{\mu\nu}_i\equiv k^\mu_i\tilde{\epsilon}^\nu_i-\tilde{\epsilon}^\mu_i k^\nu_i. \tag{49}$$

The combinatory momentum Y_i is defined as the summation of momenta of gluons at the lhs of the leg i^g in the color ordering [22].

Using the expansion (47) recursively, one can expand any ssEYM partial integrand to YM ones; the coefficients of these YM partial integrands will be studied in Sec. IV B.

Now we turn to the expansions of the GR Feynman integrands. The idea is to decompose the integrand into some independent Lorentz invariants, and solve the coefficients of such Lorentz invariants via differential operators. To find the proper decomposition, we use the conclusion in [43] that if one imposes the gauge invariance for all external legs, then the tree GR amplitude $\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\rm GR}(\boldsymbol{a}^h_n\cup\{+^h,-^h\})$ can always be decomposed into Lorentz invariants in the form

$$\omega^\epsilon(+^h,\vec{\boldsymbol{a}}^h,-^h|\text{signs})\equiv\epsilon_+\cdot v_{a_1}\left(\prod_{i=2}^{|\boldsymbol{a}|}\bar{v}_{a_{i-1}}\cdot v_{a_i}\right)\bar{v}_{a_{|\boldsymbol{a}|}}\cdot\epsilon_-. \tag{50}$$

Here $\boldsymbol{a}$ is a subset of $\boldsymbol{a}^h_n$ which is allowed to be empty, v_i denotes either k_i or ϵ_i, with $\bar{v}_i$ the other one, i.e.,

$(v_i, \bar{v}_i) = (k_i, \epsilon_i)$ or (ϵ_i, k_i), and the first/second choice is denoted by a $+$ or $-$ sign. The proof of this formula is only based on general considerations for tree amplitudes, which are the first six of seven principles/assumptions mentioned at the beginning of this section, as well as counting the number of mass dimensions. We emphasize that the power counting can be made without using Feynman rules, CHY formulas, or other tools, since the transmutation relation

$$\mathcal{T}^{\epsilon}_{+,\vec{a}_n,-}\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\rm GR}(\boldsymbol{a}^h_n \cup \{+^h,-^h\}) = \mathcal{A}^{\tilde{\epsilon}}_{\rm YM}(\vec{\boldsymbol{a}}^g_n \cup \{+^g,-^g\}) \tag{51}$$

is sufficient to determine

$$\dim(\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\rm GR}(\boldsymbol{a}^h_n \cup \{+^h,-^h\})) = \dim(\mathcal{A}^{\tilde{\epsilon}}_{\rm YM}(\vec{\boldsymbol{a}}^g_n \cup \{+^g,-^g\})) + n, \tag{52}$$

and only the number n is necessary for constructing $\omega^{\epsilon}(+^h, \vec{\boldsymbol{a}}^h, -^h|\text{signs})$ in (50). When taking the forward limit, the Lorentz invariants $\omega^{\epsilon}(+^h, \vec{\boldsymbol{a}}^h, -^h)$ behave as

$$\sum_r \epsilon^r_+ \cdot \epsilon^r_- = d-2, \quad \text{if } \boldsymbol{a} = \emptyset, \tag{53}$$

and

$$\sum_r \epsilon^r_+ \cdot v_{a_1}\left(\prod_{i=2}^{|\boldsymbol{a}|} \bar{v}_{a_{i-1}} \cdot v_{a_i}\right) \bar{v}_{a_{|\boldsymbol{a}|}} \cdot \epsilon^r_- = \prod_{i=1}^{|\boldsymbol{a}|} \bar{v}_{a_{i-1}} \cdot v_{a_i}, \quad \text{if } \boldsymbol{a} \neq \emptyset. \tag{54}$$

Thus, the one-loop GR integrand can be decomposed as

$$\begin{aligned}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm GR};\circ}(\boldsymbol{a}^h_n) &= (d-2)L^{\epsilon,\tilde{\epsilon}}_{\emptyset} + \sum_{\vec{\boldsymbol{a}}/\pi_c}\left(\prod_{i=1}^{|\boldsymbol{a}|} \bar{v}_{a_{i-1}} \cdot v_{a_i}\right) L^{\epsilon,\tilde{\epsilon}}_{\vec{\boldsymbol{a}}}(\text{sign}) \\ &= (d-2)L^{\epsilon,\tilde{\epsilon}}_{\emptyset} + \sum_{\vec{\boldsymbol{a}}/\pi_c} C^{\epsilon}_{\vec{\boldsymbol{a}}} L^{\epsilon,\tilde{\epsilon}}_{\vec{\boldsymbol{a}}}(-) + R.\end{aligned} \tag{55}$$

In the above formula, the summation is over all subsets $\boldsymbol{a} \subseteq \boldsymbol{a}^h_n$ satisfying $\boldsymbol{a} \neq \emptyset$, and all uncyclic permutations for each $\vec{\boldsymbol{a}}$. In the second line, we have collected together terms contain $C^{\epsilon}_{\vec{\boldsymbol{a}}}$, and denoted the remaining terms by R. The factor $C^{\epsilon}_{\vec{\boldsymbol{a}}}$ defined in (36) satisfies the form $\prod_{i=1}^{|\boldsymbol{a}|} \bar{v}_{a_{i-1}} \cdot v_{a_i}$, with the choice $(v_i, \bar{v}_i) = (\epsilon_i, k_i)$ for each i, i.e., the sign is $-$ for each i. This is the reason why we use the sign $-$ in $L^{\epsilon,\tilde{\epsilon}}_{\vec{\boldsymbol{a}}}(-)$. The reason for organizing $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm GR};\circ}(\boldsymbol{a}^h_n)$ as in the second line is that the term containing $(d-2)$ is detectable for the operator $\mathcal{D}$, while terms containing $C^{\epsilon}_{\vec{\boldsymbol{a}}}$ are detectable for the operators $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$.

The coefficients $L^{\epsilon,\tilde{\epsilon}}_{\emptyset}$ and $L^{\epsilon,\tilde{\epsilon}}_{\vec{\boldsymbol{a}}}(-)$ can be determined via the transmutation relation (39), as well as the relation

$$\mathcal{D}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm GR};\circ}(\boldsymbol{a}^h_n) = \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm ssEYM};\circ}(\boldsymbol{a}^h_n) \tag{56}$$

which is the special case with $\boldsymbol{a}_m = \emptyset$ in the second line of Table II. Here we have used the observation $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm ssEYM};\circ}(\boldsymbol{a}^h_n) = \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM}(+^g, -^g; \boldsymbol{a}^h_n)$, since no summation over cyclic permutations is required when $\boldsymbol{a}_m = \emptyset$. Applying (56) fixes $L^{\epsilon,\tilde{\epsilon}}_{\emptyset}$ to be[4]

$$L^{\epsilon,\tilde{\epsilon}}_{\emptyset} = \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm ssEYM};\circ}(\boldsymbol{a}^h_n). \tag{57}$$

The transmutation relation (39) indicates that

$$\begin{aligned}&C^{\epsilon}_{\vec{\boldsymbol{a}}}[\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm ssEYM};\circ}(\vec{\boldsymbol{a}}^g; \boldsymbol{a}^h_n \backslash \boldsymbol{a}^h) \\ &\quad + (-)^{|\boldsymbol{a}|}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm ssEYM};\circ}(\overleftarrow{\boldsymbol{a}}^g; \boldsymbol{a}^h_n \backslash \boldsymbol{a}^h)] \in \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm GR};\circ}(\boldsymbol{a}^h_n).\end{aligned} \tag{58}$$

This motivates us to identify $L^{\epsilon,\tilde{\epsilon}}_{\vec{\boldsymbol{a}}}(-)$ as

$$L^{\epsilon,\tilde{\epsilon}}_{\vec{\boldsymbol{a}}}(-) = \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm ssEYM};\circ}(\vec{\boldsymbol{a}}^g; \boldsymbol{a}^h_n \backslash \boldsymbol{a}^h) + (-)^{|\boldsymbol{a}|}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm ssEYM};\circ}(\overleftarrow{\boldsymbol{a}}^g; \boldsymbol{a}^h_n \backslash \boldsymbol{a}^h), \tag{59}$$

and arrive at

$$\begin{aligned}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm GR};\circ}(\boldsymbol{a}^h_n) &= (d-2)\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm ssEYM};\circ}(\boldsymbol{a}^h_n) + \sum_{\vec{\boldsymbol{a}}/\pi_c} C^{\epsilon}_{\vec{\boldsymbol{a}}}[\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm ssEYM};\circ}(\vec{\boldsymbol{a}}^g; \boldsymbol{a}^h_n \backslash \boldsymbol{a}^h) \\ &\quad + (-)^{|\boldsymbol{a}|}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{{\rm ssEYM};\circ}(\overleftarrow{\boldsymbol{a}}^g; \boldsymbol{a}^h_n \backslash \boldsymbol{a}^h)] + R.\end{aligned} \tag{60}$$

The above formula is the correct solution to Eqs. (56) and (39) if $\mathcal{D}R = 0$ and $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m} R = 0$, as verified in Appendix E.

The remaining part R can be determined by imposing the gauge invariance. Here we employ the Ward's identity operator defined in (41). If an object P is gauge invariant, i.e., $\mathcal{W}^{\epsilon}_i P = 0$, then we have

$$\partial_{\epsilon_q \cdot k_i}(\mathcal{W}^{\epsilon}_i P) = 0, \quad \text{for } \forall\, q, \tag{61}$$

therefore

$$\begin{aligned}0 &= (\partial_{\epsilon_q \cdot k_i}\mathcal{W}^{\epsilon}_i)P + \mathcal{W}^{\epsilon}_i(\partial_{\epsilon_q \cdot k_i}P) \\ &= \partial_{\epsilon_q \cdot \epsilon_i}P + \mathcal{W}^{\epsilon}_i(\partial_{\epsilon_q \cdot k_i}P).\end{aligned} \tag{62}$$

If we restrict our attention to amplitudes and Feynman integrands, we can require P to be linear in each polarization vector. Under this condition, one can immediately conclude that

$$-(\epsilon_q \cdot \epsilon_i)(\mathcal{W}^{\epsilon}_i(\partial_{\epsilon_q \cdot k_i}P)) \in P. \tag{63}$$

Now we apply the gauge invariance condition (63) to terms in (60). Manifestly,

[4]Here we used the observation that the parameter d only arises from $\sum_r \epsilon^r_+ \cdot \epsilon^r_-$, since no other Lorentz invariant depends on the number of dimensions of space-time explicitly.

$$\mathcal{W}_i^\epsilon[(d-2)\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\boldsymbol{a}_n^h)]=0,$$
$$\mathcal{W}_i^\epsilon[C^\epsilon_{\vec{\boldsymbol{a}}}[\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h)+(-)^{|\boldsymbol{a}|}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\overleftarrow{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h)]]=0,$$
$$\text{if } i\in\boldsymbol{a}_n\backslash\boldsymbol{a}, \tag{64}$$

due to the gauge invariance of the Feynman integrands $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\boldsymbol{a}_n^h)$, $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h)$ and $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\overleftarrow{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h)$. Thus, for the above parts, condition (63) is satisfied automatically. Then we move to the case $i\in\boldsymbol{a}$. Based on the definition of the coefficients $C^\epsilon_{\vec{\boldsymbol{a}}}$ in (36), we observe the following reorganization:

$$\sum_{\substack{\vec{\boldsymbol{a}}/\pi_c\\ i\in\boldsymbol{a}}} C^\epsilon_{\vec{\boldsymbol{a}}}[\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h)+(-)^{|\boldsymbol{a}|}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\overleftarrow{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h)]$$
$$=\sum_{j\neq i}(\epsilon_j\cdot k_i)(\epsilon_i\cdot B_{ij}). \tag{65}$$

It is not necessary to provide the explicit formulas of the vectors B^μ_{ij} here. The key point is

$$\mathcal{W}_i^\epsilon\partial_{\epsilon_q\cdot k_i}\left(\sum_{j\neq i}(\epsilon_j\cdot k_i)(\epsilon_i\cdot B_{ij})\right)=k_i\cdot B_{iq}, \tag{66}$$

thus from (63) we know that

$$-(\epsilon_q\cdot\epsilon_i)(k_i\cdot B_{iq})\in\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}_n^h). \tag{67}$$

In formula (60), the object $-(\epsilon_q\cdot\epsilon_i)(k_i\cdot B_{iq})$ belongs to the unknown R. In other words, we have to detect a piece of R.

To determine the full R, we first combine $(\epsilon_q\cdot k_i)(\epsilon_i\cdot B_{iq})$ in (65) and $-(\epsilon_q\cdot\epsilon_i)(k_i\cdot B_{iq})$ in (67) together as $(\epsilon_q\cdot f_i\cdot B_{iq})$, where the tensor $f_i^{\mu\nu}$ is defined in (49). Since q is chosen arbitrary, all the tensors $k_i^\mu\epsilon_i^\nu$ should be replaced by $f_i^{\mu\nu}$ at the rhs of (65). The leg i is also chosen arbitrary, and each $C^\epsilon_{\vec{\boldsymbol{a}}}$ is invariant under the cyclic permutations for $\vec{\boldsymbol{a}}$. Such symmetry requires us to replace all $k^\mu_{a_i}\epsilon^\nu_{a_i}$ in the cyclical factor $C^\epsilon_{\vec{\boldsymbol{a}}}$ by $f^{\mu\nu}_{a_i}$. Thus we find the replacement

$$C^\epsilon_{\vec{\boldsymbol{a}}}\to\mathrm{Tr}^\epsilon_{\vec{\boldsymbol{a}}}=\mathrm{Tr}(f_{a_{|\boldsymbol{a}|}}\cdots f_{a_1}). \tag{68}$$

This replacement detects various new terms in R, and all these new terms vanish under the action of $\mathcal{D}$ and $\mathcal{C}^\epsilon_{\vec{\boldsymbol{s}}_m}$. This observation supports our assumptions $\mathcal{D}R=0$ and $\mathcal{C}^\epsilon_{\vec{\boldsymbol{s}}_m}R=0$. After doing the replacement, the gauge invariance for each external graviton is manifest, since the tensor $f_i^{\mu\nu}$ vanishes under the replacement $\epsilon_i\to k_i$.

However, when doing the replacement (68), an overcounting arises. The term

$$\mathrm{Tr}^\epsilon_{\vec{\boldsymbol{a}}}[\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h)+(-)^{|\boldsymbol{a}|}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\overleftarrow{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h)] \tag{69}$$

contains not only

$$C^\epsilon_{\overleftarrow{\boldsymbol{a}}}[\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h)+(-)^{|\boldsymbol{a}|}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\overleftarrow{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h)], \tag{70}$$

but also

$$C^\epsilon_{\overleftarrow{\boldsymbol{a}}}[\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\overleftarrow{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h)+(-)^{|\boldsymbol{a}|}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h)], \tag{71}$$

and summing over $\vec{\boldsymbol{a}}$ counts both $\vec{\boldsymbol{a}}$ and $\overleftarrow{\boldsymbol{a}}$. To handle this, we recognize the second term in (69) as the first term in

$$\mathrm{Tr}^\epsilon_{\overleftarrow{\boldsymbol{a}}}[\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\overleftarrow{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h)+(-)^{|\boldsymbol{a}|}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h)], \tag{72}$$

and the first term in (69) as the second term in (72), since $\mathrm{Tr}^\epsilon_{\vec{\boldsymbol{a}}}=(-)^{|\boldsymbol{a}|}\mathrm{Tr}^\epsilon_{\overleftarrow{\boldsymbol{a}}}$, due to the antisymmetry of the tensors $f_i^{\mu\nu}$. Thus one can remove the overcounting and get the expansion

$$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}_n^h)=(d-2)\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\boldsymbol{a}_n^h)+\sum_{\vec{\boldsymbol{a}}/\pi_c}\mathrm{Tr}^\epsilon_{\vec{\boldsymbol{a}}}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM};\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h). \tag{73}$$

In expansion (73), all coefficients $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\varnothing}$ and $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\vec{\boldsymbol{a}}}(\mathrm{sign})$ in the first line of (55) are fixed, thus (73) is indeed the correct expanded formula for $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}_n^h)$, which coincides with the result found in [31]. The coefficients $\mathrm{Tr}^\epsilon_{\vec{\boldsymbol{a}}}$ vanish when the length of $\boldsymbol{a}$ is 1; this feature supports the observation in Sec. III A that the operator $\mathcal{C}^\epsilon_{\vec{\boldsymbol{a}}_m}$ does not make sense when $m=1$. Notice that without the general formula (55), one cannot conclude the solution (73) has detected all terms in the full GR integrand. For example, suppose we turn the factor $d-2$ in (73) to d, or add the tree amplitude $\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}_n^h\cup\{+^h,-^h\})$ to the rhs of (73); the obtained results are still solutions to Eqs. (56) and (39), the Lorentz and gauge invariance are also satisfied. Such modifications are excluded by the general formula (55).

The expansion (73) is equivalent to

$$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}_n^h)=(d-2)\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM}}(+^g,-^g;\boldsymbol{a}_n^h)$$
$$+\sum_{\vec{\boldsymbol{a}}}\mathrm{Tr}^\epsilon_{\vec{\boldsymbol{a}}}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g,-^g;\boldsymbol{a}_n^h\backslash\boldsymbol{a}^h), \tag{74}$$

which expands $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}_n^h)$ to ssEYM partial integrands rather than full ones. For the work in the next subsection, it is more convenient to use (74).

In expansions (73) and (74), all coefficients of partial ssEYM integrands are independent of the loop momentum ℓ, thus will not be altered by the integration over loop momentum. Thus, these expansions also hold at the level of one-loop amplitudes.

B. Coefficients of basis

From expansions (47) and (74), it is straightforward to observe that the one-loop ssEYM partial Feynman integrands and GR Feynman integrands can be expanded to one-loop YM KK basis by applying (47) recursively, formally expressed as

$$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n) = \sum_\sigma \mathbf{C}^\epsilon_1(\sigma)\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\sigma(\vec{\boldsymbol{a}}^g_n),-^g), \tag{75}$$

and

$$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM}}(+,\vec{\boldsymbol{a}}^g_m,-;\boldsymbol{a}^h_{n-m}) = \sum_\sigma \mathbf{C}^\epsilon_2(\sigma,\vec{\boldsymbol{a}}_m)\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\sigma(\vec{\boldsymbol{s}}^g_n),-^g), \tag{76}$$

where σ stands for the permutations. It is hard to find the general expressions for coefficients $\mathbf{C}^\epsilon_1(\sigma)$ and $\mathbf{C}^\epsilon_2(\sigma,\vec{\boldsymbol{a}}^g_m)$. Instead, the systematic algorithms for evaluating them can be provided, as will be shown in this subsection.

We first consider the coefficients $C^\epsilon_1(\sigma)$, which serve as the one-loop level BCJ numerators. To expand $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n)$ to $\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\sigma(\vec{\boldsymbol{a}}^g_n),-^g)$, the expansion (74) requires us to decompose the set of external legs $\boldsymbol{a}_n$ into subsects $\boldsymbol{a}$ and $\boldsymbol{a}_n\backslash\boldsymbol{a}$, while applying the expansion (47) recursively indicates further decompositions of $\boldsymbol{a}_n\backslash\boldsymbol{a}$. The successive decompositions lead to the concept which is called ordered splitting, defined for each fixed color ordering $\sigma(\vec{\boldsymbol{a}}_n) = \langle\sigma_1,\ldots,\sigma_n\rangle$ [22]. To illustrate it, we denote the color ordering as $+\dot{<}\sigma_1\dot{<}\cdots\dot{<}\sigma_n\dot{<}-$, and chose a reference ordering $j_1\prec\cdots\prec j_n$, with $j_i\in\boldsymbol{a}_n$. This reference ordering is denoted by $\mathcal{R}$. The correct ordered splittings, consistent with the given color ordering, are constructed through the following procedure:

(i) At the first step, we construct all possible ordered subsets $\vec{\boldsymbol{a}}^0 = \langle a^0_1,\ldots,a^0_{|0|}\rangle$, which satisfy two conditions: (1) $\boldsymbol{a}^0\subseteq\boldsymbol{a}_n$; (2) $a^0_1\dot{<}a^0_2\dot{<}\cdots\dot{<}a^0_{|0|}$, respective to the color ordering of the YM amplitude. We call each ordered subset $\vec{\boldsymbol{a}}^0$ a root.[5] Here $|i|$ denotes the length of the set $\boldsymbol{a}^i$.

(ii) For each root $\vec{\boldsymbol{a}}^0$, we eliminate its elements in $\boldsymbol{a}_n$ and $\mathcal{R}$, resulting in a reduced set $\boldsymbol{a}_n\backslash\boldsymbol{a}^0$, and a reduced reference ordering $\mathcal{R}\backslash\boldsymbol{a}^0$. Suppose R_1 is the lowest element in the reduce reference ordering $\mathcal{R}\backslash\boldsymbol{a}^0$, we construct all possible ordered subsets $\vec{\boldsymbol{a}}^1$ as $\vec{\boldsymbol{a}}^1 = \langle a^1_1,a^1_2,\ldots,a^1_{|1|-1},R_1\rangle$, with $a^1_1\dot{<}a^1_2\dot{<}\cdots\dot{<}a^1_{|1|-1}\dot{<}R_1$, respective to the color ordering.

(iii) By iterating the second step, one can construct $\vec{\boldsymbol{a}}^2,\vec{\boldsymbol{a}}^3,\cdots$, until $\boldsymbol{a}^0\cup\boldsymbol{a}^1\cup\cdots\cup\boldsymbol{a}^r = \boldsymbol{a}_n$.

Each ordered splitting is given as an ordered set $\vec{\mathbf{S}} = \langle\vec{\boldsymbol{a}}^0,\vec{\boldsymbol{a}}^1,\ldots,\vec{\boldsymbol{a}}^r\rangle$, where ordered sets $\vec{\boldsymbol{a}}^i$ serve as elements. For a given root $\vec{\boldsymbol{a}}^0$, an ordered set $\vec{\mathbf{B}} = \langle\vec{\boldsymbol{a}}^1,\vec{\boldsymbol{a}}^2,\ldots,\vec{\boldsymbol{a}}^r\rangle$ is called a branch. Notice that $\boldsymbol{a}^0$ can be empty, while each $\boldsymbol{a}^i$ with $i\neq 0$ contains at least one element R_i.

Now we give the corresponding kinematic factors for each ordered set $\vec{\boldsymbol{a}}^i$, by using (47) and (74). For a given ordered splitting, the root $\vec{\boldsymbol{a}}^0$ carries the factor

$$T^\epsilon_{\vec{\boldsymbol{a}}^0} = \begin{cases} \mathrm{Tr}^\epsilon_{\overleftarrow{\boldsymbol{a}^0}} = \mathrm{Tr}(f_{a^0_{|0|}},\ldots,f_{a^0_1}), & \text{if } \boldsymbol{a}^0\neq\varnothing, \\ d-2, & \text{if } \boldsymbol{a}^0=\varnothing. \end{cases} \tag{77}$$

Other ordered sets $\vec{\boldsymbol{a}}^i$ with $i\neq 0$ carry

$$K^\epsilon_{\vec{\boldsymbol{a}}^i} = \epsilon_{R_i}\cdot f_{a^i_{|i|-1}}\cdots f_{a^i_2}\cdot f_{a^i_1}\cdot Z_{a^i_1}. \tag{78}$$

The combinatory momentum $Z_{a^i_1}$ is the sum of momenta of external legs satisfying two conditions: (1) legs at the lhs of a^i_1 in the color ordering; (2) legs belong to $\vec{\boldsymbol{a}}^j$ at the lhs of $\vec{\boldsymbol{a}}^i$ in the ordered splitting, i.e., $j<i$. The coefficient of the YM partial integrand $\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\sigma(\vec{\boldsymbol{a}}^g_n),-^g)$ is the sum of contributions from all proper ordered splittings.

For the ssEYM partial integrand $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_m,-^g;\boldsymbol{a}^h_{n-m})$, the coefficient of $\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\sigma(\vec{\boldsymbol{s}}^g_n),-^g)$ is obviously the sum of contributions from all branches for the root $\vec{\boldsymbol{a}}^0 = \vec{\boldsymbol{a}}_m$.

Before ending this subsection, we point out that the differential operators transmute the one-loop GR integrand to one-loop YM partial integrands and also transmute the YM partial integrand $\mathbf{I}^\epsilon_{\mathrm{YM}}(+^g,\vec{\boldsymbol{a}}^g_n,-^g)$ to BAS double-partial integrands $\mathbf{I}_{\mathrm{BAS}}(+^s,\sigma(\vec{s}^s_n),-^s\|+^s,\vec{\boldsymbol{a}}^s_n,-^s)$, as can be seen in Table II. Furthermore, all seven principles/assumptions listed at the beginning of this section hold for the later case. The third assumption makes sense in the following way: this assumption together with the operator $\mathcal{T}^{\tilde{\epsilon}}_{+,\vec{a}_n,-}$ completely determine that each external gluon i carries the polarization vector ϵ_i, and the YM partial integrands carry the color ordering $+,\vec{\boldsymbol{a}}_n,-$, as discussed at the beginning of this section. Such characters of YM partial integrands play the role of the original third assumption. Thus one can follow the similar line for obtaining (75) to get

$$\mathbf{I}^\epsilon_{\mathrm{YM}}(+^g,\vec{\boldsymbol{a}}^g_n,-^g) = \sum_\sigma \mathbf{C}^\epsilon_1(\sigma)\mathbf{I}_{\mathrm{BAS}}(+^s,\sigma(\vec{s}^s_n),-^s\|+^s,\vec{\boldsymbol{a}}^s_n,-^s). \tag{79}$$

A similar argument for ssYMS partial Feynman integrands yields

$$\mathbf{I}^\epsilon_{\mathrm{ssYMS}}(+^s,\vec{\boldsymbol{a}}^s_m,-^s;\boldsymbol{a}^g_{n-m}\|+^A,\vec{\boldsymbol{a}}^A_n,-^A) = \sum_\sigma \mathbf{C}^\epsilon_2(\sigma,\vec{\boldsymbol{a}}_m)\mathbf{I}_{\mathrm{BAS}}(+^s,\sigma(\vec{s}^s_n),-^s\|+^s,\vec{\boldsymbol{a}}^s_n,-^s). \tag{80}$$

[5]Here we borrow the language from the framework of increasing spanning trees.

V. UNIFIED WEB FOR EXPANSIONS

The expansions found in the previous section are based on the transmutation relations provided by differential operators. Since the differential operators provide a unified web including a wide range of theories, it is natural to expect the expansions can also be extended to other theories. In this section, we discuss how to reach this goal, and establish the complete unified web for expansions.

From the expansion of one-loop GR integrand (75), one can generate various new expansions by applying differential operators. Suppose we act the operators defined via polarization vectors in $\{\tilde{\epsilon}_i\}$ at two sides of (75) simultaneously, such operators transmute both $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n)$ and $\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\sigma(\vec{\boldsymbol{a}}^g_n),-^g)$ to integrands of other theories, while keeping the coefficients $\mathbf{C}^{\epsilon}_1(\sigma)$ unmodified. For instance, using

$$\mathbf{I}^{\epsilon}_{\mathrm{BI};\circ}(\boldsymbol{a}^p_n) = \mathcal{L}^{\tilde{\epsilon}}\tilde{\mathcal{D}}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n),$$
$$\mathbf{I}_{\mathrm{NLSM}}(+^s,\sigma(\vec{\boldsymbol{a}}^s_n),-^s) = \mathcal{L}^{\tilde{\epsilon}}\tilde{\mathcal{D}}\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\sigma(\vec{\boldsymbol{a}}^g_n),-^g), \quad (81)$$

we obtain

$$\mathbf{I}^{\epsilon}_{\mathrm{BI};\circ}(\boldsymbol{a}^p_n) = \sum_{\sigma}\mathbf{C}^{\epsilon}_1(\sigma)\mathbf{I}_{\mathrm{NLSM}}(+^s,\sigma(\vec{\boldsymbol{a}}^s_n),-^s). \quad (82)$$

The set of NLSM partial integrands $\mathbf{I}_{\mathrm{NLSM}}(+^s,\sigma(\vec{\boldsymbol{a}}^s_n),-^s)$ is the generalized one-loop KK basis, due to the structure of color orderings. A more interesting case is applying the operators defined via $\{\epsilon_i\}$ rather than $\{\tilde{\epsilon}_i\}$. These operators also transmute $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n)$ at the lhs to the integrands of other theories. When acting on the rhs, they modify the coefficients $\mathbf{C}^{\epsilon}_1(\sigma)$, while keeping the YM partial integrands unaltered. The above manipulation allows us to generate the following expansions:

$$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_m,-^g;\boldsymbol{a}^h_{n-m}) = \sum_{\sigma}\mathbf{C}^{\epsilon}_2(\sigma,\vec{\boldsymbol{a}}_m)\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\sigma(\vec{\boldsymbol{a}}^g_n),-^g),$$
$$\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{BI};\circ}(\boldsymbol{a}^p_n) = \sum_{\sigma}\mathbf{C}_3(\sigma)\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\sigma(\vec{\boldsymbol{a}}^g_n),-^g),$$
$$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{EM};\circ}(\boldsymbol{a}^p_{2m};\boldsymbol{a}^h_{n-2m}) = \sum_{\sigma}\mathbf{C}^{\epsilon}_4(\sigma,X_{2m})\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\sigma(\vec{\boldsymbol{a}}^g_n),-^g),$$
$$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{EMf};\circ}(\boldsymbol{a}^p_{2m};\boldsymbol{a}^h_{n-2m}) = \sum_{\sigma}\mathbf{C}^{\epsilon}_5(\sigma,\mathcal{X}_{2m})\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\sigma(\vec{\boldsymbol{a}}^g_n),-^g). \quad (83)$$

The rule for constructing $\mathbf{C}^{\epsilon}_2(\sigma,\vec{\boldsymbol{a}}_m)$ is already provided in the previous section. In Sec. V A we will discuss the rules for constructing coefficients $\mathbf{C}^{\epsilon}_i(\sigma)$ with $i\in\{3,4,5\}$.

The full web for expansions can be established by applying differential operators further. We will not do this procedure. Instead, we use a more compact way to describe the unified web for expansions. One can replace ϵ in the expansion (79) by $\tilde{\epsilon}$, and substitute it into (75), then get the expansion of the GR integrand to BAS KK basis as

$$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR};\circ}(\boldsymbol{a}^h_n) = \sum_{\sigma}\sum_{\sigma'}\mathbf{C}^{\epsilon}_1(\sigma)\mathbf{I}_{\mathrm{BAS}}(+^s,\sigma(\vec{\boldsymbol{a}}^s_n),-^s\| +^s,\sigma'(\vec{\boldsymbol{s}}^s_n),-^s)\mathbf{C}^{\tilde{\epsilon}}_1(\sigma'), \quad (84)$$

with two coefficients $\mathbf{C}^{\epsilon}_1(\sigma)$ and $\mathbf{C}^{\tilde{\epsilon}}_1(\sigma')$. We call the expansion (84) the double expansion. The differential operators transmute the lhs of (84) to Feynman integrands of other theories, and transmute $\mathbf{C}^{\epsilon}_1(\sigma)$ or $\mathbf{C}^{\tilde{\epsilon}}_1(\sigma')$ to other $\mathbf{C}^{\epsilon}_i(\sigma)$ or $\mathbf{C}^{\tilde{\epsilon}}_j(\sigma')$ at the rhs. Thus the double expansions for all theories in Table II are obtained. The double-expanded formulas also manifest the duality between transmutation relations and expansions, as will be discussed in Sec. V B.

With the general ideas discussed above, now we begin to study the corresponding details.

A. Expansions of BI, EM and EMf to YM

As discussed above, the one-loop BI, EM and EMf Feynman integrands can also be expanded to the one-loop YM KK basis. The purpose of this subsection is to give the rules for evaluating corresponding coefficients $\mathbf{C}_3(\sigma)$, $\mathbf{C}^{\epsilon}_4(\sigma,X_{2m})$ and $\mathbf{C}^{\epsilon}_5(\sigma,\mathcal{X}_{2m})$.

We begin by considering the BI integrands, which can be generated from the GR integrands via the operator $\mathcal{L}^{\epsilon}\mathcal{D}$. Applying this operator to two sides of (75), the lhs gives $\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{BI};\circ}(\boldsymbol{a}^p_n)$. At the rhs, the coefficients $\mathbf{C}^{\epsilon}_1(\sigma)$ are transmuted to $\mathbf{C}_3(\sigma)$, while the YM partial integrands are unmodified, since the operator $\mathcal{L}^{\epsilon}\mathcal{D}$ is defined via polarization vectors in $\{\epsilon_i\}$. Thus $\mathbf{C}_3(\sigma)$ is generated from $\mathbf{C}^{\epsilon}_1(\sigma)$, namely,

$$\mathbf{C}_3(\sigma) = \mathcal{L}^{\epsilon}\mathcal{D}\mathbf{C}^{\epsilon}_1(\sigma). \quad (85)$$

We first consider the effect of the operator $\mathcal{D}$. This operator annihilates terms which do not contain the factor $d-2$, thus transmutes $\mathbf{C}^{\epsilon}_1(\sigma)$ to $\mathbf{C}^{\epsilon}_2(\sigma,\emptyset)$. It means we only need to consider ordered splittings with the root $\boldsymbol{a}^0=\emptyset$.

To continue, we perform the operator $\mathcal{L}^{\epsilon}$. There are two definitions for the operator $\mathcal{L}^{\epsilon}$, which are unequivalent at the algebraic level, but lead to the same physical result in the current case. We first consider the definition

$\mathcal{L}^\epsilon \equiv \prod_i \mathcal{L}_i^\epsilon$. The operator $\mathcal{L}^\epsilon$ turns $\epsilon_i \cdot k_j$ to $k_i \cdot k_j$, therefore only terms with the form $\prod_i \epsilon_i \cdot K_i$ can survive under the action, where K_i are combinations of external and loop momenta. In $\mathbf{C}_2^\epsilon(\sigma, \varnothing)$, such part is found to be $\prod_i \epsilon_i \cdot X_i$, where X_i is defined as the summation of k_j with $j \dot{<} i$ in the color ordering. Thus the effect of $\mathbf{C}_3(\sigma)$ is given as

$$\mathbf{C}_3(\sigma) = \mathcal{L}^\epsilon \mathbf{C}_2^\epsilon(\sigma, \varnothing) = \prod_i k_i \cdot X_i, \tag{86}$$

which is a very compact result.

Now we consider another definition of the operator $\mathcal{L}^\epsilon$:

$$\mathcal{L}^\epsilon = \sum_{\rho \in \mathrm{pair}} \prod_{\{i,j\} \in \rho} \mathcal{L}_{ij}^\epsilon. \tag{87}$$

Applying this $\mathcal{L}^\epsilon$ to $\mathbf{C}_2^\epsilon(\sigma, \varnothing)$, the survived terms are those where each polarization vector ϵ_i is contracted with another one ϵ_j. Using the definition of $\mathbf{C}_2^\epsilon(\sigma, \varnothing)$, such part is found to be

$$\sum_{\vec{\boldsymbol{B}}:|i|_{\mathrm{even}}} \left(\prod_{i=1}^{t} (-)^{\frac{|i|}{2}} M_i(\sigma, \vec{\boldsymbol{B}}) \right), \tag{88}$$

where the summation is over all possible branches $\vec{\boldsymbol{B}}$, those where the length of each subset $\boldsymbol{a}^i$ is even, and the number of subsets included in each branch is denoted by t. The monomial $M_i(\sigma, \vec{\boldsymbol{B}})$ for the subset $\vec{\boldsymbol{a}}^i$ is given as

$$\begin{aligned} M_i(\sigma, \vec{\boldsymbol{B}}) = {} & (\epsilon_{a^i_{|i|}} \cdot \epsilon_{a^i_{|i|-1}})(k_{a^i_{|i|-1}} \cdot k_{a^i_{|i|-2}})(\epsilon_{a^i_{|i|-2}} \cdot \epsilon_{a^i_{|i|-3}}) \\ & \cdots (k_{a^i_3} \cdot k_{a^i_2})(\epsilon_{a^i_2} \cdot \epsilon_{a^i_1})(k_{a^i_1} \cdot Z_{a^i_1}). \end{aligned} \tag{89}$$

Under the action of $\mathcal{L}^\epsilon$, we find

$$\mathbf{C}_3(\sigma) = \mathcal{L}^\epsilon \mathbf{C}_2^\epsilon(\sigma, \varnothing) = \sum_{\vec{\boldsymbol{B}}:|i|_{\mathrm{even}}} \left(\prod_{i=1}^{t} (-)^{\frac{|i|}{2}} N_i(\sigma, \vec{\boldsymbol{B}}) \right), \tag{90}$$

where

$$N_i(\sigma, \vec{\boldsymbol{B}}) = \left(\prod_{k=1}^{|i|-1} k_{a^i_k} \cdot k_{a^i_{k+1}} \right) (k_{a^i_1} \cdot Z_{a^i_1}). \tag{91}$$

The equivalence between (86) and (90) can be verified for simple cases, and we have checked it for the three-point integrand. The general proof is an interesting challenge, which we leave as the feature work.

Then we turn to EM and EMf Feynman integrands. The EM integrands can be generated from the GR ones via the operator $\mathcal{T}^\epsilon_{X_{2m}}(\mathcal{D}+1)$, thus the argument similar to the BI case gives

$$\mathbf{C}_4^\epsilon(\sigma, X_{2m}) = \mathcal{T}^\epsilon_{X_{2m}}(\mathcal{D}+1)\mathbf{C}_1^\epsilon(\sigma). \tag{92}$$

The operator $(\mathcal{D}+1)$ transmutes $\mathbf{C}_1^\epsilon(\sigma)$ as

$$(\mathcal{D}+1)\mathbf{C}_1^\epsilon(\sigma) = \mathbf{C}_2^\epsilon(\sigma, \varnothing) + \mathbf{C}_1^\epsilon(\sigma). \tag{93}$$

Then we need to perform $\mathcal{T}^\epsilon_{X_{2m}}$ on $\mathbf{C}_2^\epsilon(\sigma, \varnothing)$ and $\mathbf{C}_1^\epsilon(\sigma)$. Let us consider $\mathcal{T}^\epsilon_{X_{2m}}\mathbf{C}_1^\epsilon(\sigma)$ first. Recall that $\mathcal{T}^\epsilon_{X_{2m}}$ is defined as summing over $\prod_{i_k, j_k \in \rho} \mathcal{T}^\epsilon_{i_k j_k}$ for different partitions, where each partition groups the $2m$ external particles into m pairs as $\{(i_1, j_1), (i_2, j_2), \ldots (i_m, j_m)\}$, with $i_1 < i_2 < \cdots < i_m$ and $i_k < j_k$ for $\forall\, k$. Thus we can consider the effect of operator $\prod_{i_k, j_k \in \rho} \mathcal{T}^\epsilon_{i_k j_k}$ for a given partition. This operator annihilates all terms which do not contain $\prod_{i_k, j_k \in \rho} (\epsilon_{i_k} \cdot \epsilon_{j_k})$. Henceforth, one can start with ordered splittings for $\mathbf{C}_1^\epsilon(\sigma)$, and select ordered splittings by the condition that each pair in the partition appears in one subset as a single element, i.e., two particles are adjacent. Then, for a selected ordered splitting, we now consider the effect of applying $\prod_{i_k, j_k \in \rho} \mathcal{T}^\epsilon_{i_k j_k}$ to the corresponding kinematic factor $\mathbf{T}^\epsilon_{\vec{\boldsymbol{a}}^0}(\prod_{i=1}^t K^\epsilon_{\vec{\boldsymbol{a}}^i})$. For $\mathbf{T}^\epsilon_{\vec{\boldsymbol{a}}^0}$, we turn all $(f_{i_k} \cdot f_{j_k})^{\mu\nu}$ to $-k^\mu_{i_k} k^\nu_{j_k}$. For $K^\epsilon_{\vec{\boldsymbol{a}}^i}$, we turn all $(f_{i_k} \cdot f_{j_k})^{\mu\nu}$ to $-k^\mu_{i_k} k^\nu_{j_k}$ when $i_k \neq R_i$, $j_k \neq R_i$, and turn $(\epsilon_{i_k} \cdot f_{j_k})^\mu$ to $-k^\mu_{j_k}$ when $i_k = R_i$, or turn $(\epsilon_{j_k} \cdot f_{i_k})^\mu$ to $-k^\mu_{i_k}$ when $j_k = R_i$. The resulting object of $\mathcal{T}^\epsilon_{X_{2m}}\mathbf{C}_1^\epsilon(\sigma)$ is obtained by summing over contributions from selected ordered splittings, then summing over all proper partitions. Another part $\mathcal{T}^\epsilon_{X_{2m}}\mathbf{C}_2^\epsilon(\sigma, \varnothing)$ can be obtained by performing the above manipulation to branches for the root $\vec{\boldsymbol{a}}^0 = \varnothing$.

The EMf integrands are generated from the GR ones via the operator $\mathcal{T}^\epsilon_{\mathcal{X}_{2m}}(N\mathcal{D}+1)$, where N stands for the number of different flavors. Thus we have

$$\begin{aligned} \mathbf{C}_5^\epsilon(\sigma, \mathcal{X}_{2m}) &= \mathcal{T}^\epsilon_{\mathcal{X}_{2m}}(N\mathcal{D}+1)\mathbf{C}_1^\epsilon(\sigma) \\ &= N\mathcal{T}^\epsilon_{\mathcal{X}_{2m}}\mathbf{C}_2^\epsilon(\sigma, \varnothing) + \mathcal{T}^\epsilon_{\mathcal{X}_{2m}}\mathbf{C}_1^\epsilon(\sigma). \end{aligned} \tag{94}$$

The consideration for $\mathcal{T}^\epsilon_{\mathcal{X}_{2m}}\mathbf{C}_2^\epsilon(\sigma, \varnothing)$ and $\mathcal{T}^\epsilon_{\mathcal{X}_{2m}}\mathbf{C}_1^\epsilon(\sigma)$ is analogous to $\mathcal{T}^\epsilon_{X_{2m}}\mathbf{C}_2^\epsilon(\sigma, \varnothing)$ and $\mathcal{T}^\epsilon_{X_{2m}}\mathbf{C}_1^\epsilon(\sigma)$. The only difference is that the appropriate partitions are reduced: a partition is allowed if and only if $\delta_{I_{i_k} I_{j_k}} \neq 0$ for all pairs (i_k, j_k).

B. Double expansion and unified web

As discussed previously, the one-loop GR Feynman integrands can be double expanded as in (84). The BAS KK basis contributes the propagators, while coefficients $\mathbf{C}_1^\epsilon(\sigma)$ and $\mathbf{C}_1^{\tilde{\epsilon}}(\sigma')$ serve as BCJ numerators. On the other hand, we have

$$\begin{aligned}\mathcal{T}^{\epsilon}_{+\vec{a}_m-}\mathbf{C}^{\epsilon}_1(\sigma)&=\mathbf{C}^{\epsilon}_2(\sigma,\vec{a}_m),\\ \mathcal{L}^{\epsilon}\mathcal{D}\mathbf{C}^{\epsilon}_1(\sigma)&=\mathbf{C}_3(\sigma),\\ \mathcal{T}^{\epsilon}_{X_{2m}}(\mathcal{D}+1)\mathbf{C}^{\epsilon}_1(\sigma)&=\mathbf{C}^{\epsilon}_4(\sigma,X_{2m}),\\ \mathcal{T}^{\epsilon}_{\mathcal{X}_{2m}}(N\mathcal{D}+1)\mathbf{C}^{\epsilon}_1(\sigma)&=\mathbf{C}^{\epsilon}_5(\sigma,\mathcal{X}_{2m}),\\ \mathcal{T}^{\epsilon}_{+\vec{s}_n-}\mathbf{C}^{\epsilon}_1(\sigma)&=\mathbf{C}_6(\sigma,\vec{s}_n).\end{aligned}\tag{95}$$

In the first line, the length m of the set $\boldsymbol{a}_m$ is required to be $0\le m<n$. We have added $\mathbf{C}_6(\sigma,\vec{s}_n)=\delta_{\vec{s}_n\sigma(\vec{a}_n)}$ to the list, to represent

$$\begin{aligned}\mathcal{T}^{\epsilon}_{+\vec{s}_n-}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n)&=\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\vec{s}^g_n,-^g)\\ &=\sum_\sigma\delta_{\vec{s}_n\sigma(\vec{a}_n)}\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\sigma(\vec{a}^g_n),-^g).\end{aligned}\tag{96}$$

Here $\delta_{\vec{s}_n\sigma(\vec{a}_n)}$ is understood as 1 when $\vec{s}_n=\sigma(\vec{a}_n)$ and 0 otherwise. The relations among $\mathbf{C}^{\tilde{\epsilon}}_i(\sigma')$ with $i\in\{1,2,3,4,5,6\}$ are completely analogous. Let us simplify the notations as

$$\mathcal{O}^{\epsilon}_i\mathbf{C}^{\epsilon}_1(\sigma)=\mathbf{C}^{\epsilon}_i(\sigma),\tag{97}$$

where

$$\begin{aligned}&\mathcal{O}^{\epsilon}_1=\mathbb{I},\qquad \mathcal{O}^{\epsilon}_2=\mathcal{T}^{\epsilon}_{+\vec{a}_m-},\\ &\mathcal{O}^{\epsilon}_3=\mathcal{L}^{\epsilon}\mathcal{D},\qquad \mathcal{O}^{\epsilon}_4=\mathcal{T}^{\epsilon}_{X_{2m}}(\mathcal{D}+1),\\ &\mathcal{O}^{\epsilon}_5=\mathcal{T}^{\epsilon}_{\mathcal{X}_{2m}},\qquad \mathcal{O}^{\epsilon}_6=\mathcal{T}^{\epsilon}_{+\vec{s}_n-},\end{aligned}\tag{98}$$

and introduce the analogous notations $\mathcal{O}^{\tilde{\epsilon}}_i$ for $\mathbf{C}^{\tilde{\epsilon}}_i(\sigma')$. Applying the above operators to the double-expanded GR integrand in (84), we get

$$\mathbf{I}_{ij}=\sum_\sigma\sum_{\sigma'}\mathbf{C}^{\epsilon}_i(\sigma)\mathbf{I}_{\mathrm{BAS}}(+^s,\sigma(\vec{a}^s_n),-^s\|+^s\sigma'(\vec{s}^s_n),-^s)\mathbf{C}^{\tilde{\epsilon}}_j(\sigma'),\tag{99}$$

where

$$\mathbf{I}_{ij}=\mathcal{O}^{\epsilon}_i\mathcal{O}^{\tilde{\epsilon}}_j\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n).\tag{100}$$

The physical interpretation for each $\mathbf{I}_{ij}$ can be seen in Table II by using (100). For the readers' convenience, we list $\mathbf{I}_{ij}$ for different i and j in Table III. Thus (99) is indeed the double-expanded formula for Feynman integrands for theories in Table III.

The full unified web for expansions can be constructed from the double-expansion (99) and Table III, by summing over σ or σ'. To do this, we first sum over σ for $\mathbf{C}_6(\sigma,\vec{a}_n)$ to get

$$\begin{aligned}\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\vec{a}^g_n,-^g)&=\sum_{\sigma'}\mathbf{C}^{\tilde{\epsilon}}_1(\sigma')\mathbf{I}_{\mathrm{BAS}}(+^s,\vec{a}^s_n,-^s\|+^s,\sigma'(\vec{s}^s_n),-^s),\\ \mathbf{I}^{\tilde{\epsilon}}_{\mathrm{ssYMS}}(+^s,\vec{a}^s_m,-^s;\boldsymbol{a}^g_{n-m}\|+^A,\vec{a}^A_n,-^A)&=\sum_{\sigma'}\mathbf{C}^{\tilde{\epsilon}}_2(\sigma',\vec{a}^s_m)\mathbf{I}_{\mathrm{BAS}}(+^s,\vec{a}^s_n,-^s\|+^s,\sigma'(\vec{s}^s_n),-^s),\\ \mathbf{I}_{\mathrm{NLSM}}(+^s,\vec{a}^s_n,-^s)&=\sum_{\sigma'}\mathbf{C}_3(\sigma')\mathbf{I}_{\mathrm{BAS}}(+^s,\vec{a}^s_n,-^s\|+^s,\sigma'(\vec{s}^s_n),-^s),\\ \mathbf{I}^{\tilde{\epsilon}}_{\mathrm{SYMS}}(\boldsymbol{a}^s_{2m};\boldsymbol{a}^g_{n-2m}\|+^A,\vec{a}^A_n,-^A)&=\sum_{\sigma'}C^{\tilde{\epsilon}}_5(\sigma',\mathcal{X}_{2m})\mathbf{I}_{\mathrm{BAS}}(+,\vec{a}^s_n,-\|+^s,\sigma'(\vec{s}^s_n),-^s).\end{aligned}\tag{101}$$

Using the expansions in (101), one can obtain the expansions for other theories. For example, from Table III we see that

$$\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{DBI}}(\boldsymbol{a}^s_{2m};\boldsymbol{a}^p_{n-2m})=\sum_\sigma\sum_{\sigma'}\mathbf{C}_3(\sigma)\mathbf{I}_{\mathrm{BAS}}(+^s,\sigma(\vec{a}^s_n),-^s\|+^s\sigma'(\vec{s}^s_n),-^s)\mathbf{C}^{\tilde{\epsilon}}_5(\sigma',\mathcal{X}_{2m}).\tag{102}$$

Substituting the equality in the third line of (101) into (102) gives

$$\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{DBI}}(\boldsymbol{a}^s_{2m};\boldsymbol{a}^p_{n-2m})=\sum_\sigma\mathbf{C}^{\tilde{\epsilon}}_5(\sigma,\mathcal{X}_{2m})\mathbf{I}_{\mathrm{NLSM}}(+^s,\sigma(\vec{a}^s_n),-^s),\tag{103}$$

while substituting the equality in the fourth line of (101) into (102) provides

$$\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{DBI}}(\boldsymbol{a}^s_{2m};\boldsymbol{a}^p_{n-2m})=\sum_\sigma\mathbf{C}_3(\sigma)\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{SYMS}}(\boldsymbol{a}^s_{2m};\boldsymbol{a}^g_{n-2m}\|+^A,\sigma(\vec{a}^A_n),-^A).\tag{104}$$

TABLE III. $\mathbf{I}_{ij}$ for different i and j.

$\mathbf{I}_{ij}$	i	j
$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{GR}}(\boldsymbol{a}^h_n)$	1	1
$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_m,-^g;\boldsymbol{a}^h_{n-m})$	2	1
$\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{BI}}(\boldsymbol{a}^p_n)$	3	1
$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{EM}}(\boldsymbol{a}^p_{2m};\boldsymbol{a}^h_{n-2m})$	4	1
$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\mathrm{EMf}}(\boldsymbol{a}^p_{2m};\boldsymbol{a}^h_{n-2m})$	5	1
$\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{YM}}(+^g,\vec{\boldsymbol{a}}^g_n,-^g)$	6	1
$\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{ssYMS}}(+^s,\vec{\boldsymbol{a}}^s_m,-^s;\boldsymbol{a}^g_{n-m}\Vert+^A,\vec{\boldsymbol{a}}^A_n,-^A)$	6	2
$\mathbf{I}_{\mathrm{NLSM}}(+^s,\vec{\boldsymbol{a}}^s_n,-^s)$	6	3
$\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{SYMS}}(\boldsymbol{a}^s_{2m};\boldsymbol{a}^g_{n-2m}\Vert+^A,\vec{\boldsymbol{a}}^A_n,-^A)$	6	5
$\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{ssEDBI}}(\vec{\boldsymbol{a}}^s_m;\boldsymbol{a}^p_{n-m})$	3	2
$\mathbf{I}_{\mathrm{SG}}(\boldsymbol{a}^s_n)$	3	3
$\mathbf{I}^{\tilde{\epsilon}}_{\mathrm{DBI}}(\boldsymbol{a}^s_{2m};\boldsymbol{a}^p_{n-2m})$	3	5
$\mathbf{I}_{\mathrm{BAS}}(+^s,\vec{\boldsymbol{a}}^s_n,-^s\Vert+^s,\vec{\boldsymbol{s}}^s_n,-^s)$	6	6

The unified web can be established via the above method, and is represented diagrammatically in Fig. 3.

The double-expanded formula (99) is dual to the transmutation formula (19), due to the following reasons. First, the formula (99) is constructed from (19), together with some very general principles/assumptions mentioned at the beginning of Sec. IV. Second, they include the same theories. Third, operators $\mathcal{O}^{\epsilon}_i$ and coefficients $\mathbf{C}^{\epsilon}_i(\sigma)$ are

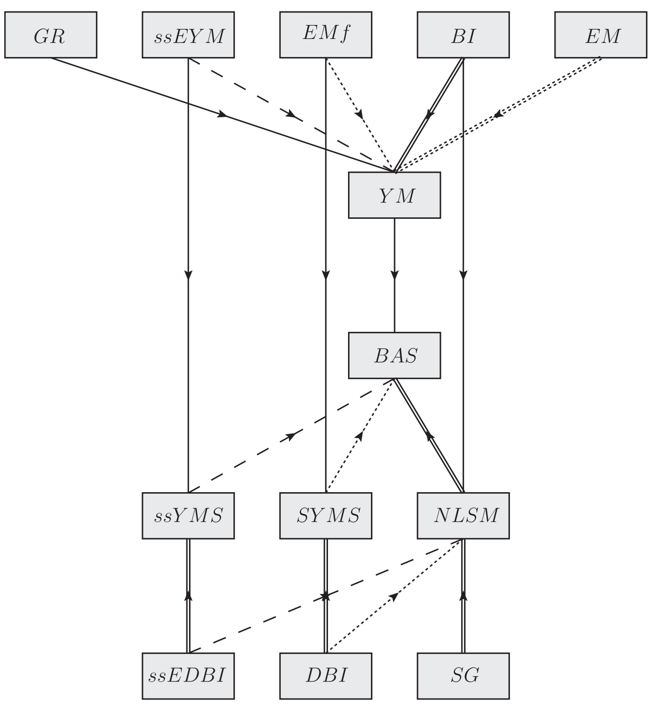

FIG. 3. Unified web for expansions of one-loop Feynman integrands. The straight lines denote the coefficients $\mathbf{C}^{\epsilon}_1(\sigma)$, the dashed lines denote $\mathbf{C}^{\epsilon}_2(\sigma,\vec{\boldsymbol{a}}_m)$, the double straight lines denote $\mathbf{C}^{\epsilon}_3(\sigma)$, the thin dashed lines denote $\mathbf{C}^{\epsilon}_5(\sigma,\mathcal{X}_{2m})$, the double thin dashed lines denote $\mathbf{C}^{\epsilon}_4(\sigma,X_{2m})$.

TABLE IV. Map between operators and coefficients.

$\mathcal{O}^{\epsilon}_i$	$\mathbf{C}^{\epsilon}_i(\sigma)$
$\mathbb{I}$	$\mathbf{C}^{\epsilon}_1(\sigma)$
$\mathcal{T}^{\epsilon}_{+\vec{\boldsymbol{a}}_m-}$	$\mathbf{C}^{\epsilon}_2(\sigma,\vec{\boldsymbol{a}}_m)$
$\mathcal{L}^{\epsilon}\mathcal{D}$	$\mathbf{C}_3(\sigma)$
$\mathcal{T}^{\epsilon}_{X_{2m}}(\mathcal{D}+1)$	$\mathbf{C}^{\epsilon}_4(\sigma,X_{2m})$
$\mathcal{T}^{\epsilon}_{\mathcal{X}_{2m}}(N\mathcal{D}+1)$	$\mathbf{C}^{\epsilon}_5(\sigma,\mathcal{X}_{2m})$
$\mathcal{T}^{\epsilon}_{+\vec{\boldsymbol{s}}_n-}$	$\mathbf{C}_6(\sigma,\vec{\boldsymbol{s}}_n)=\delta_{\vec{\boldsymbol{s}}_n\sigma(\vec{\boldsymbol{a}}_n)}$

linked by acting operators on BCJ numerators $\mathbf{C}^{\epsilon}_1(\sigma)$ as in (97), and so do operators $\mathcal{O}^{\tilde{\epsilon}}_i$ and coefficients $\mathbf{C}^{\tilde{\epsilon}}_i(\sigma')$. Thus we have a one-to-one map between operators and coefficients, as shown in Table IV. In this Table, the explicit form of $\mathbf{C}_6(\sigma,\vec{\boldsymbol{s}}_n)$ is provided, since it is as special as the operator $\mathbb{I}$ in the first column. Based on the duality discussed above, one can claim that the expansions of one-loop Feynman integrands are the dual version of transmutations relations.

VI. SUMMARY AND DISCUSSIONS

In this paper, we investigated the connections among one-loop Feynman integrands of a large variety of theories.

First, we constructed a new class of differential operators $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$, which transmute the one-loop GR Feynman integrands to one-loop ssEYM integrands.

Second, via the one-loop level transmutation relations, as well as some general principles/assumptions such as gauge invariance, we constructed the unified web for expansions of one-loop Feynman integrands for a wide range of theories including GR, ssEYM, EM, EMf, BI, YM, ssYMS, SYMS, NLSM, DBI, EDBI, and SG. We showed that the one-loop Feynman integrands of all of the above theories can be double expanded to the BAS one-loop KK basis, and provided the systematic rules for constructing the coefficients in the expansions. Throughout the whole process, we only used the knowledge of transmutation relations among one-loop Feynman integrands of different theories, as well as some very general requirements listed at the beginning of Sec. IV, without knowing any details about the Feynman integrands under consideration. Based on this character, together with the one-to-one map between transmutation operators and coefficients in expansions, we claimed that the transmutation relations and expansions are dual to each other.

In this paper and our previous work in [29], the consideration for the EYM partial Feynman integrands is not complete. We restricted ourselves to the special single-trace case that the virtual particle propagating in the loop is only a gluon. We have not considered the general case due to some technical difficulty, and leave the complete solution as the future work.

The expansions of one-loop Feynman integrands also indicate a new method for calculating the one-loop Feynman integrands of various theories. One can evaluate the BAS integrands at the first step, then use the rules for constructing coefficients to get the integrands of other theories in the double-expanded formulas. In principle, one can also calculate the GR integrands at the first step, then use the differential operators to generate others. However, in practice the GR integrands are the most complicated ones in the unified web. On the other hand, the BAS integrands are the easiest ones, since they only contain propagators, without carrying any kinematic numerator.

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APPENDIX A: CHY FORMULAS AT TREE AND ONE-LOOP LEVELS

In the CHY framework, tree amplitudes for n massless particles in arbitrary dimensions arise from a multidimensional contour integral over the moduli space of genus zero Riemann surfaces with n punctures, $\mathcal{M}_{0,n}$ [8–12], formulated as

$$\mathcal{A}_n = \int d\mu_n \mathcal{I}^L(\{k_i, \epsilon_i, z_i\})\mathcal{I}^R(\{k_i, \tilde{\epsilon}_i, z_i\}), \qquad \text{(A1)}$$

which possesses the Möbius SL$(2,\mathbb{C})$ invariance. Here k_i, ϵ_i and z_i are the momentum, polarization vector, and puncture location for the ith external particle, respectively. The measure part is defined as

$$d\mu_n \equiv \frac{d^n z}{\mathrm{volSL}(2,\mathbb{C})}\prod_i{}' \delta(\xi_i). \qquad \text{(A2)}$$

The δ functions impose the scattering equations

$$E_i \equiv \sum_{j\in\{1,2,\ldots,n\}\setminus\{i\}} \frac{k_i\cdot k_j}{z_{ij}} = 0, \qquad \text{(A3)}$$

where $z_{ij} \equiv z_i - z_j$. The scattering equations define the map from the punctures on the moduli space $\mathcal{M}_{0,n}$ to vectors on the light cone, and fully localize the integral on their solutions. The measure part is universal ,while the integrand in (A1) depends on the theory under consideration. For any theory known to have a CHY representation, the corresponding integrand factorizes into two parts $\mathcal{I}^L$ and $\mathcal{I}^R$, as can be seen in (A1). Either of them is weight-2 for each variable z_i under the Möbius transformation. In Table V, we list the tree level CHY integrands which will be used in this paper [12].[6]

TABLE V. Form of the integrands.

Theory	$\mathcal{I}^L(k_i,\epsilon_i,z_i)$	$\mathcal{I}^R(k_i,\tilde{\epsilon}_i,z_i)$
GR	$\mathbf{Pf}\, f'\Psi$	$\mathbf{Pf}'\Psi$
YM	$PT(\sigma_1,\ldots,\sigma_n)$	$\mathbf{Pf}'\Psi$
BAS	$PT(\sigma_1,\ldots,\sigma_n)$	$PT(\sigma'_1,\ldots,\sigma'_n)$

We now explain building blocks appearing in Table V in turn. The $2n\times 2n$ antisymmetric matrix Ψ is given by

$$\Psi = \left(\begin{array}{c|c} A & C \\ \hline -C^{\mathrm{T}} & B \end{array}\right), \qquad \text{(A4)}$$

where

$$A_{ij} = \begin{cases} \frac{k_i\cdot k_j}{z_{ij}} & i\neq j, \\ 0 & i=j, \end{cases} \qquad B_{ij} = \begin{cases} \frac{\epsilon_i\cdot \epsilon_j}{z_{ij}} & i\neq j, \\ 0 & i=j, \end{cases}$$

$$C_{ij} = \begin{cases} \frac{k_i\cdot \epsilon_j}{z_{ij}} & i\neq j, \\ -\sum_{l=1, l\neq j}^{n} \frac{k_l\cdot \epsilon_j}{z_{lj}} & i=j. \end{cases} \qquad \text{(A5)}$$

The notation **Pf** stands for the polynomial called Pfaffian. For a $2n\times 2n$ skew symmetric matrix S, Pfaffian is defined as

$$\mathbf{Pf}S = \frac{1}{2^n n!}\sum_{\sigma\in S_{2n}} \mathbf{sgn}(\sigma)\prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}, \qquad \text{(A6)}$$

where S_{2n} is the permutation group of $2n$ elements and $\mathbf{sgn}(\sigma)$ is the signature of σ. More explicitly, let Π be the set of all partitions of $\{1,2,\ldots,2n\}$ into pairs without regard to the order. An element α in Π can be written as

$$\alpha = \{(i_1,j_1),(i_2,j_2),\ldots,(i_n,j_n)\}, \qquad \text{(A7)}$$

with $i_k < j_k$ and $i_1 < i_2 < \cdots < i_n$. Now let

$$\sigma_\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & i_2 & j_2 & \ldots & i_n & j_n \end{pmatrix} \qquad \text{(A8)}$$

be the associated permutation of the partition α. If we define

$$S_\alpha = \mathbf{sgn}(\sigma_\alpha) a_{i_1 j_1} a_{i_2 j_2}\cdots a_{i_n j_n}, \qquad \text{(A9)}$$

[6]For theories containing gauge or flavor groups, we only show the integrands for color ordered partial amplitudes instead of full ones.

then the Pfaffian of the matrix S is given as

$$\mathbf{Pf}S = \sum_{\alpha\in\Pi} S_\alpha. \quad (A10)$$

With the definition of Pfaffian provided above, the reduced Pfaffian of the matrix Ψ is defined as

$$\mathbf{Pf}'\Psi = \frac{(-)^{i+j}}{z_{ij}}\mathbf{Pf}\Psi^{ij}_{ij}, \quad (A11)$$

where the notation Ψ^{ij}_{ij} means the ith and jth rows and columns of the matrix Ψ have been removed (with $1 \le i, j \le n$). It can be proved that this definition is independent of the choice of i and j.

The Parke-Taylor factor $PT(\sigma_1, \ldots, \sigma_n)$ is given as

$$PT(\sigma_1, \ldots, \sigma_n) = \frac{1}{z_{\sigma_1\sigma_2}z_{\sigma_2\sigma_3}\cdots z_{\sigma_{n-1}\sigma_n}z_{\sigma_n\sigma_1}}. \quad (A12)$$

It implies the color ordering $\sigma_1, \ldots, \sigma_n$ for the color ordered amplitude.

The one-loop CHY formulas can be obtained via either the underlying ambitwistor string theory [31,44–50], or the forward limit procedure [32–35]. Here we only introduce the latter one. The one-loop level scattering equations are found to be

$$E_i \equiv \sum_{j\in\{1,2,\ldots,n\}\setminus\{i\}} \frac{k_i\cdot k_j}{z_{ij}} + \frac{k_i\cdot\ell}{z_{i+}} - \frac{k_i\cdot\ell}{z_{i-}} = 0, \quad i\in\{1,\ldots,n\}$$
$$E_+ \equiv \sum_{j=1}^{n}\frac{\ell\cdot k_j}{z_{+j}} = 0, \qquad E_- \equiv \sum_{j=1}^{n}\frac{-\ell\cdot k_j}{z_{-j}} = 0. \quad (A13)$$

These equations yield the massive propagators $1/((\ell + K)^2 - \ell^2)$ in the loop, rather than the desired massless ones $1/(\ell + K)^2$. However, these massive propagators relate to the massless ones through the well known partial fraction identity

$$\frac{1}{D_1\cdots D_m} = \sum_{i=1}^{m}\frac{1}{D_i}\left[\prod_{j\neq i}\frac{1}{D_j - D_i}\right], \quad (A14)$$

which implies

$$\frac{1}{\ell^2(\ell+K_1)^2(\ell+K_1+K_2)^2\cdots(\ell+K_1+\cdots+K_{m-1})^2}$$
$$\simeq \frac{1}{\ell^2}\sum_{i=1}^{m}\left[\prod_{j=i}^{i+m-2}\frac{1}{(\ell+K_i+\cdots+K_j)^2-\ell^2}\right]. \quad (A15)$$

For each individual term at the rhs of the above relation, we have shifted the loop momentum without alternating the result of the Feynman integral. Here $\simeq$ means the lhs and rhs are not equivalent to each other at the integrand level, but are equivalent at the integration level. The lhs of (A15) is the standard propagators in the loop for an individual diagram, while each term at the rhs can be obtained via the forward limit method.

Thus, to obtain the correct one-loop Feynman integrand from the one-loop scattering equations in (A13), one needs to cut each propagator in the loop once, and sum over all resulting objects, as required by the partial fraction relation (A15). For the amplitude without any color ordering, this requirement is satisfied automatically when summing over all possible Feynman diagrams. For the color ordered amplitude, this requirement is satisfied by summing over color orderings cyclically.

As an equivalent interpretation, the forward limit method can also be understood from the dimensional reduction point of view, as studied in [33].

Let us take a brief glance at the CHY integrand at the one-loop level. In the CHY framework, the forward limit operator $\mathcal{F}$ acts on the $(n+2)$-point tree amplitude as follows:

$$\mathcal{F}\mathcal{A}_{n+2} = \mathcal{F}\int d\mu_{n+2}\mathcal{I}^L(\{k,\epsilon,z\})\mathcal{I}^R(\{k,\tilde{\epsilon},z\})$$
$$= \int d\mu'_{n+2}(\mathcal{F}\mathcal{I}^L(\{k,\epsilon,z\}))(\mathcal{F}\mathcal{I}^R(\{k,\tilde{\epsilon},z\})), \quad (A16)$$

where the measure $d\mu'_{n+2}$ is generated from $d\mu_{n+2}$ by turning the scattering equations to those in (A13). Thus the one-loop CHY integrand is determined by

$$\mathcal{I}^L_\circ(\{k,\epsilon,z\}) = \mathcal{F}\mathcal{I}^L(\{k,\epsilon,z\}),$$
$$\mathcal{I}^R_\circ(\{k,\epsilon,z\}) = \mathcal{F}\mathcal{I}^R(\{k,\epsilon,z\}). \quad (A17)$$

Using this statement, the one-loop CHY integrands for GR, YM and BAS are given in Table VI. Here Ψ is a $2(n+2)\times 2(n+2)$ matrix constituted by $\{k_1,\ldots,k_n,k_+,k_-\}$ and $\{\epsilon_1,\ldots,\epsilon_n,\epsilon_+,\epsilon_-\}$. For simplicity, we assume the nodes $+$ and $-$ are located at $(n+1)$th and $(n+2)$th rows and columns, respectively, and the reduced Pfaffian is evaluated by removing them, i.e., $\mathbf{Pf}'\Psi = \frac{(-)}{z_{+-}}\mathbf{Pf}\Psi'$, with $\Psi' = \Psi^{+-}_{+-}$. The one-loop Parke-Taylor factor $PT_\circ(\sigma_1,\ldots,\sigma_n)$ is obtained by summing over tree Parke-Taylor factors cyclically,

$$PT_\circ(\sigma_1,\ldots,\sigma_n) = \sum_{i\in\{1,\ldots,n\}} PT(+,\sigma_i,\ldots,\sigma_{i-1},-). \quad (A18)$$

Notice that since the Parke-Taylor factor only depends on the coordinates of punctures, we have

$$\mathcal{F}PT(+,\sigma_i,\ldots,\sigma_{i-1},-) = PT(+,\sigma_i,\ldots,\sigma_{i-1},-). \quad (A19)$$

TABLE VI. One-Loop Chy Integrands.

Theory	$\mathcal{I}^L_\circ(k_i,\epsilon_i,z_i)$	$\mathcal{I}^R_\circ(k_i,\tilde\epsilon_i,z_i)$
GR	$\mathcal{F}\mathbf{Pf}'\Psi$	$\mathcal{F}\mathbf{Pf}'\Psi$
YM	$PT_\circ(\sigma_1,\ldots,\sigma_n)$	$\mathcal{F}\mathbf{Pf}'\Psi$
BAS	$PT_\circ(\sigma_1,\ldots,\sigma_n)$	$PT_\circ(\sigma'_1,\ldots,\sigma'_n)$

The tree Parke Taylor factor $PT(\cdots)$ at the rhs of (A18) should be understood as $\mathcal{F}PT(\cdots)$. The integrands in Table VI can be found in [31–33,49].

The one-loop CHY formulas in (A16) suffer from the divergence in the forward limit. It was observed in [32] that the solutions of one-loop scattering equations separate into three sectors which are called regular, singular $\mathbf{I}$ and singular $\mathbf{II}$, according to the behavior of punctures $z_\pm$ in the limit $k_+ + k_- \to 0$. In this paper, we will bypass this subtle and crucial point by employing the conclusion in [33], which can be summarized as follows: as long as the CHY integrand is homogeneous in ℓ^μ, the singular solutions contribute to the scaleless integrals which vanish under the dimensional regularization. The homogeneity is manifest for the Parke-Taylor factor. For $\mathcal{F}\mathbf{Pf}'\Psi$, the only place that can violate the homogeneity in ℓ^μ is the diagonal elements in the matrix C, since the deleted rows and columns are chosen to be k_+ and k_-. Singular solutions correspond to $z_+ = z_-$, then it is direct to observe that the dependence on ℓ^μ exactly cancels away, left with a homogeneous CHY integrand. This observation allows us to ignore the problem of singular solutions.

APPENDIX B: SOME DETAILS IN SEC. III A

In this Appendix, we give some technical details omitted in Sec. III A.

We first explain the reason why the above method cannot make sense when tree sEYM amplitudes contain three external gluons. Consider the single-trace tree sEYM amplitude $\mathcal{A}^{\epsilon,\tilde\epsilon}_{\rm sEYM}(+^g,a^g,-^g;\boldsymbol{a}^h_n\backslash a^h)$, with external gluons $+^g$, $-^g$ and a^g. We use the cyclic symmetry of color orderings to rewrite it as $\mathcal{A}^{\epsilon,\tilde\epsilon}_{\rm sEYM}(a^g,-^g,+^g;\boldsymbol{a}^h_n\backslash a^h)$. The new representation indicates that such amplitude can be generated from the tree GR amplitude via differential operators as

$$\begin{aligned}&\mathcal{A}^{\epsilon,\tilde\epsilon}_{\rm sEYM}(a^g,-^g,+^g;\boldsymbol{a}^h_n\backslash a^h)\\&=\mathcal{I}^\epsilon_{a-+}\mathcal{T}^\epsilon_{a+}\mathcal{A}^{\epsilon,\tilde\epsilon}_{\rm GR}(\boldsymbol{a}^h_n\cup\{+^h,-^h\})\\&=(\partial_{\epsilon_-\cdot k_a}-\partial_{\epsilon_-\cdot l})\partial_{\epsilon_a\cdot\epsilon_+}\mathcal{A}^{\epsilon,\tilde\epsilon}_{\rm GR}(\boldsymbol{a}^h_n\cup\{+^h,-^h\}).\end{aligned}\tag{B1}$$

The color ordering is generated by choosing a and $+$ as two reference legs first, then inserting the leg $-$ between them. Here the operator $\partial_{\epsilon_-\cdot l}$ is understood as $\partial_{\epsilon_-\cdot k_+}$. Unfortunately, there is no one-loop level operator $\mathcal{O}^\epsilon_\circ$ satisfying

$$\mathcal{O}^\epsilon_\circ\mathcal{F}\mathcal{A}^{\epsilon,\tilde\epsilon}_{\rm GR}(\boldsymbol{a}^h_n\cup\{+^h,-^h\})=\mathcal{F}\mathcal{I}^\epsilon_{a-+}\mathcal{T}^\epsilon_{a+}\mathcal{A}^{\epsilon,\tilde\epsilon}_{\rm GR}(\boldsymbol{a}^h_n\cup\{+^h,-^h\}).\tag{B2}$$

To see this, we first consider the piece $\partial_{\epsilon_-\cdot k_a}\partial_{\epsilon_a\cdot\epsilon_+}$ in the operator $\mathcal{I}^\epsilon_{a-+}\mathcal{T}^\epsilon_{a+}$. At the tree level, this piece of operator turns $(\epsilon_-\cdot k_a)(\epsilon_a\cdot\epsilon_+)$ to 1, and annihilates all terms which do not contain the Lorentz invariant $(\epsilon_-\cdot k_a)(\epsilon_a\cdot\epsilon_+)$, due to the observation that the amplitude is linear in each polarization vector. Under the action of $\mathcal{E}$, tree level object $(\epsilon_-\cdot k_a)(\epsilon_a\cdot\epsilon_+)$ behaves as

$$\sum_r(\epsilon_a\cdot\epsilon^r_+)(\epsilon^r_-\cdot k_a)=\epsilon_a\cdot k_a,\tag{B3}$$

thus the on-shell condition $\epsilon_a\cdot k_a=0$ indicates that the first piece of the operator does not make sense at the one-loop level. Then we turn to another piece $\partial_{\epsilon_-\cdot l}\partial_{\epsilon_a\cdot\epsilon_+}$. At tree level, this piece turns $(\epsilon_-\cdot l)(\epsilon_a\cdot\epsilon_+)$ to 1, and annihilates all terms which do not contain $(\epsilon_-\cdot l)(\epsilon_a\cdot\epsilon_+)$. However, $\epsilon_-\cdot l$ vanishes under the action of $\mathcal{L}$, since $\epsilon_-\cdot k_-=0$ and $k_-=-k_+=-\ell$. Thus, the second piece of the operator also makes no sense at the one-loop level.

Now we use CHY formulas to prove that the operator $\partial_{\epsilon_b\cdot\ell}\partial_{\epsilon_b\rhd k_b}$ gives no contribution at the one-loop level. This operator does not act on the measure of CHY contour integration, thus can be applied to the CHY integrands directly. As noted in Appendix A, in this paper the convention for the reduced Pfaffian of Ψ is $\mathbf{Pf}'\Psi=\frac{(-)}{z_{+-}}\mathbf{Pf}\Psi'$, where $\Psi'=\Psi^{+-}_{+-}$. Because of the choice $\Psi'=\Psi^{+-}_{+-}$, when acting $\partial_{\epsilon_b\cdot\ell}$ on $\mathcal{F}\mathbf{Pf}'\Psi$, the nonvanishing contribution only from acting $\partial_{\epsilon_b\cdot\ell}$ on the diagonal element C_{bb} in the block C of the matrix Ψ. However, the operator $\partial_{\epsilon_a\rhd k_b}$ turns $\epsilon_a\rhd k_b$ to 1 and annihilates all terms which do not contain $\epsilon_a\rhd k_b$, thus eliminates bth rows and columns in Ψ'. Thus C_{bb} does not exist in $\partial_{\epsilon_a\rhd k_b}\mathcal{F}\mathbf{Pf}'\Psi$, therefore the operator $\partial_{\epsilon_b\cdot\ell}$ annihilates $\partial_{\epsilon_a\rhd k_b}\mathbf{I}^{\epsilon,\tilde\epsilon}_{{\rm GR};\circ}(\boldsymbol{a}^h_n)$.

Then, we give the derivation of (37). Let us go back to the four-gluon example, and use the cyclic symmetry of color ordering, as well as the tree level transmutation operators, to understand the four-gluon tree amplitude as

$$\begin{aligned}&\mathcal{A}^{\epsilon,\tilde\epsilon}_{\rm sEYM}(-^g,+^g,a^g,b^g;\boldsymbol{a}^h_n\backslash\{a^h,b^h\})\\&=\mathcal{I}^\epsilon_{-+a}\mathcal{I}^\epsilon_{-ab}\mathcal{T}^\epsilon_{-b}\mathcal{A}^{\epsilon,\tilde\epsilon}_{\rm GR}(\boldsymbol{a}^h_n\cup\{+^h,-^h\})\\&=(-\partial_{\epsilon_+\cdot\ell}-\partial_{\epsilon_+\cdot k_a})(-\partial_{\epsilon_a\cdot\ell}-\partial_{\epsilon_a\cdot k_b})\partial_{\epsilon_-\cdot\epsilon_b}\mathcal{A}^{\epsilon,\tilde\epsilon}_{\rm GR}(\boldsymbol{a}^h_n\cup\{+^h,-^h\}).\end{aligned}\tag{B4}$$

Here the color ordering is generated by choosing reference legs $-$ and b, inserting the leg a between $-$ and b, and inserting $+$ between $-$ and a. The operators $-\partial_{\epsilon_i\cdot\ell}$ for $i=+,a$ are understood as $\partial_{\epsilon_i\cdot k_-}$. In the expression (B4), the operator $\partial_{\epsilon_+\cdot k_a}\partial_{\epsilon_-\cdot\epsilon_b}$ acts on $(\epsilon_+\cdot k_a)(\epsilon_-\cdot\epsilon_b)$, thus yields the one-loop level operator which acts on

$\epsilon_b \lhd k_a \equiv \sum_r (\epsilon_b \cdot \epsilon_-^r)(\epsilon_+^r \cdot k_a)$. Following the discussion for the $\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\text{sEYM}}(a^g, b^g, -^g, +^g; \boldsymbol{a}_n^h \backslash \{a^h, b^h\})$ case, and using the completely similar method, one obtains

$$(-\partial_{\epsilon_a \cdot k_b})(-\partial_{\epsilon_b \lhd k_a}) \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{GR};\circ}(\boldsymbol{a}_n^h) = \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g, a^g, b^g, -^g; \boldsymbol{a}_n^h \backslash \{a^h, b^h\}), \tag{B5}$$

and equivalently,

$$\partial_{\epsilon_b \cdot k_a} \partial_{\epsilon_a \lhd k_b} \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{GR};\circ}(\boldsymbol{a}_n^h) = (-)^2 \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g, b^g, a^g, -^g; \boldsymbol{a}_n^h \backslash \{a^h, b^h\}). \tag{B6}$$

It is straightforward to generalize the relation (B6) to (37)

$$\left(\prod_{i=1}^{m-1} \partial_{\epsilon_{a_{i+1}} \cdot k_{a_i}}\right) \partial_{\epsilon_{a_1} \lhd k_{a_m}} \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{GR};\circ}(\boldsymbol{a}_n^h) = (-)^m \mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g, a_m^g, \ldots, a_1^g, -^g; \boldsymbol{a}_n^h \backslash \boldsymbol{a}_m^h), \tag{B7}$$

with $m \geq 2$.

Finally, we prove the equality (38) by using CHY formulas. For our purpose, it is sufficient to show that $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m} \mathbf{Pf}'\Psi(\epsilon_i, k_i, z_i) = 0$. Let us consider the four-gluon example, i.e., act $\partial_{\epsilon_b \cdot k_a} \partial_{\epsilon_a \cdot k_b}$ on $\mathbf{Pf}\Psi'$. The operator $\partial_{\epsilon_a \cdot k_b}$ acts on both C_{aa} and C_{ba} in the block C. For the first case, $\partial_{\epsilon_a \cdot k_b}$ transmutes $\mathbf{Pf}\Psi'$ as

$$\mathbf{Pf}\Psi' \to \frac{-1}{z_{ba}} \mathbf{Pf}(\Psi')^{a(n+a)}_{a(n+a)}, \tag{B8}$$

up to a sign, while for the second case we have

$$\mathbf{Pf}\Psi' \to \frac{1}{z_{ba}} \mathbf{Pf}(\Psi')^{b(n+a)}_{b(n+a)}, \tag{B9}$$

again up to a sign. The relative sign will be considered later. For $(\Psi')^{a(n+a)}_{a(n+a)}$, the operator $\partial_{\epsilon_b \cdot k_a}$ only acts on C_{bb}, thus provides

$$\frac{-1}{z_{ba}} \mathbf{Pf}(\Psi')^{a(n+a)}_{a(n+a)} \to \frac{1}{z_{ab} z_{ba}} \mathbf{Pf}(\Psi')^{a(n+a)b(n+b)}_{a(n+a)b(n+b)}, \tag{B10}$$

up to a sign. For $(\Psi')^{b(n+a)}_{b(n+a)}$, the operator $\partial_{\epsilon_b \cdot k_a}$ only acts on C_{ab}, thus gives

$$\frac{1}{z_{ba}} \mathbf{Pf}(\Psi')^{b(n+a)}_{b(n+a)} \to \frac{1}{z_{ab} z_{ba}} \mathbf{Pf}(\Psi')^{b(n+a)a(n+b)}_{b(n+a)a(n+b)} = \frac{1}{z_{ab} z_{ba}} \mathbf{Pf}(\Psi')^{a(n+a)b(n+b)}_{a(n+a)b(n+b)}, \tag{B11}$$

up to a sign. It seems that after performing $\partial_{\epsilon_b \cdot k_a} \partial_{\epsilon_a \cdot k_b}$, we arrive at the Pfaffians of two equivalent matrices, with the same coefficient. However, we have not considered the relative sign until now. Notice that (B10) is obtained by turning elements $(\Psi')_{a(n+a)}$ and $(\Psi')_{b(n+b)}$ to 1, and eliminating all terms in $\mathbf{Pf}\Psi'$ that do not contain both two elements, while (B11) is obtained by turning $(\Psi')_{b(n+a)}$ and $(\Psi')_{a(n+b)}$ to 1, and eliminating all terms that do not contain both of them. Comparing $(\Psi')_{a(n+a)}(\Psi')_{b(n+b)}$ with $(\Psi')_{b(n+a)}(\Psi')_{a(n+b)}$, the permutation from the ordering $a(n+a)b(n+b)$ to $b(n+a)a(n+b)$ is odd. Thus, when considering $\partial_{\epsilon_b \cdot k_a} \partial_{\epsilon_a \cdot k_b} \mathbf{Pf}\Psi'$, contributions from (B10) and (B11) cancel each other, due to the definition of Pfaffian. Consequently, $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_2} \mathbf{Pf}'\Psi = 0$ for $\vec{\boldsymbol{a}}_2 = \langle a, b \rangle$. The above argument can be generalized to the general ordered set $\vec{\boldsymbol{a}}_m = \langle a_1, \ldots, a_m \rangle$ directly, thus we get the conclusion (38).

APPENDIX C: VERIFICATION OF (39) VIA CHY FORMULAS

This Appendix is devoted to verifying the transmutation relation (39) by using CHY formulas.

The operators under consideration are in the form $\partial_{\epsilon_a \cdot V}$, thus are commutable with the CHY contour integration. Thus we can act the operators on the CHY integrands directly without altering the measure. Since the operator $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ only depends on polarization vectors in $\{\epsilon_i\}$, it only acts on $\mathcal{F}\mathbf{Pf}'\Psi(\epsilon_i, k_i, z_i)$, without altering the $\mathcal{F}\mathbf{Pf}'\Psi(\tilde{\epsilon}_i, k_i, z_i)$ part. Without lose of generality, let us assume $\vec{\boldsymbol{a}}_m = \langle 1, \ldots, m \rangle$. The nonvanishing contributions arise from acting $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ on terms containing $\epsilon_i \rhd k_{i-1}$, or $\epsilon_i \lhd k_{i-1}$ with $i \in \{1, \ldots, m\}$, as indicated by the equality (38). Based on the discussion above, now we consider the effect of acting $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ on terms containing $\epsilon_1 \rhd k_m$, the treatments for other terms are analogous.

The factor $\epsilon_1 \rhd k_m$ in $\mathcal{F}\mathbf{Pf}\Psi'$ arises from doing the summation $\sum_r (\epsilon_+^r)^\mu (\epsilon_-^r)^\nu$ for $(\epsilon_1 \cdot \epsilon_+)(\epsilon_- \cdot k_m)$ in $\mathbf{Pf}'\Psi(\epsilon_i, k_i, z_i)$. Thus, acting the operator $\partial_{\epsilon_1 \cdot k_m}$ on $\epsilon_1 \rhd k_m$ transmutes $\mathcal{F}\mathbf{Pf}\Psi'$ as

$$\mathcal{F}\mathbf{Pf}\Psi' \to \frac{1}{z_{1+} z_{m-}} \mathcal{F}\mathbf{Pf}(\Psi')^{1m(2n+1)(2n+2)}_{1m(2n+1)(2n+2)}, \tag{C1}$$

where the later matrix is obtained from Ψ' by deleting 1th, mth, $(2n+1)$th, $(2n+2)$th rows and columns. More explicitly, using the definition of Pfaffian, one can expand $\mathbf{Pf}\Psi'$ as

$$\mathbf{Pf}\Psi' = \sum_{\alpha \in \Pi} \text{sgn}(\sigma_\alpha) (\Psi')_{a_1 b_1} (\Psi')_{a_2 b_2} \cdots (\Psi')_{a_n b_n}. \tag{C2}$$

We divided terms in the summation at the rhs of (C2) into two classes, terms in the first class are those that do not contain both $(\Psi')_{1(2n+1)}$ and $(\Psi')_{m(2n+2)}$, while terms in the second class do contain both $(\Psi')_{1(2n+1)}$ and $(\Psi')_{m(2n+2)}$. The operator $\partial_{\epsilon_1 \cdot k_m}$ annihilates all terms in the first class. Notice that since the matrix Ψ' is generated from the original Ψ by removing $(n+1)$th and $(n+2)$th rows and columns, $\partial_{\epsilon_1\cdot\epsilon_+}$ and $\partial_{\epsilon_-\cdot k_m}$ cannot act on diagonal elements $C_{(n+1)(n+1)}$ and $C_{(n+2)(n+2)}$ of the block C. While acting on terms in the second class, the operator $\partial_{\epsilon_1\cdot k_m}$ transmutes $\mathcal{F}[(\Psi')_{1(2n+1)}(\Psi')_{m(2n+2)}]$ as

$$\partial_{\epsilon_1\cdot k_m}\mathcal{F}[(\Psi')_{1(2n+1)}(\Psi')_{m(2n+2)}] = \partial_{\epsilon_1\cdot k_m}\frac{\epsilon_1 \rhd k_m}{z_{1+}z_{m-}} = \frac{1}{z_{1+}z_{m-}}. \tag{C3}$$

Now we collect terms in the second class together, and remove $(\Psi')_{1(2n+1)}(\Psi')_{m(2n+2)}$ in them. By the definition of Pfaffian, the resulting terms after performing the above manipulation can be regrouped as $\mathbf{Pf}(\Psi')^{1m(2n+1)(2n+2)}_{1m(2n+1)(2n+2)}$, up to an irrelevant overall sign arises from changing $\mathbf{sgn}(\sigma_\alpha)$. Thus we get the result (C1).

Then, we act the operator $\partial_{k_1\cdot\epsilon_2}$ on $\mathcal{F}\mathbf{Pf}(\Psi')^{1m(2n+1)(2n+2)}_{1m(2n+1)(2n+2)}$. Since the matrix $(\Psi')^{1m(2n+1)(2n+2)}_{1m(2n+1)(2n+2)}$ does not include 1th rows and columns, $\partial_{k_1\cdot\epsilon_2}$ only acts on the diagonal element C_{22}, thus transmutes $\mathcal{F}\mathbf{Pf}(\Psi')^{1m(2n+1)(2n+2)}_{1m(2n+1)(2n+2)}$ as

$$\mathcal{F}\mathbf{Pf}(\Psi')^{1m(2n+1)(2n+2)}_{1m(2n+1)(2n+2)} \to \frac{1}{z_{21}}\mathcal{F}\mathbf{Pf}(\Psi')^{12m(2n+1)(2n+2)}_{12m(2n+1)(2n+2)}, \tag{C4}$$

up to an overall sign. Now the recursive pattern occurs, the operator $\partial_{k_2\cdot\epsilon_3}$ only acts on C_{33}, due to the observation $(\Psi')^{12m(2n+1)(2n+2)}_{12m(2n+1)(2n+2)}$ does not include 2th rows and columns. Thus, by iterating the above manipulation, one finally arrives at

$$\mathcal{F}\mathbf{Pf}\Psi' \to \frac{1}{z_{1+}z_{m-}}\prod_{i=1}^{m-1}\frac{1}{z_{(i+1)i}}, \tag{C5}$$

therefore

$$\mathcal{F}\mathbf{Pf}'\Psi(\epsilon_i, k_i, z_i) \to PT(+,1,\ldots,m,-), \tag{C6}$$

up to an irrelevant overall sign.

The resulting object $PT(+,1,\ldots,m,-)$, together with $\mathcal{F}\mathbf{Pf}'\Psi(\tilde{\epsilon}_i, k_i, z_i)$, gives rise to the ssEYM one-loop CHY integrand, which leads to the partial Feynman integrand $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM}(+^g, 1^g, \ldots, m^g, -^g; \boldsymbol{a}^h_n\setminus\boldsymbol{a}^h_m)$. Similar manipulation shows that acting $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ on terms containing $\epsilon_1 \lhd k_m$ yields $(-)^m\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM}(+^g, m^g, \ldots, 1^g, -^g; \boldsymbol{a}^h_n\setminus\boldsymbol{a}^h_m)$. The relative sign $(-)^m$ arises as follows. The terms containing $\epsilon_1 \rhd k_m$ are those containing

$$\mathcal{F}[(\Psi')_{1(2n+1)}(\Psi')_{m(2n+2)}] = \frac{\epsilon_1 \rhd k_m}{z_{1+}z_{m-}}, \tag{C7}$$

while terms containing $\epsilon_1 \lhd k_m$ are those containing

$$\mathcal{F}[(\Psi')_{1(2n+2)}(\Psi')_{m(2n+1)}] = \frac{\epsilon_1 \lhd k_m}{z_{1-}z_{m+}}, \tag{C8}$$

the difference between these two objects determines the relative sign. First

$$\begin{aligned}\frac{1}{z_{1+}z_{m-}}\prod_{i=1}^{m-1}\frac{1}{z_{(i+1)i}} &= (-)^m PT(+,1,\ldots,m,-),\\ \frac{1}{z_{m+}z_{1-}}\prod_{i=1}^{m-1}\frac{1}{z_{(i+1)i}} &= (-)PT(+,m,\ldots,1,-).\end{aligned} \tag{C9}$$

Comparing them gives a relative sign $(-)^{m-1}$. Second, comparing elements $(\Psi')_{1(2n+1)}(\Psi')_{m(2n+2)}$ with elements $(\Psi')_{1(2n+2)}(\Psi')_{m(2n+1)}$, the permutation from the ordering $1(2n+1)m(2n+2)$ to $1(2n+2)m(2n+1)$ contributes a relative $(-)$, due to the definition of Pfaffian.

The above results can be generalized to $\epsilon_i \rhd k_{i-1}$ or $\epsilon_i \lhd k_{i-1}$ for arbitrary $i \in \{1,\ldots,m\}$ via the replacement $1 \to i$, due to the cyclic symmetry. Collecting all pieces together, we get the desired conclusion (39). Since our method only transmutes $\mathcal{F}\mathbf{Pf}'\Psi(\epsilon_i, k_i, z_i)$, one can claim that the relation (43) which links YM and ssYMS Feynman integrands together has also been verified.

APPENDIX D: SOLVING RECURSIVE EXPANSION OF EYM

For the readers' convenience, in this Appendix we show the details of solving the recursive expansions of ssEYM Feynman integrands by employing the one-loop level differential operators.

Consider the ssEYM partial Feynman integrand $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM}(+^g, \vec{\boldsymbol{a}}^g_{n-m}, -^g; \boldsymbol{a}^h{}_m)$ with $n-m$ gluons and m gravitons. Here we denote the ordered set of gluons $\vec{\boldsymbol{a}}^g_{n-m}$ as $\vec{\boldsymbol{a}}_{n-m} = \langle 1,\ldots,n-m\rangle$, and label the gravitons in $\boldsymbol{a}^h_m$ as $\boldsymbol{a}_m = \{h_1,\ldots,h_m\}$. The Lorentz invariance, together with the assumption that polarization vectors in $\{\epsilon_i\}$ and $\{\tilde{\epsilon}_i\}$ cannot contract with each other, indicates the polarization vector ϵ_{h_m} can only appear in the following combinations, which are $\epsilon_{h_m}\cdot k_b$, $\epsilon_{h_m}\cdot k_{h_g}$, and $\epsilon_{h_m}\cdot\epsilon_{h_g}$, with $g \in \{1,\ldots,m-1\}$. Since $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM}(+^g, \vec{\boldsymbol{a}}^g_{n-m}, -^g; \boldsymbol{a}^h_m)$ is assumed to be linear in each polarization vector, the partial integrand $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM}(+^g, \vec{\boldsymbol{a}}^g_{n-m}, -^g; \boldsymbol{a}^h_m)$ can be expanded as

$$\begin{aligned}&\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM}(+^g, \vec{\boldsymbol{a}}^g_{n-m}, -^g; \boldsymbol{a}^h_m)\\ &\quad= (\epsilon_{h_m}\cdot\ell)B_+ + \sum_{b=1}^{n-m}(\epsilon_{h_m}\cdot k_b)B_b\\ &\qquad+ \sum_{g=1}^{m-1}(\epsilon_{h_m}\cdot k_{h_g})(\epsilon_{h_g}\cdot C_g) + \sum_{g=1}^{m-1}(\epsilon_{h_m}\cdot\epsilon_{h_g})D_g.\end{aligned} \tag{D1}$$

At the tree level, one can regard one of k_b or k_{h_g} as the unindependent one due to the momentum conservation, and remove the corresponding $\epsilon_{h_m}\cdot k_b$ or $\epsilon_{h_m}\cdot k_{h_g}$, as can be seen in [26]. In the current one-loop case, we do not do this procedure at the rhs of (D1), based on the following reason. At the tree level, if one replaces any $\epsilon_i\cdot k_j$ by $-\epsilon_i\cdot(\sum_{l\neq j}k_l)$, the resulting objects after performing the insertion operators will not be modified, as can be observed from the definition of operators. At the one-loop level, a similar statement does not hold for our one-loop level insertion operators $\mathcal{I}^{\epsilon}_{+a_ia_{i+1}}$, as well as the new operators $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$. Thus, in this and the next subsections, we do not use the momentum conservation to change the representations of the Feynman integrands. It means we solve the special expanded formulas with special representations of external momenta. On the other hand, we think of the momentum $k_-=-\ell$ as being removed, via $k_-=-k_+=-\ell$, and $\epsilon_{h_m}\cdot\ell$ in (D1) should be understood as $\epsilon_{h_m}\cdot k_+$, since the effects of acting $\mathcal{I}^{\epsilon}_{+a_i-}$, $\mathcal{I}^{\epsilon}_{+a_ia_{i+1}}$ and $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}$ will not be altered if replacing k_- in the Feynman integrands by $-k_+$.

Our aim is to use the differential equations provided by transmutational relations, together with the gauge invariance requirement, to solve coefficients B_+, B_b, C^{μ}_g, D_g in (D1). The desired solutions are formulas of these coefficients consisted by Lorentz invariants such as $\epsilon_{h_m}\cdot k_b$, $\epsilon_{h_m}\cdot k_{h_g}$, $\epsilon_{h_m}\cdot\epsilon_{h_g}$, and physically meaningful objects such as ssEYM Feynman integrands. We do not care about the exact expressions of these physical objects appearing in solutions. As will be seen, such solutions of B_+, B_b, C^{μ}_g, D_g naturally give the recursive expansion of $\mathbf{I}^{\epsilon,\tilde\epsilon}_{\rm ssEYM}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^h_m)$ to partial ssEYM integrands with less gravitons.

The first line on the rhs of (D1) can be detected by the insertion operators. For convenience, we denote nodes $+$ and $-$ as 0 and $n-m+1$, respectively. Acting $\mathcal{I}^{\epsilon}_{ih_m(i+1)}$ with $i\in\{0,\dots,n-m\}$ on the lhs of (D1) gives

$$\mathcal{I}^{\epsilon}_{ih_m(i+1)}\mathbf{I}^{\epsilon,\tilde\epsilon}_{\rm ssEYM}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^h_m)\\ =\mathbf{I}^{\epsilon,\tilde\epsilon}_{\rm ssEYM}(0^g,\dots,i^g,h^g_m,(i+1)^g,\dots,(n-m+1)^g;\boldsymbol{a}^h_m\backslash h^h_m). \tag{D2}$$

While acting on the rhs, these operators annihilate the second line, and transmute the first line as follows:

$$\mathcal{I}^{\epsilon}_{ih_m(i+1)}\left(\sum_{b=1}^{n-m}(\epsilon_{h_m}\cdot k_b)B_b\right)=\begin{cases}B_i-B_{i+1}, & \text{if } i\leq n-m-1,\\ B_i, & \text{if } i=n-m.\end{cases} \tag{D3}$$

When applying $\mathcal{I}^{\epsilon}_{(n-m)h_m(n-m+1)}$, we have used the assumption that $k_-=-\ell$ has been removed by using momentum conservation thus the effective part of the operator is $\partial_{\epsilon_{h_m}\cdot k_{n-m}}$. Comparing the lhs result (D2) with the rhs result (D3) provides

$$B_{n-m}=\mathbf{I}^{\epsilon,\tilde\epsilon}_{\rm ssEYM}(0^g,\dots,(n-m)^g,h^g_m,(n-m+1)^g;\boldsymbol{a}^h_m\backslash h^h_m), \tag{D4}$$

and

$$B_i=B_{i+1}+\mathbf{I}^{\epsilon,\tilde\epsilon}_{\rm ssEYM}(0^g,\dots,i^g,h^g_m,(i+1)^g,\dots,(n-m+1)^g;\boldsymbol{a}^h_m\backslash h^h_m),\\ \text{for } i\in\{0,\dots,n-m-1\}. \tag{D5}$$

Thus B_b with $b\in\{0,\dots,n-m\}$ can be calculated recursively as

$$B_b=\sum_{i=b}^{n-m}\mathbf{I}^{\epsilon,\tilde\epsilon}_{\rm ssEYM}(+^g,1^g,\dots,i^g,h^g_m,(i+1)^g,\dots,(n-m)^g,-^g;\boldsymbol{a}^h_m\backslash h^h_m). \tag{D6}$$

Substituting the above solution into (D1), the first line at the rhs is obtained as

$$\sum_{b=0}^{n-m}(\epsilon_{h_m}\cdot k_b)B_b=\sum_{\sqcup\!\sqcup}(\epsilon_{h_m}\cdot Y_{h_m})\mathbf{I}^{\epsilon,\tilde\epsilon}_{\rm ssEYM}(+^g,h^g_m\sqcup\!\sqcup\,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^h_m\backslash h^h_m). \tag{D7}$$

The combinatory momentum Y_i is defined as the summation of momenta of gluons at the lhs of the leg i^g in the color ordering [22]. The summation over all possible shuffles $\sqcup\!\sqcup$ of two ordered sets $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{s}}$ is the summation over all permutations of $\vec{\boldsymbol{a}}\cup\vec{\boldsymbol{s}}$, those preserving the orderings of $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{s}}$. For example, $\langle 1,2\rangle\sqcup\!\sqcup\langle 3,4\rangle$ includes the following ordered sets: $\langle 2,3,4,5\rangle$, $\langle 2,4,3,5\rangle$, $\langle 2,4,5,3\rangle$, $\langle 4,2,3,5\rangle$, $\langle 4,2,5,3\rangle$, $\langle 4,5,2,3\rangle$.

Now we arrive at

$$\begin{aligned}&\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^h_m)\\&=\sum_{\shuffle}(\epsilon_{h_m}\cdot Y_{h_m})\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+,h^g_m\shuffle\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^h_m\backslash h^h_m)\\&\quad+\sum_{g=1}^{m-1}(\epsilon_{h_m}\cdot k_{h_g})(\epsilon_{h_g}\cdot C_g)+\sum_{g=1}^{m-1}(\epsilon_{h_m}\cdot\epsilon_{h_g})D_g.\end{aligned}\tag{D8}$$

To continue, we study the relation between C^μ_g and D_g, by imposing the gauge invariance of the graviton h_g. Notice that direct replacing $\epsilon_{h_g}\to k_{h_g}$ makes the treatment complicated, since each term includes ϵ_{h_g}. To single out the C^μ_g and D_g terms, a convenient way is to use the Ward's identity operator defined in (41). The key point is the gauge invariance condition

$$\mathcal{W}^\epsilon_{h_g}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^h_m)=0\tag{D9}$$

indicates

$$\mathcal{I}^\epsilon_{h_gh_m-}\mathcal{W}^\epsilon_{h_g}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^h_m)=0,\tag{D10}$$

and the later one is equivalent to

$$\begin{aligned}&\mathcal{W}^\epsilon_{h_g}\mathcal{I}^\epsilon_{h_gh_m-}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^h_m)\\&\quad+(\mathcal{I}^\epsilon_{h_gh_m-}\mathcal{W}^\epsilon_{h_g})\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^h_m)=0,\end{aligned}\tag{D11}$$

then we get the equation

$$\begin{aligned}&\mathcal{W}^\epsilon_{h_g}\mathcal{I}_{h_gh_m-}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^h_m)\\&\quad+\partial_{\epsilon_{h_m}\cdot\epsilon_{h_g}}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^h_m)=0.\end{aligned}\tag{D12}$$

The assumption that $k_-=-\ell$ is removed via the momentum conservation has been used again. Substituting the rhs of (D8) into the above equation (D12), we obtain $D_g=-(k_{h_g}\cdot C_g)$, thus

$$(\epsilon_{h_m}\cdot k_{h_g})(\epsilon_{h_g}\cdot C_g)+(\epsilon_{h_m}\cdot\epsilon_{h_g})D_g=\epsilon_{h_m}\cdot f_{h_g}\cdot C_g.\tag{D13}$$

and

$$\begin{aligned}&\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^h_m)\\&=\sum_{\shuffle}(\epsilon_{h_m}\cdot Y_{h_m})\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,h^g_m\shuffle\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^h_m\backslash h^h_m)\\&\quad+\sum_{g=1}^{m-1}(\epsilon_{h_m}\cdot f_{h_g}\cdot C_g),\end{aligned}\tag{D14}$$

where the antisymmetric strength tensors are defined as $f^{\mu\nu}_i\equiv k^\mu_i\epsilon^\nu_i-\epsilon^\mu_ik^\nu_i$, $\tilde{f}^{\mu\nu}_i\equiv k^\mu_i\tilde{\epsilon}^\nu_i-\tilde{\epsilon}^\mu_ik^\nu_i$.

Until now, one remaining class of coefficients C^μ_g has not been fixed. To solve it, we first need to find the equations satisfied by C^μ_g. This goal can be reached by applying two insertion operators, one acts on $\epsilon_{h_m}\cdot k_{h_q}$ at the rhs of (D14), therefore selects the term $\epsilon_{h_q}\cdot C_q$, and another one acts on $\epsilon_{h_q}\cdot C_q$ to provide the equation for solving C^μ_q. We hope two operators transmute the lhs of (D14) in an appropriate way so that the obtained object is physically meaningful. Based on the above discussion, the combinatory operators $\mathcal{I}^\epsilon_{jh_q(j+1)}\mathcal{I}^\epsilon_{jh_mh_q}$ with $j\in\{0,\ldots,n-m\}$ are nice candidates. Again, we have denoted nodes $+$ and $-$ as 0 and $n-m+1$, respectively.

When applying $\mathcal{I}^\epsilon_{jh_q(j+1)}\mathcal{I}^\epsilon_{jh_mh_q}$ to the lhs of (D14), we use the observation $[\mathcal{I}^\epsilon_{jh_q(j+1)},\mathcal{I}^\epsilon_{jh_mh_q}]=0$ to get the physically meaningful result:

$$\begin{aligned}\mathcal{I}^\epsilon_{jh_q(j+1)}\mathcal{I}^\epsilon_{jh_mh_q}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}^g_m)&=\mathcal{I}^\epsilon_{jh_mh_q}\mathcal{I}^\epsilon_{jh_q(j+1)}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(0^g,\vec{\boldsymbol{a}}^g_{n-m},(n-m+1)^g;\boldsymbol{a}^h_m)\\&=\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{EYM}}(0^g,\ldots,j^g,h^g_m,h^g_q,(j+1)^g,\ldots,(n-m+1)^g;\boldsymbol{a}^h_m\backslash\{h^h_m,h^h_q\}).\end{aligned}\tag{D15}$$

While acting on the rhs, the operator $\mathcal{I}^\epsilon_{jh_q(j+1)}\mathcal{I}^\epsilon_{jh_mh_q}$ leads to

$$\begin{aligned}&\mathcal{I}^\epsilon_{jh_q(j+1)}\left(\sum_{i=j}^{n-m}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(0^g,\ldots,i^g,h^g_m,(i+1)^g,\ldots,(n-m+1)^g;\boldsymbol{a}^h_m\backslash h^h_m)-(\epsilon_{h_q}\cdot C_q)\right)\\&=\sum_{\shuffle}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(0^g,\ldots,j^g,h^g_q,h^g_m\shuffle\langle(j+1)^g,\ldots,(n-m)^g\rangle,(n-m+1)^g;\boldsymbol{a}^h_m\backslash\{h^h_m,h^h_q\})\\&\quad+\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(0^g,\ldots,j^g,h^g_m,h^g_q,(j+1)^g,\ldots,(n-m+1)^a;\boldsymbol{a}^h_m\backslash\{h^h_m,h^h_q\})-\mathcal{I}^\epsilon_{jh_q(j+1)}(\epsilon_{h_q}\cdot C_q),\end{aligned}\tag{D16}$$

where the splitting $\mathcal{I}^\epsilon_{jh_q(j+1)}=\mathcal{I}^\epsilon_{jh_qh_m}+\mathcal{I}^\epsilon_{h_mh_q(j+1)}$ due to the definition has been used when acting on $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(0^g,\ldots,j^g,h^g_m,(j+1)^g,\ldots,(n-m+1)^g;\boldsymbol{a}^h_m\backslash h^h_m)$. Comparing two sides gives the desired equation:

$$\mathcal{I}^{\epsilon}_{jh_q(j+1)}(\epsilon_{h_q}\cdot C_q)=\sum_{\shuffle}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(0^g,\ldots,j^g,h_q^g,h_m^g\shuffle\{(j+1)^g,\ldots,(n-1)^g\},(n-m+1)^g;\boldsymbol{a}_m^h\backslash\{h_m^h,h_q^h\}),\tag{D17}$$

which holds for arbitrary $j\in\{0,\ldots,n-m\}$. Equation (D17) bears strong similarity with the insertion relation for $(n-1)$-point partial integrands,

$$\mathcal{I}^{\epsilon}_{jh_q(j+1)}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(0^g,\ldots,(n-m+1)^g;\boldsymbol{a}_{m-1}^h)=\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(0^g,\ldots,j^g,h_q^g,(j+1)^g,\ldots,(n-m+1)^g;\boldsymbol{a}_{m-1}^h\backslash h_q^h),\tag{D18}$$

therefore it is natural to expect the previous technique can be applied to the current case. The Lorentz invariant $(\epsilon_{h_q}\cdot C_q)$ contains polarization vectors ϵ_{h_p} with $p\neq q$, thus can be divided as

$$\epsilon_{h_q}\cdot C_q=\sum_{b=0}^{n-m}(\epsilon_{h_q}\cdot k_b)B'_b+\sum_{h_p\in\boldsymbol{a}_m^h\backslash\{h_m,h_q\}}((\epsilon_{h_q}\cdot k_{h_p})(\epsilon_{h_p}\cdot C'_p)+(\epsilon_{h_q}\cdot\epsilon_{h_p})D'_p).\tag{D19}$$

It is worth emphasizing that the above expansion does not include the $\epsilon_{h_q}\cdot k_{h_m}$ term, due to the following reason. Combining this term with the coefficient of $\epsilon_{h_q}\cdot C_q$ in (D14) gives the combination $(\epsilon_{h_m}\cdot k_{h_q})(\epsilon_{h_q}\cdot k_{h_m})$, which is forbidden by observation (38), since the partial integrand $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}_m^h)$ does not include any $\epsilon_i\rhd k_j$ or $\epsilon_i\lhd k_j$. The strict proof can be seen in (E5) and the related discussions in the next Appendix, for the more general case. The coefficients B'_b can be solved by using $\mathcal{I}^{\epsilon}_{jh_q(j+1)}$, and is found to be

$$\sum_{b=0}^{n-1}(\epsilon_{h_q}\cdot k_b)B'_b=\sum_{\shuffle}(\epsilon_{h_q}\cdot Y_{h_q})\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\{h_q^g,h_m^g\}\shuffle\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}_m^h\backslash\{h_m^h,h_q^h\}).\tag{D20}$$

This equality serves as the analog of (D7). The gauge invariance condition $\mathcal{W}_{h_p}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}_m^h)=0$ requires $(\epsilon_{h_q}\cdot C_q)$ to be gauge invariant for the leg h_p, thus one can impose the gauge invariance to relate coefficients of $(\epsilon_{h_q}\cdot k_{h_p})$ and $(\epsilon_{h_q}\cdot\epsilon_{h_p})$ together as

$$(\epsilon_{h_q}\cdot k_{h_p})(\epsilon_{h_p}\cdot C'_p)+(\epsilon_{h_q}\cdot\epsilon_{h_p})D'_p=\epsilon_{h_q}\cdot f_{h_p}\cdot C'_p.\tag{D21}$$

This equality serves as the analog of (D13).

Substituting (D20) and (D21) into (D14) yields

$$\begin{aligned}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}_m^h)&=\sum_{\shuffle}(\epsilon_{h_m}\cdot Y_{h_m})\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,h_m^g\shuffle\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}_m^h\backslash h_m^h)\\&\quad+\sum_{g=1}^{m-1}\sum_{\shuffle}(\epsilon_{h_m}\cdot f_{h_g}\cdot Y_{h_g})\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\langle h_g^g,h_m^g\rangle\shuffle\vec{\boldsymbol{a}}^g_{n-m},-;\boldsymbol{a}_m^h\backslash\{h_m^h,h_g^h\})\\&\quad+\sum_{g=1}^{m-1}\sum_{h_p\in\boldsymbol{a}_m^h\backslash\{h_m,h_g\}}\epsilon_{h_m}\cdot f_{h_g}\cdot f_{h_p}\cdot C'_p.\end{aligned}\tag{D22}$$

Solving the coefficients $(C'_p)^\mu$ is completely analogous as solving C_g^μ. Now the recursive pattern is manifested. By iterating the above manipulation, the full expansion of $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}_m^h)$ can finally be obtained as

$$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\vec{\boldsymbol{a}}^g_{n-m},-^g;\boldsymbol{a}_m^h)=\sum_{\vec{\boldsymbol{s}}:\boldsymbol{s}\subseteq\boldsymbol{a}_m^h\backslash h_m}\sum_{\shuffle}K^{\epsilon}_{\vec{\boldsymbol{s}}}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\text{ssEYM}}(+^g,\langle\vec{\boldsymbol{s}}^g,h_m^g\rangle\shuffle\vec{\boldsymbol{a}}^g_{n-m},-;\boldsymbol{a}_m^h\backslash\{h_m^h,\boldsymbol{s}^h\}),\tag{D23}$$

where the summation is over all ordered sets $\vec{\boldsymbol{s}}$ with $\boldsymbol{s}\subseteq\boldsymbol{a}_m^h\backslash h_m$, and $K^{\epsilon}_{\vec{\boldsymbol{s}}}$ is defined as

$$K^{\epsilon}_{\vec{\boldsymbol{s}}}=\epsilon_{h_m}\cdot f_{s_{|\boldsymbol{s}|}}\cdots f_{s_1}\cdot Y_{s_1},\tag{D24}$$

for any $\vec{\boldsymbol{s}}=\langle s_1,\ldots,s_{|\boldsymbol{s}|}\rangle$.

To derive the main result (47), all seven principles/assumptions listed at the beginning of Sec. IV are used. The using of the first six is manifest, while the last one is necessary when excluding the $\epsilon_{h_q}\cdot k_{h_m}$ term in (D19), via the tree level equality (38). In the expansion (47), the gauge invariance for each graviton in the set $\boldsymbol{a}^h_m\backslash h_m$ is manifest, since the tensor $f_i^{\mu\nu}$ vanishes under the replacement $\epsilon_i\to k_i$. However, the gauge invariance for the graviton h_m is hidden. Similar phenomenons happen at the tree level when solving the expansions of sEYM and GR amplitudes. This is a quite general feature if we start with the expansions in the form of (D1).

APPENDIX E: VERIFICATION OF SOLUTION (60)

The verification of solution (60) is based on the following three equalities:

$$\mathcal{D}[\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}}[\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM;\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{a}^h_n\backslash\boldsymbol{a}^h)+(-)^{|\boldsymbol{a}|}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM;\circ}(\overleftarrow{\boldsymbol{a}}^g;\boldsymbol{a}^h_n\backslash\boldsymbol{a}^h)]]=0. \tag{E1}$$

$$\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}[(d-2)\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM;\circ}(\boldsymbol{a}^h_n)]=0, \tag{E2}$$

and

$$\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{s}}_m}[\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}}[\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM;\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{a}^h_n\backslash\boldsymbol{a}^h)+(-)^{|\boldsymbol{a}|}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM;\circ}(\overleftarrow{\boldsymbol{a}}^g;\boldsymbol{a}^h_n\backslash\boldsymbol{a}^h)]] \\ =\delta^{\pi}_{\vec{\boldsymbol{s}}_m,\vec{\boldsymbol{a}}}[\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM;\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{s}^h_n\backslash\boldsymbol{a}^h)+(-)^{|\boldsymbol{a}|}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM;\circ}(\overleftarrow{\boldsymbol{a}}^g;\boldsymbol{s}^h_n\backslash\boldsymbol{a}^h)]. \tag{E3}$$

Here $\delta^{\pi}_{\vec{\boldsymbol{s}}_m,\vec{\boldsymbol{a}}}$ is understood as 1 when $\vec{\boldsymbol{s}}_m=\vec{\boldsymbol{a}}$ up to a cyclic permutation, and 0 otherwise. The equality (E1) holds obviously, and ensures that $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm GR;\circ}(\boldsymbol{a}^h_n)$ in (60) satisfies Eq. (56) if $\mathcal{D}R=0$. To see (E2) and (E3), we first show that

$$\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{s}}_m}\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM;\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{a}^h_n\backslash\boldsymbol{a}^h)=0, \tag{E4}$$

for arbitrary $\vec{\boldsymbol{a}}$ (including the special case $\boldsymbol{a}=\emptyset$). If $\boldsymbol{s}_m\cap\boldsymbol{a}\neq\emptyset$, for each $i\in\boldsymbol{s}_m\cap\boldsymbol{a}$, the corresponding polarization vector ϵ_i does not appear in $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM;\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{a}^h_n\backslash\boldsymbol{a}^h)$, thus $\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM;\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{a}^h_n\backslash\boldsymbol{a}^h)$ is annihilated by $\partial_{\epsilon_i\cdot k_j}$ in $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{s}}_m}$. If $\boldsymbol{s}_m\cap\boldsymbol{a}=\emptyset$, it means $\boldsymbol{s}_m\subseteq\boldsymbol{a}_n\backslash\boldsymbol{a}$. For this case, one effective way to see (E4) is to use

$$\mathbf{I}^{\epsilon,\tilde{\epsilon}}_{\rm ssEYM;\circ}(\vec{\boldsymbol{a}}^g;\boldsymbol{a}^h_n\backslash\boldsymbol{a}^h) \\ =\sum_{\pi_c}\left[\frac{1}{\ell^2}\mathcal{F}\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\rm sEYM;\circ}(+^g,\pi_c(\vec{\boldsymbol{a}}^g),-^g;\boldsymbol{a}^h_n\backslash\boldsymbol{a}^h)\right] \\ =\frac{1}{\ell^2}\mathcal{F}\sum_{\pi_c}\mathcal{T}^{\epsilon}_{+\vec{\boldsymbol{a}}-}\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\rm GR;\circ}(\boldsymbol{a}^h_n\cup\{+^h,-^h\}). \tag{E5}$$

Since the operator $\mathcal{T}^{\epsilon}_{+\vec{\boldsymbol{a}}-}$ removes the polarization vectors ϵ_+ and ϵ_- in $\mathcal{A}^{\epsilon,\tilde{\epsilon}}_{\rm GR;\circ}(\boldsymbol{a}^h_n\cup\{+^h,-^h\})$, the manipulation $\mathcal{F}$ will not create any $\epsilon_i\rhd k_j$ or $\epsilon_i\lhd k_j$; thus $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{s}}_m}$ is commutable with $\mathcal{F}$. Since $\boldsymbol{s}_m\cap\boldsymbol{a}=\emptyset$, $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{s}}_m}$ is commutable with $\mathcal{T}^{\epsilon}_{+\vec{\boldsymbol{a}}-}$. Then, the equality (E4) is ensured by (38). On the other hand, it is straightforward to see

$$\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{s}}_m}\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}}=\delta^{\pi}_{\vec{\boldsymbol{s}}_m,\vec{\boldsymbol{a}}}. \tag{E6}$$

Combining (E4) and (E6) together gives (E2) and (E3). Using equalities (E1), (E2) and (E3), one sees that formula (60) is the solution to Eqs. (56) and (39), if $\mathcal{D}R=0$ and $\mathcal{C}^{\epsilon}_{\vec{\boldsymbol{a}}_m}R=0$ for each $\vec{\boldsymbol{a}}_m$.

[1] H. Kawai, D. C. Lewellen, and S. H. H. Tye, A relation between tree amplitudes of closed and open strings, Nucl. Phys. **B269**, 1 (1986).
[2] Z. Bern, J. J. M. Carrasco, and H. Johansson, New relations for gauge-theory amplitudes, Phys. Rev. D **78**, 085011 (2008).
[3] Z. Bern, J. J. M. Carrasco, and H. Johansson, Perturbative Quantum Gravity as a Double Copy of Gauge Theory, Phys. Rev. Lett. **105**, 061602 (2010).
[4] Z. Bern, T. Dennen, Y. t. Huang, and M. Kiermaier, Gravity as the square of gauge theory, Phys. Rev. D **82**, 065003 (2010).
[5] M. Chiodaroli, M. Günaydin, H. Johansson, and R. Roiban, Scattering amplitudes in $\mathcal{N}=2$ Maxwell-Einstein and Yang-Mills/Einstein supergravity, J. High Energy Phys. 01 (2015) 081.
[6] H. Johansson and A. Ochirov, Color-kinematics duality for QCD amplitudes, J. High Energy Phys. 01 (2016) 170.
[7] H. Johansson and A. Ochirov, Double copy for massive quantum particles with spin, J. High Energy Phys. 09 (2019) 040.
[8] F. Cachazo, S. He, and E. Y. Yuan, Scattering equations and Kawai-Lewellen-Tye orthogonality, Phys. Rev. D **90**, 065001 (2014).
[9] F. Cachazo, S. He, and E. Y. Yuan, Scattering of Massless Particles in Arbitrary Dimensions, Phys. Rev. Lett. **113**, 171601 (2014).
[10] F. Cachazo, S. He, and E. Y. Yuan, Scattering of massless particles: Scalars, gluons and gravitons, J. High Energy Phys. 07 (2014) 033.

[11] F. Cachazo, S. He, and E. Y. Yuan, Einstein-Yang-Mills scattering amplitudes from scattering equations, J. High Energy Phys. 01 (2015) 121.
[12] F. Cachazo, S. He, and E. Y. Yuan, Scattering equations and matrices: From Einstein to Yang-Mills, DBI and NLSM, J. High Energy Phys. 07 (2015) 149.
[13] C. Cheung, C. H. Shen, and C. Wen, Unifying relations for scattering amplitudes, J. High Energy Phys. 02 (2018) 095.
[14] K. Zhou and B. Feng, Note on differential operators, CHY integrands, and unifying relations for amplitudes, J. High Energy Phys. 09 (2018) 160.
[15] M. Bollmann and L. Ferro, Transmuting CHY formulae, J. High Energy Phys. 01 (2019) 180.
[16] K. Zhou and G. J. Zhou, Transmuting off-shell CHY integrals in the double-cover framework, Eur. Phys. J. C **80**, 1068 (2020).
[17] S. Stieberger and T. R. Taylor, New relations for Einstein-Yang-Mills amplitudes, Nucl. Phys. **B913**, 151 (2016).
[18] O. Schlotterer, Amplitude relations in heterotic string theory and Einstein-Yang-Mills, J. High Energy Phys. 11 (2016) 074.
[19] M. Chiodaroli, M. Gunaydin, H. Johansson, and R. Roiban, Explicit formulae for Yang-Mills-Einstein amplitudes from the double copy, J. High Energy Phys. 07 (2017) 002.
[20] D. Nandan, J. Plefka, O. Schlotterer, and C. Wen, Einstein-Yang-Mills from pure Yang-Mills amplitudes, J. High Energy Phys. 10 (2016) 070.
[21] L. de la Cruz, A. Kniss, and S. Weinzierl, Relations for Einstein-Yang-Mills amplitudes from the CHY representation, Phys. Lett. B **767**, 86 (2017).
[22] C. H. Fu, Y. J. Du, R. Huang, and B. Feng, Expansion of Einstein-Yang-Mills amplitude, J. High Energy Phys. 09 (2017) 021.
[23] F. Teng and B. Feng, Expanding Einstein-Yang-Mills by Yang-Mills in CHY frame, J. High Energy Phys. 05 (2017) 075.
[24] Y. J. Du and F. Teng, BCJ numerators from reduced Pfaffian, J. High Energy Phys. 04 (2017) 033.
[25] Y. J. Du, B. Feng, and F. Teng, Expansion of all multitrace tree level EYM amplitudes, J. High Energy Phys. 12 (2017) 038.
[26] B. Feng, X. Li, and K. Zhou, Expansion of EYM theory by differential operators, Phys. Rev. D **100**, 125012 (2019).
[27] S. Q. Hu and K. Zhou, Expansion of tree amplitudes for EM and other theories, Prog. Theor. Exp. Phys. **2020**, 073B10 (2020).
[28] K. Zhou, Unified web for expansions of amplitudes, J. High Energy Phys. 10 (2019) 195.
[29] K. Zhou, On differential operators and unifying relations for one-loop Feynman integrands, J. High Energy Phys. 10 (2021) 150.
[30] S. He, O. Schlotterer, and Y. Zhang, New BCJ representations for one-loop amplitudes in gauge theories and gravity, Nucl. Phys. **B930**, 328 (2018).
[31] Y. Geyer and R. Monteiro, Gluons and gravitons at one loop from ambitwistor strings, J. High Energy Phys. 03 (2018) 068.
[32] S. He and E. Y. Yuan, One-loop scattering equations and amplitudes from forward limit, Phys. Rev. D **92**, 105004 (2015).
[33] F. Cachazo, S. He, and E. Y. Yuan, One-loop corrections from higher dimensional tree amplitudes, J. High Energy Phys. 08 (2016) 008.
[34] B. Feng, CHY-construction of planar loop integrands of cubic scalar theory, J. High Energy Phys. 05 (2016) 061.
[35] B. Feng and C. Hu, One-loop CHY-integrand of bi-adjoint scalar theory, J. High Energy Phys. 02 (2020) 187.
[36] C. Baadsgaard, N. E. J. Bjerrum-Bohr, J. L. Bourjaily, S. Caron-Huot, P. H. Damgaard, and B. Feng, New Representations of the Perturbative S-Matrix, Phys. Rev. Lett. **116**, 061601 (2016).
[37] L. J. Dixon, Calculating scattering amplitudes efficiently, arXiv:hep-ph/9601359.
[38] V. Del Duca, L. J. Dixon, and F. Maltoni, New color decompositions for gauge amplitudes at tree and loop level, Nucl. Phys. **B571**, 51 (2000).
[39] L. J. Dixon, Scattering amplitudes: The most perfect microscopic structures in the universe, J. Phys. A **44**, 454001 (2011).
[40] Z. Bern, L. J. Dixon, and D. A. Kosower, Progress in one loop QCD computations, Annu. Rev. Nucl. Part. Sci. **46**, 109 (1996).
[41] R. Kleiss and H. Kuijf, Multigluon cross-sections and five jet production at hadron colliders, Nucl. Phys. **B312**, 616 (1989).
[42] K. A. Roehrig and D. Skinner, A gluing operator for the ambitwistor string, J. High Energy Phys. 01 (2018) 069.
[43] J. Dong, S. He, and L. Hou, Universal expansions of scattering amplitudes for gravitons, gluons and Goldstone particles, Phys. Rev. D **105**, 105007 (2022).
[44] T. Adamo, Worldsheet factorization for twistor-strings, J. High Energy Phys. 04 (2014) 080.
[45] L. Mason and D. Skinner, Ambitwistor strings and the scattering equations, J. High Energy Phys. 07 (2014) 048.
[46] T. Adamo, E. Casali, and D. Skinner, Ambitwistor strings and the scattering equations at one loop, J. High Energy Phys. 04 (2014) 104.
[47] E. Casali and P. Tourkine, Infrared behaviour of the one-loop scattering equations and supergravity integrands, J. High Energy Phys. 04 (2015) 013.
[48] Y. Geyer, L. Mason, R. Monteiro, and P. Tourkine, Loop Integrands for Scattering Amplitudes from the Riemann Sphere, Phys. Rev. Lett. **115**, 121603 (2015).
[49] Y. Geyer, L. Mason, R. Monteiro, and P. Tourkine, One-loop amplitudes on the Riemann sphere, J. High Energy Phys. 03 (2016) 114.
[50] T. Adamo and E. Casali, Scattering equations, supergravity integrands, and pure spinors, J. High Energy Phys. 05 (2015) 120.