Gauge enhanced quantum criticality beyond the standard model

Juven Wang [1,*] and Yi-Zhuang You [2,†]
[1] *Center of Mathematical Sciences and Applications, Harvard University, Cambridge, Massachusetts 02138, USA*
[2] *Department of Physics, University of California, San Diego, California 92093, USA*

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Standard lore ritualizes our quantum vacuum in the four-dimensional spacetime (4D) governed by one of the candidate Standard Models (SMs), while lifting towards some grand unification-like structure (GUT) at higher energy scales. In contrast, in our work, we introduce an alternative view that the SM is a low energy quantum vacuum arising from various neighbor vacua competition in an immense quantum phase diagram. In general, we can regard the SM arising near the gapless quantum criticality (either critical points or critical regions) between the competing neighbor vacua. In particular detail, we demonstrate how the $su(3) \times su(2) \times u(1)$ SM with 16n Weyl fermions arises near the quantum criticality between the GUT competition of Georgi-Glashow $su(5)$ and Pati-Salam $su(4) \times su(2) \times su(2)$. We propose two enveloping toy models. Model I is the conventional $so(10)$ GUT with a Spin(10) gauge group plus GUT-Higgs potential inducing various Higgs transitions. Model II modifies model I by adding a new 4D discrete torsion class of Wess-Zumino-Witten-like term built from GUT-Higgs field [which matches a non-perturbative global mixed gauge-gravity anomaly captured by a 5D invertible topological field theory $w_2 w_3(TM) = w_2 w_3(V_{\mathrm{SO}(10)})$], which manifests a beyond-Landau-Ginzburg quantum criticality between Georgi-Glashow and Pati-Salam models, with extra beyond the Standard Model excitations emerging near a hypothetical quantum critical region. If the internal symmetries were treated as global symmetries (or weakly coupled to probe background fields), we suggest a particular low-energy realization of model II as an analogous gapless 4D deconfined quantum criticality with new beyond the SM fractionalized fragmentary excitations of color-flavor separation, and gauge enhancement including a dark gauge force sector, altogether requiring a double fermionic Spin structure named DSpin. If the internal symmetries are dynamically gauged (as they are in our quantum vacuum), we suggest the model II's 4D criticality as a boundary criticality such that only appropriately gauge enhanced dynamical GUT gauge fields can propagate into an extradimensional 5D bulk. The phenomena may be regarded as SM deformation or "morphogenesis." Although our derivation is based on kinematics and global anomaly matching constraints between UV parent and various candidate IR field theories, the constraints are still highly subtle and suggestive. Future verifications on the IR dynamics will be desirable.

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I. INTRODUCTION, MOTIVATION, AND SUMMARY

It is a common ritual practice in high-energy physics (HEP) to regards our quantum vacuum in the four-dimensional spacetime [denoted as 4D or $(3+1)$D] governed by one of the candidate $su(3) \times su(2) \times u(1)$ Standard Models (SMs) [1–4] as a quantum field theory (QFT) and an effective field theory (EFT) suitable below a certain energy scale, while lifting towards one of some grand unification-like structure (GUT) [5–7] or string theory at higher energy scales,[1] see Fig. 1(a). Although many nonsupersymmetric GUT models had been ruled out by experiments due to no evidence yet on the predicted proton decay (proton lifetime $>10^{34}$ years) [8], many physicists still speculate that GUT plays a certain crucial

*jw@cmsa.fas.harvard.edu
†yzyou@ucsd.edu

[1] Throughout our article, we denote nd for n-dimensional spacetime, or $(n'+1)$D as an n'-dimensional space and one-dimensional time. We also denote the Lie algebra in the lower case such as $so(10)$, and denote the Lie group in the capital case such as Spin(10). For example, we follow the convention to call the model [7] as the $so(10)$ GUT, but it requires the Spin(10) gauge group.

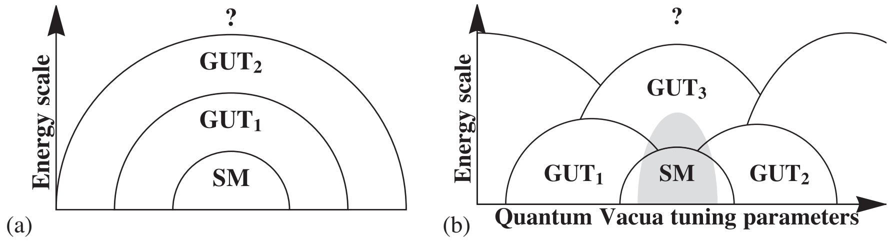

FIG. 1. (a) Standard lore seeks for a single unified dynamically gauged internal symmetry at high energy. One probes the shorter distance and higher energy scales to look for the GUT, supersymmetry, or string theory evidence. The vertical axis shows an energy scale, while the horizontal axis plays no physical role. (b) We propose an alternative view: SM is just one of many possible low energy phases of the quantum vacua of our universe. By introducing a horizontal axis that represent many possible quantum vacua tuning parameters, we can show that SM phase can tune to other GUT phases, even at a fixed energy scale (without the necessity to go to higher energy) and at zero temperature. SM arises near the gapless quantum critical region (shown as the gray area).

role in a higher energy unification [9]. How can we remedy the conventional GUTs other than seeking for their supersymmetry variants or string theory modifications at higher energy?

To address the above question, we propose to seek for a new viewpoint. In our present work, instead of viewing GUT only as some higher-energy theory of SM, we suggest that various GUTs may be neighbor quantum vacua next to SM in an immense quantum phase diagram[2] shown schematically in Fig. 1(b), with an underlying larger quantum vacua tuning parameter space [i.e., the horizontal axis in Figs. 1(b), 2, and 3]. We provide two explicit toy models in Figs. 2 and 3: SM arises near the gapless quantum critical point (for Fig. 2) or critical region (gray area for Fig. 3) between the competing neighbor GUT vacua. Readers may be puzzled: What precisely can be the quantum vacua tuning parameters? What can we gain from this viewpoint? What are the motivations? Let us address these issues one by one.

(1) Quantum vacua tuning parameters can be as familiarly simple as the tuning of the GUT-Higgs potential $[(r_{\mathbf{R}}(\Phi_{\mathbf{R}})^2 + \lambda_{\mathbf{R}}(\Phi_{\mathbf{R}})^4)]$ of some GUT-Higgs field $\Phi_{\mathbf{R}}$ that can induce a Higgs condensation[3] phase transition via tuning from $r_{\mathbf{R}} > 0$ to $r_{\mathbf{R}} < 0$. The quantum vacua tuning parameters can be those triggering a scalar condensation $\langle \Phi_{\mathbf{R}} \rangle \neq 0$ in the $r_{\mathbf{R}} < 0$ region. The possibility to access the GUT vacua from the SM vacuum by tuning certain model parameters has been largely

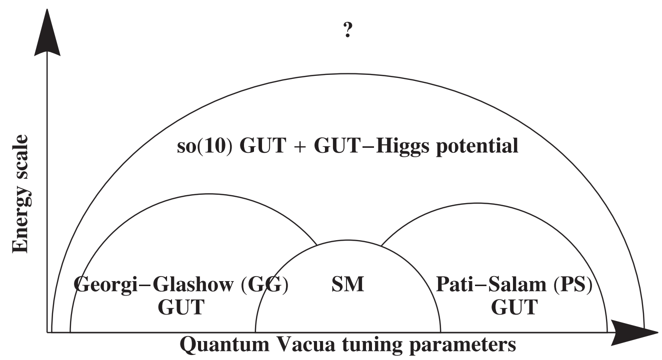

FIG. 2. Schematic phases for toy model I: The parent EFT is the conventional $so(10)$ GUT with a Spin(10) gauge group plus GUT-Higgs potential inducing various Higgs transitions to Georgi-Glashow (GG), Pati-Salam (PS), or SM.

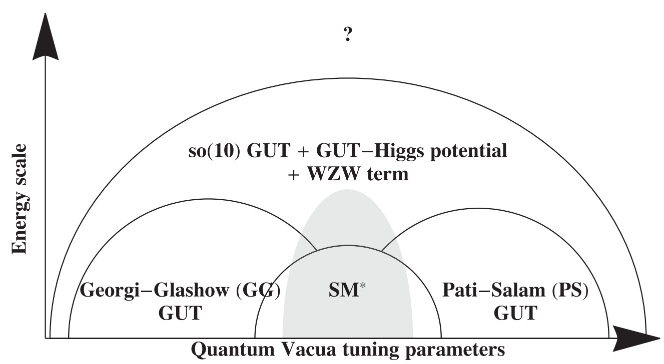

FIG. 3. Schematic phases for toy model II: The parent EFT is a modified $so(10)$ GUT with a Spin(10) gauge group, plus not only a GUT-Higgs potential but also a new 4D discrete torsion class of Wess-Zumino-Witten-like (WZW) term built from GUT-Higgs fields that saturates a nonperturbative global mixed gauge-gravity anomaly captured by a 5D invertible topological field theory $w_2w_3(TM) = w_2w_3(V_{\mathrm{SO}(10)})$, which manifests a proposed hypothetical beyond Landau-Ginzburg quantum critical region (shown in a gray area) between GG and PS models, with extra beyond the Standard Model (BSM) excitations emerging near the quantum criticality. The SM + BSM physics is denoted as SM*.

[2]Here quantum phases mean that we focus on the zero temperature physics where the quantum effect is dominant, see for example an overview [10]. The quantum phase diagram at zero temperature behaves more quantum than the thermal phase diagram at finite temperature. Of course, the two viewpoints are complementary—we can include both the energy scale axis and the zero-temperature vacuum-tuning axis as in Fig. 1(b).

[3]Throughout our work, whenever we mention Higgs field or Higgs transition, we normally mean the GUT-Higgs instead of the electroweak Higgs. Namely, we always focus on the SM gauge group $su(3) \times su(2) \times u(1)$ as above the electroweak scale instead of $su(3) \times u(1)_{\mathrm{EM}}$ below the electroweak scale.

overlooked in the existing literature, because some of these tuning parameters appear to be perturbatively irrelevant at the SM fixed point. A key proposal of this work is to investigate the nonperturbative effect of these tuning parameters in driving quantum phase transitions from the SM phase to adjacent GUT phases.

(2) Deformation class of QFT: Given the importance of symmetry and its associated 't Hooft anomaly of QFT, Seiberg [11] and others[4] conjectured that seemly different dD QFTs within the same symmetry G and same 't Hooft anomaly $\mathbf{Z}_{d+1}$ of symmetry G [14] can indeed be deformed to each other via adding degrees of freedom at short distances that preserve the same symmetry and that maintain the same overall anomaly. Namely, the whole system allows all symmetric interactions between the original QFT and any new symmetric QFTs brought down from high energy. This organization principle that connects a large class of QFTs together within the same data $(G, \mathbf{Z}_{d+1})$ via any symmetric deformation (possibly with discontinuous or continuous quantum phase transitions [10] between different phases) is called the deformation class of QFTs in dD [11], which is indeed controlled by the cobordism or deformation class of invertible topological quantum field theory $\mathbf{Z}_{d+1}$ in $(d+1)$D [15]. One can further define the deformation class for 4D SM [16,17].

As we will see, our viewpoint in Fig. 1(b) (also in Figs. 2 and 3) is not only compatible with this symmetric deformation class of QFT [11], but we also allow symmetry-breaking deformations along the quantum vacua tuning parameter space. We may refer to all these deformations of the SM to other neighbor vacua as "morphogenesis" of the SM.

(3) Proton decay: The aforementioned issue of GUT proton decay may be resolved in our framework by two ways. First, the change of viewpoint—instead of looking for GUT proton decay in our vacuum (or in a higher energy GUT along the vertical axis, as in Fig. 1), we may look for GUT proton decay by first moving to the appropriate quantum vacuum along the horizontal axis in Fig. 1(b) that already lives this specific GUT.[5] Second, a modified parent EFT that controls all possible deformation of SM in the phase diagram may give rise to a different proton decay rate.[6] The experimental bound on proton decay rate only rules out the possibility to access nonsupersymetric GUT phases from the SM phases by thermal phase transitions (i.e., by raising the energy or temperature scales), but it does not say anything about accessing these GUT phases by quantum phase transitions (by tuning parameters near ground states at low energy). This work exactly focuses on the later possibility of quantum phase transitions among the SM and GUTs.

The above three arguments summarize the motivation and philosophy behind our viewpoint. Namely, in our present work, we initiate and introduce an alternative complementary perspective—we propose that the SM vacuum can be a low energy quantum vacuum arising from the quantum competition of various neighbor GUT vacua in a quantum phase diagram. SM is just one possible phase allowed by the deformation class of SM [16]. Let us list down some key results of our work:

(1) In general, we propose that the SM may arise as one adjacent phase from the vicinity of gapless quantum criticality (either a critical point for model I in Fig. 2, or a critical region for model II in Fig. 3) between the competing neighbor GUT vacua.

(2) In particular, we demonstrate how the $su(3) \times su(2) \times u(1)$ SM [1–4] with 16n Weyl fermions (Fig. 4) could emerge near the quantum criticality between two neighbor vacua of GG $su(5)$ model [5] (Fig. 5) and PS $su(4) \times su(2) \times su(2)$ model [6] (Fig. 6), which represents two distinct Higgs phases of the further unified $so(10)$ GUT [with a Spin(10) gauge group].

(3) We propose two explicit toy models. The two models are differed by whether they can carry a 4D nonperturbative global anomaly of mixed

[4]In fact the related concept has been used in arguing that the fermion doubling problem (occurred in regularizing chiral fermions nonperturbatively on the lattice with a chiral G symmetry) can be resolved by gapping the mirror chiral fermion if and only if the chiral fermion is anomaly free in G (tautologically, the mirror fermion is also anomaly free in G), see [12,13] and references therein. The argument follows directly from the fact that the gapless anomaly-free G-symmetric chiral fermion theory is in the same deformation class of the gapped anomaly-free G-symmetric theory.

[5]Take Georgi-Glashow $su(5)$ GUT [5] as an example. The conventional viewpoint may be problematic because this specific GUT may not be the correct higher energy theory of our vacuum along the vertical axis, in Figs. 2 and 3. If we want to detect any proton decay in $su(5)$ GUT, then hypothetically we may imagine to create a small bubble within the domain wall such that inside the bubble resides any possible deformation of the SM (e.g., any models along the horizontal axis in Figs. 2 and 3). Although changing the large-scale quantum vacuum structure of our SM universe is likely energetically impossible, changing the quantum vacuum inside a small-scale bubble is possibly feasible experimentally.

[6]For example, two different toy-model parent EFTs in Figs. 2 and 3, respectively, can give different proton decay rates. We do not attempt to compute the explicit proton decay rate in this work, because so far we only have two toy models that control a p $= \{0,1\} \in \mathbb{Z}_2$ deformation class labeled by a $\mathbb{Z}_2$ nonperturbative global anomaly in 4D. The two toy models describe only a partial deformation class of the SM. There is also a $\mathbb{Z}_{16}$ deformation class for SM [16], etc. To compute a experimentally sensible proton decay rate for our vacuum, it will be the best that we (1) locate the specific point on the phase diagram that precisely labels our vacuum, and (2) compute from the general enveloping parent EFT that includes all physically relevant deformations.

gauge-gravitational (i.e., gauge-diffeomorphism) probes, captured by a 5D invertible topological quantum field theory (TQFT)[7]:

The w_2w_3 global anomaly also occurs on a certain $\mathbb{Z}_2$ gauge theory with fermionic strings [19] and all-fermion U(1) electrodynamics [20,21], which is a pure U(1) gauge theory whose electric, magnetic, and dyonic objects are all fermions. For these $\mathbb{Z}_2$ and U(1) gauge theories, they do have the spacetime tangent bundle constraints on TM, but do not have the analogous gauge bundle constraints on V_G. So this $w_2w_3 = w_2w_3(TM)$ anomaly becomes a pure gravitational anomaly for these $\mathbb{Z}_2$ and U(1) gauge theories.

We recommend the following Refs. [22–25] or this seminar video [26] for readers who wish to overview some modern perspectives about the anomalies of SM and GUT relevant gauge theories. In particular, we follow closely [25,26]. In summary, we may address anomalies with different adjectives to characterize their properties:

(i) Invertible vs noninvertible: We only focus on the invertible anomalies, which follow the standard definition of anomalies (also in high-energy physics) captured by one higher-dimensional invertible TQFT as the low energy theory of invertible topological phases. The dD invertible anomalies [also the $(d+1)$D invertible TQFTs] are classified by the cobordism group data $\Omega_G^d \equiv \mathrm{TP}_d(G)$ defined in Freed and Hopkins [27]. The partition function $\mathbf{Z}$ of a $(d+1)$D invertible TQFT satisfies $\mathbf{Z}(M^{d+1}) = 1$ on a closed M^{d+1} manifold.

In contrast, the noninvertible anomalies are nonstandard (usually not named as anomalies in high-energy physics), characterized by noninvertible topological phases with intrinsic topological orders.

(ii) Perturbative local vs nonperturbative global anomalies: Whether the anomalies are local (or global), is determined by whether the gauge or diffeomorphism transformations are infinitesimal (or large) transformations, continuously deformable (or not deformable) to the identity element. The classifications of local vs global anomalies are the integer $\mathbb{Z}$ vs the finite torsion $\mathbb{Z}_n$ classes, respectively.

(iii) Gauge anomaly vs mixed gauge-gravity anomaly vs gravitational anomaly: The adjective, gauge or gravity, refers to the types of couplings or probes that we require to detect them—whether the probes depends on the internal gauge bundle/connection or the spacetime geometry.

(iv) Background fields or dynamical fields: Anomalies of global symmetries probed by nondynamical background fields are known as 't Hooft anomalies. Anomalies coupled to dynamical fields must lead to anomaly cancellations to zero for consistency.

[7] The w_j is the jth Stiefel-Whitney (SW) characteristic class. The $w_j(TM)$ is the SW class of spacetime tangent bundle TM of manifold M. The $w_j(V_G)$ is the SW class of the principal G bundle. This mod 2 class w_2w_3 global anomaly has been checked to be absent in the $so(10)$ GUT by [12,18]. This mixed gauge-gravitational anomaly is tightly related to the new SU(2) anomaly [18] due to the bundle constraint $w_2w_3(TM) = w_2w_3(V_G)$ with G can be substituted by $\mathrm{SO}(3) \subset \mathrm{SO}(10)$ related to the embedding $\mathrm{SU}(2) = \mathrm{Spin}(3) \subset \mathrm{Spin}(10)$. However, as we will see, it is natural to introduce a new 4D WZW term [appending to the $so(10)$ GUT] with this w_2w_3 global anomaly in order to realize the SM vacuum as the quantum criticality phenomenon between the neighbor SU(5) GUT and Pati-Salam vacua.

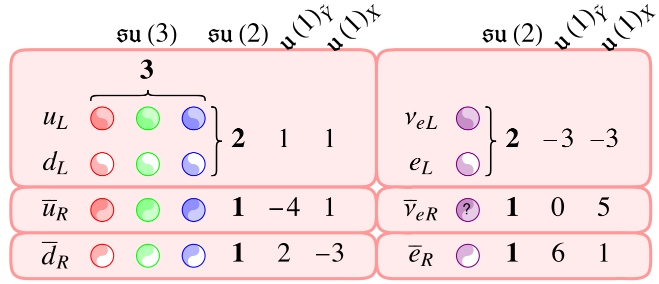

FIG. 4. SM. The 15n Weyl fermions of SM contain the representation $(\bar{\mathbf{3}},\mathbf{1})_{2,L} \oplus (\mathbf{1},\mathbf{2})_{-3,L} \oplus (\mathbf{3},\mathbf{2})_{1,L} \oplus (\bar{\mathbf{3}},\mathbf{1})_{-4,L} \oplus (\mathbf{1},\mathbf{1})_{6,L}$. The 16n Weyl fermions of SM add an extra $(\mathbf{1},\mathbf{1})_{0,L}$.

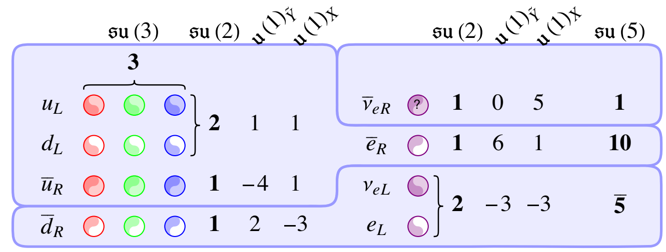

FIG. 5. Georgi-Glashow SU(5) model and the $su(5)$ GUT. The 15 Weyl fermions of SM are $\bar{\mathbf{5}} \oplus \mathbf{10}$ of SU(5); namely, $(\bar{\mathbf{3}},\mathbf{1})_{2,L} \oplus (\mathbf{1},\mathbf{2})_{-3,L} \sim \bar{\mathbf{5}}$ and $(\mathbf{3},\mathbf{2})_{1,L} \oplus (\bar{\mathbf{3}},\mathbf{1})_{-4,L} \oplus (\mathbf{1},\mathbf{1})_{6,L} \sim \mathbf{10}$ of SU(5). Also $(\mathbf{1},\mathbf{1})_{0,L} \sim \mathbf{1}$ of SU(5), so the 16 Weyl fermions of SM are $\bar{\mathbf{5}} \oplus \mathbf{10} \oplus \mathbf{1}$ of SU(5).

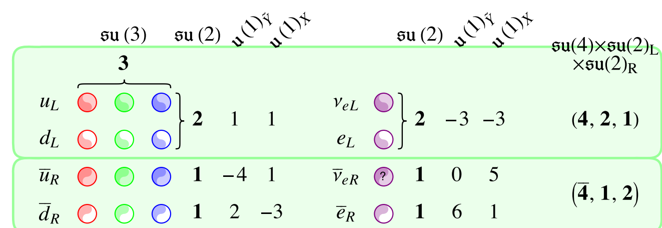

FIG. 6. PS model: $G_{\mathrm{PS}_{q'}} \equiv \frac{\mathrm{SU}(4)\times\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}}{\mathbb{Z}_{q'}} \equiv \frac{\mathrm{Spin}(6)\times\mathrm{Spin}(4)}{\mathbb{Z}_{q'}}$ with $q' = 1, 2$. The 16 Weyl fermions of SM are $(\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$ of $su(4) \times su(2)_{\mathrm{L}} \times su(2)_{\mathrm{R}}$, and the 16 of $so(10)$ [or Spin(10)]. These L and R are internal symmetry group indices. They are different from (but correlated with) the spacetime symmetry L and R. So $(\mathbf{3},\mathbf{2})_{1,L} \oplus (\mathbf{1},\mathbf{2})_{-3,L} \sim (\mathbf{4},\mathbf{2},\mathbf{1})_L$, and $(\bar{\mathbf{3}},\mathbf{1})_{2,L} \oplus (\bar{\mathbf{3}},\mathbf{1})_{-4,L} \oplus (\mathbf{1},\mathbf{1})_{6,L} \oplus (\mathbf{1},\mathbf{1})_{0,L} \sim (\bar{\mathbf{4}},\mathbf{1},\mathbf{2})_L$ of PS model.

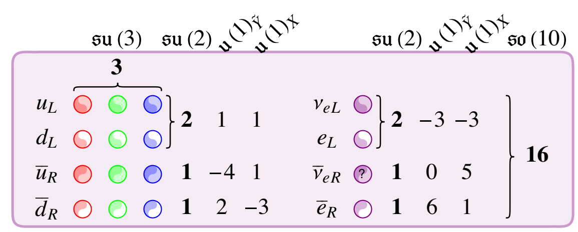

FIG. 7. The $so(10)$ GUT model: The 16 Weyl fermions of Spin(10), form the **16**-dimensional representation of Spin(10).

$$(-1)^{\int \mathrm{p} w_2 w_3(TM)} = (-1)^{\int \mathrm{p} w_2 w_3(V_{\mathrm{SO}(10)})} \quad \text{with} \quad \mathrm{p} \in \{0,1\} = \mathbb{Z}_2. \tag{1.1}$$

Toy model I as the $\mathrm{p}=0$ class without $w_2 w_3$ anomaly: Its parent EFT is the conventional $so(10)$ GUT with a Spin(10) gauge group [7] plus a GUT-Higgs potential inducing various Higgs transitions to GG, PS, or SM, schematically shown in Fig. 2. The first model has no $w_2 w_3$ or any other anomaly within the Spin(10).

Toy model II as the $\mathrm{p}=1$ class with $w_2 w_3$ anomaly and WZW term: To introduce nontrivial competitions between GG and PS phases, we consider a new parent EFT of a modified $so(10)$ GUT with a Spin(10) gauge group, which includes not only the familiar $so(10)$ GUT plus a GUT-Higgs potential, but also a new extra 4D discrete torsion class of WZW term that saturates a mod-2 class $w_2 w_3$ anomaly within the Spin(10).

The WZW term introduces nonperturbative interaction effects between different GUT-Higgs fields, which cause a substantial change of the deformation class of QFT vacuum that cannot be smoothly connected to the conventional $so(10)$ GUT vacuum. There are distinct $\mathrm{p} \in \{0,1\} = \mathbb{Z}_2$ deformation classes of QFT.

We propose a schematic quantum phase diagram, shown in Fig. 8, interpolating between different quantum vacua: the modified $so(10)$ GUT + WZW term, the $su(5)$ GG GUT, the $su(4) \times su(2)_{\mathrm{L}} \times su(2)_{\mathrm{R}}$ PS model, and the $su(3) \times su(2) \times u(1)$ SM. In fact, this $w_2 w_3$ global anomaly [hereafter $w_2 w_3$ as a shorthand for the precise bundle constraint $w_2 w_3(TM) = w_2 w_3(V_{\mathrm{SO}(10)})$] does not occur when the internal symmetry is within $su(5)$ [for the GG $su(5)$ GUT], nor does it occur within $su(4) \times su(2) \times su(2)$ (for the PS model), nor does it occur within $su(3) \times su(2) \times u(1)$ (for the SM). Alternatively, we can also regard this $w_2 w_3$ anomaly is matched in the

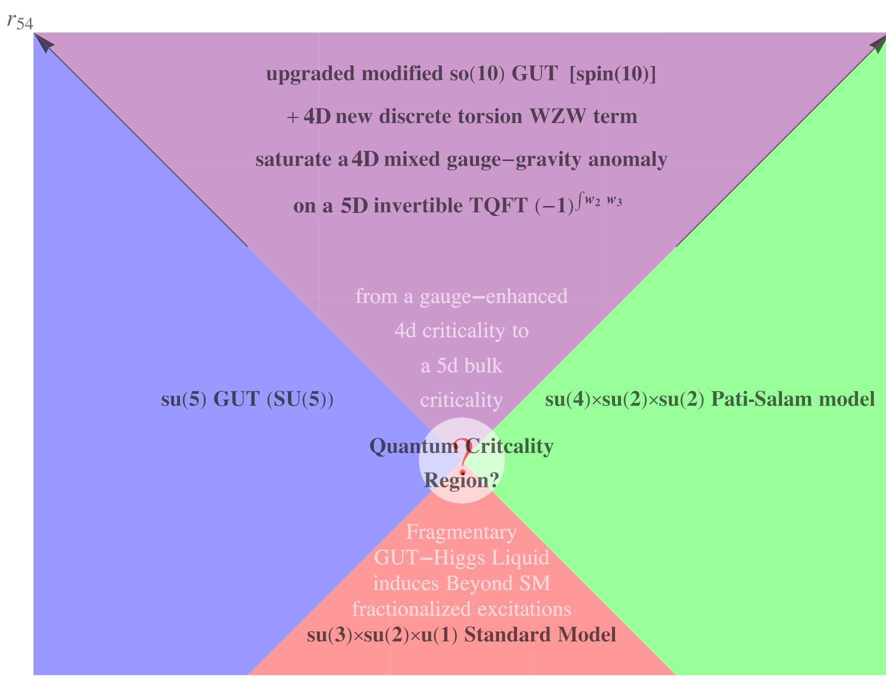

FIG. 8. One of our research motif is proposing and investigating this schematic quantum phase diagram. The phase diagram interpolates between different quantum EFT vacua: the $so(10)$ GUT [Spin(10) group], the $su(5)$ GUT [SU(5) group], the $su(4) \times su(2)_{\mathrm{L}} \times su(2)_{\mathrm{R}}$ PS model, and the $su(3) \times su(2) \times u(1)$ SM. We will explore the nature of phase transitions later in Sec. III. We propose the whitened region as a possible quantum critical region, which we explored in Secs. III and IV. Here $r_{\mathbf{R}}$ denotes the coefficient of the effective quadratic potential of $\Phi_{\mathbf{R}}$ field in the representation $\mathbf{R}$. The corresponding GUT-Higgs $\Phi_{\mathbf{R}}$ field will condense in the representation $\mathbf{R}$ if $r_{\mathbf{R}} < 0$. Relatively speaking, the IR low energy is drawn with the red color (for SM), the intermediate neighbor phases are drawn with the green or blue color [for PS or SU(5) models], while the UV higher energy is drawn with the violet purple color [for Spin(10)]; although the readers should keep in mind that we really explore the nearground-state, zero-energy, and zero-temperature quantum phase diagram. These colors are also designed to match the colors of partitions of representations in Figs. 4–7. For toy model I without WZW term and without $w_2 w_3$ anomaly, we should remove the whitened quantum critical region, but we are left with a quantum critical point at the origin. For toy model II with WZW term and with $w_2 w_3$ anomaly, we encounter the whitened quantum critical region near the origin. The quantum critical region can have dynamical consequences such as emergent deconfined dark gauge force $[\mathrm{U}(1)']^{\mathrm{emergent}}_{\mathrm{gauge}}$, see Sec. III D 2.

GG, PS, and SM via the symmetry breaking. This $w_2 w_3$ global anomaly only occurs when the internal symmetry is Spin(10) [for the modified $so(10)$ GUT + WZW term], but this anomaly still constrains the full quantum phase diagram (Fig. 8).

For toy model I without WZW term and without $w_2 w_3$ anomaly, we should remove the whitened quantum critical region in Fig. 8, but we are left with a quantum critical point at the origin.

For toy model II with WZW term and with $w_2 w_3$ anomaly, we encounter a hypothetical whitened quantum critical region near the origin constrained by the global anomaly matching in Fig. 8.

Case (1). If the internal symmetries were pretended to be global symmetries (or weakly gauged by probe background fields), then we are dealing with the quantum criticality between Landau-Ginzburg global symmetry breaking phases in 4D. Conventionally, the global symmetry breaking pattern can be triggered by the GUT-Higgs fields. Surprisingly, for model II (Fig. 3), we discover a possible gapless quantum phase with fractional excitations and deconfined emergent gauge structure in analogy to 4D deconfined quantum criticality[8] beyond the Landau-Ginzburg-Wilson-Fisher critical phenomena. Specifically, we propose a 4D mother effective field theory, where the GUT-Higgs bosonic fields can be fractionalized to new fragmentary fermionic excitations, with extra gauge enhancement. An example of such gauge enhancement introduces a new U(1) gauge sector called $[\mathrm{U}(1)']^{\text{emergent}}_{\text{gauge}}$, different from the SM electrodynamics $\mathrm{U}(1)_{\text{EM}}$. We name such a new theory as a fragmentary GUT-Higgs liquid model with emergent new fermions and new gauge fields, emergent only near the quantum criticality.

However, we should make our claim clear that our derivation on the hypothetical gapless quantum critical region for model II is primarily based on kinematics and global anomaly matching constraints between a high-energy ultraviolet (UV) parent theory [of a modified $so(10)$ GUT with WZW term] and its candidate low-energy infrared (IR) field theory. The constraints are highly subtle and suggestive. In addition, we also propose various other IR candidate phases in Sec. III D 1. But as it is widely known the limitation of any kinematic and anomaly constraint of a given QFT, we do not yet know its definite IR dynamics. The IR dynamic verifications by other analytic or numerical methods will be desirable in the future.

Case (2). If the internal symmetries are dynamically gauged (as they are not global symmetries but indeed are gauged in our quantum vacuum), we show the gauge-enhanced 4D criticality not merely has the emergent $[\mathrm{U}(1)']^{\text{emergent}}_{\text{gauge}}$, but also has the enhanced Spin(10) gauge group. The Spin(10) gauge group and $[\mathrm{U}(1)']^{\text{emergent}}_{\text{gauge}}$ forms a gauge enhancement of the smaller gauge groups of the SM, GG or PS models, only near the quantum criticality, see Fig. 8.

Because the 5D invertible TQFT has the bundle constraint $w_2 w_3(TM) = w_2 w_3(V_{\mathrm{SO}(10)})$, once the internal symmetries [such as the Spin(10)] are dynamically gauged, the 5D bulk is no longer an invertible TQFT. The Spin(10) gauge fields have also to be dynamically gauged in the 5D bulk. The Spin(10) gauge fields contribute deconfined gapless modes in 5D[9] (in contrast to the confined non-Abelian gauge fields being gapped in 4D). Remarkably, the Spin(10) gauge fields in 5D turns the previous TQFT $w_2 w_3(TM) = w_2 w_3(V_{\mathrm{SO}(10)})$ into a 5D gapless bulk criticality!

[8]The concept of deconfined quantum criticality was first developed in the condensed matter community [28], to describe a class of direct continuous transition between two distinct symmetry breaking phases with fractionalized excitations and gauge structures emerging in the low-energy spectrum at and only at the transition. It occurs when a quantum system with global symmetry G has the tendency to spontaneously break the symmetry to its distinct subgroups $G_{\text{sub},1}$ and $G_{\text{sub},2}$, while the low-energy effective field theory has G anomaly but not $G_{\text{sub},1}$ or $G_{\text{sub},2}$ anomalies in terms of 't Hooft anomalies. Then the two symmetry breaking phases cannot share a trivial G-symmetric intermediate phase, paving ways for gapless phase transition and fractionalized excitations to emerge.

Several recent works explore the possible deconfined quantum criticality in 4D spacetime (see [29–32] and references therein). A hint toward our construction of 4D deconfined quantum criticality between symmetry breaking phase is the fact that the Spin (10) (treated as global symmetry) can have a 't Hooft anomaly of gauge-gravity anomaly type (due to the aforementioned $w_2 w_3$ anomaly); while the smaller subgroups with Lie algebras $su(5)$ of GG, $su(4) \times su(2) \times su(2)$ of PS, or $su(3) \times su(2) \times u(1)$ of SM, have no such $w_2 w_3$ anomaly. So the anomalous spacetime-internal Spin(10) symmetry hints a possible fractionalization of the GUT-Higgs field as a deconfined quantum criticality.

A crucial idea of deconfined quantum criticality construction is that "the G_{PS}-symmetry-breaking defect of the GG GUT-Higgs model traps the fractionalized quantum number of unbroken GG internal symmetry group; while vice versa, the G_{GG}-symmetry-breaking defect of the PS GUT-Higgs model traps the fractionalized quantum number of unbroken PS internal symmetry group." Here G_{PS} symmetry breaking and G_{GG} symmetry breaking, respectively, refer to the internal symmetry groups G (i.e., gauge group) of PS and GG models that are partly broken.

The terminology "gauge enhanced quantum criticality" is introduced in [32]. See also a recent review [33] on this topic.

[9]The reason that the non-Abelian gauge theory can become gapless in 5D can be understood simply by analyzing the renormalization group fixed point at the 5D Yang-Mills term, the dimensional analysis says $[|F|^2] \sim [F][F] \sim [\mathrm{d}A][\mathrm{d}A] + [\mathrm{d}A][A]^2 + [A]^4$. The kinetic term $[\mathrm{d}A][\mathrm{d}A]$ has the canonical scaling dimension 5 in 5D (i.e., E^5 in energy E). The [d] has a dimension 1 and the $[A]$ has a dimension $3/2$. The $[\mathrm{d}A][A]^2$ has a dimension $11/2$, while the $[A]^4$ has a dimension 6, which means that the $[\mathrm{d}A][A]^2$ and $[A]^4$ become irrelevant at low energy. Thus, the 5D non-Abelian Yang-Mills term $|F|^2$ behaves like the gapless 5D Abelian Maxwell term $|\mathrm{d}A|^2$.

In summary, when the internal symmetries are dynamically gauged (as in our gauged quantum vacuum),

(i) 4D gauge fields: The gauge fields of SM, GG, and PS GUT [$su(3) \times su(2) \times u(1)$, $su(5)$, and $su(4) \times su(2)_{\mathrm{L}} \times su(2)_{\mathrm{R}}$] are still restricted in 4D in their respective regions of quantum phase diagram (Fig. 8). There is still some emergent $[\mathrm{U}(1)']_{\text{gauge}}^{\text{emergent}}$ gauge field, also restricted in 4D, as a 4D boundary deconfined quantum criticality [the same as the previous case (1) when internal symmetry is not gauged).

(ii) 5D gauge fields: However, when and only when the GUT gauge fields are appropriately gauge enhanced [to the Spin(10) gauge fields in our Fig. 8], then they can propagate into the extra-dimensional 5D bulk, and they can induce a 5D bulk criticality.

Indeed our proposal manifests additional BSM excitations. After all, what are these BSM excitations near the quantum criticality in our theory?

(i) Dark gauge force sector: The emergent $[\mathrm{U}(1)']_{\text{gauge}}^{\text{emergent}}$ gauge fields correspond to analogous dark photon. However, our $[\mathrm{U}(1)']_{\text{gauge}}^{\text{emergent}} \equiv [\mathrm{U}(1)']_{\text{gauge}}^{\text{dark}}$ does not directly interact with the SM gauge forces, nor interact with the SM quarks and leptons. This dark photon sector can be a light dark matter candidate. The $[\mathrm{U}(1)']_{\text{gauge}}^{\text{dark}}$ only interacts with the fractionalized new fragmentary fermionic excitations that we name "colorons" and "flavorons."

(ii) Fragmentary fermionic colorons and flavorons: These are fractionalized excitations as the fermionic patrons. We implement the parton construction, where two (or multiple) of patrons $(\xi_a, \xi_b, \ldots)$ can combine with emergent gauge fields to form the GUT-Higgs Φ:

$$\Phi_{ab} \sim \xi_a^\dagger \xi_b, \text{ or more precisely}$$
$$\Phi_{ab}(x) \sim \xi_a^\dagger(x) \exp\left(\mathrm{i} \int_x^x a_{\mu,\text{gauge}}^{\text{dark}} \mathrm{d}x^\mu\right) \xi_b(x). \tag{1.2}$$

The GUT-Higgs Φ is also the basic degrees of freedom for the 4D WZW term that saturates the $w_2 w_3$ anomaly. To rephrase what we had said, the GUT-Higgs Φ is split into the fractionalized fragmentary colorons and flavorons. Just as the GUT-Higgs Φ can interact with the SM particles and SM gauge forces, the fragmentary colorons and flavorons can also interact with the SM particles and SM gauge forces. The colorons carries the SM's $\mathrm{SU}(3)_c$ strong gauge charge, while the flavorons carries the SM's $\mathrm{SU}(2)_{\mathrm{L}}$ weak gauge charge. Just like the GUT-Higgs are made to be very heavy, these colorons and flavorons are also heavy and can also be the heavy Dark Matter candidates. This fractionalization accompanies the emergent dark gauge field $a_{\mu,\text{gauge}}^{\text{dark}}$.

(4) The number of generations/families N_f: So far we have not yet specified the role of the number of generations N_f of quarks and leptons in our theory. If each generation of 16 SM Weyl fermions associates with its own GUT-Higgs field and its WZW term, then the generation number N_f times of 16 SM Weyl fermions with N_f GUT-Higgs field requires a constraint $N_f = 1 \bmod 2$ to match the $w_2 w_3$ anomaly, where $N_f = 3$ generation indeed works. However, regardless the N_f of SM, in general, we can just introduce a single (or any odd number) of GUT-Higgs field and WZW sector to match the 1 mod 2 class of $w_2 w_3$ anomaly. In any case, it is inspiring to confirm our proposal on the gauge enhanced quantum criticality can really happen between our $N_f = 3$ SM quantum vacuum and the neighbor GUT vacua. In this article, we focus on $N_f = 1$ for simplicity, but we can also triplicate $N_f = 1$ to $N_f = 3$.

In the remaining part of Sec. I, we start from an overview on the basic required ingredients of SM and GUT in Sec. I A.

A. Various standard models and grand unifications as effective field theories

Unification, as a central theme in the modern fundamental physics, is a theoretical framework aiming to embody the "elementary" excitations and forces into a common origin. Assuming without any significant dynamical gravity effect at the subatomic scale (i.e., we are only limited to probe the underlying quantum theory by placing the quantum systems on any curved spacetime geometry, but without significant gravity backreactions), the QFT provides a suitable framework for such a unification. Furthermore, assuming that we look at the QFT description valid below a certain energy scale (thus we are ignorant above that energy scale), we shall also implement the EFT perspective.

In fact, from the EFT perspective, we should remind ourselves the "elementary" excitations are only "elementary" in respect to a given EFT quantum vacuum. Moving away from the EFT vacuum (by tuning appropriate physical parameters) to a new quantum vacuum, we shall see that the "elementary" excitations of the new vacuum may be drastically different from the original "elementary" excitations of the previous EFT. So the "elementary" excitations reveal the limitations of our EFT descriptions of quantum vacua.[10] Several examples of such $(3+1)$D QFT and EFT paradigms for HEP include SM and GUT [1–7]:

(1) SM: Glashow, Salam, and Weinberg (GSW) [1–4] proposed the electroweak theory of the unified electromagnetic and weak forces between elementary particles. The GSW theory together with the strong force [34,35] becomes the SM, which is essential to describe the subatomic particle physics. The SM gauge group can be

$$G_{\mathrm{SM}_q} \equiv \frac{\mathrm{SU}(3)_c \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\tilde{Y}}}{\mathbb{Z}_q}$$

with the mod $q = 1, 2, 3, 6$ so far undetermined by the current experiments (see an overview [36,37] on this global structure of SM Lie group issue). The subscript c is for color, the L is for the internal SU(2) (L for internal symmetry and its spinor) locked with the left-handed Weyl fermion (L for spacetime symmetry and its spinor) in the standard HEP convention, and $\tilde{Y}$ for electroweak hypercharge. The "elementary" particle excitations of this SM EFT, with 15n or 16n Weyl fermions, are constrained by the representation of $su(3) \times su(2) \times u(1)$ as (see Fig. 4)[11]:

$$(\bar{\mathbf{3}}, \mathbf{1})_{2,L} \oplus (\mathbf{1}, \mathbf{2})_{-3,L} \oplus (\mathbf{3}, \mathbf{2})_{1,L} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{-4,L} \oplus (\mathbf{1}, \mathbf{1})_{6,L} \oplus (\mathbf{1}, \mathbf{1})_{0,L}. \tag{1.3}$$

The 16th Weyl fermion $(\mathbf{1}, \mathbf{1})_{0,L}$ is an extra sterile neutrino, sterile to the SM gauge force, also called the right-handed neutrino. We will focus on the 16n Weyl fermion model in this present work.[12] In our convention, we write Weyl fermions in the left-handed (L) basis, which means that each is a 2-component $\mathbf{2}_L$ spinor of the spacetime symmetry group Spin(1,3).

(2) The $su(5)$ GUT: GG [5] hypothesized that at a higher energy, the three SM gauge interactions merged into a single electronuclear force under a simple Lie algebra $su(5)$, or precisely a Lie group

$$G_{\mathrm{GG}} \equiv \mathrm{SU}(5)$$

gauge theory. The su(5) GUT works for 15n Weyl fermions, also for 16n Weyl fermions (i.e., 15 or 16 Weyl fermions per generation). The "elementary" particle excitations of this SU(5) EFT, with 15n or 16n Weyl fermions, are constrained by the representation of SU(5) as (see Fig. 5):

$$\bar{\mathbf{5}} \oplus \mathbf{10} \oplus \mathbf{1}, \tag{1.4}$$

again written all in the left-handed (L) Weyl basis. The 16th Weyl fermion is an extra sterile neutrino, sterile to the SU(5) gauge force, also called the right-handed neutrino.

(3) The PS model: PS [6] hypothesized that the lepton forms the fourth color, extending SU(3) to SU(4). The PS model also puts the left $\mathrm{SU}(2)_{\mathrm{L}}$ and a hypothetical right $\mathrm{SU}(2)_{\mathrm{R}}$ on equal footing. The PS gauge Lie algebra is $su(4) \times su(2)_{\mathrm{L}} \times su(2)_{\mathrm{R}}$, and the PS gauge Lie group is

$$G_{\mathrm{PS}_{q'}} \equiv \frac{\mathrm{SU}(4)_c \times (\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}})}{\mathbb{Z}_{q'}} = \frac{\mathrm{Spin}(6) \times \mathrm{Spin}(4)}{\mathbb{Z}_{q'}}$$

with the mod $q' = 1, 2$ depending on the global structure of Lie group. The "elementary" particle excitations of this PS EFT, with 16n Weyl fermions, are constrained by the representation of $G_{\mathrm{PS}_{q'}}$ as (see Fig. 6):

$$(\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}), \tag{1.5}$$

written all in the left-handed (L) Weyl basis.[13]

[10]Prominent examples occur in various systems with the duality descriptions and the order/disorder operators, such as in the Ising model and Majorana fermion system in $(1+1)$D.

[11]Here we use the integer quantized $\mathrm{U}(1)_{\tilde{Y}}$. If we use the phenomenology hypercharge $\mathrm{U}(1)_Y$ which is 1/6 of $\mathrm{U}(1)_{\tilde{Y}}$, namely $q_{\mathrm{U}(1)_Y} = \frac{1}{6} q_{\mathrm{U}(1)_{\tilde{Y}}}$, to write (1.3), then we have instead

$$(\bar{\mathbf{3}}, \mathbf{1})_{\frac{1}{3},L} \oplus (\mathbf{1}, \mathbf{2})_{-\frac{1}{2},L} \oplus (\mathbf{3}, \mathbf{2})_{\frac{1}{6},L} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{-\frac{2}{3},L} \oplus (\mathbf{1}, \mathbf{1})_{1,L} \oplus (\mathbf{1}, \mathbf{1})_{0,L}.$$

[12]In our present work, we shall focus on the SM or GUT with 16n Weyl fermions.

In contrast, Refs. [38–40] consider the SM or GUT with 15n Weyl fermions and with a discrete variant of baryon minus lepton number $\mathbf{B} - \mathbf{L}$ symmetry preserved. References [38–40] then suggest that the missing 16th Weyl fermions can be substituted by additional 4D or 5D gapped TQFTs, or by 4D gapless interacting conformal field theories (CFTs) to saturate a certain $\mathbb{Z}_{16}$ global anomaly. On the other hand, our present work does not introduce these $\mathbb{Z}_{16}$-class anomalous sectors, because we already have implemented the 16n Weyl fermion models that already make the $\mathbb{Z}_{16}$ global anomaly fully canceled.

[13]To be clear, we have the Weyl spacetime spinor $\mathbf{2}_L$ of Spin(1,3) for $(\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ of $su(4) \times su(2)_{\mathrm{L}} \times su(2)_{\mathrm{R}}$. In contrast, we can also write

$$\mathbf{2}_{\mathrm{L}} \text{ of } \mathrm{Spin}(1,3) \text{ for } (\mathbf{4}, \mathbf{2}, \mathbf{1}) \text{ of } su(4) \times su(2)_{\mathrm{L}} \times su(2)_{\mathrm{R}},$$
$$\mathbf{2}_{\mathrm{R}} \text{ of } \mathrm{Spin}(1,3) \text{ for } (\mathbf{4}, \mathbf{1}, \mathbf{2}) \text{ of } su(4) \times su(2)_{\mathrm{L}} \times su(2)_{\mathrm{R}},$$

then the representations of spacetime spinor L (or R) would lock exactly with the internal spinor L (or R). Here we use the L and R to specify the left-/right-handed spacetime spinor of Spin(1,3). We use the L and R to specify the left or right internal spinor representation of $su(2)_{\mathrm{L}} \times su(2)_{\mathrm{R}}$.

(4) The $so(10)$ GUT: Georgi and Fritzsch-Minkowski [7] hypothesized that quarks and leptons become the 16-dimensional spinor representation

$$\mathbf{16}^{+} \text{ of } G_{so(10)} \equiv \text{Spin}(10) \text{ gauge group} \tag{1.6}$$

[with a local Lie algebra $so(10)$]. Thus, the 16n Weyl fermions can interact via the Spin(10) gauge fields at a higher energy. In this case, the 16th Weyl fermion, previously a sterile neutrino to the SU(5), is no longer sterile to the Spin(10) gauge fields; it also carries a charge of 1, and is thus not sterile, under the gauged center subgroup $Z(\text{Spin}(10)) = \mathbb{Z}_4$.

We relegate several tables of data relevant for SMs and GUTs into Appendix A for readers' convenience to check the quantum numbers of various elementary particles or field quanta of SMs and GUTs.

II. STANDARD MODELS FROM THE COMPETING PHASES OF GRAND UNIFICATIONS

In Sec. II, we start by enlisting and explaining some group embedding structures from some of relevant GUTs to SM in Sec. II A.

A. Spacetime-internal symmetry group embedding of SMs and GUTs, and the $w_2 w_3$ anomaly

Here we use the inclusion notation $G_{\text{large}} \hookleftarrow G_{\text{small}}$ to imply that

(1) $G_{\text{large}} \supset G_{\text{small}}$, namely the G_{large} contains G_{small} as a subgroup, or equivalently G_{small} can be embedded in G_{large}.

(2) G_{large} can be broken to G_{small} via symmetry breaking of Higgs condensation (which we will explore).

The internal symmetry group embedding structure has been explored (for example summarized in [41]):

$$\begin{array}{ccc} G_{\text{SM}_6} \equiv \frac{\text{SU}(3)_c \times \text{SU}(2)_{\text{L}} \times \text{U}(1)_Y}{\mathbb{Z}_6} & \hookrightarrow & G_{\text{GG}} \equiv \text{SU}(5) \\ \downarrow & & \downarrow \\ G_{\text{PS}_2} \equiv \frac{\text{Spin}(6) \times \text{Spin}(4)}{\mathbb{Z}_2} & \hookrightarrow & \text{Spin}(10) \end{array} \tag{2.1}$$

We further include both the complete spacetime-internal symmetry group embedding structure as follows:

$$\bar{G} \equiv G_{\text{spacetime}} \times_{N_{\text{shared}}} G_{\text{internal}} \equiv \left(\frac{G_{\text{spacetime}} \times G_{\text{internal}}}{N_{\text{shared}}} \right). \tag{2.2}$$

$$\begin{array}{ccc} \bar{G}_{\text{SM}_6} \equiv \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \frac{\text{SU}(3)_c \times \text{SU}(2)_{\text{L}} \times \text{U}(1)_Y}{\mathbb{Z}_6} & \hookrightarrow & \bar{G}_{\text{GG}} \equiv \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \text{SU}(5)\ . \\ \downarrow & & \downarrow \\ \bar{G}_{\text{PS}_2} \equiv \text{Spin} \times_{\mathbb{Z}_2^F} \frac{\text{Spin}(6) \times \text{Spin}(4)}{\mathbb{Z}_2} & \hookrightarrow & \bar{G}_{so(10)} \equiv \text{Spin} \times_{\mathbb{Z}_2^F} \text{Spin}(10) \end{array} \tag{2.3}$$

Some comments about (2.3) follow:

(1) The Spin means the spacetime rotational symmetry group $\text{Spin} \equiv \text{Spin}(1,3)$ for 4D Lorentz signature [or $\text{Spin} \equiv \text{Spin}(4)$ for 4D Euclidean signature]. The Spin contains the fermionic parity $\mathbb{Z}_2^F$ at the center subgroup thus $\text{Spin}/\mathbb{Z}_2^F = \text{SO}$ where the SO is the bosonic spacetime (special orthogonal) rotational symmetry group [similarly, $\text{SO} \equiv \text{SO}(1,3)$ for 4D Lorentz signature, or $\text{SO} \equiv \text{SO}(4)$ for 4D Euclidean signature]. The notation $G_1 \times_{N_{\text{shared}}} G_2 \equiv \frac{G_1 \times G_2}{N_{\text{shared}}}$ means modding out their common normal subgroup N_{shared}. So $\text{Spin} \times_{\mathbb{Z}_2^F} G \equiv \frac{\text{Spin} \times G}{\mathbb{Z}_2^F}$ means modding out their common normal subgroup $\mathbb{Z}_2^F$.

(2) The $\mathbb{Z}_{4,X}$ has the X-symmetry generator such that its square $(X)^2 = (-1)^F$ is the fermion parity operator, so $\mathbb{Z}_{4,X} \supset \mathbb{Z}_2^F$. Wilczek-Zee [42] firstly noticed that the $X \equiv 5(\mathbf{B}-\mathbf{L}) - 4Y$, with the baryon minus lepton number $\mathbf{B}-\mathbf{L}$ and the electroweak hypercharge Y, is a good global symmetry respected by SM and the $su(5)$ GUT. All known quarks and leptons carry a charge 1 of $\mathbb{Z}_{4,X}$, in the left-handed Weyl spinor basis. The center of Spin(10) can be chosen exactly as $Z(\text{Spin}(10)) = \mathbb{Z}_{4,X}$. We summarize how $\mathbb{Z}_{4,X}$ can be obtained in Tables III and IV. See more discussions on $\mathbb{Z}_{4,X}$ in [22,25,38–40].

(3) The $(X)^2 = (-1)^F$ relation is obeyed in the non-supersymmetric SM and GUT models, so it is natural to introduce the $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ structure in (2.3). However, it is possible to have new fermions, such as in supersymmetric SMs or GUTs, which does not necessarily obey $(X)^2 = (-1)^F$ relation. In that case, we can introduce just $\text{Spin} \times \mathbb{Z}_{4,X}$

structure. See a footnote for the alternative symmetry embedding with the $\mathrm{Spin} \times \mathbb{Z}_{4,X}$ structure.[14]

(4) In this (2.3), we keep a structure of $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$, which is essential to produce a mixed gauge-gravity nonperturbative global anomaly constraint of a $\mathbb{Z}_{16}$ class. As already mentioned in footnote 12, in this article, we keep the 16n Weyl fermions in all our SM and GUT models, thus the $\mathbb{Z}_{16}$ global anomaly is already canceled by 16n chiral fermions.

(5) In this (2.3), we also keep a structure of $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)$—the cobordism group $\Omega_G^d \equiv \mathrm{TP}_d(G)$ shows [12,23]

$$\mathrm{TP}_5(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) = \mathbb{Z}_2, \quad \text{but}\,\mathrm{TP}_5(\mathrm{Spin} \times \mathrm{Spin}(10)) = 0. \tag{2.5}$$

This implies only the $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)$ structure offers a possible $\mathbb{Z}_2$ class global anomaly in 4D that is captured by a 5D invertible TQFT with a partition function on a 5D manifold M^5[15]:

$$\mathbf{Z}(M^5) = (-1)^{\int_{M^5} w_2(TM) w_3(TM)} = (-1)^{\int_{M^5} w_2(V_{\mathrm{SO}(10)}) w_3(V_{\mathrm{SO}(10)})}. \tag{2.8}$$

But this mod 2 anomaly is absent and not allowed on the $\mathrm{Spin} \times \mathrm{Spin}(10)$ structure. The difference between $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)$ and $\mathrm{Spin} \times \mathrm{Spin}(10)$ is the following: the fermion charge under $(-1)^F$ thus odd under $\mathbb{Z}_2^F$ must be in the $\mathbb{Z}_2$ normal subgroup of the center subgroup $Z(\mathrm{Spin}(10)) = \mathbb{Z}_{4,X}$ so $(X)^2 = (-1)^F$ in order to impose the spacetime-internal $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)$ structure. However, in contrast, the $\mathrm{Spin} \times \mathrm{Spin}(10)$ allows other fermions to not obey the $(X)^2 = (-1)^F$ relation.

As mentioned in [12,18] and footnote 7, as $\mathrm{Spin}(10) \supset \mathrm{Spin}(3) = \mathrm{SU}(2)$, so

$$\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10) \supset \mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(3) = \mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{SU}(2). \tag{2.9}$$

The $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)$ structure is tightly related to the $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{SU}(2)$ also known as the Spin^h structure. We can project the $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)$ structure to the Spin^h structure. Then, in the Spin^h structure, because the fermionic wave function gains a (-1) statistical sign under a 2π self-rotation on a Spin manifold is identified with the $(-1)^F$ as the center $Z(\mathrm{SU}(2)) = \mathbb{Z}_2^F$, we can read that imposing the Spin^h structure [12,18]:

(i) The fermions must be in the half-integer isospin representation 1/2, 3/2, …, etc., of SU(2) [namely, the even-dimensional representations **2**, **4**, …, etc., of SU(2)].

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[14] Another version of the spacetime-internal symmetry group embedding (that is more suitable for supersymmetric SMs or GUTs) is

$$\begin{array}{ccc} \bar{G}_{\mathrm{SM}_6} \equiv \mathrm{Spin} \times \mathbb{Z}_{4,X} \times \frac{\mathrm{SU}(3)_c \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_Y}{\mathbb{Z}_6} & \hookrightarrow & \bar{G}_{\mathrm{GG}} \equiv \mathrm{Spin} \times \mathbb{Z}_{4,X} \times \mathrm{SU}(5) \; . \\ \downarrow & & \downarrow \\ \bar{G}_{\mathrm{PS}_2} \equiv \mathrm{Spin} \times \frac{\mathrm{Spin}(6) \times \mathrm{Spin}(4)}{\mathbb{Z}_2} & \hookrightarrow & \bar{G}_{so(10)} \equiv \mathrm{Spin} \times \mathrm{Spin}(10) \end{array} \tag{2.4}$$

[15] The invertible TQFT means that the TQFT path integral or partition function $\mathbf{Z}(M)$ on any closed manifold M has its absolute value $|\mathbf{Z}(M)| = 1$. Thus the dimension of its Hilbert space is always 1 also any closed spatial manifold, there is no topological ground state degeneracy. Here $\mathbf{Z}(M^5) = (-1)^{\int w_2 w_3} = \pm 1$ on any closed M^5 thus it is an invertible TQFT, such that when M^5 is a Dold manifold $\mathbb{CP}^2 \rtimes S^1$ or a Wu manifold $\mathrm{SU}(3)/\mathrm{SO}(3)$ generating a $\mathbf{Z}(M^5) = -1$ [18,23]. Here the $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)$ structure imposes the spacetime and gauge bundle constraint

$$w_2(TM) = w_2(V_G), \tag{2.6}$$

with $G = \mathrm{Spin}(10)/\mathbb{Z}_2^F = \mathrm{SO}(10)$. Moreover, the Steenrod square Sq^1 is an operation sending the second cohomology to the third cohomology class: H^2 to H^3, which we can regard $\mathrm{Sq}^1 = \frac{1}{2}\delta$ with δ as a coboundary operator (see for example [23]). Then, in the case $G = \mathrm{SO}(10)$, we can deduce another bundle constraint:

$$w_3(TM) + w_1(TM) w_2(TM) = \mathrm{Sq}^1 w_2(TM) = \mathrm{Sq}^1 w_2(V_G) = w_3(V_G). \tag{2.7}$$

On the orientable spacetime, the first Stiefel-Whitney class $w_1(TM) = 0$, so

$$w_3(TM) = w_3(V_G).$$

Thus, combining the above formulas, on the orientable $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)$ structure, we derive that $w_2(TM) w_3(TM) = w_2(V_G) w_3(V_G)$ in (2.8), shorthand as $w_2 w_3 = w_2 w_3(TM) = w_2 w_3(V_G)$. This derivation also works for other $G = \mathrm{Spin}(n)/\mathbb{Z}_2^F = \mathrm{SO}(n)$ for $n \geq 3$.

(ii) The bosons must be in the integer isospin representation 0, 1, 2, …, etc., of SU(2) [namely, the odd-dimensional representations $\mathbf{0}, \mathbf{1}, \mathbf{3}, \ldots$, etc., of SU(2)].

(6) The last but the most important comment above all, is that in order to realize a possible continuous deconfined quantum phase transition, we do require to use the $w_2 w_3$ anomaly in (2.8), such that this anomaly occurs in the phase transition between the GG and PS models in Fig. 8. So we do aim to impose the $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)$ structure as in (2.3) in order to implement the $w_2 w_3$ anomaly. In short, the readers can ask: Why do we need the $w_2 w_3$ anomaly near the criticality for establishing a possible continuous quantum phase transition between the GG and PS models? The answer is the following:

(i) The GG and PS models are Landau-Ginzburg symmetry breaking type of phases (when we treat the internal symmetry as global symmetry) or the gauge-symmetry breaking type of phases (when we treat the internal symmetry group as gauge group). The $w_2 w_3$ anomaly is matched on two sides of phases by GG and PS models via symmetry breaking. (In fact, no $w_2 w_3$ anomaly is allowed in GG and PS models.)

(ii) But the $w_2 w_3$ anomaly can protect a gapless quantum phase transition (or a gapless intermediate quantum critical region) between the GG and PS models when the Spin(10) symmetry is restored at their phase transition. Their phase transition can be protected to be Spin(10) symmetry preserving gapless due to the $w_2 w_3$ anomaly exists only in the enlarged Spin(10) internal symmetry group.

Because the conventional $so(10)$ GUT is free from the $w_2 w_3$ anomaly [12,18], we will need to explicitly introduce a new WZW-like term built out of GUT-Higgs field in the mother EFT, which allows the GUT-Higgs sector (beyond the SM sector) to saturate the $w_2 w_3$ anomaly. To this end, we will start from writing down a GUT-Higgs model in the context of $so(10)$ GUT, and then trying to modifying the GUT-Higgs model to saturate the $w_2 w_3$ anomaly. (That mother EFT will be the main achievement later in Sec. III.)

B. Branching rule of SMs and GUTs, and a GUT-Higgs model

In the following, we motivate the GUT model with the GUT-Higgs model as the gauge symmetry breaking pattern to go to the lower energy EFT (such as SM). Most of these breaking patterns are well-established and overviewed in [43]. The additional new input is that we try to unify several models into a GUT-Higgs model with as minimum amount of GUT-Higgs fields as possible. In Appendix B, we try to go through the logic again, and carefully examine the consequences and possibilities of the types of required GUT-Higgs models. Later we will motivate the possible Lagrangian of the GUT-Higgs potential.

Here we summarize what we need from the analysis done in Appendix B:

(1) We can use a Lorentz scalar boson with a 45-dimensional real representation of $so(10)$ or Spin(10):

$$\Phi_{so(10),\mathbf{45}} \equiv \Phi_{\mathbf{45}} \in \mathbb{R} \tag{2.10}$$

to break the Spin(10) of $so(10)$ GUT to the SU(5) of GG model, also we can use this same $\Phi_{\mathbf{45}}$ to break $G_{\mathrm{PS}_2} \equiv \frac{\mathrm{Spin}(6)\times\mathrm{Spin}(4)}{\mathbb{Z}_2}$ of PS model to the $G_{\mathrm{SM}_6} \equiv \frac{\mathrm{SU}(3)_c\times\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{U}(1)_Y}{\mathbb{Z}_6}$ of the SM.

(2) We can use a Lorentz scalar boson with a 54-dimensional real representation of $so(10)$ or Spin(10):

$$\Phi_{so(10),\mathbf{54}} \equiv \Phi_{\mathbf{54}} \in \mathbb{R}, \tag{2.11}$$

to break the Spin(10) of $so(10)$ GUT to the $G_{\mathrm{PS}_2} \equiv \frac{\mathrm{Spin}(6)\times\mathrm{Spin}(4)}{\mathbb{Z}_2}$ of PS model, also we can use this same $\Phi_{\mathbf{54}}$ to break SU(5) of GG model to the $G_{\mathrm{SM}_6} \equiv \frac{\mathrm{SU}(3)_c\times\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{U}(1)_Y}{\mathbb{Z}_6}$ of the SM.

(3) The combinations of the two facts above is summarized in Fig. 9, where we can use the $\Phi_{\mathbf{45}}$ and $\Phi_{\mathbf{54}}$ to write the GUT-Higgs model, that can induce the qualitative phase diagram similar to Fig. 8.

Given the $so(10)$ GUT, to induce the three other models in Fig. 9, we can add the GUT-Higgs potential $\mathrm{U}(\Phi_{\mathbf{R}})$ with $\Phi_{\mathbf{R}}$ of some representation $\mathbf{R}$. The $\mathrm{U}(\Phi_{\mathbf{R}})$ is chosen to have positive Φ^4 coefficients (thus $\lambda_{\mathbf{45}}, \lambda_{\mathbf{54}} > 0$), while the $r_{\mathbf{45}}$ and $r_{\mathbf{54}}$ are real-number tunable parameters shown in Figs. 8 and 10:

$$\mathrm{U}(\Phi_{\mathbf{R}}) = (r_{\mathbf{45}}(\Phi_{\mathbf{45}})^2 + \lambda_{\mathbf{45}}(\Phi_{\mathbf{45}})^4) + (r_{\mathbf{54}}(\Phi_{\mathbf{54}})^2 + \lambda_{\mathbf{54}}(\Phi_{\mathbf{54}})^4). \tag{2.12}$$

A slice of Fig. 10 becomes Fig. 8. (Temporarily now we get rid of the GUT-Higgs $\Phi_{\mathbf{1}}$, and thus get rid of r_1 axis in Fig. 10. More on this $\Phi_{\mathbf{1}}$ later.) We can use this $\mathrm{U}(\Phi_{\mathbf{R}})$ potential in (2.12) to induce these interior parts of four phases [the $so(10)$ GUT, the $su(5)$ GUT, the PS model, and the SM].

(1) If $\langle\Phi_{\mathbf{45}}\rangle$ condenses, namely if $r_{\mathbf{45}} < 0$ so $\langle\Phi_{\mathbf{45}}\rangle \neq 0$, then the $so(10)$ GUT becomes Higgs down to the $su(5)$ GUT.

(2) If $\langle\Phi_{\mathbf{54}}\rangle$ condenses, namely if $r_{\mathbf{54}} < 0$ so $\langle\Phi_{\mathbf{54}}\rangle \neq 0$, then the $so(10)$ GUT becomes Higgs down to the PS model.

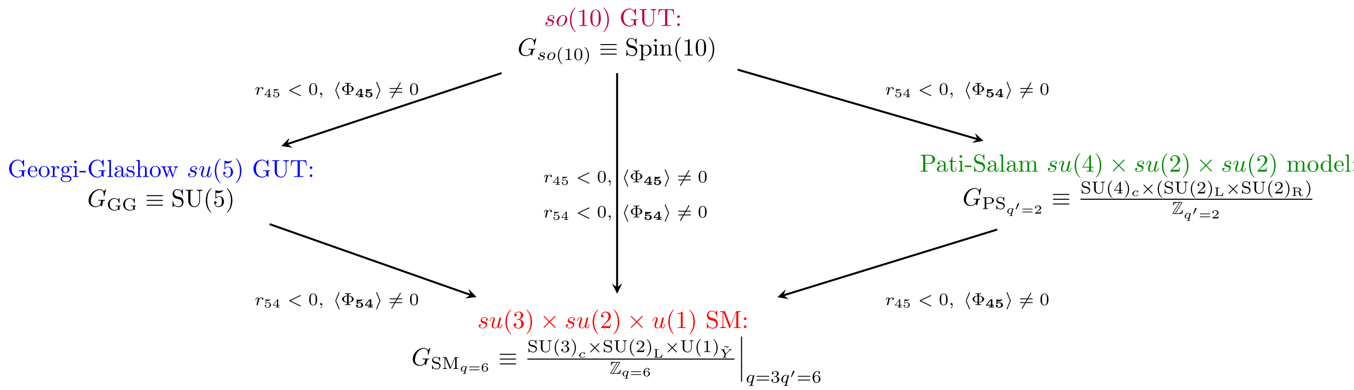

FIG. 9. Beware that the direction of the group symmetry breaking "→" is the opposite direction to the group inclusion "↩." (These colors are also designed to match the colors in Figs. 4–7, 8.).

(3) If $\langle \Phi_{\mathbf{45}} \rangle$ and $\langle \Phi_{\mathbf{54}} \rangle$ both condense, namely if $r_{\mathbf{45}} < 0$ and $r_{\mathbf{54}} < 0$ so that $\langle \Phi_{\mathbf{45}} \rangle \neq 0$ and $\langle \Phi_{\mathbf{54}} \rangle \neq 0$. The theory becomes Higgs down to the SM.

All these above Higgs condensations induce continuous phase transitions.

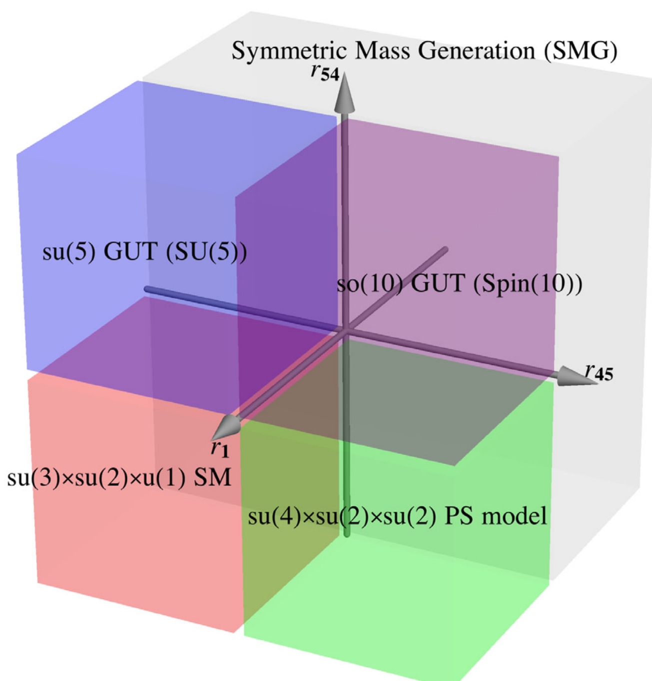

FIG. 10. Schematic quantum phase diagram interpolating between the $so(10)$ GUT [Spin(10) group], the Georgi-Glashow $su(5)$ GUT [SU(5) group], the $su(4) \times su(2)_{\mathrm{L}} \times su(2)_{\mathrm{R}}$ PS model, and the $su(3) \times su(2) \times u(1)$ SM, and the symmetric mass generation (SMG). Here the real parameter $r_{\mathbf{R}} \in \mathbb{R}$ denotes the coefficient of the effective quadratic potential of Φ field in the representation $\mathbf{R}$. The corresponding GUT-Higgs Φ field will condense in the representation $\mathbf{R}$ if $r_{\mathbf{R}} < 0$. Relatively speaking, the IR low energy is drawn with the red color (for SM), the intermediate neighbor phases are drawn with the green or blue color [for PS or SU(5) models], while the UV higher energy is drawn with the violet purple color [for Spin(10)]. These colors are also designed to match the colors of partitions of representations in Figs. 4–7.

The purpose of the next Sec. III is to design various EFT and to explore the possible phase structures and phase transitions (of Figs. 8 and 10). In particular, we will write down a mother EFT such that it saturates the $w_2 w_3$ global anomaly and it realizes an exotic quantum phase transition between the GG $su(5)$ GUT and the PS model.

III. MOTHER EFFECTIVE FIELD THEORY WITH COMPETING GUT-HIGGS FIELDS

A. Elementary GUT-Higgs model induces the SM

In Sec. II (especially Sec. II B), we write down a GUT-Higgs potential $\mathrm{U}(\Phi_{\mathbf{R}})$ in (2.12) appending to the $so(10)$ GUT with 16n complex Weyl fermions ψ_L. Let us write down the full path integral $\mathbf{Z}_{\mathrm{GUT}}$ of such $so(10)$ GUT plus $\mathrm{U}(\Phi_{\mathbf{R}})$, in a Lorentzian signature, evaluated on a four manifold M^4:

$$\mathbf{Z}_{\mathrm{GUT}} \equiv \int [\mathcal{D}\psi_L][\mathcal{D}\psi_L^\dagger][\mathcal{D}A][\mathcal{D}\Phi_{\mathbf{R}}]\ldots \exp(\mathrm{i}S_{\mathrm{GUT}}[\psi_L, \psi_L^\dagger, A, \Phi_{\mathbf{R}}, \ldots]|_{M^4}). \quad (3.1)$$

The action S_{GUT} is

$$S_{\mathrm{GUT}} = \int_{M^4} \left(\mathrm{Tr}(F \wedge \star F) - \frac{\theta}{8\pi^2} g^2 \mathrm{Tr}(F \wedge F) \right) + \int_{M^4} (\psi_L^\dagger (\mathrm{i}\bar{\sigma}^\mu D_{\mu,A}) \psi_L + |D_{\mu,A} \Phi_{\mathbf{R}}|^2 - \mathrm{U}(\Phi_{\mathbf{R}}) - ((\Phi_{\mathbf{R}})(\psi_L^\dagger \cdots)(\psi_L \cdots) + \mathrm{H.c.}) + \ldots) \mathrm{d}^4 x. \quad (3.2)$$

The $S_{\mathrm{YM}} = \int \mathrm{Tr}(F \wedge \star F)$ part is the Yang-Mills gauge theory, with Lie algebra valued field strength curvature 2-form $F = \mathrm{d}A - \mathrm{i}gA \wedge A$. Here $(\psi_L^\dagger \cdots)$ and $(\psi_L \cdots)$ imply indefinite multiple numbers of Weyl fermion fields, so as to properly match the representation $\mathbf{R}$ of the Higgs field $\Phi_{\mathbf{R}}$. For the $so(10)$ GUT, we have to sum over the Spin(10) gauge bundle, whose 1-form connection is the spin-1 Lorentz vector and Spin(10) gauge field, written as

$$A = \left(\sum_{\mathrm{a}=1}^{45} \mathrm{T}^{\mathrm{a}} A^{\mathrm{a}}_{\mathrm{Spin}(10),\mu}\right) \mathrm{d}x^{\mu}. \quad (3.3)$$

There are 45 such Lie algebra generators, T^{a}, with

(i) rank-16 matrix representations that act on the quark-and-lepton matter representation $\mathbf{16}^{+}$ of Spin(10),

(ii) rank-45 matrix representations that act on the $\Phi_{\mathbf{45}}$ as the **45** of Spin(10),

(iii) rank-54 matrix representations that act on the $\Phi_{\mathbf{54}}$ as the **54** of Spin(10).

Locally the Spin(10) Lie algebra is the same as the $so(10)$ Lie algebra, but globally we really need to define the principal Spin(10) gauge bundle P_A to sum over. So more precisely the path integral over the gauge field measure really means $\int [\mathcal{D}A] \ldots \equiv \sum_{\text{gauge bundle} P_A} \int [\mathcal{D}\tilde{A}] \ldots$, where $\tilde{A}$ are gauge connections over each specific gauge bundle choice P_A. The θ term, $\theta \mathrm{Tr}(F \wedge F)$ can be added or removed depending on the model. In this work, we shall set $\theta = 0$ or close to zero.

The ψ_L is a 2-component spin-1/2 Weyl fermion $\mathbf{2}_L$ of Spin(1,3). The $\dagger$ is the standard complex conjugate transpose. The $\bar{\sigma}^{\mu} = (\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3)$ and $\sigma^{\mu} = (\sigma^0, \sigma^1, \sigma^2, \sigma^3)$ are the standard spacetime spinor rotational $su(2)$ Lie algebra generators for L and R Weyl spinors. The action S_{GUT} also includes the Weyl spinor kinetic term and GUT-Higgs kinetic term, coupling to gauge fields via the covariant derivative operator $D_{\mu,A} \equiv \nabla_{\mu} - \mathrm{i}gA_{\mu}$. The ∇_{μ} can contain the curve-spacetime covariant derivative data such as Christoffel symbols or the spinor's spin-connection if needed. The … are possible extra deformation terms to be added later.

This subsection mostly treats the spin-0 Lorentz scalar Higgs field $\Phi_{\mathbf{R}}$ with some representation $\mathbf{R}$ as the elementary Higgs field. We will however fractionalize this elementary Higgs field $\Phi_{\mathbf{R}}$ to other further elementary fermionic fields in the later Secs. III C and III D.

1. Model I: Without the Wess-Zumino-Witten term and symmetric mass generation

Following the choice in Sec. II B and in (2.12), we can further adjust it to

$$\begin{aligned} \mathrm{U}(\Phi_{\mathbf{R}}) = &(r_{\mathbf{45}}(\Phi_{\mathbf{45}})^2 + \lambda_{\mathbf{45}}(\Phi_{\mathbf{45}})^4) \\ &+ (r_{\mathbf{54}}(\Phi_{\mathbf{54}})^2 + \lambda_{\mathbf{54}}(\Phi_{\mathbf{54}})^4) \\ &+ (r_{\mathbf{1}}(\Phi_{\mathbf{1}})^2 + \lambda_{\mathbf{1}}(\Phi_{\mathbf{1}})^4). \end{aligned} \quad (3.4)$$

The property (whether $\langle \Phi_{\mathbf{45}} \rangle \neq 0$ or $\langle \Phi_{\mathbf{54}} \rangle \neq 0$ condenses, or both condense, namely whether $r_{\mathbf{54}} < 0$ or $r_{\mathbf{54}} < 0$) still follows Sec. II B. The theory becomes Higgs down to the $su(5)$ GUT, or the PS model, or the SM, see Fig. 9. Here are some extra comments for adding $\Phi_{\mathbf{1}}$ or other $\Phi_{\mathbf{R}}$ terms to Fig. 10:

(1) We can introduce a Lorentz scalar boson with a one-dimensional trivial but real representation of $so(10)$ or Spin(10):

$$\mathbf{\Phi}_{so(10),1} \equiv \Phi_{\mathbf{1}} \in \mathbb{R}. \quad (3.5)$$

(i) If $\langle \Phi_{\mathbf{1}} \rangle = 0$ does not condense, namely if $r_{\mathbf{1}} > 0$, then the theory remains in the $so(10)$ GUT.

(ii) If $\langle \Phi_{\mathbf{1}} \rangle \neq 0$ condenses, namely if $r_{\mathbf{1}} < 0$, for a small $\langle \Phi_{\mathbf{1}} \rangle < \Phi_{\mathbf{1},c}$, then the theory still remains in the $so(10)$ GUT (as $\langle \Phi_{\mathbf{1}} \rangle$ is an irrelevant perturbation).

(iii) However, not only $\langle \Phi_{\mathbf{1}} \rangle \neq 0$ condenses, but when $\langle \Phi_{\mathbf{1}} \rangle > \Phi_{\mathbf{1},c}$ exceeds a critical value, then it can drive to the SMG phase and gap out all fermions while preserving the G symmetry (if the theory is free from all 't Hooft anomalies in G).[16]

How do we associate $\langle \Phi_{\mathbf{1}} \rangle > \Phi_{\mathbf{1},c}$ with the SMG effect? First notice that the four of the spinor representations $\mathbf{16}^{+}$ of Spin(10) can produce the tensor product decomposition [58]

$$\begin{aligned} \mathbf{16} \otimes \mathbf{16} \otimes \mathbf{16} \otimes \mathbf{16} &= (\mathbf{10} \oplus \mathbf{120} \oplus \overline{\mathbf{126}}) \otimes (\mathbf{10} \oplus \mathbf{120} \oplus \overline{\mathbf{126}}) \\ &= (\mathbf{10} \otimes \mathbf{10}) \oplus (\mathbf{120} \otimes \mathbf{120}) \oplus (\overline{\mathbf{126}} \otimes \overline{\mathbf{126}}) \oplus 2(\mathbf{10} \otimes \mathbf{120}) \oplus 2(\mathbf{10} \otimes \overline{\mathbf{126}}) \oplus 2(\mathbf{120} \otimes \overline{\mathbf{126}}) \\ &= (\mathbf{1} \oplus \mathbf{45} \oplus \mathbf{54}) \oplus (\mathbf{1} \oplus \mathbf{45} \oplus \mathbf{54} \oplus 2(\mathbf{210}) \oplus \mathbf{770} \oplus \mathbf{945} \oplus \mathbf{1050} \oplus \overline{\mathbf{1050}} \oplus \mathbf{4125} \oplus \mathbf{5940}) \\ &\quad \oplus (\mathbf{54} \oplus \mathbf{945} \oplus \overline{\mathbf{1050}} \oplus \overline{\mathbf{2772}} \oplus \mathbf{4125} \oplus \overline{\mathbf{6930}}) \oplus 2(\mathbf{45} \oplus \mathbf{210} \oplus \mathbf{945}) \oplus 2(\mathbf{210} \oplus \overline{\mathbf{1050}}) \\ &\quad \oplus 2(\mathbf{45} \oplus \mathbf{210} \oplus \mathbf{945} \oplus \overline{\mathbf{1050}} \oplus \mathbf{5940} \oplus \overline{\mathbf{6930}}) \end{aligned} \quad (3.6)$$

[16] The SMG mechanism is explored in various references, for some selective examples, by Fidkowski and Kitaev [44] in (0 + 1)D, by Wang and Wen [45,46] for gapping chiral fermions in (1 + 1)D, You *et al.* [47,48] in (2 + 1)D, and notable examples in (3 + 1)D by Eichten and Preskill [49], Wen [50], You, BenTov, and Xu [51,52], BenTov and Zee [53], Kikukawa [54], Wang and Wen [12], Catterall *et al.* [55,56], Razamat and Tong [13,57], etc.

More systematically, with the symmetric (S) or antisymmetric (A) matrix representation subscript indicated on the right-hand side:

$$\begin{aligned}
\mathbf{16}\otimes\mathbf{16} &= \mathbf{10}_{\rm S}\oplus\mathbf{120}_{\rm A}\oplus\overline{\mathbf{126}}_{\rm S}.\\
\mathbf{10}\otimes\mathbf{10} &= \mathbf{1}_{\rm S}\oplus\mathbf{45}_{\rm A}\oplus\mathbf{54}_{\rm S}.\\
\mathbf{120}\otimes\mathbf{120} &= \mathbf{1}_{\rm S}\oplus\mathbf{45}_{\rm A}\oplus\mathbf{54}_{\rm S}\oplus\mathbf{210}_{\rm S}\\
&\oplus\mathbf{210}_{\rm A}\oplus\mathbf{770}_{\rm S}\oplus\mathbf{945}_{\rm A}\\
&\oplus\mathbf{1050}_{\rm S}\oplus\overline{\mathbf{1050}}_{\rm S}\oplus\mathbf{4125}_{\rm S}\\
&\oplus\mathbf{5940}_{\rm A}.\\
\mathbf{126}\otimes\mathbf{126} &= \mathbf{54}_{\rm S}\oplus\mathbf{945}_{\rm A}\oplus\mathbf{1050}_{\rm S}\oplus\mathbf{2772}_{\rm S}\\
&\oplus\mathbf{4125}_{\rm S}\oplus\mathbf{6930}_{\rm A}.\\
\mathbf{10}\otimes\mathbf{120} &= \mathbf{45}\oplus\mathbf{210}\oplus\mathbf{945}.\\
\mathbf{10}\otimes\mathbf{126} &= \mathbf{210}\oplus\mathbf{1050}.\\
\mathbf{120}\otimes\mathbf{126} &= \mathbf{45}\oplus\mathbf{210}\oplus\mathbf{945}\oplus\mathbf{1050}\\
&\oplus\mathbf{5940}\oplus\mathbf{6930}. \qquad (3.7)
\end{aligned}$$

From (3.6), we learn that four of **16** can produce two trivial representations **1** of $so(10)$ or Spin (10), one from $\mathbf{10}\otimes\mathbf{10}$ and one from $\mathbf{120}\otimes\mathbf{120}$. Therefore, on the mean field level, we can deduce the expectation of the GUT-Higgs $\Phi_{\mathbf{1}}$ from some schematic effective four-fermion interactions of ψ in **16** of Spin(10)[17]:

$$\langle\Phi_{\mathbf{1}}\rangle \simeq \langle\psi\psi\psi\psi\rangle \neq 0. \qquad (3.8)$$

But we do not wish to impose the ordinary Anderson-Higgs quadratic mass term induced by $\langle\psi\psi\rangle \neq 0$, otherwise this $\langle\psi\psi\rangle \neq 0$ will lead to Spin(10) symmetry breaking, instead of the Spin(10) symmetry preserving SMG. This means that we have to impose $\langle\psi\psi\rangle = 0$, so

$$\langle\psi\psi\rangle\psi\psi = 0,$$
$$\text{no conventional mass due to } \langle\psi\psi\rangle = 0. \qquad (3.9)$$

Thus the above argument implies that above a critical condensation value $\langle\Phi_{\mathbf{1}}\rangle > \Phi_{\mathbf{1},c}$ as the interaction strength goes above a critical value, we do obtain the SMG effect in Fig. 10.

To implement the SMG to gap out the 16 Weyl fermions in **16**, a necessary check is that the fermions are free from all 't Hooft anomalies in the Spin(10), or more precisely free from all 't Hooft anomalies in the spacetime-internal Spin $\times_{\mathbb{Z}_2^F}$ Spin(10) structure. This is true based on (2.5), because there is only a mod 2 class w_2w_3 global anomaly, which the 16 Weyl fermions in **16** do not carry any w_2w_3 global anomaly. So we are able to gap out the 16 Weyl fermions while preserving Spin $\times_{\mathbb{Z}_2^F}$ Spin(10) symmetry.

To strengthen and improve Ref. [50]'s argument, we may regard our Φ as a bivector of two 10-dimensional vector $\phi_{so(10),\mathbf{10}} \equiv \phi_{\mathbf{10}}$ in **10** (or regard Φ as a bivector of two **120**-dimensional vector $\phi_{so(10),\mathbf{120}} \equiv \phi_{\mathbf{120}}$ in 120). Thus, schematically

$$\begin{aligned}
\langle\Phi_{\mathbf{1}}\rangle &\simeq \langle\phi_{\mathbf{10}}\phi_{\mathbf{10}}\rangle + \langle\phi_{\mathbf{120}}\phi_{\mathbf{120}}\rangle + \ldots\\
&\simeq \langle\psi\psi\psi\psi\rangle + \ldots \neq 0. \qquad (3.10)
\end{aligned}$$

This $\langle\Phi_{\mathbf{1}}\rangle > \Phi_{\mathbf{1},c} \neq 0$ implies that the bilinear of vectors (bivector) condense: $\langle\phi_{\mathbf{10}}\phi_{\mathbf{10}}\rangle \neq 0$ and/or $\langle\phi_{\mathbf{120}}\phi_{\mathbf{120}}\rangle \neq 0$, but the $\langle\phi_{\mathbf{10}}\rangle = \langle\phi_{\mathbf{120}}\rangle = 0$. So no ordinary quadratic fermion mass term is induced, but only the SMG is induced. The SMG causes the symmetry-preserving disordered mass.

But one of the mother EFTs (model II) that we will propose later in Sec. III A 2, indeed have an extra new bosonic sector carrying the mod 2 class w_2w_3 global anomaly. This bosonic sector includes the WZW term built out of GUT-Higgs fields. To reiterate, there is no conflict about gapping the 16 Weyl fermions, but having the extra bosonic sector carry another anomaly. This simply implies that if we demand to preserve Spin $\times_{\mathbb{Z}_2^F}$ Spin(10) symmetry, although we can gap out the Weyl fermions in **16**, the extra GUT-Higgs WZW bosonic sectors will still induce additional symmetry-preserving gapless modes.

(2) In the standard Anderson-Higgs electroweak symmetry breaking mechanism, Higgs coupling $(\psi_L^\dagger\Phi_{\mathbf{R}}(\mathrm{i}\sigma^2\psi_L'^*) + \text{H.c.})$ is introduced in order to give quadratic masses to Weyl fermions. In this work, we may need to introduce more general GUT-Higgs fields $\Phi_{\mathbf{R}}$ with various representations **R**. For a generic representation **R**, the Higgs field may couple to a product of even number (not limited to two) of fermion operators (e.g., $\psi^\dagger\psi^\dagger\psi\psi$ or $\psi\psi\psi\psi$), such that the fermion representation can combine to match the corresponding Higgs field representation. (We shall not get distracted to handle the Anderson-Higgs electroweak symmetry breaking masses of Weyl fermions in this article, as this effect is well studied. But we make some comments in Appendix B.)

[17]Here fermions are anticommuting Grassman variables, so this expression $\langle\psi\psi\psi\psi\rangle$ is only schematic. The precise expression of $\langle\psi\psi\psi\psi\rangle$ includes additional spacetime-internal representation indices and also includes possible additional spacetime derivatives (for point splitting the fermions to neighbor sites if writing them on a regularized lattice).

(3) Scaling dimensions of tuning parameters. $r_{\mathbf{R}}$. Because the GUT-Higgs field $\Phi_{\mathbf{45}}$, $\Phi_{\mathbf{54}}$, and $\Phi_{\mathbf{1}}$ all couple to four fermion operators (e.g., $\psi^\dagger\psi^\dagger\psi\psi$ or $\psi\psi\psi\psi + \text{H.c.}$), the term $r_{\mathbf{R}}\Phi_{\mathbf{R}}^2$ that tunes the Higgs transition will correspond to a eight-fermion interaction. At the SM fixed point, the matter fermion ψ has a scaling dimension $3/2$. So the eight-fermion interaction that drives the Higgs transition will have a scaling dimension $3/2 \times 8 = 12$, which is much higher than the spacetime dimension of 4. For this reason, such interaction is often ignored in the existing study of the SM. Although such an interaction is perturbatively irrelevant at the SM fixed point, a strong enough interaction will lead to a nonperturbative effect that modifies the tuning parameters $r_{\mathbf{R}}$ and eventually drives the Higgs transitions between the SM phase and its adjacent GUT phases (such as the PS and GG phases).

So taking into account the GUT-Higgs condensation or noncondensation, we obtain a qualitative phase diagram in Fig. 10.

2. Model II: With the Wess-Zumino-Witten term and deconfined quantum criticality

Now we propose a new mother EFT path integral by modifying the action S_{GUT} to $S_{\text{GUT}}^{\text{WZW}}$ via adding the WZW term and other terms, in a Lorentzian signature path integral:

$$\mathbf{Z}_{\text{GUT}}^{\text{WZW}} \equiv \int [\mathcal{D}\psi_L][\mathcal{D}\psi_L^\dagger][\mathcal{D}A][\mathcal{D}\Phi_{\mathbf{R}}][\mathcal{D}\Phi^{\text{bi}}][\mathcal{D}\phi]\ldots \exp(\mathrm{i}S_{\text{GUT}}^{\text{WZW}}[\psi_L,\psi_L^\dagger,A,\Phi_{\mathbf{R}},\Phi^{\text{bi}},\phi,\ldots]|_{M^4}), \tag{3.11}$$

$$\begin{aligned} S_{\text{GUT}}^{\text{WZW}} \equiv {}& \int_{M^4} \text{Tr}(F\wedge\star F) + \int_{M^4}\bigg(\psi_L^\dagger(\mathrm{i}\bar\sigma^\mu D_{\mu,A})\psi_L + |D_{\mu,A}\Phi_{\mathbf{R}}|^2 - \text{U}(\Phi_{\mathbf{R}}) \\ & + \frac{1}{2}\phi^\intercal\Phi^{\text{bi}}\phi + \frac{1}{2}\sum_{a=1}^{5}(\psi_L^\intercal \mathrm{i}\sigma^2(\phi_{2a-1}\Gamma_{2a-1} - \mathrm{i}\phi_{2a}\Gamma_{2a})\psi_L + \text{H.c.})\bigg)\mathrm{d}^4x + S^{\text{WZW}}[\Phi^{\text{bi}}]. \end{aligned} \tag{3.12}$$

The purpose of the new discrete torsion class 4D WZW-like term (written on a 5D manifold with 4D boundary), which we will introduce in details later, is to saturate the w_2w_3 global anomaly. The mother EFT contains the following detailed ingredients:

(1) There are 16n complex Weyl fermions, each ψ_L is the **16** of Spin(10) minimally coupled to Spin(10) gauge field in the covariant derivative. Properties of the Spin(10) gauge field A and other familiar terms in S_{GUT} had been explained in the earlier Sec. III A.

(2) An SO(10) real vector field $\phi \in \mathbb{R}$ is in **10** of $so(10)$ also of Spin(10). To be explicit, ϕ contains one vector index, ϕ_a with $a \in \{1,2,\ldots,10\}$.

(3) An SO(10) real bivector field $\Phi^{\text{bi}} \in \mathbb{R}$ is obtained from the tensor product of the two ϕ, in the $\mathbf{10}\otimes\mathbf{10} = \mathbf{1}_{\text{S}} \oplus \mathbf{45}_{\text{A}} \oplus \mathbf{54}_{\text{S}}$ of $so(10)$ also of Spin(10). To be explicit, Φ^{bi} contains two vector indices, Φ^{bi}_{ab} with $a,b \in \{1,2,\ldots,10\}$. We can arrange Φ^{bi}_{ab} into three different representations $\mathbf{R}$ of $\Phi_{\mathbf{R}}$ as the three GUT-Higgs fields $\Phi_{\mathbf{1}}$, $\Phi_{\mathbf{45}}$ and $\Phi_{\mathbf{54}}$ (which appeared in Sec. III A 1):

$$\Phi^{\text{bi}}_{ab} = \phi_a\phi_b \text{ includes} \begin{cases} \text{Tr}\Phi^{\text{bi}} = \sum_a \Phi^{\text{bi}}_{aa} \text{ gives } \Phi_{\mathbf{R}} = \Phi_{\mathbf{1}} \text{ in } \mathbf{1}_{\text{S}} \\ \hat\Phi^{\text{bi}} \equiv \Phi^{\text{bi}}_{[a,b]} = \frac{1}{2}(\Phi^{\text{bi}}_{ab} - \Phi^{\text{bi}}_{ba}) = \frac{1}{2}(\phi_a\phi_b - \phi_b\phi_a) = \frac{1}{2}[\phi_a,\phi_b] \text{ gives } \Phi_{\mathbf{R}} = \Phi_{\mathbf{45}} \text{ in } \mathbf{45}_{\text{A}} \\ \tilde\Phi^{\text{bi}} \equiv \Phi^{\text{bi}}_{\{a,b\}} = \frac{1}{2}(\Phi^{\text{bi}}_{ab} + \Phi^{\text{bi}}_{ba}) = \frac{1}{2}(\phi_a\phi_b + \phi_b\phi_a) = \frac{1}{2}\{\phi_a,\phi_b\} \text{ gives } \Phi_{\mathbf{R}} = \Phi_{\mathbf{54}} \text{ in } \mathbf{54}_{\text{S}} \end{cases}. \tag{3.13}$$

For brevity, we also denote the antisymmetric bivector $\Phi^{\text{bi}}_{[a,b]}$ or $\Phi_{\mathbf{45}}$ as $\hat\Phi^{\text{bi}}$, and denote the symmetric bivector $\Phi^{\text{bi}}_{\{a,b\}}$ or $\Phi_{\mathbf{54}}$ as $\tilde\Phi^{\text{bi}}$.

(4) GUT-Higgs field kinetic term and covariant derivative: The kinetic term for the GUT-Higgs fields is written as $|D_{\mu,A}\Phi_{\mathbf{R}}|^2 \equiv (D_A^\mu\Phi_{\mathbf{R}})^\dagger(D_{\mu,A}\Phi_{\mathbf{R}})$, with the complex conjugate transpose written as dagger †.

Moreover, we can also combine the kinetic terms for $\Phi_{\mathbf{1}}$, $\Phi_{\mathbf{45}}$, and $\Phi_{\mathbf{54}}$ in terms of the kinetic term for the bivector Φ^{bi}. This kinetic term becomes $\text{Tr}((D_A^\mu\Phi^{\text{bi}})^\intercal(D_{\mu,A}\Phi^{\text{bi}}))$, with the matrix transpose written as $\intercal$, where the trace (Tr) is over the 10-dimensional Lie algebra representation of $so(10)$. We can write down the explicit form $(D_{\mu,A}\Phi^{\text{bi}})_{ab} \equiv \nabla_\mu\Phi^{\text{bi}}_{ab} - \mathrm{i}g[A_\mu,\Phi^{\text{bi}}]_{ab} = \nabla_\mu\Phi^{\text{bi}}_{ab} - \mathrm{i}g(A_{\mu,ab}\Phi^{\text{bi}}_{bc} - \Phi^{\text{bi}}_{ab}A_{\mu,bc})$

with $a, b, c \in \{1, 2, ..., 10\}$,[18] where $A_{\mu,ab} = \sum_\alpha A^\alpha_\mu \mathrm{T}'^\alpha_{ab}$ with another 45 pieces of the rank-10 matrix representation T'^α.

In general, the Lie algebra generator T^α is Hermitian. In the case of the real representation **10**, the T'^α is not only Hermitian but also an imaginary and antisymmetric matrix.

In summary, for our purpose, the two expressions of GUT-Higgs kinetic terms are both correct: $\sum_{\mathbf{R}=\mathbf{1},\mathbf{45},\mathbf{54}} |D_{\mu,A}\Phi_{\mathbf{R}}|^2 \equiv (D^\mu_A \Phi_{\mathbf{1}})^\dagger (D_{\mu,A}\Phi_{\mathbf{1}}) + (D^\mu_A \Phi_{\mathbf{45}})^\dagger (D_{\mu,A}\Phi_{\mathbf{45}}) + (D^\mu_A \Phi_{\mathbf{54}})^\dagger (D_{\mu,A}\Phi_{\mathbf{54}})$, and the bivector field expression: $\mathrm{Tr}((D^\mu_A \Phi^{\mathrm{bi}})^\intercal (D_{\mu,A}\Phi^{\mathrm{bi}}))$.

All these above GUT-Higgs fields (in the vector or bivector representations) also coupled to the $so(10)$ gauge fields in the standard way.

(5) Yukawa-like coupling terms: We also have several Yukawa-like coupling terms:
 (i) between the GUT-Higgs bivectors Φ^{bi} and the vectors ϕ, explicitly, $\phi^\intercal \Phi^{\mathrm{bi}} \phi \equiv \sum_{a,b} \phi^\intercal_a \Phi^{\mathrm{bi}}_{ab} \phi_b$.
 (ii) between the GUT-Higgs vectors ϕ and the Weyl spinor ψ_L, the $(\psi_L^\intercal \mathrm{i}\sigma^2 (\phi_{2a-1}\Gamma_{2a-1} - \mathrm{i}\phi_{2a}\Gamma_{2a})\psi_L + \mathrm{H.c.})$ is apparently a Hermitian scalar. The σ^2 matrix acts on the 2-component spacetime Weyl spinor ψ_L. Γ_a (with $a \in \{1, 2, ..., 10\}$) are ten rank-16 matrices satisfying $\{\Gamma_{2a-1}, \Gamma_{2b-1}\} = 2\delta_{ab}$, $\{\Gamma_{2a}, \Gamma_{2b}\} = 2\delta_{ab}$, $[\Gamma_{2a-1}, \Gamma_{2b}] = 0$ (for $a, b = 1, 2, ..., 5$).

(6) Mean-field approximation: If for a moment, we neglect the gauge field A coupling in the covariant derivative, neglect the GUT-Higgs potential $\mathrm{U}(\Phi_{\mathbf{R}})$, and neglect the possible WZW term $S^{\mathrm{WZW}}[\Phi^{\mathrm{bi}}]$, then we only have the quadratic Lagrangian in between GUT-Higgs bivectors Φ^{bi}, vectors ϕ, and the Weyl spinor ψ_L. Then this quadratic Lagrangian, $\frac{1}{2}\phi^\intercal \Phi^{\mathrm{bi}} \phi + \frac{1}{2}\sum_{a=1}^5 (\psi_L^\intercal \mathrm{i}\sigma^2 (\phi_{2a-1}\Gamma_{2a-1} - \mathrm{i}\phi_{2a}\Gamma_{2a}) \psi_L + \mathrm{H.c.})$, at the mean-field level, can be integrated out to impose constraints and relations between the bivectors Φ^{bi}, vectors ϕ, and the Weyl spinor ψ_L. In some sense, what is integrated out becomes a Lagrange multiplier to impose a constraint on the remained fields. In this limit, we only need to regard the Weyl spinor ψ_L as the elementary fields, the vectors ϕ is the **10** from the tensor product of two ψ_L since $\mathbf{16} \otimes \mathbf{16} = (\mathbf{10} \oplus \mathbf{120} \oplus \overline{\mathbf{126}})$. Then the bivector Φ^{bi} is from the tensor product of two ϕ as the $\mathbf{10} \otimes \mathbf{10}$, out of the quartic ψ_Ls $\mathbf{16} \otimes \mathbf{16} \otimes \mathbf{16} \otimes \mathbf{16}$.

(7) Wess-Zumino-Witten-like discrete torsion term: For now, we directly provide our endgame answer to the WZW term, later we will backup and derive this WZW term in details from scratch in Sec. III B.

The schematic WZW action that we propose to match the mod 2 class $w_2 w_3$ global anomaly is

$$S^{\mathrm{WZW}}[\Phi] = \pi \int_{M^5} B(\Phi) \wedge \mathrm{d}B'(\Phi), \tag{3.14}$$

in terms of differential form with mod 2 valued forms of B and B' fields, in the de Rham cohomology. The theory is defined on the 5D manifold M^5 whose boundary is the 4D space time $M^4 = \partial M^5$.[19] The B and B' are constructed out of some GUT-Higgs field Φ [such as the bivector $\tilde{\Phi}^{\mathrm{bi}}$ or $\hat{\Phi}^{\mathrm{bi}}$, for $\Phi^{\mathrm{bi}}_{\{a,b\}}$ or $\Phi^{\mathrm{bi}}_{[a,b]}$, respectively, organized in (3.13)]. More precisely, the WZW term is written in the singular cohomology class of B and B' cochain fields:

$$\begin{aligned} S^{\mathrm{WZW}}[\Phi] &= \pi \int_{M^5} B(\tilde{\Phi}^{\mathrm{bi}}) \smile \delta B'(\hat{\Phi}^{\mathrm{bi}}) \\ &= 2\pi \int_{M^5} B(\tilde{\Phi}^{\mathrm{bi}}) \smile \frac{\delta}{2} B'(\hat{\Phi}^{\mathrm{bi}}) \\ &= 2\pi \int_{M^5} B(\tilde{\Phi}^{\mathrm{bi}}) \smile \mathrm{Sq}^1 B'(\hat{\Phi}^{\mathrm{bi}}). \end{aligned} \tag{3.15}$$

Here the 2-cochain fields are $\mathbb{Z}_2$ valued, they can be chosen as cohomology classes thus $B \in \mathrm{H}^2(M, \mathbb{Z}_2)$ and $B' \in \mathrm{H}^2(M, \mathbb{Z}_2)$. The δ is the coboundary operator, and the Steenrod square $\mathrm{Sq}^1 \equiv \frac{\delta}{2}$ mod 2 here maps the singular cohomology

[18]The reason that $(D_{\mu,A}\Phi^{\mathrm{bi}})_{ab} \equiv \nabla_\mu \Phi^{\mathrm{bi}}_{ab} - \mathrm{i}g[A_\mu, \Phi^{\mathrm{bi}}]_{ab}$ has a matrix commutator $[A_\mu, \Phi^{\mathrm{bi}}]$ in contrast with the familiar form $D_{\mu,A}\phi \equiv \nabla_\mu \phi - \mathrm{i}gA_\mu \phi$, is due to the following fact: The Lie group G transformation for some $U \in G$ acts on the gauge field A as $A \mapsto U(A + \frac{\mathrm{i}}{g}\mathrm{d})U^\dagger$ [or $A \mapsto U(A + \frac{\mathrm{i}}{g}\mathrm{d})U^\intercal$ when U is real valued]. However, the Lie group transformation acts on the vector field ϕ as $\phi \mapsto U\phi$, while acts on the rank-10 matrix bivector field Φ^{bi} as $\Phi^{\mathrm{bi}}_{ab} \mapsto U\Phi^{\mathrm{bi}}_{ab}U^\intercal$.

[19]Here we normalize the usual differential form $B(\tilde{\Phi}^{\mathrm{bi}})/\pi \mapsto B(\tilde{\Phi}^{\mathrm{bi}})$ and $B'(\hat{\Phi}^{\mathrm{bi}})/\pi \mapsto B'(\hat{\Phi}^{\mathrm{bi}})$, so the usual differential form partition function $\exp(\mathrm{i}\frac{2}{2\pi}\int_{M^5} B(\tilde{\Phi}^{\mathrm{bi}}) \wedge \mathrm{d}B'(\hat{\Phi}^{\mathrm{bi}}))$ maps to $\exp(\mathrm{i}\pi \int_{M^5} B(\tilde{\Phi}^{\mathrm{bi}}) \wedge \mathrm{d}B'(\hat{\Phi}^{\mathrm{bi}}))$. See a related discussion on the 5D $B\mathrm{d}B'$ theory in [21]. The quantization conditions on the closed cycles, also map from $\oiint B(\tilde{\Phi}^{\mathrm{bi}})$ or $\oiint B'(\hat{\Phi}^{\mathrm{bi}}) = n\pi \bmod 2\pi \mapsto \oiint B(\tilde{\Phi}^{\mathrm{bi}})$ or $\oiint B'(\hat{\Phi}^{\mathrm{bi}}) = n \bmod 2$. It can be verified that this WZW has two properties: (1) The invertible $|\mathbf{Z}(M^5)| = 1$ on a closed five manifold, but a specific manifold $\mathbf{Z}(M^5) = -1$ can possibly sign the underlying bulk 5D invertible TQFT $w_2 w_3$. (2) This WZW term really is a 4D theory, having physical impacts only on the 4D M^4—it is a 4D boundary theory of the 5D bulk invertible TQFT on the extended M^5.

$H^2(M,\mathbb{Z}_2)\mapsto H^3(M,\mathbb{Z}_2)$, on some triangulable manifold M.[20] The wedge product $\wedge$ of differential form in (3.14) becomes the cup product $\smile$ of cochains or cohomology classes in (3.15). Note that the triangulable manifold M is always a smooth differentiable manifold, thus we can downgrade the singular cohomology result (3.15) to reproduce the de Rham cohomology expression (3.14).

(8) GUT-Higgs potential $\mathrm{U}(\Phi_{\mathbf{R}})$, and a relation to the nonlinear sigma model (NLSM): Mostly we shall simply choose the GUT-Higgs potential written in (3.4),

$$\begin{aligned}\mathrm{U}(\Phi_{\mathbf{R}}) &= (r_{\mathbf{45}}(\Phi_{\mathbf{45}})^2+\lambda_{\mathbf{45}}(\Phi_{\mathbf{45}})^4)\\ &\quad+(r_{\mathbf{54}}(\Phi_{\mathbf{54}})^2+\lambda_{\mathbf{54}}(\Phi_{\mathbf{54}})^4)\\ &\quad+(r_{\mathbf{1}}(\Phi_{\mathbf{1}})^2+\lambda_{\mathbf{1}}(\Phi_{\mathbf{1}})^4),\end{aligned}$$

which is sufficient for a continuum QFT description. Some lattice or condensed matter based theorists may wonder whether there is a NLSM description at a deeper UV. One approach is to write down a potential with a NLSM constraint $(\mathrm{Tr}(\Phi^{\intercal}\Phi)-\mathrm{R}^2)$ with the norm of GUT-Higgs centered around a radius R, and introduce a Lagrange multiplier λ, such that integrating out $\int[\mathcal{D}\lambda]\ldots$ gives the fixed radius constraint at UV. With appropriate deformations, we anticipate a renormalization group flow from UV to IR gives the GUT-Higgs potential. One reason to introduce a NLSM is that it is natural to adding the WZW term to NLSM. However, an NLSM description turns out to be not necessary for writing our WZW term.

(9) Deconfined quantum criticality (DQC): The motivation to add this 4D $S^{\mathrm{WZW}}[\Phi]$ into our 4D mother EFT is to induce the analogous phenomenon called the deconfined quantum criticality [28]. The original deconfined quantum criticality [28] is proposed as a continuous quantum phase transition between two kinds of Landau symmetry breaking orders: Néel antiferromagnet order and valence-bond solid (VBS) order in 3D [namely, $(2+1)$D].

Here in out gauge theory context in 4D [namely, $(3+1)$D], between the GG $su(5)$ GUT and the PS $su(4)\times su(2)\times su(2)$ model, we do not really have the conventional Landau symmetry breaking orders as both the $su(5)$ and $su(4)\times su(2)\times su(2)$ are dynamically gauged as gauge theories. But if we regard the $su(5)$ and $su(4)\times su(2)\times su(2)$ are internal global symmetries that are not yet gauged, then we are able to seek for a deconfined quantum criticality construction between the GG and PS models, as we will verify in the next Sec. III B.

[20]Generally, given a chain complex $\mathrm{C}_\bullet$ and a short exact sequence of Abelian groups:

$$0\to \mathrm{A}'\to\mathrm{A}\to\mathrm{A}''\to 0,$$

we have a short exact sequence of cochain complexes:

$$0\to\mathrm{Hom}(\mathrm{C}_\bullet,\mathrm{A}')\to\mathrm{Hom}(\mathrm{C}_\bullet,\mathrm{A})\to\mathrm{Hom}(\mathrm{C}_\bullet,\mathrm{A}'')\to 0.$$

Hence we can obtain a long exact sequence of cohomology groups:

$$\cdots\to H^n(\mathrm{C}_\bullet,\mathrm{A}')\to H^n(\mathrm{C}_\bullet,\mathrm{A})\to H^n(\mathrm{C}_\bullet,\mathrm{A}'')\xrightarrow{\partial} H^{n+1}(\mathrm{C}_\bullet,\mathrm{A}')\\ \to\cdots,$$

the connecting homomorphism ∂ is called Bockstein homomorphism. For instance, $\beta_{(n,m)}:H^*(-,\mathbb{Z}_m)\to H^{*+1}(-,\mathbb{Z}_n)$ is the Bockstein homomorphism associated with the extension $\mathbb{Z}_n\xrightarrow{\cdot m}\mathbb{Z}_{nm}\to\mathbb{Z}_m$ where $\cdot m$ is the group homomorphism given by multiplication by m. Specifically, $\beta_{(2,2^n)}=\frac{1}{2^n}\delta \bmod 2$, thus the Steenrod square obeys $\mathrm{Sq}^1\equiv\beta_{(2,2)}\equiv\frac{\delta}{2} \bmod 2$.

B. Homotopy and cohomology group arguments to induce a WZW term

We review the 3D WZW term construction in the familiar DQC in 3D [namely, $(2+1)$D] [28], in Appendix C, based on more nonperturbative arguments from homotopy and cohomology groups, and anomaly classifications from cobordism. Here we proceed with the same logic, to construct the 4D WZW term in the new DQC in 4D [namely, $(3+1)$D] to justify what we claimed in (3.15).

Below we write G as the original larger symmetry group, while G_{sub} is the remained preserved unbroken symmetry in the corresponding order (i.e., Néel or VBS orders for 3D DQC; the GG or PS for the 4D DQC we will propose). Then we have the following fibration structure:

$$G_{\mathrm{sub}}\hookrightarrow G\to\frac{G}{G_{\mathrm{sub}}}, \tag{3.16}$$

where the quotient space $\frac{G}{G_{\mathrm{sub}}}$ is the base manifold (i.e., the orbit) as the symmetry-breaking order parameter space. The G is the total space obtained from the fibration of the G_{sub} fiber (i.e., the stabilizer) over the base $\frac{G}{G_{\mathrm{sub}}}$.

Now we follow the similar logic for the 3D DQC summarized in Appendix C, generalizing the idea to deal with our 4D DQC.

1. Induce a 4D WZW term between Georgi-Glashow $su(5)$ and Pati-Salam $su(4)\times su(2)\times su(2)$ models on a 5D bulk $w_2(V_{\mathrm{SO}(10)})w_3(V_{\mathrm{SO}(10)})$

Following the principle in Appendix C, we aim to induce a 4D WZW term between Georgi-Glashow $su(5)$ and Pati-Salam $su(4)\times su(2)\times su(2)$ models on a 5D bulk $w_2(V_{\mathrm{SO}(10)})w_3(V_{\mathrm{SO}(10)})$. First we look at the order-parameter target manifold via the fibration structure (3.16), formed by the bosonic GUT-Higgs fields. For the bosonic GUT-Higgs fields, we only have the internal SO(10) symmetry not the Spin(10) symmetry, but we can include the orientation reversal which gives an $\mathrm{O}(10) = \mathrm{SO}(10) \rtimes \mathbb{Z}_2$ symmetry. Then the fibration (3.16) becomes

$$\mathrm{GG}\, su(5)\, \mathrm{GUT}\colon (G_{\mathrm{sub}} = \mathrm{U}(5)) \hookrightarrow (G = \mathrm{O}(10)) \to \left(\frac{G}{G_{\mathrm{sub}}} = \frac{\mathrm{O}(10)}{\mathrm{U}(5)}\right). \quad (3.17)$$

Here we can keep the larger U(5) instead of SU(5) as the preserved internal symmetry of the $su(5)$ GUT.

$$\mathrm{PS}\, su(4) \times su(2) \times su(2)\colon (G_{\mathrm{sub}} = \mathrm{O}(6) \times \mathrm{O}(4)) \hookrightarrow (G = \mathrm{O}(10)) \to \left(\frac{G}{G_{\mathrm{sub}}} = \frac{\mathrm{O}(10)}{\mathrm{O}(6) \times \mathrm{O}(4)}\right). \quad (3.18)$$

Recall that $su(4) \times su(2) \times su(2)$ has the same Lie algebra as $so(6) \times so(4)$. Here we also keep the larger $\mathrm{O}(6) \times \mathrm{O}(4)$ instead of $\mathrm{SO}(6) \times \mathrm{SO}(4)$ as the preserved internal symmetry of the PS model. Homotopy groups for these target manifolds of GUT-Higgs fields are in the table:

	π_0	π_1	π_2	π_3	π_4	π_5
$GG\,\frac{\mathrm{O}(10)}{\mathrm{U}(5)}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
$PS\,\frac{\mathrm{O}(10)}{\mathrm{O}(6)\times\mathrm{O}(4)}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$
O(10)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
O(4)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
O(6)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
U(5)	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
SO(10)	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
SO(4)	0	$\mathbb{Z}_2$	0	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
SO(6)	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
SU(5)	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$

(3.19)

Let us comment about the construction of 4D WZW and its 4D 't Hooft anomaly, step by step:

(1) Start with the hint from homotopy groups, we need to find defects trapped in the order-parameter target manifold of bosonic GUT-Higgs fields in the GG and PS models,[21] classified by $\pi_{n_{\mathrm{GG}}}(\frac{\mathrm{O}(10)}{\mathrm{U}(5)})$ and $\pi_{n_{\mathrm{PS}}}(\frac{\mathrm{O}(10)}{\mathrm{O}(6)\times\mathrm{O}(4)})$ such that the dimensionality $n_{\mathrm{GG}} + n_{\mathrm{PS}} = d$ where the d is the total spacetime dimension thus $d = 4$ (or one lower dimension compared with the 5D where the WZW is extended to put on). This suggests that we take

$$\pi_2\left(\frac{\mathrm{O}(10)}{\mathrm{U}(5)}\right) = \mathbb{Z}, \qquad \pi_2\left(\frac{\mathrm{O}(10)}{\mathrm{O}(6)\times\mathrm{O}(4)}\right) = \mathbb{Z}_2,$$
$$n_{\mathrm{GG}} + n_{\mathrm{PS}} = 2 + 2 = 4. \quad (3.20)$$

Note that $(\frac{\mathrm{O}(m+n)}{\mathrm{O}(m)\times\mathrm{O}(n)}) \equiv \mathrm{Gr}(m, m+n)$ is a Grassmannian manifold. Here we need $\mathrm{Gr}(6,10) = \mathrm{Gr}(4,10)$. Equation (3.20) indeed corresponds to the homotopy classes of 't Hooft-Polyakov monopoles of the GG model and the PS model respectively, broken down from the $so(10)$ GUT.

(2) We will use the cohomology construction of the WZW term, furnished by the hints of homotopy groups. Then we need a relation between homotopy group and cohomology group.

In algebraic topology, an Eilenberg-MacLane space $K(G,n)$ is a topological space with a single nontrivial homotopy group, such that $\pi_n(K(G,n)) \cong G$ and $\pi_m(K(G,n)) = 0$ if $m \neq n$. It can be regarded as a building block for homotopy theory, also it provides a bridge between homotopy and cohomology. Let X be a topological space or a manifold. The set $[X, K(G,n)]$ of based homotopy classes of based maps from X to $K(G,n)$ is a natural bijection with the nth singular cohomology group $\mathrm{H}^n(X,G)$. In particular, when $\pi_n(X) \cong G$,

$$\mathrm{H}^n(X,G) = \mathrm{Hom}(\pi_n(X), G) = \mathrm{Hom}(G,G). \quad (3.21)$$

There is a distinguished element $\omega \in \mathrm{H}^n(X,G)$, as the generator of the cohomology group $\mathrm{H}^n(X,G)$, corresponding to the identity morphism in $\mathrm{Hom}(G,G)$. The morphism is realized as

$$\omega\colon \pi_n(X) \to G, \qquad f \in \pi_n(X) \mapsto \int_{x\in S^n} \omega(f(x)) \in G. \quad (3.22)$$

(3) With the above homotopy group (3.19) in mind, we can use the Serre spectral sequence to derive the following[22]:

[21]Caveat: We emphasize again that here we are considering defects in the order-parameter target manifold of *bosonic* GUT-Higgs fields. We are *not* talking about the objects of *fermionic* sectors (quarks/leptons) or gauge theory sectors in SMs. For example, there are magnetic monopoles from $\pi_1(G_{\mathrm{SM}_6}) = \pi_2(G_{\mathrm{GG}}/G_{\mathrm{SM}_6}) = \pi_2(G_{\mathrm{PS}_2}/G_{\mathrm{SM}_6}) = \mathbb{Z}$, also from $\pi_1(G_{\mathrm{SM}_3}) = \pi_2(G_{\mathrm{PS}_1}/G_{\mathrm{SM}_3}) = \mathbb{Z}$ or from any $\pi_1(G_{\mathrm{SM}_q}) = \mathbb{Z}$ with $q = 1,2,3,6$. But we are talking about different objects in the order-parameter target manifold of *bosonic* GUT-Higgs fields—they are however related to the 't Hooft-Polyakov monopoles of the GG model and the PS model respectively, broken down from the $so(10)$ GUT.

[22]We can answer in more general case $\mathrm{O}(2n)/\mathrm{U}(n)$. We will need the universal coefficient theorem, so that $H^2(X,\mathrm{A}) = \mathrm{Hom}(H_2(X),\mathrm{A}) \oplus \mathrm{Ext}(H_1(X),\mathrm{A})$, for some topological space X and any Abelian group coefficient A.

The space $\mathrm{O}(2n)/\mathrm{U}(n)$ has two connected components, each of which is diffeomorphic to $\mathrm{SO}(2n)/\mathrm{U}(n)$, so $H^k(\mathrm{O}(2n)/\mathrm{U}(n),\mathrm{A}) = H^k(\mathrm{SO}(2n)/\mathrm{U}(n),\mathrm{A}) \oplus H^k(\mathrm{SO}(2n)/\mathrm{U}(n),\mathrm{A})$.

For $n > 1$, the space $\mathrm{SO}(2n)/\mathrm{U}(n)$ is simply connected with $\pi_2(\mathrm{SO}(2n)/\mathrm{U}(n)) = \mathbb{Z}$, so by the Hurewicz theorem we have $H_1(\mathrm{SO}(2n)/\mathrm{U}(n),\mathbb{Z}) = 0$ and $H_2(\mathrm{SO}(2n)/\mathrm{U}(n),\mathbb{Z}) = \mathbb{Z}$. Therefore by the universal coefficient theorem, so we have $H^2(\mathrm{SO}(2n)/\mathrm{U}(n),\mathrm{A}) = \mathrm{Hom}(\mathbb{Z},\mathrm{A}) \oplus \mathrm{Ext}(0,\mathrm{A}) = \mathrm{A}$. Thus, $H^2(\mathrm{O}(2n)/\mathrm{U}(n),\mathrm{A}) = \mathrm{A}^2$.

$$H^2(\mathrm{O}(10)/\mathrm{U}(5), \mathbb{Z}) = \mathbb{Z}^2.$$
$$H^2(\mathrm{O}(10)/\mathrm{U}(5), \mathbb{Z}_2) = \mathbb{Z}_2^2. \quad (3.23)$$

In fact, we just need one of the two components from SO(10)/U(5), whose cohomology group:

$$H^2(\mathrm{SO}(10)/\mathrm{U}(5), \mathbb{Z}) = \mathbb{Z},$$
$$H^2(\mathrm{SO}(10)/\mathrm{U}(5), \mathbb{Z}_2) = \mathbb{Z}_2. \quad (3.24)$$

(4) We can also derive

$$H^2(\mathrm{O}(10)/(\mathrm{O}(6)\times\mathrm{O}(4)), \mathbb{Z}) = \mathbb{Z}_2,$$
$$H^2(\mathrm{O}(10)/(\mathrm{O}(6)\times\mathrm{O}(4)), \mathbb{Z}_2) = \mathbb{Z}_2^2. \quad (3.25)$$

The mod 2 cohomology of real Grassmannian manifold is well known from the theory of Stiefel-Whitney characteristic classes. The integral cohomology is trickier but it can be worked out.

(5) We now take a $\mathbb{Z}_2$ cohomology class called $B(\tilde{\Phi}^{\mathrm{bi}})$ out of

$$B(\tilde{\Phi}^{\mathrm{bi}}) \in H^2(\mathrm{O}(10)/(\mathrm{O}(6)\times\mathrm{O}(4)), \mathbb{Z}_2), \quad (3.26)$$

and another $\mathbb{Z}_2$ cohomology class called $B'(\hat{\Phi}^{\mathrm{bi}})$ out of

$$B'(\hat{\Phi}^{\mathrm{bi}}) \in H^2(\mathrm{O}(10)/\mathrm{U}(5), \mathbb{Z}_2). \quad (3.27)$$

(a) The $B(\tilde{\Phi}^{\mathrm{bi}})$ field as a second cohomology class, can be constructed out of the GUT-Higgs field $\Phi_{\mathbf{54}}$ in the **54** representation of $so(10)$. In particular, we can also write $\Phi_{\mathbf{54}}$ as a bivector GUT-Higgs field symmetric representation, $\mathbf{54}_{\mathrm{S}}$ out of $\mathbf{10}\otimes\mathbf{10}$, called $\tilde{\Phi}^{\mathrm{bi}}$ that we detail in Sec. III C.

(b) The $B'(\hat{\Phi}^{\mathrm{bi}})$ field as a second cohomology class, can be constructed out of the GUT-Higgs field $\Phi_{\mathbf{45}}$ in the **45** representation of $so(10)$. In particular, we can also write $\Phi_{\mathbf{45}}$ as a bivector GUT-Higgs field antisymmetric representation, $\mathbf{45}_{\mathrm{A}}$ out of $\mathbf{10}\otimes\mathbf{10}$, called $\hat{\Phi}^{\mathrm{bi}}$ that we detail in Sec. III C.

Similar to the familiar 3D DQC in Appendix C, we can also provide the physical intuitions on the link invariants between various defects: between the charged objects and the charge operators constructed from homotopy groups and cohomology groups. For example:

(i) Georgi-Glashow GUT-Higgs target manifold and defects: The $B'(\hat{\Phi}^{\mathrm{bi}}) \in H^2(\mathrm{O}(10)/\mathrm{U}(5), \mathbb{Z}_2)$ can be placed on a 2-surface called $\hat{\varrho}^2$, as a charge operator $\exp(\mathrm{i}\pi \oiint_{\hat{\varrho}^2} B'(\hat{\Phi}^{\mathrm{bi}})) = \exp(\mathrm{i}\pi \oiint_{\hat{\varrho}^2} c_1(V_{\mathrm{U}(5)}))$ (i.e., symmetry generator as a topological operator) measures the charge of a preserved U(5) symmetry in the defect trapped in the target manifold O(10)/U(5). The first Chern class $c_1(V_{\mathrm{U}(5)})$ of the associated vector bundle of U(5) evaluates a magnetic flux mod 2 on this 2-surface $\hat{\varrho}^2$. There is a defect line along a 1D loop called $\varsigma^1_{\mathrm{GG}}$, paired up with a one connection called $\hat{v}$ gives a 1D line operator $\exp(\mathrm{i}\pi \oint_{\varsigma^1_{\mathrm{GG}}} \hat{v})$ as a charged object. The charge operator two-surface $\hat{\varrho}^2$ can be linked with a charged 1D loop $\varsigma^1_{\mathrm{GG}}$ in the 4D spacetime. Follow the generalized higher global symmetry language [59], this nontrivial linking number Lk implies a measurement of U(5) symmetry on the defect. Precisely, the linking number Lk, manifested as a statistical Berry phase, is evaluated via the expectation value of path integral:

$$\left\langle \exp\left(\mathrm{i}\pi \oiint_{\hat{\varrho}^2} B'(\hat{\Phi}^{\mathrm{bi}})\right) \cdot \exp\left(\mathrm{i}\pi \oint_{\varsigma^1_{\mathrm{GG}}} \hat{v}\right) \right\rangle = (-1)^{\mathrm{Lk}(\hat{\varrho}^2, \varsigma^1_{\mathrm{GG}})}|_{M^4} \cdot \langle \exp(\mathrm{i}\pi \oint_{\varsigma^1_{\mathrm{GG}}} \hat{v})\rangle. \quad (3.28)$$

Related descriptions of link invariants of QFTs can be found in [60,61] and references therein.

(ii) Pati-Salam GUT-Higgs target manifold and defects: The $B(\tilde{\Phi}^{\mathrm{bi}}) \in H^2(\mathrm{O}(10)/(\mathrm{O}(6)\times\mathrm{O}(4)), \mathbb{Z}_2)$ can be placed on a two surface called $\tilde{\varrho}^2$, as a charge operator $\exp(\mathrm{i}\pi \oiint_{\tilde{\varrho}^2} B(\tilde{\Phi}^{\mathrm{bi}})) = \exp(\mathrm{i}\pi \oiint_{\tilde{\varrho}^2} w_2(V_{(\mathrm{O}(6)\times\mathrm{O}(4))}))$ [23] (i.e., symmetry generator as a topological operator) measures the charge of a preserved $(\mathrm{O}(6)\times\mathrm{O}(4))$ symmetry in the defect trapped in the target manifold $\mathrm{O}(10)/(\mathrm{O}(6)\times\mathrm{O}(4))$. There is a defect line along a 1D loop called $\varsigma^1_{\mathrm{PS}}$, paired up with a one connection called $\tilde{v}$ gives a 1D line operator $\exp(\mathrm{i}\pi \oint_{\varsigma^1_{\mathrm{PS}}} \tilde{v})$ as a charged object. The charge operator two-surface $\tilde{\varrho}^2$ can be linked with a charged 1D loop $\varsigma^1_{\mathrm{PS}}$ in the 4D spacetime. Follow the generalized higher global symmetry language [59], this nontrivial linking number Lk implies a measurement of $(\mathrm{O}(6)\times\mathrm{O}(4))$ symmetry on the defect. Precisely, the linking number Lk, manifested as a statistical Berry

[23] Note that the second Stiefel-Whitney class of associated vector bundle of the product of orthogonal groups satisfies $w_2(V_{(\mathrm{O}(n)\times\mathrm{O}(m))}) = w_2(V_{\mathrm{O}(n)}) + w_2(V_{\mathrm{O}(m)}) + w_1(V_{\mathrm{O}(n)})w_1(V_{\mathrm{O}(m)})$.

phase, is evaluated via the expectation value of path integral:

$$\left\langle \exp\left(\mathrm{i}\pi \oiint_{\tilde{\varrho}^2} B(\tilde{\Phi}^{\mathrm{bi}})\right)\cdot\exp\left(\mathrm{i}\pi\oint_{\varsigma^1_{\mathrm{PS}}}\tilde{v}\right)\right\rangle = (-1)^{\mathrm{Lk}(\tilde{\varrho}^2,\varsigma^1_{\mathrm{PS}})}|_{M^3}\cdot\langle\exp(\mathrm{i}\pi\oint_{\varsigma^1_{\mathrm{PS}}}\tilde{v})\rangle. \quad (3.29)$$

(iii) If we extend the 4D spacetime t, x, y, z to an extra fifth dimension ϖ, the previous 1D loop $\varsigma^1_{\mathrm{GG}}$ trajectory can be a 2D pseudo–world sheet $\varsigma'^2_{\mathrm{GG}}$ in the 5D M^5. Similarly, the previous 1D loop $\varsigma^1_{\mathrm{PS}}$ trajectory can be a 2D pseudo–world sheet $\varsigma'^2_{\mathrm{PS}}$ in the 5D M^5. Such two 2D configurations can be linked in 5D, with a linking number

$$\mathrm{Lk}(\varsigma'^2_{\mathrm{GG}},\varsigma'^2_{\mathrm{PS}})|_{M^5}.$$

This describes the link in the extended 5D spacetime of two charged objects, charged under U(5) and $(\mathrm{O}(6)\times\mathrm{O}(4))$, respectively.

(iv) In a parallel story, the charge operators (of the above charged objects) are the 2D $B'(\hat{\Phi}^{\mathrm{bi}})$ operator on $\hat{\varrho}^2$, and 2D $B(\tilde{\Phi}^{\mathrm{bi}})$ surface operator on $\tilde{\varrho}^2$. Such two configurations can be linked in 5D, with a linking number:

$$\mathrm{Lk}(B'(\hat{\Phi}^{\mathrm{bi}})\ \mathrm{on}\ \hat{\varrho}^2, B(\tilde{\Phi}^{\mathrm{bi}})\ \mathrm{on}\ \tilde{\varrho}^2)|_{M^5}.$$

This describes the link in the extended 5D spacetime of two charge operators:

If we open up the closed $\oiint_{\tilde{\varrho}^2} B(\tilde{\Phi}^{\mathrm{bi}})$ on $\tilde{\varrho}^2$ with an open end on the 4D boundary M^4 of the bulk M^5, then this open end carries a closed 1D loop $\oint_{\varsigma^1_{\mathrm{GG}}}\hat{v}$. Their link configuration in 4D corresponds to the earlier (3.28):

$$\mathrm{Lk}(\hat{\varrho}^2,\varsigma^1_{\mathrm{GG}})|_{M^4}.$$

If we open up the closed $\oiint_{\hat{\varrho}^2} B'(\hat{\Phi}^{\mathrm{bi}})$ on $\hat{\varrho}^2$ with an open end on the 4D boundary M^4 of the bulk M^5, then this open end carries a closed 1D loop $\oint_{\varsigma^1_{\mathrm{PS}}}\tilde{v}$. Their link configuration in 4D corresponds to the earlier (3.29):

$$\mathrm{Lk}(\tilde{\varrho}^2,\varsigma^1_{\mathrm{PS}})|_{M^4}.$$

We leave more of these picturesque discussions and imaginative figures, in a companion work.

(6) Based on the above observations about the link invariants, follow Appendix C's logic, our 4D DQC construction is valid if we introduce a mod 2 class 4D WZW term, defined on a 4D boundary M^4 of a 5D manifold M^5, schematically in a differential form or de Rham cohomology,

$$\exp(\mathrm{i}S^{\mathrm{WZW}}[\Phi]) = \exp\left(\mathrm{i}\pi\int_{M^5} B(\tilde{\Phi}^{\mathrm{bi}})\wedge \mathrm{d}B'(\hat{\Phi}^{\mathrm{bi}})\right)\Big|_{M^4=\partial M^5}. \quad (3.30)$$

Recall footnote 19 about our normalizations of differential forms and cohomology classes. More precisely, we can improve this to construct WZW in the singular cohomology class:

$$\exp(\mathrm{i}S^{\mathrm{WZW}}[\Phi]) = \exp\left(\mathrm{i}\pi\int_{M^5} B(\tilde{\Phi}^{\mathrm{bi}})\smile\delta B'(\hat{\Phi}^{\mathrm{bi}})\right)\Big|_{M^4=\partial M^5} = \exp\left(\mathrm{i}2\pi\int_{M^5} B(\tilde{\Phi}^{\mathrm{bi}})\smile \mathrm{Sq}^1 B'(\hat{\Phi}^{\mathrm{bi}})\right)\Big|_{M^4=\partial M^5}. \quad (3.31)$$

We thus succeed to verify our claims in (3.14) and (3.15), while all notations here follow there in Sec. III A 2.

(7) Our 4D DQC construction will be supported by a 4D 't Hooft anomaly in the spacetime-internal global symmetry $(\mathrm{Spin}\times_{\mathbb{Z}_2^F}\mathrm{Spin}(10))$ on a four manifold M^4, captured by a 5D bulk invertible TQFT [12,18] living on a five manifold M^5 with $\partial M^5 = M^4$:

$$\exp\left(\mathrm{i}\pi\int_{M^5} w_2(TM)w_3(TM)\right) = \exp\left(\mathrm{i}\pi\int_{M^5} w_2(V_{\mathrm{SO}(10)})w_3(V_{\mathrm{SO}(10)})\right). \quad (3.32)$$

This 4D 't Hooft anomaly is a mod 2 class global anomaly, mentioned already in (2.5) and (2.8). We comment more about the cobordism group data on perturbative local and nonperturbative global anomalies in various SMs and GUTs in Appendix D.

These conclude our derivation of 4D WZW and 't Hooft anomaly for a candidate 4D DQC for GG-PS GUT transition.

C. Composite GUT-Higgs model within the SM

Before analyzing the effect of the 4D WZW term, we will first review how $so(10)$ GUT, GG, PS, and SM can be unified in the same quantum phase diagram by the different condensation pattern of the SO(10) bivector GUT-Higgs field. Following Sec. II B, for this discussion, we will first turn off the WZW term, assuming that the theory has no additional w_2w_3 anomaly. Starting from the $so(10)$ GUT phase, which has the largest internal symmetry group Spin(10), the GUT-Higgs field can be unified as an SO(10) bivector field

$$\Phi^{\text{bi}}_{ab} \sim \phi_a \phi_b \quad (\text{for } a, b = 1, 2, \ldots, 10), \tag{3.33}$$

which can be considered as a composition of two SO(10) vector fields ϕ_a, where the SO(10) vector ϕ_a can be further considered as a composition of two Weyl fermions ψ

$$\phi_{2a-1} \sim \frac{1}{2}(\psi^\intercal \mathrm{i}\sigma^2 \Gamma_{2a-1}\psi + \text{H.c.}),$$
$$\phi_{2a} \sim \frac{1}{2\mathrm{i}}(\psi^\intercal \mathrm{i}\sigma^2 \Gamma_{2a}\psi - \text{H.c.}), \ (\text{for } a = 1, 2, \ldots, 5). \tag{3.34}$$

Here when two quantum fields Φ_A and Φ_B are linearly coupled with each other in the field theory (as source and original fields), we denote them in this notation $\Phi_A \sim \Phi_B$, such that they are "dual" to each other and share exactly the same symmetry properties. There are 16×16 real symmetric matrices Γ_a acting in the fermion flavor space, which are determined by the following algebraic relations (for $a, b = 1, 2, \ldots, 5$):

$$\{\Gamma_{2a-1}, \Gamma_{2b-1}\} = 2\delta_{ab}, \qquad \{\Gamma_{2a}, \Gamma_{2b}\} = 2\delta_{ab},$$
$$[\Gamma_{2a-1}, \Gamma_{2b}] = 0. \tag{3.35}$$

In view of the above composite construction, we refer to the bivector representation Φ^{bi} as the composite GUT-Higgs field.

The composite Higgs field contains elementary Higgs components of both $\Phi_{\mathbf{45}}$ and $\Phi_{\mathbf{54}}$, since $\mathbf{10} \otimes \mathbf{10} = \mathbf{1} \oplus \mathbf{45}_{\text{A}} \oplus \mathbf{54}_{\text{S}}$. Follow (3.13), we introduce the following notations to denote different irreducible representations of the composite GUT-Higgs field [in terms of SO(10) vector bilinears]:

(i) $\text{Tr}\Phi^{\text{bi}} \sim \sum_a \phi_a \phi_a$ is equivalent to $\Phi_{\mathbf{1}}$ as the $\mathbf{1}_{\text{S}}$ of SO(10).

(ii) $\hat{\Phi}^{\text{bi}}: \Phi^{\text{bi}}_{[a,b]} \sim \frac{1}{2}[\phi_a, \phi_b]$ is equivalent to $\Phi_{\mathbf{45}}$ as the $\mathbf{45}_{\text{A}}$, antisymmetric (A) part of $\mathbf{10} \otimes \mathbf{10}$, of SO(10).

(iii) $\tilde{\Phi}^{\text{bi}}: \Phi^{\text{bi}}_{\{a,b\}} - \frac{1}{10}\text{Tr}\Phi^{\text{bi}}\delta_{ab} \sim \frac{1}{2}\{\phi_a, \phi_b\} - \frac{1}{10}\sum_c \phi_c \phi_c \delta_{ab}$ is equivalent to $\Phi_{\mathbf{54}}$ as the $\mathbf{54}_{\text{S}}$, symmetric (S) part of $\mathbf{10} \otimes \mathbf{10}$, of SO(10).

The competition between $\tilde{\Phi}^{\text{bi}}$ and $\hat{\Phi}^{\text{bi}}$ condensation leads to different GUT or SM phases in the phase diagram. We enumerate all the symmetry breaking patterns (below "→" means "breaking to") as follows:

(1) $\text{Spin}(10) \to \frac{\text{Spin}(6)\times\text{Spin}(4)}{\mathbb{Z}_2} = \frac{\text{SU}(4)\times\text{SU}(2)_{\text{L}}\times\text{SU}(2)_{\text{R}}}{\mathbb{Z}_2}$ by condensing $\tilde{\Phi}^{\text{bi}}$ (the $\mathbf{54}_{\text{S}}$ symmetric representation) to the following specific configuration in the symmetric rank-10 bivector matrix form:

$$\langle \tilde{\Phi}^{\text{bi}} \rangle = \left(-3\sum_{a=1}^{4} + 2\sum_{a=5}^{10}\right)\Phi^{\text{bi}}_{\{a,a\}} = \phi^\intercal \begin{pmatrix} -3 \cdot \mathbf{1}_{2\times 2} & \\ & 2 \cdot \mathbf{1}_{3\times 3} \end{pmatrix} \otimes \sigma^0 \phi \in \frac{\text{O}(10)}{\text{O}(6)\times\text{O}(4)}. \tag{3.36}$$

The GUT-Higgs field $\tilde{\Phi}$ discriminates the SO(4) vector $(\phi_1, \phi_2, \phi_3, \phi_4)$ from the SO(6) vector $(\phi_5, \phi_6, \phi_7, \phi_8, \phi_9, \phi_{10})$, which breaks Spin(10) down to $\frac{\text{Spin}(6)\times\text{Spin}(4)}{\mathbb{Z}_2}$ realizing the Pati-Salam symmetry G_{PS_2}. The 16 Weyl fermions split as $\mathbf{16} \sim (\mathbf{4}, \mathbf{2}, \mathbf{1})_{\text{L}} \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})_{\text{R}}$ under $\frac{\text{SU}(4)\times\text{SU}(2)_{\text{L}}\times\text{SU}(2)_{\text{R}}}{\mathbb{Z}_2}$.[24] The L/R sectors are distinguished by the operator

$$\chi = \psi^\dagger \left(\prod_{a=1}^{10} \Gamma_a\right)\psi = \pm 1. \tag{3.37}$$

Let $\rho_{[a,b]} = \frac{1}{2\mathrm{i}}[\phi_a, \phi_b]$ be the SU(4) generators (for $a, b = 5, 6, \ldots, 10$). Using algebraic relations, we can check that in the L sector, SU(4) acts as $\psi_{\text{L}} \mapsto e^{\mathrm{i}\rho_{[a,b]}}\psi_{\text{L}} e^{-\mathrm{i}\rho_{[a,b]}}$, matching the $\mathbf{4}$ representation; and in the R sector, SU(4) acts as $\psi_{\text{R}} \mapsto e^{-\mathrm{i}\rho^*_{[a,b]}}\psi_{\text{R}} e^{\mathrm{i}\rho^*_{[a,b]}}$, matching the $\bar{\mathbf{4}}$ representation.

(2) $\text{Spin}(10) \to \text{SU}(5) \times \mathbb{Z}_{4,X}$ by condensing $\hat{\Phi}^{\text{bi}}$ (the $\mathbf{45}_{\text{A}}$ antisymmetric representation) to the following specific configuration in the antisymmetric rank-10 bivector matrix form:

$$\langle \hat{\Phi}^{\text{bi}} \rangle = \sum_{a=1}^{5} \Phi^{\text{bi}}_{[2a-1,2a]} = -\frac{1}{2}\phi^\intercal \mathbf{1}_{5\times 5} \otimes \mathrm{i}\sigma^2 \phi \in \frac{\text{O}(10)}{\text{U}(5)}. \tag{3.38}$$

If we combine the SO(10) vector ϕ_b (for $b = 1, 2, \ldots, 10$) into a five-component complex vector $\varphi_a = (\phi_{2a-1} + \mathrm{i}\phi_{2a})/\sqrt{2}$ (for $a = 1, 2, \ldots, 5$), φ would transform as the $\mathbf{5}_1$ under $\text{U}(5) = \frac{\text{SU}(5)\times\text{U}(1)}{\mathbb{Z}_5}$[25] in SO(10). The GUT-Higgs field

[24]Recall in footnote 13 about the left or right spinors, the L/R notations here are for the internal-symmetry's spinors, while the L/R notations are for the spacetime-symmetry's Weyl spinors.

[25]References [62,63] point out the subtle differences between different nonisomorphic versions of U(5) Lie groups (and their corresponding gauge theories) that we should refine and redefine them as several $\text{U}(5)_{\hat{q}}$ with $\hat{q} \in \mathbb{Z}$:

$$\text{U}(5)_{\hat{q}} \equiv \frac{\text{SU}(5)\times\text{U}(1)_{\hat{q}}}{\mathbb{Z}_5} \equiv \{(g, \mathrm{e}^{\mathrm{i}\theta}) \in \text{SU}(5)\times\text{U}(1) | (\mathrm{e}^{\mathrm{i}\frac{2\pi n}{5}}\mathbb{I}, 1) \sim (\mathbb{I}, \mathrm{e}^{\mathrm{i}\frac{2\pi n\hat{q}}{5}}), n \in \mathbb{Z}_5\}, \tag{3.39}$$

where we use two data $(g, \mathrm{e}^{\mathrm{i}\theta})$ to label the $\text{SU}(5)\times\text{U}(1)$ group elements, respectively, while we identify $(\mathrm{e}^{\mathrm{i}\frac{2\pi n}{5}}\mathbb{I}, 1) \sim (\mathbb{I}, \mathrm{e}^{\mathrm{i}\frac{2\pi n\hat{q}}{5}})$ for $n \in \mathbb{Z}_5$, with a rank-5 identity matrix $\mathbb{I}$. They have the group isomorphisms between different $\hat{q}$ as

$$\text{U}(5)_{\hat{q}} \cong \text{U}(5)_{-\hat{q}} \cong \text{U}(5)_{5m\pm\hat{q}}.$$

See further discussions in footnote 37. Whenever we mention $\text{U}(5) \subset \text{SO}(10)$, we really require $\text{U}(5)_{\hat{q}=1,4} \subset \text{SO}(10)$. In contrast, whenever we mention $\text{U}(5) \subset \text{Spin}(10)$, we really require $\text{U}(5)_{\hat{q}=2,3} \subset \text{Spin}(10)$.

$\hat{\Phi}^{\mathrm{bi}} = \sum_{a=1}^{5} \varphi_a^\dagger \varphi_a$ itself defines the generator of the $\mathrm{U}(1)_X$ group, whose $\mathbb{Z}_4$ subgroup defines $\mathbb{Z}_{4,X}$. The 16 Weyl fermions split as $\mathbf{16} \sim \bar{\mathbf{5}}_1 \oplus \mathbf{10}_1 \oplus \mathbf{1}_1$ under $\mathrm{SU}(5) \times \mathbb{Z}_{4,X}$. The $\mathbb{Z}_{4,X}$ generator in the Spin(10) spinor representation is given by

$$q_X = \sum_{a=1}^{5} \psi^\dagger \mathrm{i}\Gamma_{2a-1}\Gamma_{2a}\psi. \qquad (3.40)$$

By diagonalizing q_X operator, we indeed found fivefold eigenvalues of -3, tenfold eigenvalues of 1 and a onefold eigenvalue of 5. After mod 4, they all correspond to charge 1 under $\mathbb{Z}_{4,X}$. Further investigate the representation of SU(5) generators in each q_X-charge sectors, we can confirm that the $q_X = -3$ sector is indeed in the antifundamental representation $\bar{\mathbf{5}}$ and so on to form $\mathbf{16} \sim \bar{\mathbf{5}}_{-3} \oplus \mathbf{10}_1 \oplus \mathbf{1}_5$.

(3) $\mathrm{Spin}(10) \to \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_{\tilde{Y}}}{\mathbb{Z}_6} \times \mathbb{Z}_{4,X}$ by simultaneously condensing $\tilde{\Phi}^{\mathrm{bi}}$ and $\hat{\Phi}^{\mathrm{bi}}$ (both $\mathbf{54}_{\mathrm{S}}$ and $\mathbf{45}_{\mathrm{A}}$ representations) to configurations specified in Eqs. (3.36) and (3.38). The unbroken symmetry group is generated by the subalgebra of $so(10)$ that commute with both GUT-Higgs condensates $\langle\tilde{\Phi}^{\mathrm{bi}}\rangle$ and $\langle\hat{\Phi}^{\mathrm{bi}}\rangle$, which must take the form of

$$\phi^\intercal \begin{pmatrix} \mathrm{iA}_{2\times 2} & \\ & \mathrm{iA}_{3\times 3} \end{pmatrix} \otimes \sigma^0 \phi \quad \text{or}$$
$$\phi^\intercal \begin{pmatrix} \mathrm{S}_{2\times 2} & \\ & \mathrm{S}_{3\times 3} \end{pmatrix} \otimes \sigma^2 \phi, \qquad (3.41)$$

where $\mathrm{A}_{n\times n} = -\mathrm{A}_{n\times n}^\intercal \in \mathbb{R}_{n\times n}$ are real antisymmetric matrices and $\mathrm{S}_{n\times n} = \mathrm{S}_{n\times n}^\intercal \in \mathbb{R}_{n\times n}$ are real symmetric matrices. They can be combined in the complex representation as

$$\varphi^\dagger \begin{pmatrix} \mathrm{S}_{2\times 2} + \mathrm{iA}_{2\times 2} & \\ & \mathrm{S}_{3\times 3} + \mathrm{iA}_{3\times 3} \end{pmatrix} \varphi = \varphi^\dagger \begin{pmatrix} \mathrm{H}_{2\times 2} & \\ & \mathrm{H}_{3\times 3} \end{pmatrix} \varphi, \qquad (3.42)$$

such that $\mathrm{H}_{n\times n} = \mathrm{H}_{n\times n}^\dagger \in \mathbb{C}_{n\times n}$ are complex Hermitian matrices. There is no traceless condition imposed on $\mathrm{H}_{3\times 3}$ and $\mathrm{H}_{2\times 2}$ and they act independently in each subspace, so they generate the $\mathrm{U}(3) \times \mathrm{U}(2)$ subgroup of U(5), which is further a subgroup of SO(10). The two U(1) subgroups of U(3) and U(2) are generated by $\sum_{a=3}^{5} \varphi_a^\dagger \varphi_a$ and $\sum_{a=1}^{2} \varphi_a^\dagger \varphi_a$, respectively. Since the $\mathrm{U}(1)_X$ (or $\mathbb{Z}_{4,X}$) generator has already been identified as $\sum_{a=1}^{5} \varphi_a^\dagger \varphi_a$, so the $\mathrm{U}(1)_{\tilde{Y}}$ generator must be given by the remaining U(1) generator $\frac{1}{2}(-3\sum_{a=1}^{2} + 2\sum_{a=3}^{5})\varphi_a^\dagger \varphi_a$, which is represented in the Spin(10) spinor representation as

$$q_{\tilde{Y}} = \frac{1}{2}\left(-3\sum_{a=1}^{2} + 2\sum_{a=3}^{5}\right)\psi^\dagger \mathrm{i}\Gamma_{2a-1}\Gamma_{2a}\psi. \qquad (3.43)$$

By diagonalizing χ, $q_{\tilde{Y}}$, and q_X operators jointly [defined in Eqs. (3.37), (3.43), (3.40)], we can classify the 16 Weyl fermions ψ (actually they are all in the left-handed spacetime Weyl spinor ψ_L basis) by the quantum numbers as follows:

$\mathrm{U}(1)_{\tilde{Y}}$	$\mathrm{U}(1)_X$	internal L/R	$\mathrm{SU}(2)_{\mathrm{L}}^z$	$\mathrm{SU}(2)_{\mathrm{R}}^z$	ψ
2	-3	R	0	1	$\bar{d}_R$
-3	-3	L	1	0	ν_L
-3	-3	L	-1	0	e_L
1	1	L	1	0	u_L ,
1	1	L	-1	0	d_L
-4	1	R	0	-1	$\bar{u}_R$
6	1	R	0	1	$\bar{e}_R$
0	5	R	0	-1	$\bar{\nu}_R$

(3.44)

matching all the fermion contents in the SM (see Table III).

No bilinear mass generation by bivector GUT-Higgs.— Unlike the SM-Higgs that generates a bilinear mass for SM Weyl fermions, the GUT-Higgs in **45** and **54** do not generate a bilinear mass for SM Weyl fermions. Because the SO(10) bivector GUT-Higgs field Φ^{bi} corresponds to four-fermion operators, which is supposed to be perturbatively irrelevant. Even if it condenses, it is not expected to gap out the Weyl fermions if its vacuum expectation value is small (but it will Higgs down the gauge group), so the theory remains gapless in the fermion sector in all phases. However, sufficiently strong Higgs condensation of $\mathrm{Tr}\Phi^{\mathrm{bi}}$ (or $\Phi_{\mathbf{1}}$ equivalently) can lead to SMG [13,44–57] as discussed previously.

D. Fragmentary GUT-Higgs liquid model beyond the SM

1. Candidate low-energy dynamics of the WZW theory

The WZW term and its associated $w_2 w_3$ global anomaly can significantly modify the dynamics in the GUT-Higgs sector. There are several possibilities for the low-energy IR dynamical fate of the WZW theory. We shall enumerate the IR candidate phases as much as we could (logically this is similar to the Lieb-Schultz-Mattis theorem constraint now applied to the SM physics):

(1) Spontaneous symmetry breaking: The SO(10) internal symmetry of WZW term [or Spin(10) for the full modified $so(10)$ GUT] is spontaneously broken by GUT-Higgs condensation. Within this scenario,

there are a few different symmetry breaking patterns relevant to our discussion (recall Sec. II B):
(a) $\langle \Phi_{\mathbf{45}} \rangle \neq 0$, the $so(10)$ GUT is Higgs down to the $su(5)$ GUT.
(b) $\langle \Phi_{\mathbf{54}} \rangle \neq 0$, the $so(10)$ GUT is Higgs down to the PS model.
(c) $\langle \Phi_{\mathbf{45}} \rangle \neq 0$ and $\langle \Phi_{\mathbf{54}} \rangle \neq 0$, the $so(10)$ GUT is Higgs down to the SM.

In all three cases, the $w_2 w_3(V_{\mathrm{SO}(10)})$ anomaly is matched by symmetry breaking the Spin(10) down to the GG, PS, and SM groups.[26] The resulting vacua is in the same quantum phase as the corresponding vacua in the absence of the WZW term.

(2) The SO(10) symmetry remains unbroken, and the $w_2 w_3$ anomaly persists to low energy. The low-energy effective theory must saturate the anomaly requirement, which further leads to several different possibilities:
(a) WZW CFT: The WZW theory flows to a nontrivial CFT fixed point, where the GUT-Higgs field Φ remains gapless and disordered (not condensing), and also does not deconfine into fragmented excitations.
(b) DQC: The GUT-Higgs field Φ deconfines into fragmented excitations: partons and emergent gauge fields, which are new particles beyond the SM. The low-energy physics will be described by new quantum electrodynamics (QED′) or quantum chromodynamics (QCD′) sectors. In any case, the total gauge group must be enlarged to include the emergent gauge structure of partons, which is a phenomenon called gauge enhanced quantum criticality (GEQC) [32]. This can be viewed as the generalization of the DQC [28,64–66] to gauge-Higgs models. Possible field theory descriptions of the DQC can be classified by the parton statistics as:
(i) Fermionic parton theory, where the fractionalized particles in the emergent matter sector are fermions, which is the focus of our following work.
(ii) Bosonic parton theory, where the fractionalized particles in the emergent matter sector are bosons.

It is possible that two seemly different descriptions (e.g., fermionic vs bosonic parton theories) may be related by dualities, as discussed in [66,67]. In this scenario, the $w_2 w_3$ anomaly should be matched either by the anomalous fermionic matter or by a nontrivial θ term of the emergent gauge field.
(c) Topological order with low-energy noninvertible TQFT: The $w_2 w_3$ anomaly could also be matched by a certain 4D topological order. A simplest possibility is the $\mathbb{Z}_2$-gauge theory topological order (more precisely, generated by dynamical Spin structures), which can be considered as a descendent of the DQC when the emergent gauge group is reduced to $\mathbb{Z}_2$ by some further Higgsing.

Among the above possibilities are the following: (1) The spontaneous symmetry breaking scenario in the WZW theory has no substantial difference with our previous discussions without the WZW term, which will not be repeated here. (2a) The WZW CFT is a nontrivial possibility, which the authors are not aware of suitable theoretical tools to study it, and which will thus be left for future exploration. (2b) The DQC scenario will be the focus of the following discussion. In particular, we will consider a QED_4' theory with fermionic partons as the effective field theory description. The WZW theory could potentially admits dual bosonic parton descriptions as well, but we will also leave this possibility for future study. (2c) The topological order scenario could be derived from the DQC scenario, which will also be left for future study.

2. Dirac fermionic parton theory and a double-Spin structures DSpin within a modified so(10) GUT

Here we propose a fermionic parton construction for the WZW term in Sec. III B. We propose that WZW term Eq. (3.14) can also be viewed as a low-energy description of this Dirac fermionic parton theory with an action:

$$S_{\mathrm{QED}_4'}[\xi, \bar{\xi}, a, \Phi] = \int_{M^4} \bar{\xi}(\mathrm{i}\gamma^{\mu} D_{\mu} - \tilde{\Phi}^{\mathrm{bi}} - \mathrm{i}\gamma^{\mathrm{FIVE}} \hat{\Phi}^{\mathrm{bi}})\xi \mathrm{d}^4 x. \tag{3.45}$$

We will soon argue that importantly the fermion parity $\mathbb{Z}_2^{F'}$ of this Dirac fermionic parton ξ is required to be different from the original fermion parity $\mathbb{Z}_2^F$ of the standard model or GUT fermions ψ. Namely, we will soon introduce a new kind of Spin structure with two distinct fermion parities, which we name it formally a double Spin structure:

$$\mathrm{DSpin} \equiv (\mathbb{Z}_2^F \times \mathbb{Z}_2^{F'}) \rtimes \mathrm{SO}. \tag{3.46}$$

The theory contains the following ingredients:

[26] However, the $\mathbb{Z}_2$ class $w_2 w_3(V_{\mathrm{SO}(10)})$ anomaly of SO(10) bundle is split to different kinds of $w_2 w_3$ anomalies of SO(6) and SO(4) bundles in the PS symmetry group: More precisely, see Appendix D in detail, $w_2(V_{\mathrm{SO}(10)}) w_3(V_{\mathrm{SO}(10)}) = w_2(V_{\mathrm{SO}(6)}) w_3(V_{\mathrm{SO}(6)}) + w_2(V_{\mathrm{SO}(4)}) w_3(V_{\mathrm{SO}(4)}) + w_2(V_{\mathrm{SO}(6)}) \times w_3(V_{\mathrm{SO}(4)}) + w_2(V_{\mathrm{SO}(4)}) w_3(V_{\mathrm{SO}(6)}) \mod 2$, where the crossing term $w_2(V_{\mathrm{SO}(6)}) w_3(V_{\mathrm{SO}(4)}) + w_2(V_{\mathrm{SO}(4)}) w_3(V_{\mathrm{SO}(6)})$ may or may not survive depending on whether we include additional time-reversal T or CP type of discrete symmetries protection or not.

(1) There are 10 Dirac fermions ξ forming the **10** (vector representation) of $\mathrm{SO}(10)$. Here γ^μ are the standard rank-4γ matrices of four-component Dirac fermions with $\gamma^{\mathrm{FIVE}} = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ and $\bar{\xi} = \xi^\dagger\gamma^0$.

(2) The covariant derivative $D_\mu = \nabla_\mu - \mathrm{i}a_\mu - \mathrm{i}gA_\mu$ contains the minimal coupling of the fermionic parton ξ to a new emergent dynamical $\mathrm{U}(1)'$ gauge field a_μ, as well as the minimal coupling to the $\mathrm{SO}(10)$ gauge field A_μ [which is part of the $\mathrm{Spin}(10)$ gauge field in the conventional $so(10)$ GUT in Sec. II B]. We may treat the $\mathrm{SO}(10)$ gauge field A_μ as a background field for now, and discuss how it can be gauged later.

(3) The GUT-Higgs field Φ is written as its 10×10 matrix representation Φ^{bi} of the $\mathrm{SO}(10)$ bivector form. It couples to the fermionic partons by taking its traceless symmetric component $\tilde{\Phi}^{\mathrm{bi}}$ [the **54** of $\mathrm{SO}(10)$] as the vector mass of ξ and its antisymmetric component $\hat{\Phi}^{\mathrm{bi}}$ [the **45** of $\mathrm{SO}(10)$] as the axial mass of ξ. In this way, the $\mathrm{SO}(10)$ bivector GUT-Higgs boson effectively deconfines into two $\mathrm{SO}(10)$ vector fermions: $\Phi^{\mathrm{bi}}_{ab} \sim \xi^\dagger_a\xi_b$.[27]

(4) In the QED_4' theory $S_{\mathrm{QED}_4'}$, the GUT-Higgs field fractionalizes into gapless fermionic partons with emergent $\mathrm{U}(1)'$ gauge interactions. The situation is similar to the $\mathrm{U}(1)$ Dirac spin liquid [68,69] discussed in the condensed matter physics context. Therefore we may also call this QED_4' theory as the fragmentary GUT-Higgs liquid model.[28]

(5) The name of "fragmentary" GUT-Higgs liquid (Sec. III D 2) is meant to distinguish and emphasize the fractionalization of bivector field as $\Phi_{ab} \sim \xi^\dagger_a\xi_b$ of fermionic partons in (3.45), instead of $\Phi^{\mathrm{bi}}_{ab} \sim \phi_a\phi_b$ of the bosonic partons in (3.13) and (3.33) for the "composite" GUT-Higgs model (Sec. III C).

[27]If this theory has 't Hooft anomaly in G, it cannot be trivially gapped by preserving the G symmetry. Since we like to construct fermion parton theory QED_4' (3.45) to saturate the w_2w_3 anomaly of $\mathrm{SO}(10)$ symmetry [or $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)$ symmetry], we should forbid the (3.45) to get any quadratic mass term that preserves the $\mathrm{SO}(10)$. It turns out that the QED_4' have $\mathrm{U}(1)'$, CP', and T' symmetries that can forbid any $\mathrm{SO}(10)$ symmetric quadratic mass term:

(i) The $\mathrm{U}(1)'$ symmetry: $\xi \to e^{\mathrm{i}\theta}\xi$ forbids any Majorana mass of the form $\xi^{\mathrm{T}}_{\mathrm{L/R}}\mathrm{i}\sigma^2\xi_{\mathrm{L/R}}$ that potentially gaps out the Dirac fermion (written as two Weyl fermions: $\xi = \xi_{\mathrm{L}} + \xi_{\mathrm{R}}$).

(ii) The CP' symmetry $\mathbb{Z}_2^{\mathrm{CP}'}$: $\xi(t,\vec{x}) \to \gamma^0\gamma^{\mathrm{FIVE}}\xi^*(t,-\vec{x})$ forbids the vector $\bar{\xi}\xi$ mass: $\bar{\xi}\xi \to -\bar{\xi}\xi$.

(iii) The T' symmetry $\mathbb{Z}_2^{\mathrm{T}'}$: $\xi(t,\vec{x}) \to \mathcal{K}\gamma^0\gamma^{\mathrm{FIVE}}\xi(-t,\vec{x})$ forbids the axial $\mathrm{i}\bar{\xi}\gamma^{\mathrm{FIVE}}\xi$ mass: $\mathrm{i}\bar{\xi}\gamma^{\mathrm{FIVE}}\xi \to -\mathrm{i}\bar{\xi}\gamma^{\mathrm{FIVE}}\xi$.

[28]Because the order-parameter target manifold in our construction involves a Grassmannian manifold $(\frac{\mathrm{O}(m+n)}{\mathrm{O}(m)\times\mathrm{O}(n)}) \equiv \mathrm{Gr}(m, m+n)$, the corresponding GUT-Higgs liquid may also be called Grassmannian liquid by some condensed matter people.

We first argue that the QED_4' theory (without a θ term) in Eq. (3.45) saturates the same w_2w_3 anomaly as the WZW term in Sec. III B. The starting point is to identify that the spacetime-internal symmetry [here $\mathrm{Spin}' \times_{\mathbb{Z}_2^{F'}} \mathrm{U}(1)'$] and the gauge group [here $\mathrm{SO}(10)$] of the fermionic parton theory is

$$G_{\mathrm{QED}_4'} \equiv \mathrm{Spin}' \times_{[\mathbb{Z}_2^{F'}]} [\mathrm{U}(1)'] \times \mathrm{SO}(10) \equiv \mathrm{Spin}^{c'} \times \mathrm{SO}(10), \tag{3.47}$$

with fermions in the $\mathbf{10}_1$ representation of $\mathrm{SO}(10)$ and $\mathrm{U}(1)'$. Notice that we use the prime notation to indicate that those groups contain the new fermion parity $\mathbb{Z}_2^{F'}$. Such that $\mathrm{U}(1)' \supset \mathbb{Z}_2^{F'}$, $\mathrm{Spin}' \supset \mathbb{Z}_2^{F'}$, and $\mathrm{Spin}^{c'} \supset \mathbb{Z}_2^{F'}$. Here we use the bracket notation around $[\mathrm{U}(1)']$ to indicate that this $\mathrm{U}(1)'$ is dynamically gauged eventually in terms of the emergent gauge fields near the quantum criticality. In other words, the new fermion parity $\mathbb{Z}_2^{F'}$ must also be dynamically gauged because $[\mathrm{U}(1)'] \supset [\mathbb{Z}_2^{F'}]$.

How do we reconcile the Spin structure (of the familiar SM and GUT in Sec. III) and the Spin' structure [of this new fermion parton theory (3.45)] in the full theory? After all, we have to place a full theory on some curved spacetime with a single unified geometric structure. The full spacetime-internal structure of this modified $so(10)$ GUT, that we require to include $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)$ of (2.3) and $\mathrm{Spin}^{c'} \times \mathrm{SO}(10)$ of (3.47) as subgroups, turns out to be[29]

$$G^{\mathrm{modified}}_{so(10)\text{-GUT}} \equiv (\mathrm{DSpin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) \times_{[\mathbb{Z}_2^{F'}]} [\mathrm{U}(1)'], \tag{3.48}$$

where we implement the early advertised double Spin structure $\mathrm{DSpin} \equiv (\mathbb{Z}_2^F \times \mathbb{Z}_2^{F'}) \rtimes \mathrm{SO}$ structure. We leave the detail construction of this full spacetime-internal

[29]Again we use the bracket notation around $[\mathrm{U}(1)']$ and $[\mathbb{Z}_2^{F'}]$ to indicate that they must be dynamically gauged. Although the $\mathrm{Spin}(10)$ is also dynamically gauged in the GUT, the $\mathrm{Spin}(10)$ may still be treated as a global symmetry in the context of quantum criticality of the internal flavor symmetry of fermions in the condensed matter system. However, the $[\mathrm{U}(1)']$ and $[\mathbb{Z}_2^{F'}]$ must be dynamically gauged due to their roles at quantum criticality, regardless whether the $\mathrm{Spin}(10)$ is gauged or not. In summary, there is a hierarchy of gauging: the brackets [..] imply those degrees of freedom have a higher priority to be gauged.

$G^{\text{modified}}_{so(10)\text{-GUT}}$ symmetry based on the group extension in the footnote remark[30] and Appendix E.

The $\mathrm{U}(1)'$ group is free of anomaly, which is consistent with the fact that this emergent $\mathrm{U}(1)'$ structure can be gauged. Gauging $\mathrm{U}(1)'$ out of $\mathrm{Spin}^{c'}\times\mathrm{SO}(10)$ removes the spin structure of the fermion theory, allowing the gauge theory to be placed on nonspin manifolds. So the resulting theory is a bosonic theory with an $\mathrm{SO}\times\mathrm{SO}(10)$ symmetry. It is expected that the spacetime SO group should carry the w_2w_3 anomaly, and the anomaly could only originate from the fermionic partons in the QED_4' theory.

To check the anomaly in the fermion sector, we first turn off the Higgs coupling (as it does not affect the anomaly analysis), such that the theory becomes as simple as $\int_{M^4}\bar{\xi}\gamma^\mu D_\mu\xi \mathrm{d}^4x$. Without coupling to the GUT-Higgs field, the theory has an enlarged $\mathrm{SU}(2)'$ gauge group, generated by $\xi^\dagger\xi$, $\mathrm{Re}\xi^\intercal\gamma^5\xi$, $\mathrm{Im}\xi^\intercal\gamma^5\xi$, among which $\xi^\dagger\xi$ generates the $\mathrm{U}(1)'$ gauge group as a subgroup of $\mathrm{SU}(2)'$. With the enlarged $\mathrm{SU}(2)'$ gauge group, the fermionic parton theory is promoted from a QED_4' theory to a QCD_4' theory (without enlarging the fermion content), whose group structure is[31]

$$
\begin{aligned}
G_{\mathrm{QCD}_4'} &= \mathrm{Spin}'\times_{[\mathbb{Z}_2^{F'}]}[\mathrm{SU}(2)']\times\mathrm{SO}(10)\\
&\equiv \mathrm{Spin}^{h'}\times\mathrm{SO}(10),
\end{aligned}
\tag{3.51}
$$

the original Dirac fermion ξ is in $\mathbf{2}_L\oplus\mathbf{2}_R$ of Spin(1, 3) and (1, **10**) of $\mathrm{U}(1)'\times\mathrm{SO}(10)$, while now the fermion ξ becomes in $\mathbf{2}_L$ of Spin(1,3) and in the $(\mathbf{2},\mathbf{10})$ representation of $\mathrm{SU}(2)'\times\mathrm{SO}(10)$. Again we use the bracket notation around $[\mathrm{SU}(2)']$ and $[\mathbb{Z}_2^{F'}]$ to indicate that they must be dynamically gauged near the criticality. This QED_4' to QED_4' promotion does not change the anomaly structure, because the $\mathrm{SU}(2)'$ group is still anomaly free.

[30]Here are some comments about our construction of spacetime-internal symmetry. More details are in Appendix E. First, the ψ fermion in the **16** of Spin(10) requires a fermion parity $\mathbb{Z}_2^F$, while the ξ fermion in the **10** of SO(10) requires another new fermion parity $\mathbb{Z}_2^{F'}$. Next, both ψ and ξ fermions require the common $\mathrm{SO}\times\mathrm{SO}(10)$ structure (as the quotient group of the total symmetry group), because they share the same bosonic part of spacetime rotational special orthogonal symmetry group SO, and their SO(10) gauge fields are the same. However, the ψ fermion requires a total structure $\mathrm{Spin}\times_{\mathbb{Z}_2^F}\mathrm{Spin}(10)$ under the short exact sequence: $1\to\mathbb{Z}_2^F\to\mathrm{Spin}\times_{\mathbb{Z}_2^F}\mathrm{Spin}(10)\to\mathrm{SO}\times\mathrm{SO}(10)\to 1$; the ξ fermion requires a different total structure $\mathrm{Spin}'\times\mathrm{SO}(10)$ under the short exact sequence: $1\to\mathbb{Z}_2^{F'}\to\mathrm{Spin}'\times\mathrm{SO}(10)\to\mathrm{SO}\times\mathrm{SO}(10)\to 1$. Their structures cannot be compatible under the same fermion parity, thus we require to introduce two fermion parities with the $\mathrm{DSpin}\equiv(\mathbb{Z}_2^F\times\mathbb{Z}_2^{F'})\rtimes\mathrm{SO}$ structure under $1\to\mathbb{Z}_2^F\times\mathbb{Z}_2^{F'}\to\mathrm{DSpin}\to\mathrm{SO}\to 1$ such that $\mathrm{DSpin}\supset\mathrm{Spin}=\mathbb{Z}_2^F\rtimes\mathrm{SO}$ and $\mathrm{DSpin}\supset\mathrm{Spin}'=\mathbb{Z}_2^{F'}\rtimes\mathrm{SO}$. The above short exact sequences can be combined into the following group extensions:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{Z}_2^{F'} & & \mathbb{Z}_2^{F'} & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z}_2^F & \longrightarrow & (\mathrm{DSpin}\times_{\mathbb{Z}_2^F}\mathrm{Spin}(10)) & \longrightarrow & \mathrm{Spin}'\times\mathrm{SO}(10) & \longrightarrow & 1\\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z}_2^F & \longrightarrow & \mathrm{Spin}\times_{\mathbb{Z}_2^F}\mathrm{Spin}(10) & \longrightarrow & \mathrm{SO}\times\mathrm{SO}(10) & \longrightarrow & 1\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
\tag{3.49}
$$

This total extended spacetime-internal $(\mathrm{DSpin}\times_{\mathbb{Z}_2^F}\mathrm{Spin}(10))$ group is compatible with both fermionic spectrum restrictions for ψ and ξ. By modifying the $\mathbb{Z}_2^{F'}$ into $\mathrm{U}(1)'$ in the web of (3.49), we thus obtain the $G^{\text{modified}}_{so(10)\text{-GUT}}\equiv(\mathrm{DSpin}\times_{\mathbb{Z}_2^F}\mathrm{Spin}(10))\times_{\mathbb{Z}_2^{F'}}\mathrm{U}(1)'$ in (3.48).

Related to the DSpin structure, by including an extra discrete symmetry such as a time-reversal symmetry, the literatures also discover the structures known as DPin [70] and EPin [37] structures, see also an interpretation via the regularized quantum many-body model [71]. See more elaborations in Appendix E.

[31]Similar to (3.49), by modifying the $\mathbb{Z}_2^{F'}$ into $\mathrm{SU}(2)'$ in the web, we thus obtain a modification on (3.48) into

$$
G^{\text{modified}}_{so(10)\text{-GUT}}\equiv(\mathrm{DSpin}\times_{\mathbb{Z}_2^F}\mathrm{Spin}(10))\times_{[\mathbb{Z}_2^{F'}]}[\mathrm{SU}(2)'],
\tag{3.50}
$$

which has a quotient group $G_{\mathrm{QCD}_4'}\equiv\mathrm{Spin}^{h'}\times\mathrm{SO}(10)$ in (3.51). See more elaborations in Appendix E.

Namely, there are only two possible combinations of nonperturbative global anomalies out of the cobordism classification for $\mathrm{Spin}' \times_{\mathbb{Z}_2^{F'}} \mathrm{SU}(2)'$ symmetry given by $\mathrm{TP}_5(\mathrm{Spin}' \times_{\mathbb{Z}_2^{F'}} \mathrm{SU}(2)') = \mathbb{Z}_2^2$ [12,18,23]:

(1) No Witten $\mathrm{SU}(2)'$ anomaly [72]: Given that there are even number (ten) of fundamental fermions **2** of $\mathrm{SU}(2)'$, so 10 mod 2 = 0.
(2) No new $\mathrm{SU}(2)'$ anomaly [12]: Given that there is no **4** of $\mathrm{SU}(2)'$ fermions, so 0 mod 2 = 0.

So the anomaly is still contained in the SO(10) group out of $G_{\mathrm{QCD}'_4} = \mathrm{Spin}^{h'} \times \mathrm{SO}(10)$. To match the $w_2 w_3$ anomaly, we make a connection to the recently discovered new SU(2) anomaly [18] by the following trick on the SO × SO(10) sector: We first embed $\mathrm{SU}(2)' \times \mathrm{SO}(10)$ in Sp(10) and use a sequence of maximal special (S) or regular (R) Lie subalgebra [58] decomposition $\mathrm{Sp}(10) \hookleftarrow \mathrm{Sp}(2) \times \mathrm{Sp}(8) \hookleftarrow \mathrm{SU}(2)'' \times \mathrm{Sp}(8)$ to show that a different $\mathrm{SU}(2)''$ subgroup carries the $w_2 w_3$ anomaly. Under the embedding, the representation of the fermionic parton ξ splits as[32]

$$\begin{array}{ccccccccc} \mathrm{U}(1)' \times \mathrm{SO}(10) & \hookrightarrow & \mathrm{SU}(2)' \times \mathrm{SO}(10) & \hookrightarrow & \mathrm{Sp}(10) & \hookleftarrow & \mathrm{Sp}(2) \times \mathrm{Sp}(8) & \hookleftarrow & \mathrm{SU}(2)'' \times \mathrm{Sp}(8) \\ \mathbf{10}_1 & & (\mathbf{2},\mathbf{10}) & \overset{\mathrm{S}}{\sim} & \mathbf{20} & \overset{\mathrm{R}}{\sim} & (\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{16}) & \overset{\mathrm{S}}{\sim} & (\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{16}). \end{array} \tag{3.52}$$

Some comments on (3.52):

(1) The $(\mathbf{1},\mathbf{16})$ is free from both the old Witten's $\mathrm{SU}(2)'$ and the new $\mathrm{SU}(2)'$ anomaly, but the $(\mathbf{4},\mathbf{1})$ has the new $\mathrm{SU}(2)''$ anomaly $w_2 w_3(V_{\mathrm{SO}(3)''})$ [18].
(2) Since we have argued that $(\mathbf{2},\mathbf{10})$ in $\mathrm{SU}(2)' \times \mathrm{SO}(10)$ has no Witten or the new $\mathrm{SU}(2)'$ anomalies in the $\mathrm{SU}(2)'$ sector, so the new-$\mathrm{SU}(2)''$ anomaly must come from the remained SO(10), or more precisely the remained SO × SO(10) out of the full $\mathrm{Spin}^{h'} \times \mathrm{SO}(10)$ in (3.51). According to [23,25], the classification of 't Hooft anomaly of SO × SO(10) symmetry is generated respectively by the cobordism group:

$$\mathrm{TP}_5(\mathrm{SO} \times \mathrm{SO}(10)) = \mathbb{Z}_2^2, \begin{cases} (-1)^{\int w_2 w_3(TM)} & \text{out of the tangent bundle } TM \text{ of SO}, \\ (-1)^{\int w_2 w_3(V_{\mathrm{SO}(10)})} & \text{out of the associated vector bundle of SO(10)}. \end{cases} \tag{3.53}$$

Therefore, we claim that the new-$\mathrm{SU}(2)''$ anomaly can be identified by $w_2 w_3(V_{\mathrm{SO}(10)})$, come from the remained SO(10) out of the $\mathrm{Spin}^{h'} \times \mathrm{SO}(10)$.
(3) We can further extend the $\mathrm{Spin}^{h'} \times \mathrm{SO}(10)$ structure of the fermionic parton theory QCD'_4 to the full $(\mathrm{DSpin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) \times_{[\mathbb{Z}_2^{F'}]} [\mathrm{SU}(2)']$ structure of the modified $so(10)$ GUT, under the pullback:

$$1 \to \mathbb{Z}_2^F \to (\mathrm{DSpin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) \times_{[\mathbb{Z}_2^{F'}]} [\mathrm{SU}(2)'] \to \mathrm{Spin}^{h'} \times \mathrm{SO}(10) \to 1. \tag{3.54}$$

In terms of the interpretation of the anomaly [we can gauge the anomaly-free $\mathrm{SU}(2)'$], we are left with

$$1 \to \mathbb{Z}_2^F \to \mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10) \to \mathrm{SO} \times \mathrm{SO}(10) \to 1. \tag{3.55}$$

The two $w_2 w_3(TM)$ and $w_2 w_3(V_{\mathrm{SO}(10)})$ anomalies in the $\mathrm{TP}_5(\mathrm{SO} \times \mathrm{SO}(10)) = \mathbb{Z}_2^2$ becomes identified as the same anomaly in the $\mathrm{TP}_5(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) = \mathbb{Z}_2$ of (2.5). Thus, of course, now we can also interpret as the gauge anomaly $w_2 w_3(V_{\mathrm{SO}(10)})$ as the gravitational anomaly $w_2 w_3(TM)$ due to the relation $(-1)^{\int w_2 w_3(TM)} = (-1)^{\int w_2 w_3(V_{\mathrm{SO}(10)})}$ as mentioned before. The analysis establishes that the proposed QED'_4 or QCD'_4 theory in Eq. (3.45) at least has the same 4D nonperturbative global mixed gauge-gravitational $w_2 w_3$ anomaly as the proposed 4D WZW term in (3.15).

To reproduce the WZW term more explicitly, we extend the QED'_4 theory to the 5D bulk

$$S_{\mathrm{QED}'_5}[\xi,\bar{\xi},a,\Phi] = \int_{M^5} \bar{\xi}(\mathrm{i}\gamma^\mu D_\mu - m - \gamma^5 \tilde{\Phi}^{\mathrm{bi}} - \gamma^6 \mathrm{i}\hat{\Phi}^{\mathrm{bi}})\xi \mathrm{d}^5 x, \tag{3.56}$$

where ξ still forms the $\mathbf{10}_1$ under $\mathrm{U}(1)' \times \mathrm{SO}(10)$. Note that, in 5D, each Dirac fermion already defines five gamma matrices γ^0, γ^1, γ^2, γ^3, γ^4, which are rank-4 matrices. By doubling the fermion content (which means we need two

[32] Here we apply the symplectic group notation under $\mathrm{Sp}(n) = \mathrm{USp}(2n) = \mathrm{Sp}(2n,\mathbb{C}) \cup \mathrm{U}(2n)$, such that $\mathrm{Sp}(1) = \mathrm{USp}(2) = \mathrm{SU}(2) = \mathrm{Spin}(3)$ and $\mathrm{Sp}(2) = \mathrm{USp}(4) = \mathrm{Spin}(5)$. The $G_1 \hookrightarrow G_2$ means that the inclusion $G_1 \subset G_2$ as a subgroup. The representations on two sides of "∼" shows their decomposition relation.

sets of 5D Dirac fermions in **10**, thus there are $2 \times \mathbf{10}$ Dirac fermions in 5D), we are able to introduce two more gamma matrices, denoted γ^5 and γ^6, such that all seven gamma matrices $\gamma^0, \ldots, \gamma^6$ are rank-8 matrices satisfying the Clifford algebra relation $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$. The bulk fermions are gapped by the mass term m. The boundary QED'_4 theory (with massless fermions) is reduced from the bulk QED'_5 theory (with massive fermions) as the effective domain wall theory, which lives on the 4D domain wall separating the $m > 0$ and $m < 0$ phases in 5D.[33]

To show that the QED'_4 theory is equivalent to the WZW theory, we only need to show that the bulk QED'_5 theory can reproduce the WZW term (3.15). For this purpose, we introduce two 2-form $\mathbb{R}$ gauge fields $\mathcal{B} = \mathcal{B}_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ and $\mathcal{B}' = \mathcal{B}'_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ that couple to the fermionic parton as

$$S_{\mathrm{QED}'_5}[\xi, \bar{\xi}, a, \Phi, \mathcal{B}, \mathcal{B}'] = \int_{M^5} \bar{\xi}(\mathrm{i}\gamma^\mu D_\mu - m - \gamma^5 \tilde{\Phi}^{\mathrm{bi}} - \gamma^6 \mathrm{i} \hat{\Phi}^{\mathrm{bi}} - \mathrm{i}\gamma^5 \gamma^\mu \gamma^\nu \mathcal{B}_{\mu\nu} - \mathrm{i}\gamma^6 \gamma^\mu \gamma^\nu \mathcal{B}'_{\mu\nu}) \xi \mathrm{d}^5 x. \tag{3.57}$$

Integrating out the massive fermion ξ, we obtain the BF 5-form term with 2-form $\mathcal{B}$ and $\mathcal{B}'$ fields:

$$S_{\mathrm{BF}_5}[\mathcal{B}, \mathcal{B}'] = \frac{1}{\pi} \int_{M^5} \mathcal{B} \wedge \mathrm{d}\mathcal{B}', \tag{3.58}$$

with the constraint that the 2-form gauge fields $\mathcal{B}$ and $\mathcal{B}'$ are locked to the cohomology classes that measure the defects in $\tilde{\Phi}^{\mathrm{bi}}$ and $\hat{\Phi}^{\mathrm{bi}}$, respectively,

$$B(\tilde{\Phi}^{\mathrm{bi}}) = \frac{\mathcal{B}}{\pi} = \frac{\mathcal{B}(\tilde{\Phi}^{\mathrm{bi}})}{\pi} \in H^2(\mathrm{O}(10)/(\mathrm{O}(6) \times \mathrm{O}(4)), \mathbb{Z}_2),$$
$$B'(\hat{\Phi}^{\mathrm{bi}}) = \frac{\mathcal{B}'}{\pi} = \frac{\mathcal{B}'(\hat{\Phi}^{\mathrm{bi}})}{\pi} \in H^2(\mathrm{O}(10)/\mathrm{U}(5), \mathbb{Z}_2). \tag{3.59}$$

The emergent $\mathrm{U}(1)'$ gauge field a decouples from the GUT-Higgs field Φ and the 2-form gauge fields $\mathcal{B}, \mathcal{B}'$, which can be integrated out independently. Further integrate out the 2-form gauge fields $\mathcal{B}, \mathcal{B}'$, we obtain an action for Φ (simply by substituting the constraint), $S_{\mathrm{WZW}}[\Phi] = \frac{1}{\pi} \int_{M^5} \mathcal{B}(\tilde{\Phi}^{\mathrm{bi}}) \wedge \mathrm{d}\mathcal{B}'(\hat{\Phi}^{\mathrm{bi}})$. Recall the footnote about our normalizations of differential forms and cohomology classes. This leads to the proposed WZW term in Eq. (3.15)

$$S_{\mathrm{WZW}}[\Phi] = \pi \int_{M^5} B(\tilde{\Phi}^{\mathrm{bi}}) \smile \delta B'(\hat{\Phi}^{\mathrm{bi}}), \tag{3.60}$$

which is expected to be placed on the 5D manifold M^5 whose boundary is the 4D spacetime $M^4 = \partial M^5$.

3. Color-flavor separation and dark gauge sector: 4D deconfined quantum criticality

The QED'_4 theory describes the DQC scenario of the 4D WZW-term like theory at low energy. In this scenario, the GUT-Higgs field deconfines into fragmentary excitations, which are new 0D particles beyond the SM:

(i) 10 new fermions ξ in the $\mathbf{10}_1$ of $\mathrm{U}(1)' \times \mathrm{SO}(10)$, as fermionic partons that fractionalize the GUT-Higgs field;
(ii) a new $\mathrm{U}(1)'$ photon a_μ in the $\mathbf{1}_0$ of $\mathrm{U}(1)' \times \mathrm{SO}(10)$, which mediates a new gauge force that exists between and only between fermionic partons. It does not couple to any particle in the SM sector, hence appears dark to us. Therefore, we will call it the "dark photon."

The GUT-Higgs boson can be considered as the bound state of two fermionic partons [of opposite emergent $\mathrm{U}(1)'$ gauge charges] bind together by the emergent $\mathrm{U}(1)'$ gauge force mediated by dark photons:

(i) From particle physic perspective, the fermionic partons and dark photons are more fundamental constituents of the GUT-Higgs bosons.
(ii) From condensed matter physics perspective, these fragmentary excitations are emergent collective modes of the GUT-Higgs field instead. The two complementary viewpoints are a matter of culture. The readers can take whichever interpretation that is more favorable to their mindset.

Because the QED'_4 theory is deconfined in 4D, the fragmentary GUT-Higgs liquid is expected to be a stable phase in the phase diagram Fig. 8. It covers the quantum critical region (critical in the sense that excitations are gapless), and may possibly extend into the modified $so(10)$ GUT phase (as long as fermionic partons remain deconfined). Starting from the fragmentary GUT-Higgs liquid phase, we can access the adjacent phases by GUT-Higgs condensation:

(i) $\langle \tilde{\Phi}^{\mathrm{bi}} \rangle \neq 0$, the system enters the PS GUT phase, where fermionic partons are fully gapped by the vector mass.
(ii) $\langle \hat{\Phi}^{\mathrm{bi}} \rangle \neq 0$, the system enters the $su(5)$ GUT phase, where fermionic partons are fully gapped by the axial mass.
(iii) $\langle \tilde{\Phi}^{\mathrm{bi}} \rangle \neq 0$ and $\langle \hat{\Phi}^{\mathrm{bi}} \rangle \neq 0$, the system enters the SM phase, where fermionic partons are fully gapped by both vector and axial masses.

In all phases, the dark photon will remain gapless and decoupled from all the other particles, which provides a new candidate for the light dark matter.

A substantial difference of fermionic partons ξ in the fragmentary GUT-Higgs liquid from quarks and leptons ψ in the SM, lies in their distinct assignment of quantum

[33]The 5D theory has the $2 \times \mathbf{10}$ Dirac fermions of four complex components (alternatively, **10** of 8 complex components), while the domain wall theory in 4D has **10** Dirac fermions of 4 complex components, in one lower dimension. The 4D domain wall fermion has only half of degrees of freedom of the 5D bulk.

TABLE I. The Dirac fermionic parton ξ contains flavorons f and colorons c as Grassmann numbers. Please beware that the $\mathrm{U}(1)'^{\text{dark}}_{\text{gauge}}$ is for the dark gauge (dark photon) sector, which is totally distinct from the $\mathrm{U}(1)_{\text{EM}}$. The $\mathrm{U}(1)_{\text{EM}}$ is from the electroweak Higgs symmetry breaking of the $\mathrm{SU}(2)_{\text{L,flavor}} \times \mathrm{U}(1)_{\tilde{Y}}$ down to a subgroup $\mathrm{U}(1)_{\text{EM}}$.

	$\mathrm{U}(1)'^{\text{dark}}_{\text{gauge}}$	$\mathrm{SU}(3)_{c,\text{color}}$	$\mathrm{SU}(2)_{\text{L,flavor}}$	$\mathrm{U}(1)_{\tilde{Y}}$	$\mathrm{U}(1)_X$	$\mathrm{U}(1)_{\text{EM}}$
f	1	$\mathbf{1}$	$\mathbf{2}$	3	−2	1 or 0
c	1	$\mathbf{3}$	$\mathbf{1}$	−2	−2	−1/3
f'	1	$\mathbf{1}$	$\mathbf{2}$	−3	2	0 or −1
c'	1	$\bar{\mathbf{3}}$	$\mathbf{1}$	2	2	1/3

numbers. For the spacetime symmetry representation, the Dirac fermion partons ξ is in the complex $\mathbf{2}_L \oplus \mathbf{2}_R$ of $\mathrm{Spin}(1,3)$; the SM's Weyl fermion is in the complex $\mathbf{2}_L$ of $\mathrm{Spin}(1,3)$.

For the internal symmetry representation, consider entering the SM phase from the fragmentary GUT-Higgs liquid, the Dirac fermionic partons, apart from the gap opening, also has its representation split from $\mathbf{10}_1$ under $\mathrm{U}(1)' \times \mathrm{SO}(10)$ to[34]

$$(\mathbf{1},\mathbf{2})_{1,3,-2} \oplus (\mathbf{3},\mathbf{1})_{1,-2,-2} \oplus (\mathbf{1},\mathbf{2})_{1,-3,2} \oplus (\bar{\mathbf{3}},\mathbf{1})_{1,2,2} \text{ under } \mathrm{SU}(3)_c \times \mathrm{SU}(2)_{\text{L}} \times \mathrm{U}(1)'^{\text{dark}}_{\text{gauge}} \times \mathrm{U}(1)_{\tilde{Y}} \times \mathrm{U}(1)_X$$

of the SM. The weak SU(2) flavor and the strong SU(3) color quantum numbers separate to different fermions, called flavoron and coloron, denoted by the f and c Dirac fermions as Grassmann numbers respectively, as summarized in Table I. We shall name this phenomenon as color-flavor separation, as it is analogous to the spin-charge separation [73–75] in condensed matter physics.

The flavoron can participate SU(2) weak interaction but not SU(3) strong interaction. On the contrary, the coloron can participate SU(3) strong interaction but not SU(2) electroweak interaction. Many of them also carry electromagnetic charge, such that they can also participate electromagnetic interaction. Beyond the SM interactions, the flavoron and coloron also interact among themselves by the emergent $\mathrm{U}(1)'$ gauge force mediated by the dark photon. Note that there exists a flavoron (in the f_L sector) which does not participate in SU(3) strong and electromagnetic interactions. It only participate SU(2) weak interaction (like left-handed neutrinos) and dark gauge interaction (unlike neutrinos), which makes it especially a potential better candidate for heavy dark matter.

[34]Here we use the branching rule of the Lie algebra representations for the following inclusion: $so(10) \hookleftarrow su(5) \times u(1)_X$ (R regular subalgebra), so that $\mathbf{10} \sim \mathbf{5}_{-2} \oplus \bar{\mathbf{5}}_2$; and also the $su(5) \hookleftarrow su(3) \times su(2) \times u(1)_{\tilde{Y}}$ (R regular subalgebra) so that $\mathbf{5} \sim (1,\mathbf{2})_3 \oplus (\mathbf{3},1)_{-2}$ and $\bar{\mathbf{5}} \sim (1,\mathbf{2})_{-3} \oplus (\bar{\mathbf{3}},1)_2$.

IV. CONCLUSION: MOTHER EFFECTIVE FIELD THEORY FOR BSM GAUGE ENHANCED QUANTUM CRITICALITY

A. Summary of main results: EFT for internal Spin(10) global symmetry or dynamical gauge theory

To conclude, here in Table II, we summarize the quantum field content of the mother effective field theory of the 4D $so(10)$ GUT + GUT-Higgs potential + with or without WZW term. We summarize our physical findings on the various quantum vacua of mother effective field theory. Although there are various possible IR fates of the UV modified $so(10)$ GUT + WZW theory listed in Sec. III D 1, we will focus on the deconfined quantum criticality scenario here.

Based on three binary conditions:

(i) Without or with the GUT-Higgs potential $\mathrm{U}(\Phi_{\mathbf{R}})$ and GUT-Higgs condensation $\langle\Phi_{\mathbf{R}}\rangle \neq 0$ of Eq. (3.4): (i) Whether we stays in the Spin(10) group of $so(10)$ GUT, or (ii) add the GUT-Higgs potential to Higgs down the Spin(10) deforming it to G_{GG}, G_{PS}, and G_{SM}.

(2) Without or with the WZW term $S^{\text{WZW}}[\Phi] = \pi \int_{M^5} B(\tilde{\Phi}^{\text{bi}}) \smile \delta B'(\hat{\Phi}^{\text{bi}})$ of Eq. (3.15): Namely (i) whether we stay in model I—an $so(10)$ GUT without the $w_2 w_3$ anomaly—or (ii) model II—a modified so(10) GUT + WZW matches the $w_2 w_3$ anomaly.

(3) Without or with the dynamically gauged internal symmetry group $G = G_{\text{internal}}$: (i) whether we keep the $[G_{\text{internal}}]$ symmetry as a global symmetry, or (ii) we gauge the $[G_{\text{internal}}]$,[35] namely gauging $[\mathrm{Spin}(10)]$, $[G_{\text{GG}}]$, $[G_{\text{PS}}]$, and $[G_{\text{SM}}]$.

The three binary conditions enumerate totally eight possibilities (where below we can use 3-bits, "???", each bit "?" labels a "x" or "o" to specify without or with that binary condition holds), which we enlist their physics interpretations, one by one:

(1) xxx—Without $\mathrm{U}(\Phi_{\mathbf{R}})$, without WZW, without gauged $[G_{\text{internal}}]$: We stay in the Landau-Ginzburg phase of the Spin(10) global symmetry.

(2) oxx—With $\mathrm{U}(\Phi_{\mathbf{R}})$, without WZW, without gauged $[G_{\text{internal}}]$: We stay in the Landau-Ginzburg phases, but the $\mathrm{U}(\Phi_{\mathbf{R}})$ potentially breaks the Spin(10) global symmetry to other continuous Lie group global symmetries G_{GG}, G_{PS}, and G_{SM}, via spontaneous global symmetry breaking. There are 45, 24, 21, and 12 Lie algebra generators for each of these groups. So there are corresponding numbers of the low energy Nambu-Goldstone modes, matching the number of the broken Lie algebra generators based on the Goldstone's theorem. In principle,

[35]We may use the bracket notation on a group $[G_{\text{internal}}]$ to emphasize that group is dynamically gauged.

TABLE II. Quantum field representations (reps) for two toy models. Model I contains the Weyl spacetime-spinor ψ, the Spin(10) gauge field A (45 Lie algebra generators denoted as $45_{\text{adj.}}$, but not the **45** rep), the SO(10)-bivector spacetime-scalar Φ^{bi}, and the SO(10)-vector spacetime-scalar ϕ as an auxiliary field (Lagrange multiplier with no dynamics). Model II contains all the field contents of model I, in addition, model II contains extra fields: the 4D WZW term $\pi \int_{M^5} B(\tilde{\Phi}^{\text{bi}}) \smile \delta B'(\hat{\Phi}^{\text{bi}})$ lives on the boundary of a 5D bulk can induce a candidate low energy QED_4' theory with a Dirac spacetime-spinor ξ (as a fermionic parton) and a $\text{U}(1)'$ emergent dark gauge field a [1 Lie algebra generator denoted as $1_{\text{adj.}}$, which carries no $\text{U}(1)'$ charge]. The rep of fermionic parton ξ in $su(3) \times su(2) \times u(1)_{\tilde{Y}} \times u(1)_X$ is given in Table I. There are two types of fermion parities in a double spin structure $\text{DSpin} \equiv (\mathbb{Z}_2^F \times \mathbb{Z}_2^{F'}) \rtimes \text{SO}$.

Field content	Spin ≡ Spin(1,3)	Spin(10)	$\mathbb{Z}_2^F$	$\mathbb{Z}_2^{F'}$	$\text{U}(1)'^{\text{dark}}_{\text{gauge}}$
		Model I			
ψ	$\mathbf{2}_L$	**16**	1	0	0
A	**4**	$45_{\text{adj.}}$	0	0	0
$\Phi^{\text{bi}} = \Phi_{\mathbf{1}} \oplus \hat{\Phi}^{\text{bi}} \oplus \tilde{\Phi}^{\text{bi}}$	**1**	$\mathbf{10} \otimes \mathbf{10} = \mathbf{100} = \mathbf{1} \oplus \mathbf{45} \oplus \mathbf{54}$	0	0	0
ϕ	**1**	**10**	0	0	0
		Model II (include model I's above + extra below)			
ξ	$\mathbf{2}_L \oplus \mathbf{2}_R$	**10**	0	1	1
a	**4**	**1**	0	0	$1_{\text{adj.}}$

because there is no 't Hooft anomaly for the 16n chiral fermions with these G_{internal} internal global symmetries, we can gap out all chiral fermions while preserving G_{internal} via a symmetric mass generation through appropriate interactions [12,13].

(3) xxo—Without $\text{U}(\Phi_{\mathbf{R}})$, without WZW, with gauged $[G_{\text{internal}}]$: We obtain the familiar $so(10)$ GUT with the [Spin(10)] gauged. At a deep UV higher energy, there shows the asymptotic freedom of 16n Weyl fermions (quarks and leptons are liberated with a weaker coupling at a shorter distance for such a non-Abelian Lie group gauge force [34,35]). At an IR lower energy, the Spin(10) gauge fields confine the 16n Weyl fermions, which is a strongly coupled gauge theory with all fermions can gain an energy gap (i.e., "mass" due to the confinement).

(4) oxo—With $\text{U}(\Phi_{\mathbf{R}})$, without WZW, with gauged $[G_{\text{internal}}]$: Then we are in the dynamical gauge theory phases but with gauge symmetry breaking. The $\text{U}(\Phi_{\mathbf{R}})$ potentially breaks the Spin(10) gauge group to other continuous Lie gauge group G_{GG}, G_{PS}, and G_{SM}, via Anderson-Higgs mechanism of spontaneous gauge symmetry breaking. There are 45, 24, 21, and 12 Lie algebra generators for each of these groups. Recall in the global symmetry story, there are corresponding numbers of the low energy Nambu-Goldstone modes, matching the number of the broken Lie algebra generators based on the Goldstone's theorem. But now some massless gauge fields can "eat" the degrees of freedom of Goldstone bosons, so to become the massive gauge field with extra degrees of freedom.

Note that again, at a deep UV higher energy, there shows the asymptotic freedom of Weyl fermions; while at an IR lower energy, the non-Abelian Lie gauge forces of G_{GG}, G_{PS}, and G_{SM}) can confine some of the Weyl fermions. In this strongly coupled gauge theory, some fermions can gain an energy gap (i.e., "mass") due to the confinement. But we do still have the electroweak-Higgs causing spontaneous gauge symmetry breaking $su(2)_{\text{L}} \times u(1)_Y \to u(1)_{\text{EM}}$. The $u(1)_{\text{EM}}$ stays deconfined and propagate the gapless electromagnetic waves in our vacuum.

Here the fermion mass can come from a combination of mechanism from the confinement mass, the Anderson-Higgs (gauge-) symmetry-breaking mass, or the gauge theory analog of the symmetric mass generation.

(5) xox—Without $\text{U}(\Phi_{\mathbf{R}})$, with WZW, without gauged $[G_{\text{internal}}]$: We stay in the Landau-Ginzburg phase of the Spin(10) global symmetry, but the 4D WZW term causes the 4D DQC with fractionalized fragmentary excitations.

This DQC is also a GEQC because we have a new gauge force [which we call dark gauge force with $\text{U}(1)'^{\text{dark}}_{\text{gauge}}$ dark photons] emergent near the criticality. The fractionalized fragmentary excitations carry the $\text{U}(1)'^{\text{dark}}_{\text{gauge}}$ gauge charge. If the $\text{U}(1)'^{\text{dark}}_{\text{gauge}}$ dark photons stay gapless dynamically at deep IR, then it is due to the protection of $w_2 w_3$ anomaly.

(6) oox—With $\text{U}(\Phi_{\mathbf{R}})$, with WZW, without gauged $[G_{\text{internal}}]$: We stay in the Landau-Ginzburg phases, but the $\text{U}(\Phi_{\mathbf{R}})$ potentially breaks the Spin(10) global symmetry to other continuous Lie group global symmetries G_{GG}, G_{PS}, and G_{SM}, via spontaneous global symmetry breaking. Other than the low energy Nambu-Goldstone modes matching the number of the broken Lie algebra generators in the

neighbor phases, we still have the fractionalized fragmentary excitations that also carries $\mathrm{U}(1)'^{\mathrm{dark}}_{\mathrm{gauge}}$ gauge charge, with $\mathrm{U}(1)'^{\mathrm{dark}}_{\mathrm{gauge}}$ dark photons.

(7) xoo—Without $\mathrm{U}(\Phi_{\mathbf{R}})$, with WZW, with gauged $[G_{\mathrm{internal}}]$: We obtain the modified $so(10)$ GUT + WZW with the [Spin(10)] gauged. At a deep UV higher energy, the GUT-Higgs potential + WZW term may affect the renormalizability of EFT; however, what we concern is the EFT that works below certain energy cutoff scale such as GUT scale M_{GUT} or the 5D bulk invertible TQFT energy gap Δ_{iTQFT}. Other than the DQC and GEQC phenomena described above in scenario xox, the theory shows the following:
 (a) The Spin(10) gauge bosons can propagate or leak to the 5D bulk.
 (b) The 16n Weyl fermions are gappable (because there is no anomaly protection for these 16n fermions).
 (c) We have again the 10 fractionalized fragmentary fermions, gauge charged under $\mathrm{U}(1)'^{\mathrm{dark}}_{\mathrm{gauge}}$ dark photon. Furthermore, the 10 fractionalized fragmentary fermions carry also the strong $\mathrm{SU}(3)_c$ gauge charge, and the weak $\mathrm{SU}(2)_{\mathrm{L}}$ gauge charge, recall from Table I.
 (d) Here we are doing the fragmentary GUT-Higgs liquid model beyond the SM (with 10 fractionalized fragmentary fermions coupled to $\mathrm{U}(1)'^{\mathrm{dark}}_{\mathrm{gauge}}$ dark photon) of Sec. III D that can match the w_2w_3 anomaly. In contrast, we are not thinking of the 10 gauge neutral bosons from composite GUT-Higgs model within the SM of Sec. III C that does not have the w_2w_3 anomaly.

(8) ooo—With $\mathrm{U}(\Phi_{\mathbf{R}})$, with WZW, with gauged $[G_{\mathrm{internal}}]$: This scenario follows directly from the scenario xoo, but with a GUT-Higgs potential triggering (gauge) symmetry breaking. All statements in scenario xoo follow also here. Moreover,
 (a) There is a sequence of various possibilities at various energy scales from the UV to the IR dynamical fates of this QFT. We do not know the definite answer of quantum dynamics. Here we only enlist the possibilities of quantum dynamical fates of the modified $so(10)$ GUT + 4D WZW term (with 16n Weyl fermions) based on the w_2w_3 anomaly matching constraints:
  (i) Spin(10) gauge group can be broken down to contain an SU(2) gauge subgroup such that there is a new SU(2) anomaly of mixed gauge-gravity type $w_2w_3(TM) = w_2w_3(V_{\mathrm{SO}(3)})$ within the $\mathrm{Spin}\times_{\mathbb{Z}_2^F}\mathrm{SU}(2)\equiv\mathrm{Spin}^h$ symmetry [18], again dynamically gauging SU(2) makes the SU(2) gauge bosons can propagate to the 5D bulk.
  (ii) The gauge group can be broken down to contain a U(1) gauge subgroup, which can also have a pure gravitational $w_2w_3(TM)$ anomaly if the theory is all-fermion U(1) gauge theory [20,21]. The $\mathrm{Spin}\times_{\mathbb{Z}_2^F}\mathrm{U}(1)\equiv\mathrm{Spin}^c$ structure trivializes the $w_2w_3(TM)$ anomaly.
  (iii) The gauge group can be broken down to contain a $\mathbb{Z}_2$ gauge subgroup which can also have a pure gravitational $w_2w_3(TM)$ anomaly if the theory has fermionic strings [19,76–78]. The Spin structure trivializes the $w_2w_3(TM)$ anomaly.
 (b) However, the WZW dynamics in the quantum critical region that we propose in Sec. III D 2 shows none of the above. Instead, we suggest a different IR low energy fate of WZW theory: The Spin(10) symmetry can be fully preserved, while the mixed gauge-gravity anomaly $w_2w_3(TM) = w_2w_3(V_{\mathrm{SO}(10)})$ is matched by a Dirac fermionic parton theory QED_4' with emergent $\mathrm{U}(1)'$ dark gauge force and with a DSpin structure. Figure 11 shows a schematic phase diagram. For model I, without a WZW term, there is no deconfined QED_4' within the dashed circle region. For model II, with a WZW term, there is a deconfined QED_4' within the dashed circle region.

4D boundary criticality and a 5D bulk bosonic invertible TQFT.—Notice that we can interpret the above 4D criticality as a boundary

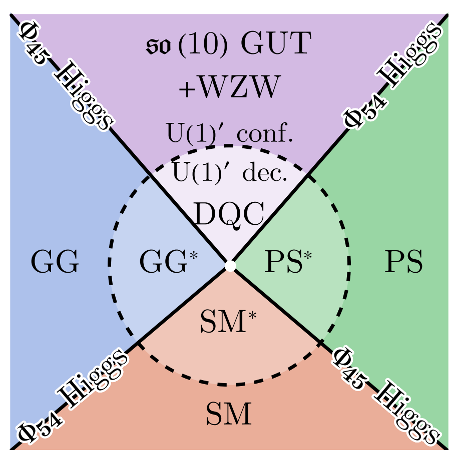

FIG. 11. Follow Fig. 8, here we show the same phase diagram in the presence of the WZW term if its low energy consequence is the fermionic parton theory QED_4' (Sec. III D 2). The dashed circle denotes the confine-deconfine phase transition of the emergent $\mathrm{U}(1)'$ gauge field. The solid-line phase boundaries between two neighbor phases all are described by GUT-Higgs condensation continuous phase transitions. The SM^* phase means a modification of SM plus additional BSM fields due to QED_4', within the $\mathrm{U}(1)'$ deconfined region inside the dashed circle. Similar situations for GG^* and PS^*.

criticality with the $w_2 w_3$ anomaly on the 5D bulk of a mod 2 class invertible TQFT. The 4D WZW, that can be built from the GUT-Higgs fields, can saturate 4D $w_2 w_3(TM) = w_2 w_3(V_{\mathrm{SO}(10)})$ anomaly. So we only require the 5D bulk as some 5D invertible topological order or symmetry-protected topological states if we require an on site Spin(10) symmetry on the 4D boundary and on the 5D bulk; see an overview of modern quantum matter terminology and definitions in [79,80].

Bosonic UV completion.—For this 16n Weyl fermion models, once the $[\mathrm{Spin}(10)] \supset [\mathbb{Z}_2^F]$ is dynamically gauged, the whole UV completion of the full 4D and 5D system requires only the bosons, as the local on site Hilbert space with gauge-invariant bosonic operators.

Although above we focus on the 16n-Weyl-fermion SMs or GUTs, we can consider the 15n-Weyl-fermion models, especially for the $su(5)$ GUT and the SM + 4D WZW term, see Sec. IV B.

B. 16n vs 15n Weyl fermions: Give "mass" to "right-handed sterile" neutrinos, canceling mod 2 and mod 16 anomalies, and topological quantum criticality

Although we mostly focus on the 16n-Weyl-fermion SMs or GUTs in this work, here we comment about several ways to obtain the low-energy 15n-Weyl-fermion models (since the real-world experiments only observed the 15n-Weyl-fermion so far) by giving a large mass to the 16th Weyl fermions, the so-called "right-handed sterile" neutrinos (in any of the three generations of leptons).[36]

What are examples of conventional ways [43] to give a large (Anderson-Higgs type quadratic) mass to the 16th Weyl fermions? We can pair Weyl fermion to itself (i.e., Majorana mass) or to another Weyl fermion (e.g., Dirac mass):

(1) Introduce a Higgs $\Phi_{so(10),\mathbf{126}}$ which can be paired with $\overline{\mathbf{126}}$ out of two Weyl fermions in $\mathbf{16} \otimes \mathbf{16} = \mathbf{10} \oplus \mathbf{120} \oplus \overline{\mathbf{126}}$.

(2) Introduce a Higgs $\Phi_{so(10),\mathbf{16}}$ and add an extra Weyl fermion (17th Weyl fermion) singlet **1** under Spin (10). This works only if some of the following holds:

  (i) The 17th Weyl fermion is not charged under the $\mathbb{Z}_{4,X}$ symmetry, so we have the $\mathbb{Z}_{16}$ anomaly canceled already by 16n Weyl fermions. This is likely to be true because this 17th Weyl fermion is singlet **1** under Spin(10), thus is also not acted by the center $Z(\mathrm{Spin}(10)) = \mathbb{Z}_{4,X}$.

  (ii) If the 17th Weyl fermion is also charged under the $\mathbb{Z}_{4,X}$ symmetry, then we require the $\mathbb{Z}_{4,X}$ symmetry is broken (thus the $\mathbb{Z}_{16}$ anomaly is removed), or the $\mathbb{Z}_{4,X}$ symmetry is preserved but 17 mod 16 anomaly is canceled again by additional new sectors with -1 mod 16 anomaly.

What are other new ways to leave only the observed 15n Weyl fermions at low energy, but the $\mathbb{Z}_{16}$ global anomaly can still be canceled in the full quantum system? To begin with, to characterize the full 4D anomaly of this 15n SMs or GUTs, we should combine the two types of anomalies: First, a potential global $\mathbb{Z}_2$ anomaly, the $w_2 w_3$ for our 4D WZW term, such as in the fragmentary GUT-Higgs liquid model in Sec. III D. Second, the $\mathbb{Z}_{16}$ global anomaly captured by a 5D version of Atiyah-Patodi-Singer (APS) eta invariant for the $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ structure from $\mathrm{TP}_5(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}) = \mathbb{Z}_{16}$. We can write that 5D APS invariant in terms of the 4D APS invariant of Pin^+ structure from $\mathrm{TP}_4(\mathrm{Pin}^+) = \mathbb{Z}_{16}$. The two combined invertible TQFT, labeled by $\mathrm{p} \in \mathbb{Z}_2$ and $\nu \in \mathbb{Z}_{16}$, has a partition function $\mathbf{Z}$ on M^5, which together labels a deformation class of SM [16]:

$$\mathbf{Z}_{\text{5d-iTQFT}}^{(\mathrm{p},\nu)} \equiv \exp\left(\mathrm{i}\pi \cdot \mathrm{p} \cdot \int_{M^5} w_2 w_3\right) \cdot \exp\left(\frac{2\pi \mathrm{i}}{16} \cdot \nu \cdot \eta(\mathrm{PD}(\mathcal{A}_{\mathbb{Z}_2}))\Big|_{M^5}\right),$$
$$\text{with} \quad \mathrm{p} \in \mathbb{Z}_2, \text{a 4D Atiyah-Patodi-Singer}\ \eta\ \text{invariant} \equiv \eta_{\mathrm{Pin}^+} \in \mathbb{Z}_{16}, \quad \nu \in \mathbb{Z}_{16}. \tag{4.1}$$

The cohomology classes of background gauge field $\mathcal{A}_{\mathbb{Z}_2} \in H^1(M, \frac{\mathbb{Z}_{4,X}}{\mathbb{Z}_2^F})$ is defined on a $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$-manifold M obeys a constraint: $w_2(TM) = \mathcal{A}_{\mathbb{Z}_2}^2$.

Inspired by highly entangled interacting quantum matter recent developments (see reviews in [79,80]), Refs. [38–40] proposed additional new sectors to cancel the anomalies, for example,

(3) Symmetry-preserving anomalous gapped 4D TQFT.

(4) Symmetric-preserving 5D invertible TQFT in the extra dimension.

(5) Symmetry-breaking gapped phase of Landau-Ginzburg kinds.

[36] Note that the "right-handed sterile ν_R" neutrino is just the conventional name used in the HEP phenomenology. We would mostly write this ν_R in the left-handed Weyl fermion basis. Also the ν_R although is sterile to the G_{SM} and SU(5), the ν_R is not sterile to Spin(10) and $\mathbb{Z}_{4,X}$.

(6) Symmetry-preserving (or breaking) 5D topological gravity theory.
(7) Symmetry-preserving or symmetry-breaking gapless phase, e.g., extra massless theories, free or interacting CFTs. The interacting CFT can also be related to unparticle physics [81] in the high-energy phenomenology community.

The heavy gapped new sectors above can be heavy dark matter candidates. The interesting constraints from mod 2 and mod 16 global anomalies on our 4D DQC model are as follows:

(1) $\mathbb{Z}_{16}$ anomaly constraints on the GG and SM of 15n Weyl fermions: On the Georgi-Glashow $su(5)$ GUT and the Standard Model $\mathrm{SM}_{q=6}$ side, we can have 15n Weyl fermions, plus additional new sectors enlisted (above and in [38–40]) to match the $\mathbb{Z}_{16}$ anomaly.
(2) $\mathbb{Z}_2$ $w_2 w_3$ anomaly constraints on the $so(10)$ GUT and PS of 16n Weyl fermions: On the $so(10)$ GUT and the Pati-Salam model sides, there are various types of $\mathbb{Z}_2$ class $w_2 w_3$ anomalies, of the SO(10), SO(6), or SO(4) bundles. The $\mathbb{Z}_2$ $w_2 w_3$ anomaly is meant to be canceled by our 4D WZW term.
(3) At the vicinity of the 4D DQC we have proposed, there can be another interplay between the 15n Weyl fermions (GG and SM) to 16n Weyl fermions [the $so(10)$ GUT and PS], such that the DQC becomes a topological quantum phase transition or topological quantum criticality.

4D boundary criticality to a 5D bulk criticality.— Compare with the phase diagram in Fig. 8. Notice that we can interpret the above 4D criticality as a boundary criticality:

(1) On the modified $so(10)$ GUT and the PS model + WZW term side with 16n Weyl fermions in Fig. 8: with the $w_2 w_3$ $\mathbb{Z}_2$-class anomaly on the 5D bulk of a mod 2 class invertible TQFT.
(2) On the modified $su(5)$ GUT and the SM + WZW term side with 15n Weyl fermions in Fig. 8: with the $\eta(\mathrm{PD}(\mathcal{A}_{\mathbb{Z}_2}))\mathbb{Z}_{16}$-class anomaly on the 5D bulk of a mod 2 class invertible TQFT.

Once the [Spin(10)] is dynamically gauged,

(1) The 5D bulk on the modified $so(10)$ GUT and the PS model side (16n Weyl fermions): The [Spin(10)] dynamical gauge fields can propagate and leak to the 5D bulk are deconfined and gapless.
(2) The 5D bulk on the modified $su(5)$ GUT and the SM side (15n Weyl fermions): Only the $[\mathbb{Z}_{4,X}]$ subgroup ($Z(\mathrm{Spin}(10)) = \mathbb{Z}_{4,X}$) are dynamically gauged in the 5D bulk of the original fermionic invertible TQFT $\eta(\mathrm{PD}(\mathcal{A}_{\mathbb{Z}_2}))$. Gauging $[\mathbb{Z}_{4,X}]$ turns the 5D fermionic bulk to a 5D bosonic bulk TQFT (with long-range entanglement, gapped topological order, and described by gauged cohomology, gauged cobordism, or higher category theory). The 5D bulk can remain to be gapped.

Thus there is a phase transition between the deconfined and gapless 5D bulk to another side of gapped 5D bulk. This phase transition can be interpreted as a 5D bulk topological quantum criticality.

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APPENDIX A: QUANTUM NUMBERS AND REPRESENTATIONS OF SMS AND GUTS IN TABLES

Here we summarize the representations of "elementary" chiral fermionic particles of quarks and leptons of SMs and GUTs in the following tables.

Spacetime symmetry representation: Here Weyl fermions are spacetime Weyl spinors, which we prefer to write all Weyl fermions as

$$\mathbf{2}_L \text{ of } \mathrm{Spin}(1,3) = \mathrm{SL}(2,\mathbb{C}) \tag{A1}$$

with a complex representation in the 4D Lorentz signature. On the other hand, the Weyl spinor is

$$\mathbf{2}_L \text{ of } \mathrm{Spin}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \tag{A2}$$

with a pseudoreal representation in the 4D Euclidean signature.

Internal symmetry representation: Below we provide Tables III and IV to organize the internal symmetry representations of particle contents of the SM, the $su(5)$ GUT, the Pati-Salam model, and the $so(10)$ GUT.

1. Embed the SM into the $su(5)$ GUT, then into the $so(10)$ GUT

There is a QFT embedding, the $so(10)$ GUT ⊃ the $su(5)$ GUT ⊃ the SM_6 only for $G_{\mathrm{SM}_{q=6}}$ via an internal symmetry group embedding:

$$\begin{aligned}\mathrm{Spin}(10) &\supset G_{\mathrm{GG}} \equiv \mathrm{SU}(5)\\ &\supset G_{\mathrm{SM}_6} \equiv \frac{\mathrm{SU}(3)_c \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\tilde{Y}}}{\mathbb{Z}_6}.\end{aligned} \tag{A3}$$

The representations of quarks and leptons for these models are organized in Table III. There are two versions of

TABLE III. Embed the $su(3) \times su(2) \times u(1)$ SM into the Georgi-Glashow $su(5)$ GUT, then into the $so(10)$ GUT. We show the quantum numbers of $15 + 1 = 16$ left-handed Weyl fermion [spacetime spinors $\mathbf{2}_L$ in $\mathrm{Spin}(1,3)$] in each of three generations of matter fields in SM. The 15 of 16 Weyl fermion are $\bar{\mathbf{5}} \oplus \mathbf{10}$ of SU(5); namely, $(\bar{\mathbf{3}}, \mathbf{1}, 1/3)_L \oplus (\mathbf{1}, \mathbf{2}, -1/2)_L \sim \bar{\mathbf{5}}$ and $(\mathbf{3}, \mathbf{2}, 1/6)_L \oplus (\bar{\mathbf{3}}, \mathbf{1}, -2/3)_L \oplus (\mathbf{1}, \mathbf{1}, 1)_L \sim \mathbf{10}$ of SU(5). The 1 of 16 is presented neither in the standard GSW SM nor in the $su(5)$ GUT, but it is within **16** of the $so(10)$ GUT. The numbers in the table entries indicate the quantum numbers associated with the representation of the groups given in the top row. We show a generation of SM fermion matter fields in Table III. There are three generations, triplicating Table III, in SM. All fermions have the fermion parity $\mathbb{Z}_2^F$ representation charge of 1. In the $su(5)$ GUT, by including the $\mathrm{U}(1)_X$, we have the $(\mathrm{SU}(5) \times \mathrm{U}(1)_X)/\mathbb{Z}_5 = \mathrm{U}(5)_{\hat{q}=2}$ structure described in [62,63]. Here $\mathrm{U}(1)_X \supset \mathbb{Z}_{4,X} \supset \mathbb{Z}_2^F$ and $\mathrm{SU}(5) \supset \mathrm{U}(1)_Y$. Both $\mathrm{U}(1)_X$ and $\mathrm{U}(1)_{\mathbf{B-L}}$ are outside the SU(5).

SM fermion spinor field	SU(3)	SU(2)	$\mathrm{U}(1)_Y$	$\mathrm{U}(1)_{\tilde{Y}}$	$\mathrm{U}(1)_{\mathrm{EM}}$	$\mathrm{U}(1)_{\mathbf{B-L}}$	$\mathrm{U}(1)_X$	$\mathbb{Z}_{5,X}$	$\mathbb{Z}_{4,X}$	$\mathbb{Z}_2^F$	SU(5)	Spin(10)
$\bar{d}_R$	$\bar{\mathbf{3}}$	$\mathbf{1}$	1/3	2	1/3	−1/3	−3	−3	1	1	$\bar{\mathbf{5}}$	**16**
l_L	$\mathbf{1}$	$\mathbf{2}$	−1/2	−3	0 or −1	−1	−3	−3	1	1		
q_L	$\mathbf{3}$	$\mathbf{2}$	1/6	1	2/3 or −1/3	1/3	1	1	1	1	**10**	
$\bar{u}_R$	$\bar{\mathbf{3}}$	$\mathbf{1}$	−2/3	−4	−2/3	−1/3	1	1	1	1		
$\bar{e}_R = e_L^+$	$\mathbf{1}$	$\mathbf{1}$	1	6	1	1	1	1	1	1		
$\bar{\nu}_R = \nu_L$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	1	5	0	1	1	**1**	

TABLE IV. Embed the $su(3) \times su(2) \times u(1)$ SM into the Pati-Salam model $su(4) \times su(2) \times su(2)$, then into the $so(10)$ GUT. We have $T_{3,\mathrm{L}} + Y = Q_{\mathrm{EM}}$, the Lie algebra linear combination $\mathrm{SU}(2)_{\mathrm{L}}$ (the third generator) and $\mathrm{U}(1)_Y$ gives the $\mathrm{U}(1)_{\mathrm{EM}}$ charge. We have $T_{3,\mathrm{R}} + Y = \frac{\mathbf{B-L}}{2}$, the Lie algebra linear combination of $\mathrm{SU}(2)_{\mathrm{R}}$ (the third generator) and $\mathrm{U}(1)_Y$ gives the $\mathrm{U}(1)_{\mathbf{B-L}}$. We choose the right-handed antiparticle to be in **2** of $\mathrm{SU}(2)_{\mathrm{R}}$ [so its right-handed particle to be in $\bar{\mathbf{2}}$ of $\mathrm{SU}(2)_{\mathrm{R}}$] that makes a specific assignment on the ± sign of its $T_{3,\mathrm{R}}$ charge. So we have the formula, $T_{3,\mathrm{L}} - T_{3,\mathrm{R}} = Q_{\mathrm{EM}} - \frac{\mathbf{B-L}}{2}$.

SM fermion spinor field	SU(3)	$\mathrm{SU}(2)_{\mathrm{L}}$	$\mathrm{SU}(2)_{\mathrm{R}}$	$\mathrm{U}(1)_{\frac{\mathbf{B-L}}{2}}$	$\mathrm{U}(1)_Y$	$\mathrm{U}(1)_{Y_R}$	$\mathrm{U}(1)_{\mathrm{EM}}$	$\mathrm{U}(1)_X$	$\mathbb{Z}_{4,X}$	$\mathbb{Z}_2^F$	Spin(10)
u_L	$\mathbf{3}$	q_L:**2**	$\mathbf{1}$	1/6	1/6	2/3	2/3	1	1	1	**16**
d_L	$\mathbf{3}$		$\mathbf{1}$	1/6	1/6	−1/3	−1/3	1	1	1	
ν_L	$\mathbf{1}$	l_L:**2**	$\mathbf{1}$	−1/2	−1/2	0	0	−3	1	1	
e_L	$\mathbf{1}$		$\mathbf{1}$	−1/2	−1/2	−1	−1	−3	1	1	
$\bar{u}_R$	$\bar{\mathbf{3}}$	$\mathbf{1}$	q_R:**2**	−1/6	−2/3	−1/6	−2/3	1	1	1	
$\bar{d}_R$	$\bar{\mathbf{3}}$	$\mathbf{1}$		−1/6	1/3	−1/6	1/3	−3	1	1	
$\bar{\nu}_R = \nu_L$	$\mathbf{1}$	$\mathbf{1}$	l_R:**2**	1/2	0	1/2	0	5	1	1	
$\bar{e}_R = e_L^+$	$\mathbf{1}$	$\mathbf{1}$		1/2	1	1/2	1	1	1	1	

electroweak hypercharge normalization listed in Table III, such that the charge of $\mathrm{U}(1)_Y$ is $\frac{1}{6}$ of the charge of $\mathrm{U}(1)_{\tilde{Y}}$.

2. Embed the SM into the left-right and Pati-Salam models and into the $so(10)$ GUT

There are two version of internal symmetry groups for PS model [6]:

$$G_{\mathrm{PS}_{q'}} \equiv \frac{\mathrm{SU}(4) \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}}{\mathbb{Z}_{q'}} \equiv \frac{\mathrm{Spin}(6) \times \mathrm{Spin}(4)}{\mathbb{Z}_{q'}},$$

with $q' = 1, 2$. There are two version of internal symmetry groups for Senjanovic-Mohapatra's left-right (LR) model [82],

$$G_{\mathrm{LR}_{q'}} \equiv \frac{\mathrm{SU}(3)_c \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{U}(1)_{\frac{\mathbf{B-L}}{2}}}{\mathbb{Z}_{3q'}}$$

with $q' = 1, 2$. In general, there is a QFT embedding, the PS model ⊃ the LR model ⊃ the SM for both $q' = 1, 2$ via the internal symmetry group embedding:

$$G_{\mathrm{PS}_{q'}} \supset G_{\mathrm{LR}_{q'}} \supset G_{\mathrm{SM}_{q=3q'}} \equiv \frac{\mathrm{SU}(3)_c \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\tilde{Y}}}{\mathbb{Z}_{q=3q'}}. \tag{A4}$$

Namely, when $q' = 1$, we have

$$G_{\mathrm{PS}_1} \supset G_{\mathrm{LR}_1} \supset G_{\mathrm{SM}_3}. \tag{A5}$$

Furthermore, only when $q' = 2$, we can have the whole embedded into the Spin(10) for the $so(10)$ GUT:

$$\mathrm{Spin}(10) \supset G_{\mathrm{PS}_2} \supset G_{\mathrm{LR}_2} \supset G_{\mathrm{SM}_6}. \tag{A6}$$

The representations of quarks and leptons for these models are organized in Table IV.

APPENDIX B: REPRESENTATION AND BRANCHING RULE FOR GUT-HIGGS SYMMETRY BREAKING

Here are we organize the set of branching rules of representations following the symmetry breaking pattern of various GUTs to SM (these rules are used in Sec. II A):

(1) $\mathrm{Spin}(10) \hookleftarrow \mathrm{SU}(5) \hookleftarrow \frac{\mathrm{SU}(3)_c\times\mathrm{SU}(2)_\mathrm{L}\times\mathrm{U}(1)_Y}{\mathbb{Z}_6}$ branching rules:

(a) For $\mathrm{Spin}(10) \hookleftarrow \mathrm{SU}(5)$, also for $\mathrm{SO}(10) \hookleftarrow \mathrm{U}(5)_{\hat{q}=1} = \frac{\mathrm{SU}(5)\times\mathrm{U}(1)_{\hat{q}=1}}{\mathbb{Z}_5}$ or $\mathrm{Spin}(10) \hookleftarrow \mathrm{U}(5)_{\hat{q}=2} = \frac{\mathrm{SU}(5)\times\mathrm{U}(1)_{\hat{q}=2}}{\mathbb{Z}_5}$ (or in terms of Lie algebra $so(10) \hookleftarrow su(5)\times u(1)$ with a regular Lie subalgebra in [58]),[37] the branching rule says

$$\begin{cases} \mathbf{10} \sim \mathbf{5} \oplus \bar{\mathbf{5}} & \text{or} \quad \mathbf{10} \sim \mathbf{5}_2 \oplus \bar{\mathbf{5}}_{-2} \\ \mathbf{16} \sim \mathbf{1} \oplus \bar{\mathbf{5}} \oplus \mathbf{10} & \text{or} \quad \mathbf{16} \sim \mathbf{1}_{-5} \oplus \bar{\mathbf{5}}_3 \oplus \mathbf{10}_{-1} \\ \mathbf{45} \sim \mathbf{1} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{24} & \text{or} \quad \mathbf{45} \sim \mathbf{1}_0 \oplus \mathbf{10}_4 \oplus \overline{\mathbf{10}}_{-4} \oplus \mathbf{24}_0 \\ \mathbf{54} \sim \mathbf{15} \oplus \overline{\mathbf{15}} \oplus \mathbf{24} & \text{or} \quad \mathbf{54} \sim \mathbf{15}_4 \oplus \overline{\mathbf{15}}_{-4} \oplus \mathbf{24}_0 \\ \mathbf{120} \sim \mathbf{5} \oplus \bar{\mathbf{5}} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{45} \oplus \overline{\mathbf{45}} & \text{or} \quad \mathbf{5}_2 \oplus \bar{\mathbf{5}}_{-2} \oplus \mathbf{10}_{-6} \oplus \overline{\mathbf{10}}_6 \oplus \mathbf{45}_2 \oplus \overline{\mathbf{45}}_{-2} \\ \mathbf{126} \sim \mathbf{1} \oplus \mathbf{5} \oplus \overline{\mathbf{10}} \oplus \mathbf{15} \oplus \overline{\mathbf{45}} \oplus \mathbf{50} & \text{or} \quad \mathbf{1}_{10} \oplus \mathbf{5}_2 \oplus \overline{\mathbf{10}}_6 \oplus \mathbf{15}_{-6} \oplus \overline{\mathbf{45}}_{-2} \oplus \mathbf{50}_2 \end{cases}. \tag{B1}$$

(b) For $\mathrm{SU}(5) \hookleftarrow \frac{\mathrm{SU}(3)_c\times\mathrm{SU}(2)_\mathrm{L}\times\mathrm{U}(1)_Y}{\mathbb{Z}_6}$ [or in terms of Lie algebra $su(5) \hookleftarrow su(3)\times su(2)\times u(1)$ with a regular Lie subalgebra in [58]], the branching rule says

$$\begin{cases} \mathbf{5} \sim (\mathbf{1},\mathbf{2})_{-3} \oplus (\mathbf{3},\mathbf{1})_2 \\ \mathbf{10} \sim (\mathbf{1},\mathbf{1})_{-6} \oplus (\bar{\mathbf{3}},\mathbf{1})_4 \oplus (\mathbf{3},\mathbf{2})_{-1} \\ \mathbf{15} \sim (\mathbf{1},\mathbf{3})_{-6} \oplus (\mathbf{3},\mathbf{2})_{-1} \oplus (\mathbf{6},\mathbf{1})_4 \\ \mathbf{24} \sim (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{3},\mathbf{2})_5 \oplus (\bar{\mathbf{3}},\mathbf{2})_{-5} \oplus (\mathbf{8},\mathbf{1})_0 \\ \dots \\ \mathbf{45} \sim (\mathbf{1},\mathbf{2})_{-3} \oplus (\mathbf{3},\mathbf{1})_2 \oplus (\bar{\mathbf{3}},\mathbf{1})_{-8} \oplus (\bar{\mathbf{3}},\mathbf{2})_7 \oplus (\mathbf{3},\mathbf{3})_2 \oplus (\bar{\mathbf{6}},\mathbf{1})_2 \oplus (\mathbf{8},\mathbf{2})_{-3} \\ \mathbf{50} \sim (\mathbf{1},\mathbf{1})_{12} \oplus (\mathbf{3},\mathbf{1})_2 \oplus (\bar{\mathbf{3}},\mathbf{2})_7 \oplus (\mathbf{6},\mathbf{1})_{-8} \oplus (\bar{\mathbf{6}},\mathbf{3})_2 \oplus (\mathbf{8},\mathbf{2})_{-3} \end{cases}. \tag{B2}$$

(i) First, in order to break the Spin(10) or SO(10) down to SU(5), we take the representation whose branching rule in (B1) contains the **1** of SU(5) or $\mathbf{1}_0$ of U(5) on the right-handed side so that SU(5) or U(5) is left unbroken. This means that we may take a GUT-Higgs **45** that we name it as (2.10):

$$\Phi_{so(10),\mathbf{45}} \equiv \Phi_{\mathbf{45}}. \tag{B3}$$

(ii) Second, in order to break SU(5) further down to $G_{\mathrm{SM}_6} \equiv \frac{\mathrm{SU}(3)_c\times\mathrm{SU}(2)_\mathrm{L}\times\mathrm{U}(1)_Y}{\mathbb{Z}_6}$, we take the representation whose branching rule in (B2) contains the $(\mathbf{1},\mathbf{1})_0$ of G_{SM_6}. This means that we can take the **24** of SU(5) as the second GUT-Higgs called $\Phi_{su(5),\mathbf{24}}$. But if we want to obtain this second GUT-Higgs from a higher-energy $so(10)$ GUT, it turns out that we can find $\Phi_{su(5),\mathbf{24}}$ within (2.11):

[37]Follow footnote 25 for different nonisomorphic versions of U(5) Lie groups defined as $\mathrm{U}(5)_{\hat{q}} \equiv \frac{\mathrm{SU}(5)\times\mathrm{U}(1)_{\hat{q}}}{\mathbb{Z}_5} \equiv \{(g, e^{i\theta}) \in \mathrm{SU}(5)\times\mathrm{U}(1) | (e^{i\frac{2\pi n}{5}}\mathbb{I}, 1) \sim (\mathbb{I}, e^{i\frac{2\pi n\hat{q}}{5}}), n \in \mathbb{Z}_5\}$, the Lie group embedding shows (the proof is given in [62,63])

$$\mathrm{Spin}(10) \supset \mathrm{SU}(5) \text{ and } \mathrm{Spin}(10) \supset \mathrm{U}(5)_{\hat{q}=2,3},$$
$$\text{but } \mathrm{Spin}(10) \not\supset \mathrm{U}(5)_{\hat{q}=1,4},$$

while

$$\mathrm{SO}(10) \supset \mathrm{SU}(5) \text{ and } \mathrm{SO}(10) \supset \mathrm{U}(5)_{\hat{q}=1,4},$$
$$\text{but } \mathrm{SO}(10) \not\supset \mathrm{U}(5)_{\hat{q}=2,3}.$$

The embedding $\mathrm{SO}(10) \supset \mathrm{U}(5)_{\hat{q}=1,4}$ cannot be lifted to Spin(10) thus $\mathrm{Spin}(10) \not\supset \mathrm{U}(5)_{\hat{q}=1,4}$; but $\mathrm{Spin}(10) \supset \mathrm{U}(5)_{\hat{q}=2,3}$.

$$\Phi_{so(10),\mathbf{54}} \equiv \Phi_{\mathbf{54}}, \tag{B4}$$

from (B1) more naturally, as we will soon see.[38]

(2) Spin(10) $\hookleftarrow \frac{\mathrm{Spin}(6)\times\mathrm{Spin}(4)}{\mathbb{Z}_2} \hookleftarrow \frac{\mathrm{SU}(3)_c\times\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{U}(1)_Y}{\mathbb{Z}_6}$ branching rules:

(a) For Spin(10) $\hookleftarrow \frac{\mathrm{Spin}(6)\times\mathrm{Spin}(4)}{\mathbb{Z}_2} = \frac{\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathbb{Z}_2}$, also for SO(10) $\hookleftarrow$ SO(6) × SO(4) [or in terms of Lie algebra $so(10) \hookleftarrow so(6)\times so(4)$ or $su(4)\times su(2)\times su(2)$ with a regular Lie subalgebra in [58]], we find that

$$\begin{cases} \mathbf{10} \sim (\mathbf{1},\mathbf{2},\mathbf{2}) \oplus (\mathbf{6},\mathbf{1},\mathbf{1}) \\ \mathbf{16} \sim (\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2}) \\ \mathbf{45} \sim (\mathbf{1},\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{3}) \oplus (\mathbf{6},\mathbf{2},\mathbf{2}) \oplus (\mathbf{15},\mathbf{1},\mathbf{1}) \\ \mathbf{54} \sim (\mathbf{1},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3},\mathbf{3}) \oplus (\mathbf{6},\mathbf{2},\mathbf{2}) \oplus (\mathbf{20}',\mathbf{1},\mathbf{1}) \\ \mathbf{120} \sim (\mathbf{1},\mathbf{2},\mathbf{2}) \oplus (\mathbf{6},\mathbf{3},\mathbf{1}) \oplus (\mathbf{6},\mathbf{1},\mathbf{3}) \oplus (\mathbf{10},\mathbf{1},\mathbf{1}) \oplus (\overline{\mathbf{10}},\mathbf{1},\mathbf{1}) \oplus (\mathbf{15},\mathbf{2},\mathbf{2}) \\ \mathbf{126} \sim (\mathbf{6},\mathbf{1},\mathbf{1}) \oplus (\overline{\mathbf{10}},\mathbf{3},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1},\mathbf{3}) \oplus (\mathbf{15},\mathbf{2},\mathbf{2}) \end{cases}. \tag{B5}$$

(b) For $\frac{\mathrm{Spin}(6)\times\mathrm{Spin}(4)}{\mathbb{Z}_2} = \frac{\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathbb{Z}_2} \hookleftarrow \frac{\mathrm{SU}(3)_c\times\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{U}(1)_Y}{\mathbb{Z}_6}$ [or in terms of Lie algebra $so(6)\times so(4)$ or $su(4)\times su(2)\times su(2) \hookleftarrow su(3)\times su(2)\times u(1)$], we find that the $su(4) \hookleftarrow su(3)\times u(1)$ (with a regular Lie subalgebra in [58]) branching rule says

$$\begin{cases} \mathbf{4} \sim \mathbf{1}_{-3} \oplus \mathbf{3}_1 \\ \mathbf{6} \sim \mathbf{3}_{-2} \oplus \bar{\mathbf{3}}_2 \\ \mathbf{10} \sim \mathbf{1}_{-6} \oplus \mathbf{3}_{-2} \oplus \mathbf{6}_2 \\ \mathbf{15} \sim \mathbf{1}_0 \oplus \mathbf{3}_4 \oplus \bar{\mathbf{3}}_{-4} \oplus \mathbf{8}_0 \end{cases}. \tag{B6}$$

(i) First, in order to break the Spin(10) down to $G_{\mathrm{PS}_2} \equiv \frac{\mathrm{Spin}(6)\times\mathrm{Spin}(4)}{\mathbb{Z}_2} = \frac{\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathbb{Z}_2}$, we take the representation whose branching rule in (B5) contains the $(\mathbf{1},\mathbf{1},\mathbf{1})$ on the right-handed side so that G_{PS_2} is left unbroken. This means that we may take a GUT-Higgs **54** that we had named it in (2.11) as

$$\Phi_{so(10),\mathbf{54}} \equiv \Phi_{\mathbf{54}}.$$

(ii) Second, in order to break G_{PS_2} further down to $G_{\mathrm{SM}_6} \equiv \frac{\mathrm{SU}(3)_c\times\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{U}(1)_Y}{\mathbb{Z}_6}$, we take the representation whose branching rule in (B2) contains the $(\mathbf{1},\mathbf{1})_0$ of G_{SM_6}. This means that we can take the **15** of SU(4) as the second GUT-Higgs called $\Phi_{su(4),\mathbf{15}}$. But if we want to obtain this second GUT-Higgs from a higher-energy $so(10)$ GUT, it turns out that we can find $\Phi_{su(4),\mathbf{15}}$ from what we had named in (2.10) called

$$\Phi_{so(10),\mathbf{45}} \equiv \Phi_{\mathbf{45}},$$

from (B5) more naturally, as we will soon see.[39]

(3) $\frac{\mathrm{SU}(3)_c\times\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{U}(1)_Y}{\mathbb{Z}_6} \hookleftarrow \frac{\mathrm{SU}(3)_c\times\mathrm{U}(1)_{\mathrm{EM}}}{\mathbb{Z}_3}$ branching rules: The SM electroweak Higgs in the representation

$$\Phi_{\mathrm{SM}}\mathrm{in}(\mathbf{1},\mathbf{2})_{Y=\frac{1}{2}} = (\mathbf{1},\mathbf{2})_{Y_W=1} = (\mathbf{1},\mathbf{2})_{\tilde{Y}=3} \text{ of } su(3)\times su(2)\times u(1) \tag{B7}$$

does the job to break $G_{\mathrm{SM}_6} \equiv \frac{\mathrm{SU}(3)_c\times\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{U}(1)_Y}{\mathbb{Z}_6}$ to $\frac{\mathrm{SU}(3)_c\times\mathrm{U}(1)_{\mathrm{EM}}}{\mathbb{Z}_3}$. Then next, we can ask how to find Φ_{SM} from the representation of $su(5)$, or $su(4)\times su(2)\times su(2)$, or $so(10)$:

(a) Φ_{SM} from $su(5)$: From the branching rule in (B2), one can try to take the $\Phi_{su(5),\mathbf{5}}$ and $\Phi_{su(5),\mathbf{45}}$ which contains $(\mathbf{1},\mathbf{2})_{-3}$ of $su(3)\times su(2)\times u(1)$

[38]It may be also possible to introduce the second GUT-Higgs model of $\Phi'_{so(10),\mathbf{45}} \equiv \Phi'_{\mathbf{45}}$ (different from $\Phi_{\mathbf{45}}$), which also contains the $\Phi_{su(5),\mathbf{24}}$ that can break SU(5) down to G_{SM_6}.

Another possible choice proposed in Georgi's textbook [43] is that in addition to the first GUT-Higgs $\Phi_{so(10),\mathbf{45}} \equiv \Phi_{\mathbf{45}}$, one may also introduce a scalar Higgs of a **16** or a **126** of Spin(10) in order to Higgs down to G_{SM}.

However, these choices are not ideal for us, due to the reason of quantum criticality that we pursue later. The quantum criticality that we pursue only require $\Phi_{so(10),\mathbf{45}} \equiv \Phi_{\mathbf{45}}$ and $\Phi_{so(10),\mathbf{54}} \equiv \Phi_{\mathbf{54}}$, from (2.10) and (2.11).

[39]Another possible choice proposed in Georgi's textbook [43] is that in addition to the first GUT-Higgs model, $\Phi_{so(10),\mathbf{54}} \equiv \Phi_{\mathbf{54}}$, one may also introduce a scalar Higgs of a **16** or a **126** of Spin(10) in order to Higgs down to G_{SM}.

However, these choices are not ideal for us, due to the reason of quantum criticality that we pursue later. The quantum criticality that we pursue only require $\Phi_{so(10),\mathbf{45}} \equiv \Phi_{\mathbf{45}}$ and $\Phi_{so(10),\mathbf{54}} \equiv \Phi_{\mathbf{54}}$, from (2.10) and (2.11).

which is the complex conjugation of Φ_{SM}'s $(\mathbf{1},\mathbf{2})_{\tilde{Y}=3}$.

(b) Φ_{SM} from $su(4)\times su(2)\times su(2)$: From the branching rule in (B6), one can try to take the $\Phi_{su(4)\times su(2)\times su(2),(\mathbf{4},\mathbf{2},\mathbf{1})}$ that contains $(\mathbf{1},\mathbf{2})_{-3}$ of $su(3)\times su(2)\times u(1)$, which is also the complex conjugation of Φ_{SM}'s $(\mathbf{1},\mathbf{2})_{\tilde{Y}=3}$. We may also need $\Phi_{su(4)\times su(2)\times su(2),(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})}$ if we wish to break the $\mathrm{SU}(2)_{\mathrm{R}}$ completely.

(c) Φ_{SM} from $so(10)$: From the branching rule in (B1), we can get the $\Phi_{su(5),\mathbf{5}}$ and $\Phi_{su(5),\mathbf{45}}$ out of $\mathbf{10}$, $\mathbf{120}$, or $\overline{\mathbf{126}}$ of $so(10)$, which we can call $\Phi_{so(10),\mathbf{10}}$, $\Phi_{so(10),\mathbf{120}}$, and $\Phi_{so(10),\overline{\mathbf{126}}}$. These $\mathbf{10}$, $\mathbf{120}$, or $\overline{\mathbf{126}}$ are particular sensible according to [43], because these Higgs can be paired up with the fermion bilinear operators $\psi_i\psi_j$ whose representations are also in the tensor product $\mathbf{16}\otimes\mathbf{16}=\mathbf{10}\oplus\mathbf{120}\oplus\overline{\mathbf{126}}$.

From the branching rule in (B5), we can get the $\Phi_{su(4)\times su(2)\times su(2),(\mathbf{4},\mathbf{2},\mathbf{1})}$ and $\Phi_{su(4)\times su(2)\times su(2),(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})}$ out of $\mathbf{16}$ of Spin(10), which we can call $\Phi_{so(10),\mathbf{16}}$.

APPENDIX C: INDUCE A 3D WZW TERM BETWEEN NÉEL $so(2)$ AND VBS $so(3)$ ON A 4D BULK $w_2(V_{\mathrm{SO}(3)})w_2(V_{\mathrm{SO}(2)})$

This appendix provides a logical pedagogical account on the familiar 3D DQC [28] proposed as a continuous quantum phase transition, on a $(2+1)$D bosonic lattice model with an internal nonrelativistic (iso)spin-$1/2$ bosons,[40] between two kinds of Landau-Ginzburg symmetry breaking orders on each lattice site:

(1) One side has the Néel antiferromagnet order: This order breaks the $\mathbb{Z}^2$-spatial lattice translation to $(\mathbb{Z}_2)^2$ on a lattice. It also breaks the SO(3) internal (iso)spin rotational symmetry [actually, breaking SO(3) faithfully, not SU(2)[41]]. But it respects the spatial rotational symmetry, which is $\mathbb{Z}_4$ spatial rotational symmetry on a square lattice, but it preserves an enhanced SO(2) spatial rotational symmetry in the continuum.

(2) Another side has the VBS order, which preserves a faithful SO(3) (iso)spin rotational symmetry (again, see footnote 41), because the VBS order pairs the two neighbor-site (iso)spin-$1/2$ bosons to an (iso) spin-0 state $\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle-|\downarrow\rangle|\uparrow\rangle)$. But the pattern of VBS breaks the $\mathbb{Z}_4$ spatial rotational symmetry on a square lattice, so the VBS breaks an SO(2) spatial rotational symmetry in the continuum.

If we take into account the discrete $\mathbb{Z}_2$ symmetry (a time reversal or a spatial reflection symmetry), the above SO(2) symmetry becomes an $\mathrm{O}(2)=\mathrm{SO}(2)\rtimes\mathbb{Z}_2$ symmetry, while the above SO(3) symmetry becomes an $\mathrm{O}(3)=\mathrm{SO}(3)\times\mathbb{Z}_2$ symmetry.

Below we write G as the original symmetry group [such as $\mathrm{SO}(3)\times\mathrm{SO}(2)$ valid to the UV lattice scale], while G_{sub} is the remained preserved unbroken symmetry in the corresponding order (Néel or VBS orders). Then we have the following fibration structure:

$$G_{\mathrm{sub}}\hookrightarrow G\to\frac{G}{G_{\mathrm{sub}}},\tag{C1}$$

where the quotient space $\frac{G}{G_{\mathrm{sub}}}$ is the base manifold (i.e., the orbit) as the symmetry-breaking order parameter space. The G is the total space obtained from the fibration of the G_{sub} fiber (i.e., the stabilizer) over the base $\frac{G}{G_{\mathrm{sub}}}$. Here is a systematic table computation on the homotopy group π_k of $(\frac{G}{G_{\mathrm{sub}}})$ for Néel or VBS orders,

	π_0	π_1	π_2	π_3	π_4	π_5
Néel $S^2=\frac{\mathrm{O}(3)\times\mathrm{O}(2)}{\mathrm{O}(2)\times\mathrm{O}(2)}=\frac{\mathrm{O}(3)}{\mathrm{O}(2)}=\frac{\mathrm{SO}(3)\times\mathrm{SO}(2)}{\mathrm{SO}(2)\times\mathrm{SO}(2)}=\frac{\mathrm{SO}(3)}{\mathrm{SO}(2)}$	0	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
VBS $S^1=\frac{\mathrm{O}(3)\times\mathrm{O}(2)}{\mathrm{O}(3)\times\mathrm{O}(1)}=\frac{\mathrm{O}(2)}{\mathrm{O}(1)}=\frac{\mathrm{SO}(3)\times\mathrm{SO}(2)}{\mathrm{SO}(3)\times\mathrm{SO}(1)}=\frac{\mathrm{SO}(2)}{\mathrm{SO}(1)}$	0	$\mathbb{Z}$	0	0	0	0
O(5)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
SO(5)	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

(C2)

To our knowledge, the most systematic, physically intuitive, and mathematically transparent construction of the 3D DQC and its 3D WZW term can be based on the following arguments:

(1) The Néel order breaks an SO(3) (iso)spin rotational symmetry down to an $\mathrm{U}(1)=\mathrm{SO}(2)$ (iso)spin rotational symmetry such as along the z axis, such that (3.16) in the Néel order becomes

$$(G_{\mathrm{sub}}=\mathrm{SO}(2)\times\mathrm{SO}(2))\hookrightarrow(G=\mathrm{SO}(3)\times\mathrm{SO}(2))\to\left(\frac{G}{G_{\mathrm{sub}}}=S^2\right).\tag{C3}$$

[40] What condensed matter people call the spin-$1/2$ bosons on site is actually the isospin-$1/2$ boson, which is in the representation $\mathbf{2}$ of the internal symmetry SU(2), as the internal SU(2) doublet, or namely the qubit. The spin up $|\uparrow\rangle$ and down $|\downarrow\rangle$ are mapped to $|1\rangle$ and $|0\rangle$ of qubit. To emphasize again, the internal SU(2) here is not the spacetime SU(2) from the spacetime Spin group.

[41] There is an internal SU(2) spin rotational symmetry, but the center $Z(SU(2))=\mathbb{Z}_2$ does not act on the Hilbert space in a physical faithful or meaningful way. What faithful representation means physically here is that whether we can find states as that representation, being acted by any physical operator such that these states can be distinguished from each other. The answer is that we cannot distinguish the two states charged under $Z(SU(2))=\mathbb{Z}_2$ physically in this bosonic system.

(i) Hedgehog core, instanton, and magnetic monopole: The SO(3) symmetry breaking hedgehog core has a 0D singularity in the spacetime. This 0D singularity of this hedgehog core in the 3D spacetime can be also regarded an instanton in the 3D spacetime. We can couple this whole configuration to SO(3) background gauge field; this means that we can use the $w_2(V_{SO(3)})$ to measure the magnetic charge of SO(3). We evaluate the $w_2(V_{SO(3)})$ over the Néel's SO(3) symmetry-breaking target space S^2, and it turns out that there is a 2π flux over S^2. Therefore, the hedgehog core is not only an instanton event but also an SO(3) magnetic monopole, living on a 0D open end of some nondynamical 1d 't Hooft line defect of SO(3) background gauge field.

(ii) This SO(3) symmetry-breaking hedgehog core traps a "fractionalized charge-1/2 object charged under the preserved SO(2) symmetry (or $\mathbb{Z}_4$ symmetry on a lattice scale)," namely in the projective representation of $\mathbb{Z}_4$, which is in the unit integer representation $\mathbb{Z}_8$. Namely, the SO(3)-symmetry-breaking defect, hedgehog core in the Néel phase, traps the $\frac{1}{2}$ fractionalization of the unbroken SO(2), or $\mathbb{Z}_4$, charged object of VBS order.

(iii) The winding number of such Néel hedgehog configuration can be classified by

$$\pi_2\left(\frac{\mathrm{SO}(3)\times\mathrm{SO}(2)}{\mathrm{SO}(2)\times\mathrm{SO}(2)}\right)=\pi_2\left(\frac{\mathrm{SO}(3)}{\mathrm{SO}(2)}\right)=\pi_2(S^2)=\mathbb{Z}. \tag{C4}$$

This says the S^2 as a 2D surface in 3D spacetime wrapping around the target S^2 of the Néel's SO(3) symmetry-breaking target space [the base manifold and stabilizer in (C3)]. The spatial S^2 circle as a homology class [in $H_2(M,\mathbb{Z})$, called this 2D sphere ϱ^2] can be paired up with a cohomology class $\mathcal{B}\in H^2(M,\mathbb{Z})$. To make sense the unit generator of the winding $\mathbb{Z}$ class, the $\mathcal{B}$ evaluated on ϱ^2 (bounding a three disk Σ^3 by ϱ^2 so $\partial\Sigma^3=\varrho^2$) must have the following:

$$\oiint_{\varrho^2=\partial\Sigma^3}\mathcal{B}=\oiint_{\varrho^2}w_2(V_{\mathrm{SO}(3)})=1 \mod 2. \tag{C5}$$

(iv) Now imagine in a 3D spacetime picture, we can regard:

(a) the 0D hedgehog core $\varsigma^0_{\text{Néel hedgehog}}$ as the charged object, fractionalized charged under the preserved SO(2) (a projective representation in $\mathbb{Z}_4$, precisely a linear representation in $\mathbb{Z}_8$).

(b) the 2D S^2 called ϱ^2 with $\mathcal{B}\in H^2(M,\mathbb{Z})$ on the ϱ^2, as the charge operator, or the symmetry generator of the SO(2).

Then, follow the higher symmetry or generalized global symmetry language [59], the measurement of the symmetry is exactly performed by evaluating the linking between the $\varsigma^0_{\text{Néel hedgehog}}$ and ϱ^2 in a 3D spacetime M^3. Precisely, the linking number Lk, manifested as a statistical Berry phase, is evaluated via the expectation value of path integral:

$$\left\langle\exp\left(\mathrm{i}\pi\oiint_{\varrho^2=\partial\Sigma^3}\mathcal{B}\right)\cdot\exp(\mathrm{i}\pi\varphi|_{\varsigma^0_{\text{Néel hedgehog}}})\right\rangle =(-1)^{\mathrm{Lk}(\varrho^2,\varsigma^0_{\text{Néel hedgehog}})}\Big|_{M^3}\cdot\langle\exp(\mathrm{i}\pi\varphi\big|_{\varsigma^0_{\text{Néel hedgehog}}})\rangle. \tag{C6}$$

Here $\varphi|_{\varsigma^0_{\text{Néel hedgehog}}}$ is the 0D vertex operator evaluated around the 0D hedgehog core, which is again the 0D magnetic monopole at the open end of the SO(3) background-gauged 1D 't Hooft line. Related descriptions of link invariants of QFTs can be found in [60,61] and references therein.

(2) The VBS order breaks an SO(2) spatial rotational symmetry in the continuum (or breaks $\mathbb{Z}_4$ rotational symmetry on a lattice), such that (3.16) in the VBS order becomes

$$(G_{\text{sub}}=\mathrm{SO}(3)\times\mathrm{SO}(1))\hookrightarrow(G=\mathrm{SO}(3)\times\mathrm{SO}(2))\to\left(\frac{G}{G_{\text{sub}}}=S^1\right). \tag{C7}$$

(i) The SO(2) symmetry-breaking VBS vortex core has a 0D singularity trapping an (iso) spin-1/2 object called the (iso)spinon in the space (famously popularized by Levin-Senthil [83]), which indeed is a 1D vortex loop (called this 1D loop $\varsigma^1_{\text{VBS vortex}}$) in the spacetime.

(ii) The (iso)spinon with (iso)spin-1/2 trapped at the VBS order parameter vortex core is a "fractionalized charge-1/2 object charged under the preserved symmetry SO(3)," namely in the projective representation of SO(3), which is in the fundamental representation **2** of SU(2). Namely, the SO(2)-symmetry-breaking defect, the vortex in the VBS phase, traps the $\frac{1}{2}$ fractionalization of SO(3) charged object of Néel order.

(iii) The winding number of such VBS vortex configuration can be classified by

$$\pi_1\left(\frac{\mathrm{SO}(3)\times\mathrm{SO}(2)}{\mathrm{SO}(3)\times\mathrm{SO}(1)}\right)=\pi_1\left(\frac{\mathrm{SO}(2)}{\mathrm{SO}(1)}\right)=\pi_1(S^1)=\mathbb{Z}. \tag{C8}$$

This says the spatial S^1 wrapping around the target S^1 of the VBS's SO(2) symmetry-breaking target space [the base manifold and stabilizer in

(C7)]. The spatial S^1 circle as a homology class [in $H_1(M,\mathbb{Z})$, called this 1D circle ϱ^1] can be paired up with a cohomology class $\mathcal{A}\in H^1(M,\mathbb{Z})$. To make sense the unit generator of the winding $\mathbb{Z}$ class, the $\mathrm{d}\mathcal{A}$ evaluated on a two-disk Σ^2 (bounded by ϱ^1 so $\partial\Sigma^2=\varrho^1$) must have the following Stoke theorem:

$$\oint_{\varrho^1=\partial\Sigma^2}\mathcal{A}=\int_{\Sigma^2}\mathrm{d}\mathcal{A}=\int_{\Sigma^2}w_2(V_{\mathrm{SO}(2)}) = 1 \mod 2. \tag{C9}$$

(iv) Now imagine in a 3D spacetime picture, we can regard the following:
(a) the 1D vortex loop $\varsigma^1_{\text{VBS vortex}}$ as the charged object, fractionalized charged under the preserved SO(3) [a projective representation in SO(3), precisely a linear representation in SU(2)].
(b) the 1D S^1 circle ϱ^1 with $\mathcal{A}\in H^1(M,\mathbb{Z})$ on the loop, as the charge operator, or the symmetry generator of the SO(3).

Then, the measurement of the symmetry is exactly performed by evaluating the linking between the $\varsigma^1_{\text{VBS vortex}}$ and ϱ^1 in 3D spacetime. Precisely, the linking number Lk, manifested as a statistical Berry phase, is evaluated via the expectation value of path integral:

$$\left\langle \exp\left(\mathrm{i}\pi\oint_{\varrho^1=\partial\Sigma^2}\mathcal{A}\right)\cdot\exp\left(\mathrm{i}\pi\oint_{\varsigma^1_{\text{VBS vortex}}}a\right)\right\rangle = (-1)^{\mathrm{Lk}(\varrho^1,\varsigma^1_{\text{VBS vortex}})}\Big|_{M^3}\cdot\langle\exp(\mathrm{i}\pi\oint_{\varsigma^1_{\text{VBS vortex}}}a)\rangle. \tag{C10}$$

Here a is a 1D background-gauged SO(2) connection evaluated around the 1D vortex loop. Related descriptions of link invariants of QFTs can be found in [60,61] and references therein.

(3) Overall, combined with the above data, we have learned that the 3D DQC construction can be induced by the linking number $\mathrm{Lk}(\varrho^2,\varsigma^0_{\text{Néel hedgehog}})=1$ and $\mathrm{Lk}(\varrho^1,\varsigma^1_{\text{VBS vortex}})=1$ in the 3D spacetime. To furnish more physical intuitions, we can deduce the following:
(i) If we extend the 3D spacetime t, x, y to an extra fourth dimension z, the previous 0D hedgehog core $\varsigma^0_{\text{Néel hedgehog}}$ trajectory can be a 1D pseudoworldline $\varsigma'^1_{\text{Néel hedgehog}}$ in the 4D spacetime M^4. Similarly, the previous 1D vortex loop $\varsigma^1_{\text{VBS vortex}}$ trajectory can be a 2D pseudo–world sheet $\varsigma'^2_{\text{VBS vortex}}$ in the 4D spacetime M^4. Such two configurations can be linked in 4D, with a linking number:

$$\mathrm{Lk}(\varsigma'^1_{\text{Néel hedgehog}},\varsigma'^2_{\text{VBS vortex}})|_{M^4}. \tag{C11}$$

This describes the link in the extended 4D spacetime of two charged objects, charged under SO(2) and SO(3), respectively.
(ii) In a parallel story, the charge operators (of the above charged objects) are the 1D SO(2)-background gauged $\mathcal{A}$ line operator on ϱ^1, and 2D SO(3)-background gauged $\mathcal{B}$ surface operator on ϱ^2. Such two configurations can be linked in 4D with a linking number:

$$\mathrm{Lk}(\mathcal{A}\,\text{on}\,\varrho^1,\mathcal{B}\,\text{on}\,\varrho^2)|_{M^4}. \tag{C12}$$

This describes the link in the extended 4D spacetime of two charge operators, of SO(2) and SO(3) respectively.
(a) If we open up the closed $\oint_{\varrho^1}\mathcal{A}$ on ϱ^1 with two open ends on the 3D boundary M^3 of the bulk M^4, then one open end carries a $\varphi|_{\varsigma^0_{\text{Néel hedgehog}}}$. Their link configuration in 3D corresponds to the earlier (C6):

$$\mathrm{Lk}(\varsigma^0_{\text{Néel hedgehog}},\varrho^2)|_{M^3}.$$

(ii) If we open up the closed $\oiint_{\varrho^2}\mathcal{B}$ on ϱ^2 with an open end on the 3D boundary M^3 of the bulk M^4, then this open end carries a closed 1d vortex loop $\oint_{\varsigma^1_{\text{VBS vortex}}}a$. Their link configuration in 3D corresponds to the earlier (C10):

$$\mathrm{Lk}(\varsigma^1_{\text{VBS vortex}},\varrho^1)|_{M^3}.$$

These above facts together imply that:
(i) The 3D DQC construction [28] is valid if we introduce a mod 2 class 3D WZW term defined on a 3D boundary M^3 of a 4D manifold M^4. Based on the homotopy data $\pi_1(S^1)=\mathbb{Z}$ and $\pi_2(S^2)=\mathbb{Z}$, schematically the WZW in a differential form or de Rham cohomology is.[42]

$$\exp(\mathrm{i}S^{\text{WZW}})=\exp\left(\mathrm{i}\pi\int_{M^4}\mathcal{A}\wedge\mathrm{d}\mathcal{B}\right).\Big|_{M^3=\partial M^4}. \tag{C13}$$

More precisely, we can improve this to construct the cohomology class relying on $\mathcal{A}\in H^1(S^1,\mathbb{Z})=\mathbb{Z}$ and $\mathcal{B}\in H^2(S^2,\mathbb{Z})=\mathbb{Z}$

[42]Here our differential form normalization follows the footnote 19 So we send $\mathcal{A}/\pi\mapsto\mathcal{A}$ and $\mathcal{B}/\pi\mapsto\mathcal{B}$. It can again be easily verified that this WZW has two properties: (1) invertible on $|\mathbf{Z}(M^4)|=1$ on a closed four manifold, (2) this WZW term really is a 3D boundary theory on M^3 of the extended M^4. This WZW term is meant to capture the 3D boundary anomaly of the 4D bulk invertible TQFT: $(-1)^{\int_{M^4}w_2(V_{\mathrm{SO}(3)})w_2(V_{\mathrm{SO}(2)})}$.

classes, the WZW term is written in the singular cohomology class of $\mathcal{A}$ and $\mathcal{B}$:

$$\exp(\mathrm{i}S^{\mathrm{WZW}}) = \exp\left(\mathrm{i}\pi \int_{M^4} \mathcal{A} \smile \delta\mathcal{B}\right)\Big|_{M^3=\partial M^4} = \exp\left(\mathrm{i}2\pi \int_{M^4} \mathcal{A} \smile \mathrm{Sq}^1\mathcal{B}\right)\Big|_{M^3=\partial M^4}, \tag{C14}$$

with the coboundary operator δ, and the Steenrod square $\mathrm{Sq}^1 \equiv \frac{\delta}{2}$ mod 2 here maps the singular cohomology $H^2(M,\mathbb{Z}_2) \mapsto H^3(M,\mathbb{Z}_2)$, on some triangulable manifold M.[43]

(ii) The 3D DQC construction [28] is supported by a 3D 't Hooft anomaly in the $\mathrm{SO}(3)\times\mathrm{SO}(2)$ global symmetry on a three manifold M^3, captured by a 4D bulk invertible TQFT [66] living on a four manifold M^4 with a boundary $\partial M^4 = M^3$:

$$\exp\left(\mathrm{i}\pi \int_{M^4} w_2(V_{\mathrm{SO}(3)}) w_2(V_{\mathrm{SO}(2)})\right). \tag{C15}$$

This 3D 't Hooft anomaly is a mod 2 class global anomaly, whose 4D invertible TQFT corresponds to a $\mathbb{Z}_2$ generator in the following cobordism group $\Omega_G^d \equiv \mathrm{TP}_d(G)$ (see the detailed computations in [63]):

$$\text{a } \mathbb{Z}_2 \text{ generator} w_4(V_{\mathrm{SO}(5)}) \text{ in } \mathrm{TP}_4(\mathrm{SO}\times\mathrm{SO}(5)) = \mathbb{Z}_2,$$
$$\text{a } \mathbb{Z}_2 \text{ generator } w_2(V_{\mathrm{SO}(3)}) w_2(V_{\mathrm{SO}(2)})) \text{ in } \mathrm{TP}_4(\mathrm{SO}\times\mathrm{SO}(3)\times\mathrm{SO}(2)) = \mathbb{Z}_2. \tag{C16}$$

With (C14) and (C15), these conclude our derivation of 3D WZW and 't Hooft anomaly for 3D DQC for Néel-VBS transition.

APPENDIX D: PERTURBATIVE LOCAL AND NONPERTURBATIVE GLOBAL ANOMALIZES VIA COBORDISM: WITHOUT OR WITH T OR CP SYMMETRY

Here we enlist the results of perturbative local and nonperturbative global anomalies via cobordism mostly obtained from [23,25]. Some of these results are used in (2.5). For some spacetime-internal symmetry group $\bar{G}$ of the SM or GUT models, we denote

$$\bar{G} \equiv G_{\mathrm{spacetime}} \times_{N_{\mathrm{shared}}} G_{\mathrm{internal}} \equiv \left(\frac{G_{\mathrm{spacetime}} \times G_{\mathrm{internal}}}{N_{\mathrm{shared}}}\right).$$

We apply a version of cobordism group $\Omega_{\bar{G}}^d \equiv \mathrm{TP}_d(\bar{G})$ from Freed and Hopkins [27]. References [12,23,25,63] had computed some of these fifth cobordism group TP_5 classifications of the 4D anomalies (via the Thom-Madsen-Tillmann spectra [84,85], Adams spectral sequence [86], and Freed-Hopkins's theorem [27]), to obtain

$$\begin{aligned}
\mathrm{TP}_5(\mathrm{Spin}\times_{\mathbb{Z}_2^F}\mathbb{Z}_{4,X}\times G_{\mathrm{SM}_q}) &= \begin{cases} \mathbb{Z}^5\times\mathbb{Z}_2\times\mathbb{Z}_4^2\times\mathbb{Z}_{16}, & q=1,3 \\ \mathbb{Z}^5\times\mathbb{Z}_2^2\times\mathbb{Z}_4\times\mathbb{Z}_{16}, & q=2,6 \end{cases}, \\
\mathrm{TP}_5(\mathrm{Spin}\times_{\mathbb{Z}_2^F}\mathbb{Z}_{4,X}\times\mathrm{SU}(5)) &= \mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_{16}, \\
\mathrm{TP}_5(\mathrm{Spin}\times_{\mathbb{Z}_2^F} G_{\mathrm{PS}_2}) &= \mathrm{TP}_5\left(\mathrm{Spin}\times_{\mathbb{Z}_2^F}\frac{\mathrm{Spin}(6)\times\mathrm{Spin}(4)}{\mathbb{Z}_2}\right) = \mathbb{Z}\times\mathbb{Z}_2^2, \\
\mathrm{TP}_5(\mathrm{Spin}\times_{\mathbb{Z}_2^F} G_{\mathrm{PS}_1}) &= \mathrm{TP}_5(\mathrm{Spin}\times_{\mathbb{Z}_2^F}\mathrm{Spin}(6)\times\mathrm{Spin}(4)) = \mathbb{Z}\times\mathbb{Z}_2^3, \\
\mathrm{TP}_5(\mathrm{Spin}\times_{\mathbb{Z}_2^F}\mathrm{Spin}(10)) &= \mathbb{Z}_2, \\
\mathrm{TP}_5(\mathrm{Spin}\times\mathrm{Spin}(10)) &= 0.
\end{aligned} \tag{D1}$$

For details about their 5D manifold generators and 5D invertible TQFTs, see Ref. [25]. Comments on these perturbative local and nonperturbative global anomalies are in order:

(1) Perturbative local anomalies are classified by integer $\mathbb{Z}$ classes, detectable via the infinitesimal or small gauge or diffeomorphism transformations deformable to the identity element. Given the chiral fermion (quarks and leptons) contents in Appendix A, we can check that all the perturbative local anomalies (all $\mathbb{Z}$ classes) are cancelled in SMs and GUTs. These perturbative local anomaly cancellations are well known, verified in any standard text books on SMs and GUTs.

[43] The $\mathbb{Z}_2$ classification of the WZW term also comes from another quantum matter intuitive argument: When two copies of the WZW terms are put together, the system can be trivialized by an interlayer large coupling without breaking symmetry.

(2) Nonperturbative global anomalies are classified by finite torsion $\mathbb{Z}_n$ classes, detectable via the large gauge or diffeomorphism transformations, not deformable to the identity element.

(i) The $\mathbb{Z}_2$ and $\mathbb{Z}_4$ anomalies in $\mathrm{TP}_5(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times G_{\mathrm{SM}_q})$ or $\mathrm{TP}_5(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \mathrm{SU}(5))$ include the variants or mutated versions of the Witten anomaly [72], by modifying the original SU(2) bundle to some principal SU(n) bundles. Also there is a $\mathbb{Z}_4$ class anomaly from the hypercharge $\mathrm{U}(1)_Y^2$ paired with a X-background field with $(X)^2 = (-1)^F$. All these $\mathbb{Z}_2$ and $\mathbb{Z}_4$ anomalies are checked to be canceled [38–40].

(ii) The $\mathbb{Z}_{16}$ anomaly in $\mathrm{TP}_5(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times G_{\mathrm{SM}_q})$ or $\mathrm{TP}_5(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \mathrm{SU}(5))$ can be canceled if there are 16n Weyl fermions, each is charged under $\mathbb{Z}_{4,X}$ with $(X)^2 = (-1)^F$. Since we only observe 15n Weyl fermions so far by experiments, Refs. [38–40] proposed alternative scenarios to cancel $\mathbb{Z}_{16}$ anomaly with 15n Weyl fermions at low energy—we revisit this issue separately in Sec. IV B.

(iii) Several $\mathbb{Z}_2$ anomalies in $\mathrm{TP}_5(\mathrm{Spin} \times_{\mathbb{Z}_2^F} G_{\mathrm{PS}_{q'=1,2}})$ or $\mathrm{TP}_5(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10))$ come from either the variants of the Witten SU(2) anomaly [72] [modifying the SU(2) gauge bundle to other bundles] or the variants of the new SU(2) anomaly [18] [modifying the $w_2(TM)w_3(TM) = w_2(V_{\mathrm{SO}(3)})w_3(V_{\mathrm{SO}(3)})$ of SO(3) bundle to other SO(n) bundles]. Following [12,18], we can check that the chiral fermion sectors (of quarks and leptons) of PS and $so(10)$ GUTs do not suffer from any of these $\mathbb{Z}_2$ global anomalies.

However, the hallmark of our 4D WZW term, and the fragmentary GUT-Higgs liquid model in Sec. III D, relies on matching them with the w_2w_3 anomaly. So, below, we walk through the distinct properties of the various kinds of w_2w_3 anomalies listed in (D1), in more details.

(1) $\mathrm{TP}_5(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) = \mathbb{Z}_2$ is generated by a 5D invertible TQFT, explained in [12,18,23,25],

$$(-1)^{\int w_2(TM)w_3(TM)} = (-1)^{\int w_2(V_{\mathrm{SO}(10)})w_3(V_{\mathrm{SO}(10)})}.$$

(2) $\mathrm{TP}_5(\mathrm{Spin} \times_{\mathbb{Z}_2^F} G_{\mathrm{PS}_1})$ includes $(\mathbb{Z}_2)^3$. One $\mathbb{Z}_2$ is closely related to the Witten SU(2) anomaly, see [25]. The other $(\mathbb{Z}_2)^2$ are generated by 5D invertible TQFTs:

$$(-1)^{\int w_2(V_{\mathrm{SO}(6)})w_3(V_{\mathrm{SO}(6)})} \quad \text{and} \quad (-1)^{\int \tilde{\eta}(\mathrm{PD}(w_4(V_{\mathrm{SO}(4)})))}.$$

The $\tilde{\eta}$ is a mod 2 index of 1D Dirac operator as a real massive 1D fermion, as a 1D cobordism invariant of $\mathrm{TP}_1(\mathrm{Spin}) = \mathbb{Z}_2$.

(3) $\mathrm{TP}_5(\mathrm{Spin} \times_{\mathbb{Z}_2^F} G_{\mathrm{PS}_2})$ includes $(\mathbb{Z}_2)^2$, which are generated by 5D invertible TQFTs:

$$(-1)^{\int w_2(V_{\mathrm{SO}(6)})w_3(V_{\mathrm{SO}(6)})} \quad \text{and} \quad (-1)^{\int w_2(V_{\mathrm{SO}(4)})w_3(V_{\mathrm{SO}(4)})}.$$

(4) Now we can ask what are the relations between the w_2w_3 of SO(10) bundle [for the $so(10)$ GUT], and that of SO(6) and SO(4) bundles (for the PS model)? We find that

$$\begin{aligned} &w_2(V_{\mathrm{SO}(n+m)})w_3(V_{\mathrm{SO}(n+m)}) \\ &\quad = w_2(V_{\mathrm{SO}(n)})w_3(V_{\mathrm{SO}(n)}) \\ &\qquad + w_2(V_{\mathrm{SO}(m)})w_3(V_{\mathrm{SO}(m)}) \mod 2, \end{aligned} \tag{D2}$$

where the crossing terms become

$$\begin{aligned} &w_2(V_{\mathrm{SO}(n)})w_3(V_{\mathrm{SO}(m)}) + w_2(V_{\mathrm{SO}(m)})w_3(V_{\mathrm{SO}(n)}) \\ &\quad = \mathrm{Sq}^1(w_2(V_{\mathrm{SO}(n)})w_2(V_{\mathrm{SO}(m)})) \\ &\quad = w_1(TM)(w_2(V_{\mathrm{SO}(n)})w_2(V_{\mathrm{SO}(m)})), \end{aligned} \tag{D3}$$

based on the Wu formula using the Steenrod square Sq^1. This (D3) vanishes if we restrict to the system without time-reversal T symmetry (i.e., charge-conjugation-parity CP symmetry) or on orientable manifolds so $w_1(TM) = 0$ (i.e., here we only require Spin structures instead of $\mathrm{Pin}^{\pm}$ structures). So with no T or CP symmetry, we simply relate a mod 2 anomaly of the $so(10)$, to two mod 2 anomalies of PS model:

$$\begin{aligned} &w_2(V_{\mathrm{SO}(10)})w_3(V_{\mathrm{SO}(10)}) \\ &\quad = w_2(V_{\mathrm{SO}(6)})w_3(V_{\mathrm{SO}(6)}) \\ &\qquad + w_2(V_{\mathrm{SO}(4)})w_3(V_{\mathrm{SO}(4)}) \mod 2. \end{aligned} \tag{D4}$$

(5) With a time-reversal T or CP symmetry, or a generic T' such as CT symmetry: If we hope to have the crossing term

$$w_2(V_{\mathrm{SO}(6)})w_3(V_{\mathrm{SO}(4)}) + w_2(V_{\mathrm{SO}(4)})w_3(V_{\mathrm{SO}(6)}) \tag{D5}$$

to enter the anomaly constraint in the PS models, we need to have $\mathrm{Sq}^1(w_2(V_{\mathrm{SO}(6)})w_2(V_{\mathrm{SO}(4)})) = w_1(TM)(w_2(V_{\mathrm{SO}(6)})w_2(V_{\mathrm{SO}(4)})) \neq 0$, this means that we need to include the time-reversal T (or CP) symmetry, or a generic T' such as CT symmetry.

In the $so(10)$ GUT, there are actually two kinds of time-reversal symmetry square:

$$T^2 = (-1)^F \ \text{for} \, \mathrm{Pin}^+, \quad T^2 = +1 \ \text{for} \, \mathrm{Pin}^-. \tag{D6}$$

There are two kinds of commutation relations between time-reversal T and the Spin(10) generators: either commute (direct product "$\times$") or non-commute (semidirect product "$\ltimes$").

So if we include the time-reversal T into the $(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10))$ structure, then there are totally (at least) four kinds of time-reversal symmetries for the $so(10)$ GUT. Based on the computation in Ref. [63], we summarize the four versions of the $so(10)$ GUT with time-reversal symmetries, and their cobordism group TP_5:

$$\begin{aligned} \mathrm{TP}_5(\mathrm{Pin}^+ \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) &= \mathbb{Z}_2, \\ \mathrm{TP}_5(\mathrm{Pin}^- \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) &= \mathbb{Z}_2, \\ \mathrm{TP}_5(\mathrm{Pin}^+ \ltimes_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) &= \mathbb{Z}_2, \\ \mathrm{TP}_5(\mathrm{Pin}^- \ltimes_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) &= \mathbb{Z}_2. \end{aligned} \tag{D7}$$

Interestingly, for the cases of $\mathrm{TP}_5(\mathrm{Pin}^+ \ltimes_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) = \mathbb{Z}_2$ and $\mathrm{TP}_5(\mathrm{Pin}^- \ltimes_{\mathbb{Z}_2^F} \mathrm{Spin}(10)) = \mathbb{Z}_2$, their 4D anomalies are generated by a subtly distinct 5D invertible TQFT

$$(-1)^{\int w_2(TM) w_3(TM)} = (-1)^{\int w_2(V_{\mathrm{O}(10)}) w_3(V_{\mathrm{O}(10)})}. \tag{D8}$$

Notice now we have $w_2(V_{\mathrm{O}(10)}) w_3(V_{\mathrm{O}(10)})$ instead of $w_2(V_{\mathrm{SO}(10)}) w_3(V_{\mathrm{SO}(10)})$. The bundle constraints for $(\mathrm{Pin}^+ \ltimes_{\mathbb{Z}_2^F} \mathrm{Spin}(10))$ and $(\mathrm{Pin}^- \ltimes_{\mathbb{Z}_2^F} \mathrm{Spin}(10))$ are also different:

$$\begin{aligned} &\text{i. } \mathrm{Pin}^+ \ltimes_{\mathbb{Z}_2^F} \mathrm{Spin}(10) \text{constraint}: \ w_2(V_{\mathrm{O}(10)}) = w_2(TM), \qquad w_3(V_{\mathrm{O}(10)}) = w_3(TM). \\ &\text{ii. } \mathrm{Pin}^- \ltimes_{\mathbb{Z}_2^F} \mathrm{Spin}(10) \text{constraint}: \ w_2(V_{\mathrm{O}(10)}) = w_2(TM) + w_1(TM)^2, \\ &\quad w_3(V_{\mathrm{O}(10)}) + w_1(V_{\mathrm{O}(10)}) w_2(V_{\mathrm{O}(10)}) = \mathrm{Sq}^1 w_2(V_{\mathrm{O}(10)}) = \mathrm{Sq}^1 w_2(TM) = w_3(TM) + w_1(TM) w_2(TM). \end{aligned} \tag{D9}$$

The punchline here in (D9) is that because time-reversal T (or CP) or some T' is a valid global symmetry, we can put the theory on an unorientable manifold with $w_1(TM) \neq 0$ also $w_1(V_{\mathrm{O}(10)}) \neq 0$. Therefore, the crossing term in (D5) can still contribute a potential anomaly. This crossing term anomaly $w_2(V_{\mathrm{SO}(6)}) w_3(V_{\mathrm{SO}(4)}) + w_2(V_{\mathrm{SO}(4)}) w_3(V_{\mathrm{SO}(6)})$ turns out to play a possible crucial role in our construction of Sec. III D. See more discussions in a companion work.

Similar stories apply to a larger gauge group unification for three generations of fermions, such as the $so(18)$ GUT with a Spin(18) gauge group. We simply replace all above discussions of $so(10)$ to $so(18)$, and replace Spin(10) to Spin(18).

APPENDIX E: FERMIONIC DOUBLE SPIN STRUCTURE DSPIN FOR A MODIFIED $so(10)$ GUT-HIGGS LIQUID MODEL

Here are detailed comments about our construction of spacetime-internal symmetry that involves the fermionic double Spin structure DSpin given in Sec. III D 2.

(1) First, we recall that we have introduced

$$\begin{cases} \text{Weyl fermion } \psi \text{ in the } \mathbf{16} \text{ of } \mathrm{Spin}(10) \text{ for the } so(10) \text{ GUT}, \\ \text{Dirac fermion } \xi \text{ in the } \mathbf{10} \text{ of } \mathrm{SO}(10) \text{ [also of } \mathrm{Spin}(10)] \text{ for the fermionic parton } \mathrm{QED}_4' \text{ theory}. \end{cases}$$

(2) The modified $so(10)$ GUT requires a $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)$ structure in order to manifest a $w_2 w_3$ anomaly. In this structure, the fermion ψ in $\mathbf{16}$ is charged with $(-1)^F$ odd under the fermion parity $\mathbb{Z}_2^F$. This meanwhile implies the constraint on the matter field spectrum under the $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10)$ structure: There is a short exact sequence: $1 \to \mathbb{Z}_2^F \to Z(\mathrm{Spin}(10)) = \mathbb{Z}_{4,X} \to Z(\mathrm{SO}(10)) = \mathbb{Z}_2 \to 1$. Given the $\mathbb{Z}_{4,X}$ charge state $|X\rangle$ with $X = 0, 1, 2, 3$, we have its representation z^X such that $z \in \mathrm{U}(1)$ with $|z| = 1$, where we embed the normal subgroup $\mathbb{Z}_2^F \subset \mathbb{Z}_{4,X} \subset \mathrm{U}(1)$.

(i) The $\mathbb{Z}_{4,X}$ symmetry generator $\mathrm{U}_{\mathbb{Z}_{4,X}}$ acts on $|X\rangle$, which becomes $\mathrm{U}_{\mathbb{Z}_{4,X}} |X\rangle = \mathrm{i}^X |X\rangle$ with $z = \mathrm{i}$.

(ii) The subgroup $\mathbb{Z}_2^F$ symmetry generator $\mathrm{U}_{\mathbb{Z}_2^F} = (\mathrm{U}_{\mathbb{Z}_{4,X}})^2$ can also act on $|X\rangle$, which becomes $\mathrm{U}_{\mathbb{Z}_2^F} |X\rangle = (\mathrm{U}_{\mathbb{Z}_{4,X}})^2 |X\rangle = \mathrm{i}^{2X} |X\rangle = (-1)^X |X\rangle$. Thus, we read the fermion parity $(-1)^F$, the $|1\rangle$

and $|3\rangle$ are fermionic with -1 (thus odd in $\mathbb{Z}_2^F$), while the $|0\rangle$ and $|2\rangle$ are bosonic with $+1$ (thus even in $\mathbb{Z}_2^F$).

(iii) Any fermion charged under $\mathbb{Z}_2^F$ must have the $(-1)^F = -1$ also identified as the $\mathbb{Z}_2$ normal subgroup of the center $Z(\text{Spin}(10)) = \mathbb{Z}_{4,X}$. Thus these fermions must have a $Z(\text{Spin}(10)) = \mathbb{Z}_{4,X}$ charge either 1 or 3 mod 4.

(iv) Any boson not charged under $\mathbb{Z}_2^F$ must have a $Z(\text{Spin}(10)) = \mathbb{Z}_{4,X}$ charge either 0 or 2 mod 4.

(3) The ξ fermion in the **10** of SO(10) has a charge 1 mod 2 under $Z(\text{SO}(10)) = \mathbb{Z}_2$. The ξ fermion has a charge 2 mod 4 under $Z(\text{Spin}(10)) = \mathbb{Z}_{4,X}$, thus the ξ is "bosonic under the $\mathbb{Z}_2^F$." Thus the ξ fermion is not compatible with the fermion parity required in $\text{Spin} \times_{\mathbb{Z}_2^F} \text{Spin}(10)$ described earlier. Thus, we must introduce a new fermion parity $\mathbb{Z}_2^{F'}$ for ξ.

(4) We construct the full spacetime-internal symmetry group by including the bosonic spacetime rotational symmetry SO, the bosonic internal symmetry SO(10), and the two fermion parities $\mathbb{Z}_2^F \times \mathbb{Z}_2^{F'}$, then we combine the group extensions

$$
\begin{aligned}
&1 \to \mathbb{Z}_2^F \to \text{Spin} = \mathbb{Z}_2^F \rtimes \text{SO} \to \text{SO} \to 1,\\
&1 \to \mathbb{Z}_2^{F'} \to \text{Spin}' = \mathbb{Z}_2^{F'} \rtimes \text{SO} \to \text{SO} \to 1,\\
&1 \to \mathbb{Z}_2^F \times \mathbb{Z}_2^{F'} \to \text{DSpin} \to \text{SO} \to 1,\\
&1 \to \mathbb{Z}_2^F \to \text{Spin}(10) \to \text{SO}(10) \to 1,\\
&1 \to \mathbb{Z}_2^{F'} \to \mathbb{Z}_2^{F'} \times \text{SO}(10) \to \text{SO}(10) \to 1,
\end{aligned}
\tag{E1}
$$

to obtain the full web (3.49),

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G'_{\text{int}\supseteq\mathbb{Z}_2^{F'}} & & G'_{\text{int}\supseteq\mathbb{Z}_2^{F'}} & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z}_2^F & \longrightarrow & (\text{DSpin} \times_{\mathbb{Z}_2^F} \text{Spin}(10)) \times_{\mathbb{Z}_2^{F'}} G'_{\text{int}} & \longrightarrow & (\text{Spin}' \times \text{SO}(10)) \times_{\mathbb{Z}_2^{F'}} G'_{\text{int}} & \longrightarrow & 1\\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z}_2^F & \longrightarrow & \text{Spin} \times_{\mathbb{Z}_2^F} \text{Spin}(10) & \longrightarrow & \text{SO} \times \text{SO}(10) & \longrightarrow & 1\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
\tag{E2}
$$

where we can choose $G'_{\text{int}} = \mathbb{Z}_2^{F'}, \text{U}(1)'$, or $\text{SU}(2)'$ to reproduce the required structure in Sec. III D 2. In all cases, we have $G'_{\text{int}} \supseteq \mathbb{Z}_2^{F'}$ contains the new fermion parity as its normal subgroup.

In addition to the DSpin structure, by including an extra discrete symmetry (such as a time-reversal symmetry), the literature also discovers the structure known as DPin [70] and EPin [37] structures.

(1) The DPin [70] is known as introducing two types of fermions with $\mathbb{Z}_2^{F_+}$ and $\mathbb{Z}_2^{F_-}$, such that an extra discrete $\mathbb{Z}_2^T$ symmetry (e.g., called a time-reversal symmetry) exchanges this two types of fermions. The $\text{DPin}(d)$ contains a discrete dihedral group of order 8, known as $\mathbb{D}_8 = (\mathbb{Z}_2^{F_+} \times \mathbb{Z}_2^{F_-}) \rtimes_{\rho,0} \mathbb{Z}_2^T$, where ρ is a nontrivial $\mathbb{Z}_2^T$ action on $\text{Aut}(\mathbb{Z}_2^{F_+} \times \mathbb{Z}_2^{F_-})$ with two kinds of fermion parity $\mathbb{Z}_2^{F_+} \times \mathbb{Z}_2^{F_-}$ at the $\mathbb{D}_8$'s center. Overall, the $\mathbb{D}_8$ structure sits at the group extension $1 \to (\mathbb{Z}_2^{F_+} \times \mathbb{Z}_2^{F_-}) \to \mathbb{D}_8 \to \mathbb{Z}_2^T \to 1$.

(2) The EPin [37] is known as simultaneously imposing both Pin^+ and Pin^- structure, via introducing two types of fermions (with $\mathbb{Z}_2^{F_+}$ and $\mathbb{Z}_2^{F_-}$) with the time-reversal symmetry acting differently on fermions, $T^2 = (-1)^{F_+}$ and $T^2 = +1$ respectively (via the group extension $1 \to \mathbb{Z}_2^{F_+} \to \mathbb{Z}_4^{TF_+} \to \mathbb{Z}_2^T \to 1$ and $1 \to \mathbb{Z}_2^{F_-} \to \mathbb{Z}_2^T \times \mathbb{Z}_2^{F_-} \to \mathbb{Z}_2^T \to 1$).

See also the interpretations via the regularized quantum many-body model [71].

[1] S. L. Glashow, Partial symmetries of weak interactions, Nucl. Phys. **22**, 579 (1961).
[2] A. Salam and J. C. Ward, Electromagnetic and weak interactions, Phys. Lett. **13**, 168 (1964).
[3] A. Salam, Weak and electromagnetic interactions, Conf. Proc. C **680519**, 367 (1968).
[4] S. Weinberg, A Model of Leptons, Phys. Rev. Lett. **19**, 1264 (1967).
[5] H. Georgi and S. L. Glashow, Unity of All Elementary Particle Forces, Phys. Rev. Lett. **32**, 438 (1974).
[6] J. C. Pati and A. Salam, Lepton number as the fourth color, Phys. Rev. D **10**, 275 (1974).
[7] H. Fritzsch and P. Minkowski, Unified interactions of leptons and hadrons, Ann. Phys. (N.Y.) **93**, 193 (1975).
[8] P. A. Zyla *et al.* (Particle Data Group Collaboration), Review of particle physics, Prog. Theor. Exp. Phys. **2020**, 083C01 (2020).
[9] A. Zee, *Quantum Field Theory in a Nutshell* (Princeton University Press, Princeton, UK, 2010), p. 576.
[10] S. Sachdev, *Quantum Phase Transitions*, 2 ed. (Cambridge University Press, Cambridge, England, 2011).
[11] N. Seiberg, Thoughts about quantum field theory, *Proceedings of the Strings 2019* (2019), https://www.ias.edu/sites/default/files/sns/files/QFT.pdf.
[12] J. Wang and X.-G. Wen, A non-perturbative definition of the standard models, Phys. Rev. Research **2**, 023356 (2020).
[13] S. S. Razamat and D. Tong, Gapped Chiral Fermions, Phys. Rev. X **11**, 011063 (2021).
[14] G. 't Hooft, Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking, NATO Sci. Ser. B **59**, 135 (1980).
[15] D. S. Freed and M. J. Hopkins, Reflection positivity and invertible topological phases, arXiv:1604.06527.
[16] J. Wang, Z. Wan, and Y.-Z. You, Cobordism and deformation class of the standard model, arXiv:2112.14765.
[17] J. Wang, Z. Wan, and Y.-Z. You, Proton stability: From the standard model to beyond grand nification, Phys. Rev. D **106**, 025016 (2022).
[18] J. Wang, X.-G. Wen, and E. Witten, A new SU(2) anomaly, J. Math. Phys. (N.Y.) **60**, 052301 (2019).
[19] R. Thorngren, Framed Wilson operators, fermionic strings, and gravitational anomaly in 4d, J. High Energy Phys. 02 (2015) 152.
[20] C. Wang, A. C. Potter, and T. Senthil, Classification of interacting electronic topological insulators in three dimensions, Science **343**, 629 (2014).
[21] S. M. Kravec, J. McGreevy, and B. Swingle, All-fermion electrodynamics and fermion number anomaly inflow, Phys. Rev. D **92**, 085024 (2015).
[22] I. Garcia-Etxebarria and M. Montero, Dai-Freed anomalies in particle physics, J. High Energy Phys. 08 (2019) 003.
[23] Z. Wan and J. Wang, Higher anomalies, higher symmetries, and cobordisms I: Classification of higher-symmetry-protected topological states and their boundary fermionic/bosonic anomalies via a generalized cobordism theory, Ann. Math. Sci. Appl. **4**, 107 (2019).
[24] J. Davighi, B. Gripaios, and N. Lohitsiri, Global anomalies in the Standard Model(s) and beyond, J. High Energy Phys. 07 (2020) 232.
[25] Z. Wan and J. Wang, Beyond standard models and grand unifications: Anomalies, topological terms, and dynamical constraints via cobordisms, J. High Energy Phys. 07 (2020) 062.
[26] J. Wang, Ultra Unification: Quantum Fields Beyond the Standard Model, https://www.youtube.com/watch?v=vPuNULvtkCI, Seminar at Harvard CMSA Quantum Matter in Mathematics and Physics (2021).
[27] D. S. Freed and M. J. Hopkins, Reflection positivity and invertible topological phases, Geom. Topol. **25**, 1165 (2021).
[28] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. A. Fisher, Deconfined quantum critical points, Science **303**, 1490 (2004).
[29] Z. Bi and T. Senthil, Adventure in Topological Phase Transitions in $3+1$ -D: Non-Abelian Deconfined Quantum Criticalities and a Possible Duality, Phys. Rev. X **9**, 021034 (2019).
[30] Z. Wan and J. Wang, Adjoint QCD_4, deconfined critical phenomena, symmetry-enriched topological quantum field theory, and higher symmetry-extension, Phys. Rev. D **99**, 065013 (2019).
[31] Z. Bi, E. Lake, and T. Senthil, Landau ordering phase transitions beyond the Landau paradigm, Phys. Rev. Research **2**, 023031 (2020).
[32] J. Wang, Y.-Z. You, and Y. Zheng, Gauge enhanced quantum criticality and time reversal deconfined domain wall: SU(2) Yang-Mills dynamics with topological terms, Phys. Rev. Research **2**, 013189 (2020).
[33] Y.-Z. You and J. Wang, Deconfined quantum criticality among grand unified theories, arXiv:2202.13498.
[34] D. J. Gross and F. Wilczek, Ultraviolet Behavior of Non-abelian Gauge Theories, Phys. Rev. Lett. **30**, 1343 (1973).
[35] H. D. Politzer, Reliable Perturbative Results for Strong Interactions?, Phys. Rev. Lett. **30**, 1346 (1973).
[36] D. Tong, Line operators in the standard model, J. High Energy Phys. 07 (2017) 104.
[37] Z. Wan, J. Wang, and Y. Zheng, Higher anomalies, higher symmetries, and cobordisms II: Lorentz symmetry extension and enriched bosonic / fermionic quantum gauge theory, Ann. Math. Sci. Appl. **5**, 171 (2020).
[38] J. Wang, Anomaly and cobordism constraints beyond the standard model: Topological force, arXiv:2006.16996.
[39] J. Wang, Anomaly and cobordism constraints beyond grand unification: Energy hierarchy, arXiv:2008.06499.
[40] J. Wang, Ultra unification, Phys. Rev. D **103**, 105024 (2021).
[41] J. C. Baez and J. Huerta, The algebra of grand unified theories, Bull. Am. Math. Soc. **47**, 483 (2010).
[42] F. Wilczek and A. Zee, Operator Analysis of Nucleon Decay, Phys. Rev. Lett. **43**, 1571 (1979).
[43] H. Georgi, *Lie Algebras In Particle Physics: From Isospin To Unified Theories* (CRC Press, Reading, MA, 2018), p. 320.
[44] L. Fidkowski and A. Kitaev, Effects of interactions on the topological classification of free fermion systems, Phys. Rev. B **81**, 134509 (2010).
[45] J. Wang and X.-G. Wen, Non-perturbative regularization of $1+1$D anomaly-free chiral fermions and bosons: On the equivalence of anomaly matching conditions and boundary gapping rules, arXiv:1307.7480.
[46] J. Wang and X.-G. Wen, A solution to the $1+1$D gauged chiral fermion problem, Phys. Rev. D **99**, 111501 (2019).

[47] Y.-Z. You, Y.-C. He, C. Xu, and A. Vishwanath, Symmetric Fermion Mass Generation as Deconfined Quantum Criticality, Phys. Rev. X **8**, 011026 (2018).

[48] Y.-Z. You, Y.-C. He, A. Vishwanath, and C. Xu, From bosonic topological transition to symmetric fermion mass generation, Phys. Rev. B **97**, 125112 (2018).

[49] E. Eichten and J. Preskill, Chiral gauge theories on the lattice, Nucl. Phys. **B268**, 179 (1986).

[50] X.-G. Wen, A lattice non-perturbative definition of an SO(10) chiral gauge theory and its induced standard model, Chin. Phys. Lett. **30**, 111101 (2013).

[51] Y. You, Y. BenTov, and C. Xu, Interacting topological superconductors and possible origin of $16n$ chiral fermions in the standard model, arXiv:1402.4151.

[52] Y.-Z. You and C. Xu, Interacting topological insulator and emergent grand unified theory, Phys. Rev. B **91**, 125147 (2015).

[53] Y. BenTov and A. Zee, Origin of families and $SO(18)$ grand unification, Phys. Rev. D **93**, 065036 (2016).

[54] Y. Kikukawa, On the gauge invariant path-integral measure for the overlap Weyl fermions in $\underline{16}$ of SO(10), Prog. Theor. Exp. Phys. **2019**, 113B03 (2019).

[55] S. Catterall, Chiral lattice theories from staggered fermions, Phys. Rev. D **104**, 014503 (2021).

[56] S. Catterall, G. C. Toga, and N. Butt, Symmetric mass generation for Kähler-Dirac fermions, Phys. Rev. D **104**, 094504 (2021).

[57] D. Tong, Comments on symmetric mass generation in 2d and 4d, arXiv:2104.03997.

[58] R. Feger, T. W. Kephart, and R. J. Saskowski, LieART2.0—A mathematica application for Lie algebras and representation theory, Comput. Phys. Commun. **257**, 107490 (2020).

[59] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, Generalized global symmetries, J. High Energy Phys. 02 (2015) 172.

[60] P. Putrov, J. Wang, and S.-T. Yau, Braiding statistics and link invariants of bosonic/fermionic topological quantum matter in $2+1$ and $3+1$ dimensions, Ann. Phys. (Amsterdam) **384**, 254 (2017).

[61] M. Guo, K. Ohmori, P. Putrov, Z. Wan, and J. Wang, Fermionic finite-group gauge theories and interacting symmetric/crystalline orders via cobordisms, Commun. Math. Phys. **376**, 1073 (2020).

[62] J. Wang and Y.-Z. You, Gauge enhanced quantum criticality between grand unifications: categorical higher symmetry retraction, arXiv:2111.10369.

[63] Z. Wan and J. Wang, Higher anomalies, higher symmetries, and cobordisms IV: Beyond Standard Models, Grand Unification, and Ultra Unification (2021).

[64] A. Vishwanath and T. Senthil, Physics of Three-Dimensional Bosonic Topological Insulators: Surface-Deconfined Criticality and Quantized Magnetoelectric Effect, Phys. Rev. X **3**, 011016 (2013).

[65] A. W. Sandvik, Evidence for Deconfined Quantum Criticality in a Two-Dimensional Heisenberg Model with Four-Spin Interactions, Phys. Rev. Lett. **98**, 227202 (2007).

[66] C. Wang, A. Nahum, M. A. Metlitski, C. Xu, and T. Senthil, Deconfined Quantum Critical Points: Symmetries and Dualities, Phys. Rev. X **7**, 031051 (2017).

[67] L. Zou, C. Wang, and T. Senthil, Symmetry enriched U(1) quantum spin liquids, Phys. Rev. B **97**, 195126 (2018).

[68] Y. Zhou, K. Kanoda, and T.-K. Ng, Quantum spin liquid states, Rev. Mod. Phys. **89**, 025003 (2017 Apr).

[69] L. Savary and L. Balents, Quantum spin liquids: A review, Rep. Prog. Phys. **80**, 016502 (2017).

[70] J. Kaidi, J. Parra-Martinez, Y. Tachikawa, and w. a. m. a. b. A. Debray, Topological superconductors on superstring worldsheets, SciPost Phys. **9**, 10 (2020).

[71] A. Prakash and J. Wang, Unwinding fermionic SPT phases: Supersymmetry extension, Phys. Rev. B **103**, 085130 (2021).

[72] E. Witten, An SU(2) anomaly, Phys. Lett. **117B**, 324 (1982).

[73] S. Tomonaga, Remarks on Bloch's method of sound waves applied to many-fermion problems, Prog. Theor. Phys. **5**, 544 (1950).

[74] J. M. Luttinger, An exactly soluble model of a many-fermion system, J. Math. Phys. (N.Y.) **4**, 1154 (1963).

[75] P. W. Anderson, Spin-charge separation is the key to the high Tc cuprates, Physica (Amsterdam) **341C**, 9 (2000).

[76] Y.-A. Chen and P.-S. Hsin, Exactly solvable lattice hamiltonians and gravitational anomalies, arXiv:2110.14644.

[77] L. Fidkowski, J. Haah, and M. B. Hastings, Gravitational anomaly of $3+1$ dimensional $\mathbb{Z}_2$ toric code with fermionic charges and fermionic loop self-statistics, arXiv: 2110.14654.

[78] Z. Wan, J. Wang, and X.-G. Wen, companion paper, $3+1$d boundaries with gravitational anomaly of $4+1$d invertible topological order for branch-independent bosonic systems, Phys. Rev. B **106**, 045127 (2022).

[79] X.-G. Wen, Zoo of quantum-topological phases of matter, Rev. Mod. Phys. **89**, 041004 (2017).

[80] T. Senthil, Symmetry-protected topological phases of quantum matter, Annu. Rev. Condens. Matter Phys. **6**, 299 (2015).

[81] H. Georgi, Unparticle Physics, Phys. Rev. Lett. **98**, 221601 (2007).

[82] G. Senjanovic and R. N. Mohapatra, Exact left-right symmetry and spontaneous violation of parity, Phys. Rev. D **12**, 1502 (1975).

[83] M. Levin and T. Senthil, Deconfined quantum criticality and Néel order via dimer disorder, Phys. Rev. B **70**, 220403 (2004).

[84] R. Thom, Quelques propriétés globales des variétés différentiables, Commentarii mathematici Helvetici **28**, 17 (1954).

[85] S. Galatius, I. Madsen, U. Tillmann, and M. Weiss, The homotopy type of the cobordism category, Acta Math. **202**, 195 (2009).

[86] J. F. Adams, On the structure and applications of the Steenrod algebra, Commentarii mathematici Helvetici **32**, 180 (1958).