Large N correlators of Chern-Simons theory in lens spaces

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We compute the invariants for a class of knots and links in arbitrary representations in $S^3/\mathbb{Z}_p$ in the large k (level), large N (rank) limit, keeping $N/(k+N) = \lambda$ fixed, in $U(N)$ and $Sp(N)$ Chern-Simons theories. Using the relation between the saddle-point description and collective field theory, we first find that the invariants for the Hopf link and unknot are given by the on-shell collective field theory action. We next show that the results of these two invariants can be used to compute the invariants of other torus knots and links. We also discuss the large N phase structure of the Hopf link invariant and observe that the same may admit a Douglas-Kazakov type phase transition depending on the choice of representations and λ.

I. INTRODUCTION

Knot theory is an interesting field in mathematics. A knot is a smooth embedding of a circle $\mathcal{S}^1$ to a three manifold. A link is a collection of disjoint knot. An important question in knot theory is whether two knots (or links in general) are equivalent by ambient isotopy (i.e., if one can continuously deform one knot into the other without breaking it). *Knot invariants*[1] are used to distinguish between different inequivalent knots.

A variety of knot invariants in three dimensions can be obtained from the correlation functions of Wilson loop (WL) operators in Chern-Simons (CS) theory [1]. The CS theory is topological at classical as well as quantum level. The only interesting observables in this theory are WLs along oriented knots and links. Since WLs are independent of metric, their expectation values are topological and hence they are *bona fide* candidates for the knot invariants. Witten proved that these topologically invariant correlation functions are precisely the generalized knot invariants in three manifolds [1]. For $G \equiv SU(N)$, $G \equiv SU(2)$ and $G \equiv SO(N)$ these correlation functions give the HOMFLY-PT polynomial, the Jones polynomial, and the Kauffman polynomials, respectively, when the representations associated with the WLs are in fundamental representations. In general the correlation functions of WLs in arbitrary representations provide a wide class of topological invariants. These are known as *coloured polynomials*. From the CS theory perspective such invariants are well defined and we also understand the source of representations associated with the invariants. In mathematics such colored invariants have been defined by Reshetikhin and Turaev [2] and used to construct new quantum invariants of three manifolds.

Although CS theory can be solved exactly, the calculation of knot invariants is difficult. There exists a large literature on the computation of generalized colored invariants for different links and knots. Some of them can be found in [3–18]. In this paper we consider $U(N)$ and $Sp(N)$ CS theories and compute the knot invariants in the following limit:

$$N, k \to \infty \quad \text{keeping } \lambda = \frac{N}{k+N} \text{ fixed.} \quad (1.1)$$

Here, N is the rank of the gauge group and k is the level of the CS theory. This limit is called a *double scaling limit*. We use the saddle-point technique to calculate the invariant for the Hopf link, shown in Fig. 1 in lens space $S^3/\mathbb{Z}_p$ for any representations $\mathcal{R}_1$ and $\mathcal{R}_2$ associated with two unknots.[2] We then show that with this result in hand one can compute the invariants for the class of torus knots $(2, k = \text{odd})$ and links $(2, k = \text{even})$ following the method developed in [3].

Since the representations $\mathcal{R}_{1/2}$ are integrable, in the large N limit we describe these representations in terms of eigenvalue distributions of unitary matrices. As a result,

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[1]Knot invariants in general mean invariants for both knots and links.

[2]The computation presented in this paper could be extended to find the invariants for hyperbolic knots and links. For example Borromean links (for three given representations $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$). Therefore such calculations might be helpful in understanding the volume conjecture [18–21] and its generalized version.
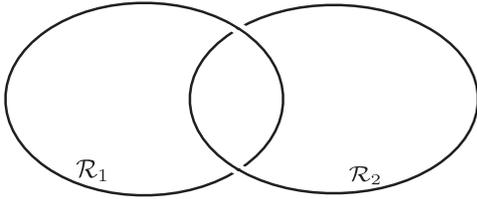
FIG. 1.   Hopf link.

in the double scaling limit the problem can be studied with the aid of a collective field theory[3] [24]. We find that the Hopf link invariant satisfies the Hamilton-Jacobi equation with $p\lambda$ playing the role of time and the Hamiltonian being the free collective field theory Hamiltonian. Therefore, large N (classical) value of the Hopf link invariant, evaluated on the classical solutions, is given by the on-shell action of the free collective field theory. The Hamilton's equations are similar to the continuity and Navier-Stokes equations of an incompressible fluid with negative pressure. Thus the study of a Hopf link invariant boils down to the study of time evolution of a one-dimensional incompressible fluid with the boundary conditions depending on $\mathcal{R}_{1/2}$.

The knowledge of the Hopf link invariant should be enough to find the invariants for the class of other torus knots $(2, k = \text{odd})$ and links $(2, k = \text{even})$ using the generalized Alexander-Conway skein relations recursively. However when we place representations other than fundamental it becomes difficult. In [3] Ramadevi *et.al.* developed a technique to get the invariants for links made of braids up to four strands. A class of invariants for links and knots including torus knots can be obtained from the eigenvalues of half-twist braid matrices. We use these relations and show that in the double scaling limit the invariants for a wide class of knots and links can be computed from the result of the Hopf link invariant. We also emphasize that our results are valid for any arbitrary large N representations associated with the knots.

We also study the large N phase structure of two-point correlation functions of CS theory. We show that in the double scaling limit the two-point correlation function (Hopf link invariant) admits a third-order phase transition depending on the choice of $\mathcal{R}_{1/2}$ and λ. Such a phase transition is similar to the Douglas-Kazakov phase transition [25].

A. Main results

The main results of this paper are the following:

(i) We show that in the large N limit the computation of two-point correlation functions of CS theory for Hopf link boils down to computation of an on-shell action of a free collective field theory. To be precise, the calculation of a Hopf link invariant turns out to

be equivalent to the time evolution of a one-dimensional incompressible fluid with the boundary conditions depending on $\mathcal{R}_{1/2}$.

(ii) The Hopf link invariant in the large N limit is equal to the on-shell partition function of a collective field theory. The result is given by Eq. (3.32). The functions $\sigma_{1/2}(\theta)$ contain information of the representations $\mathcal{R}_{1/2}$ associated with the Hopf link.

(iii) We also show that in the large N limit the invariants for torus knots $(2, k = \text{odd})$ and links $(2, k = \text{even})$ can be computed from the Hopf link invariants and are given by (3.42).

(iv) We study the large N phase transition in two-point correlation functions of CS theory.

The plan of the paper is as follows. In Sec. II we review the correlation functions of WLs in CS theory in Seifert manifold in different framings. Section III provides the detailed calculation of the Hopf link invariant and other torus knot invariants in $U(N)$ CS theory. We also discuss the large N phase structure of the Hopf link invariant in this section. We conclude Sec. III with an example. Large N phase structure of $Sp(N)$ CS theory and correlations functions are discussed in Sec. IV. We end the paper with a discussion in Sec. V. In Appendix A we discuss how to obtain the correlation functions in different manifolds using surgery and their framing dependence. In other appendixes we elaborate various technical details used in the main text.

II. PRELIMINARY: CORRELATION FUNCTIONS IN CHERN-SIMONS THEORY

In this section we review the correlation functions in CS theory on Seifert manifolds in different framings. Experts may skip this section.

The topological nature of classical CS theory is preserved even at the quantum level (correlation functions) but at the cost of a choice of *framing*. Physical observables (correlations of WLs) are completely determined in terms of topological data of the three manifold $\mathbb{M}$ up to a framing. In order to understand the framing dependence in detail we first state an important connection between CS theory and the Wess-Zumino-Witten (WZW) model [1]. Quantization of CS theory with gauge group G and level k in $\mathbb{M}$ with boundary $\Sigma (= \partial\mathbb{M})$ renders a physical Hilbert space $\mathcal{H}(\Sigma)$ which is isomorphic to space of conformal blocks of the WZW model with an affine Lie algebra $\mathbf{g}_k$. Using this connection, correlations of WL operators in CS theory can be written in terms of observables of the WZW model.

To show the framing dependence explicitly we consider the CS theory on a Seifert manifold $\mathcal{M}_{(g,p)}$. A Seifert manifold is a nontrivial circle bundle over genus g Riemann surface[4] Σ_g with the first Chern class p. Seifert manifolds

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for a generic p can be obtained from $\mathcal{M}_{(g,0)}$ (which is a product of genus g Riemann surface and a circle $\Sigma_g \times S^1$) by surgery. Different choices of surgeries give different framings of $\mathcal{M}_{(g,p)}$. The n-point correlation functions in CS theory are defined as

$$\langle \mathcal{W}^{\mathrm{K}_1,\ldots,\mathrm{K}_n}_{\mathcal{R}_1,\ldots\mathcal{R}_n} \rangle = \int [DA] e^{iS_{CS}} \prod_{a=1}^{n} \mathcal{W}^{\mathrm{K}_a}_{\mathcal{R}_a}(A) \qquad (2.1)$$

where $\mathcal{W}^{\mathrm{K}}_{\mathcal{R}}(A) = \mathrm{Tr}_{\mathcal{R}} U_{\mathrm{K}}$ with $U_{\mathrm{K}} = P \exp \oint_{\mathrm{K}} A$ and S_{CS} is the standard CS action with gauge field A. Denoting the n-point correlations of WLs by $\mathcal{W}^{(g,p)}_{\mathcal{R}_1\cdots\mathcal{R}_n}[G,k]$ one can write them in the following form [26–28]

$$\mathcal{W}^{(g,p)}_{\mathcal{R}_1\cdots\mathcal{R}_n}[G,k] = \sum_{\mathcal{R}} \mathcal{K}^{(p)}_{\mathcal{R}_1\mathcal{R}} \mathcal{W}_{\mathcal{R}\mathcal{R}_2\cdots\mathcal{R}_n}(\Sigma_g \times S^1, G, k) \quad (2.2)$$

where $\mathcal{K}^{(p)}$ depends on the framing. Note that being topological $\mathcal{W}^{(g,p)}_{\mathcal{R}_1\cdots\mathcal{R}_n}[G,k]$ does not explicitly depend on the geometry of the knots, it only depends on the topology up to a framing. $\mathcal{W}_{\mathcal{R}\mathcal{R}_2\cdots\mathcal{R}_n}(\Sigma_g \times S^1, G, k)$ is the n-point correlation function of n vertical lines (wrapped around S^1) in $\Sigma_g \times S^1$ carrying representation $\mathcal{R}_i$, given by

$$\mathcal{W}_{\mathcal{R}\mathcal{R}_2\cdots\mathcal{R}_n}(\Sigma_g \times S^1, G, k) = \sum_{\mathcal{R}} \mathcal{S}^{2-n-2g}_{0\mathcal{R}} \prod_{a=1}^{n} \mathcal{S}_{\mathcal{R}\mathcal{R}_a}. \quad (2.3)$$

See Appendix A for a detailed discussion. Here $\mathcal{S}_{\mathcal{R}\mathcal{R}'}$ and $\mathcal{T}_{\mathcal{R}\mathcal{R}'}$ are the modular transform matrices associated with the highest weight representations of the affine Lie algebra $\mathbf{g}_k$ under inversion and translation of modular parameter, respectively (see [29] for details). They satisfy

$$\mathcal{S}^2 = (\mathcal{S}\mathcal{T})^3 = \mathbb{I}. \quad (2.4)$$

The summation on the right-hand side of (2.2) runs over integrable representations of $\mathbf{g}_k$. $\mathcal{W}^{(g,p)}_{\mathcal{R}_1\cdots\mathcal{R}_n}[G,k]$ in (2.2) represents a topological invariant of links of n unknots as shown[5] in Fig. 2.

When $p = 1$ and $g = 0$ the Seifert manifold is a three sphere S^3. On S^3, there exists a canonical choice of framing, given by $\mathcal{K}^{(1)} = \mathcal{S}$. In canonical framing the correlation of n WLs (2.2) is given by
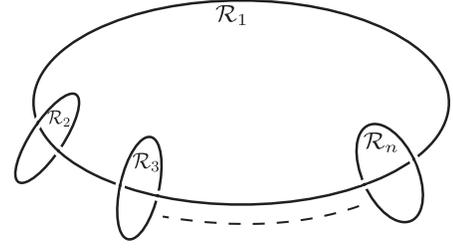

FIG. 2. Links of unknots.

$$\mathcal{W}_{\mathcal{R}_1\cdots\mathcal{R}_n}[S^3, G, k] = \mathcal{S}^{2-n}_{0\mathcal{R}_1} \prod_{a=2}^{n} \mathcal{S}_{\mathcal{R}_1\mathcal{R}_a}. \quad (2.5)$$

The invariant for the link diagram in Fig. 2 in a Seifert manifold can be obtained from $\Sigma_g \times S^1$ by surgery with the choice $\mathcal{K}^{(p)} = (TST)^p$. This particular choice is called Seifert framing. In Seifert framing, therefore, the invariant is given by [26–28]

$$\mathcal{W}^{(g,p)}_{\mathcal{R}_1\cdots\mathcal{R}_n}[G,k] = \sum_{\mathcal{R}} \mathcal{T}^{-p}_{\mathcal{R}\mathcal{R}} \mathcal{S}^{2-n-2g}_{0\mathcal{R}} \prod_{a=1}^{n} \mathcal{S}_{\mathcal{R}\mathcal{R}_a}. \quad (2.6)$$

The main focus of this paper is to study the large N structure of these knot invariants.

III. KNOT INVARIANTS IN $U(N)$ CHERN-SIMONS THEORY

In this section we consider the gauge group $G = U(N)$. For our purpose, we define a modified WL operator $\tilde{W}_{\mathcal{R}}$

$$\tilde{W}_{\mathcal{R}} = \frac{W_{\mathcal{R}}}{\mathcal{S}_{0\mathcal{R}}}. \quad (3.1)$$

The n-point correlation functions of these modified WL operators are given by

$$\tilde{\mathcal{W}}^{(g,p)}_{\mathcal{R}_1\cdots\mathcal{R}_n}(N,k) = \sum_{\mathcal{R}} \mathcal{S}^{2-2g-n}_{0\mathcal{R}} T^{-p}_{\mathcal{R}\mathcal{R}} \prod_{a=1}^{n} \frac{\mathcal{S}_{\mathcal{R}\mathcal{R}_a}}{\mathcal{S}_{0\mathcal{R}_a}}. \quad (3.2)$$

$\mathcal{S}$ and $\mathcal{T}$ for $u(N)_k$ are given by (B2) and (B5). For a given representation $\mathcal{R}_a$ we define a set of N variables $\{\theta^{(a)}_1, \cdots \theta^{(a)}_N\}$

$$\theta^{(a)}_i = \frac{2\pi}{N+K} \left(h^{(a)}_i - \frac{N-1}{2} \right), \quad (3.3)$$

where $h^{(a)}_i = n^{(a)}_i + N - i$ are the hook numbers of the Young diagram associated with $\mathcal{R}_a$ and $n^{(a)}_i$ is number of boxes in the ith row. Since $\mathcal{R}_a$'s are integrable representations of $u(N)_k$, $n^{(a)}_i$'s satisfy

$$-\frac{k}{2} \le n_N^{(a)} \le \cdots \le n_1^{(a)} \le \frac{k}{2}. \tag{3.4}$$

It is easy to check that in the double scaling limit the variables $\theta_i^{(a)}$ ranges from $-\pi$ to π. These $\theta_i^{(a)}$ can be thought of as eigenvalues of unitary matrices. However, there is a difference. The minimum gap between two eigenvalues is $2\pi/(k+N)$. Therefore, in large N limit an integrable representation can be described in terms of a distribution of these eigenvalues, denoted by $\sigma_{(a)}(\theta)$ and defined as

$$\sigma_a(\theta) = \frac{1}{N}\sum_{i=1}^N \delta(\theta - \theta_i^{(a)}). \tag{3.5}$$

In the double scaling limit (1.1), $\sigma_a(\theta)$ satisfies the constraint

$$\sigma_a(\theta) \le \frac{1}{2\pi\lambda} \tag{3.6}$$

which follows from the fact that two eigenvalues have a minimum separation $2\pi/(k+N)$. Considering the above change of variables one can compute the ratio of $\mathcal{S}_{\mathcal{R}\mathcal{R}_a}$ and $\mathcal{S}_{0\mathcal{R}_a}$. It turns out to be

$$\frac{\mathcal{S}_{\mathcal{R}\mathcal{R}_a}}{\mathcal{S}_{0\mathcal{R}_a}} = e^{-\frac{2i}{N}\sum_i (h_i - \frac{N-1}{2})\sum_i \theta_i^{(a)}} \chi_R(\theta_i^{(a)}) \tag{3.7}$$

where h_i's are the hook numbers associated with $\mathcal{R}$ and $\chi_\mathcal{R}(\theta)$ is the character of the $U(N)$ in the representation $\mathcal{R}$. Using the expression for $\mathcal{T}$ (B5) the n-point correlation function can be written as

$$\tilde{\mathcal{W}}_{\mathcal{R}_1\cdots\mathcal{R}_n}^{(g,p)}(N,k) = \sum_{\mathcal{R}} \frac{q^{-\frac{p}{2}\mathcal{C}_2(\mathcal{R})}e^{-i\Theta\mathcal{C}_1(\mathcal{R})}}{\mathcal{S}_{0\mathcal{R}}^{2g+n-2}}\prod_{a=1}^n \chi_\mathcal{R}(\theta^{(a)}), \tag{3.8}$$

where

$$q = e^{2\pi i/(k+N)} \tag{3.9}$$

and

$$\Theta = 2\sum_{a=1}^n \left(\frac{1}{N}\sum_{i=1}^N \theta_i^{(a)}\right). \tag{3.10}$$

This correlation function is similar to the partition function of a q-deformed Yang-Mills theory with a Θ-term on a genus g Riemann surface with n boundaries [22,26]. The distributions of holonomies on those boundaries are given by $\{\theta_i^{(a)}\}$. The only difference between (3.8) and the partition function of a q-deformed 2d Yang-Mills theory is that the sum in (3.8) runs over integrable representations.

When $\mathcal{R}_1 = \cdots = \mathcal{R}_n = 0$, the correlation function (3.8) gives the partition function of $U(N)$ CS theory of level k on the Seifert manifold. The large N phase structure of this theory for $g = 0$ was studied in [30]. The theory undergoes a third-order phase transition in the double scaling limit (1.1). We have reviewed the result in Appendix B.

In this section we explicitly compute the two-point correlation functions ($n = 2$) in the *double scaling limit* (1.1) and show that using the result of two-point correlation function one can compute the invariants for a class of torus knots ($2, k = $ odd) and links ($2, k = $ even).

Two-point correlators in $S^3/\mathbb{Z}_p$, which is a Seifert manifold with $g = 0$ for any p, is given by

$$\mathcal{W}_{\mathcal{R}_1,\mathcal{R}_2}[S^3/\mathbb{Z}_p, N, k] = \sum_{\mathcal{R}}\mathcal{T}_{\mathcal{R}\mathcal{R}}^{-p}\mathcal{S}_{\mathcal{R}\mathcal{R}_1}\mathcal{S}_{\mathcal{R}\mathcal{R}_2} \tag{3.11}$$

and represents a topological invariant for the Hopf link (Fig. 1). In the large N limit the two-point function admits a genus expansion (perturbative part) [31]

$$\ln[\mathcal{W}_{\mathcal{R}_1,\mathcal{R}_2}[S^3/\mathbb{Z}_p, N, k]] = \sum_{h=0}^\infty N^{2-2h}\mathrm{W}_{\mathcal{R}_1,\mathcal{R}_2}^{(h)}(\lambda). \tag{3.12}$$

Our goal is to compute the leading contribution $\mathrm{W}_{\mathcal{R}_1,\mathcal{R}_2}^{(0)}(\lambda)$ for any $\mathcal{R}_{1/2}$ using the saddle-point technique.

The modified two-point correlation function (3.1) is given by

$$\tilde{\mathcal{W}}_{\mathcal{R}_1\mathcal{R}_2}(S^3/\mathbb{Z}_p, N, k)$$
$$= \sum_{\mathcal{R}} q^{-\frac{p}{2}\mathcal{C}_2(\mathcal{R})}e^{-i\Theta\mathcal{C}_1(\mathcal{R})}\chi_\mathcal{R}(\theta^{(1)})\chi_\mathcal{R}(\theta^{(2)}), \tag{3.13}$$

where $\theta^{(1/2)}$ are eigenvalues corresponding to the representation $\mathcal{R}_{1/2}$. We consider the representations $\mathcal{R}_1$ and $\mathcal{R}_2$ such that the eigenvalues are symmetrically distributed about *zero* and hence the Θ term drops out from (3.13). Expressing the characters of $U(N)$ in terms of a Schur polynomial we can explicitly write down the two-point correlation functions in the following form:

$$\tilde{\mathcal{W}}_{\mathcal{R}_1\mathcal{R}_2}(S^3/\mathbb{Z}_p, N, k) = \sum_{\{y_i\}} \frac{e^{2\pi i p\frac{N^2}{24}}}{2^{N(N-1)}}\frac{\det||e^{iNy_j\theta_k^{(1)}}||}{\prod_{i<j}^N \sin\left(\frac{\theta_i^{(1)}-\theta_j^{(1)}}{2}\right)}\frac{\det||e^{iNy_j\theta_k^{(2)}}||}{\prod_{i<j}^N \sin\left(\frac{\theta_i^{(2)}-\theta_j^{(2)}}{2}\right)}e^{-p\pi iN\lambda\sum_{j=1}^N y_j^2}, \tag{3.14}$$

where

$$y_j = \frac{1}{N}\left(h_j - \frac{N-1}{2}\right) \tag{3.15}$$

is the shifted hook number for the representation $\mathcal{R}$.

In general it is difficult to find an exact expression for the two-point function. However, in the large N limit we see that the right-hand side of (3.14) is dominated by a single representation for a specific class of $\mathcal{R}_1$ and $\mathcal{R}_2$ and it is indeed possible to find an exact expression for $\tilde{\mathcal{W}}_{\mathcal{R}_1\mathcal{R}_2}(S^3/\mathbb{Z}_p, N, k)$.

A. Large N analysis of two-point function: Invariant for the Hopf link

We follow the work of Gross and Matytsin [22,23] to do the large N analysis of (3.14). In the large N limit the integrable representation on the right-hand side of (3.13) can be denoted by a density function $\rho(y)$,

$$\rho(y) = -\frac{dx}{dy(x)}, \quad \text{where } y(x) = \frac{y_i}{N}, \quad x = \frac{i}{N}. \tag{3.16}$$

Since i runs from 1 to N, $x \in [0,1]$. Also from the discreteness of y_is [Eq. (3.15)] it follows that

$$\rho(y) \leq 1. \tag{3.17}$$

Following [30], after a Wick rotation in complex p plane $(p \rightarrow -ip)$ we introduce[6]

$$\tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2} = 2^{N(N-1)}e^{-\pi pN^2/12}\tilde{\mathcal{W}}_{\mathcal{R}_1\mathcal{R}_2}(S^3/\mathbb{Z}_p, N, k) \tag{3.18}$$

and a new variable A in place of $p\lambda$

$$A = 2\pi p\lambda. \tag{3.19}$$

One can show from (3.14) that $\tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2}$ satisfies

$$2N\frac{\partial\tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2}}{\partial A} = \frac{1}{D[\theta^{(a)}]}\sum_{k=1}^{N}\frac{\partial^2}{\partial\theta_k^{(a)2}}[D[\theta^{(a)}]\tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2}], \tag{3.20}$$

where

$$D[\theta^{(a)}] = \prod_{i<j}^{N}\sin\left(\frac{\theta_i^{(a)} - \theta_j^{(a)}}{2}\right) \quad \text{for } a = 1, 2. \tag{3.21}$$

Assuming $\tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2}$ is dominated by a single representation in the large N limit, we choose

$$\tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2} = e^{N^2F_N} \quad \text{and} \quad \lim_{N\rightarrow\infty} F_N = F \tag{3.22}$$

and after a little algebra we find that F satisfies (see Appendix C for detailed calculation)

$$2\frac{\partial F}{\partial A} = \int\sigma_a(\theta)\left[\left(\frac{\partial}{\partial\theta}\frac{\delta F}{\delta\sigma_a(\theta)}\right) + 2U(\theta)\frac{\partial}{\partial\theta}\frac{\delta F}{\delta\sigma_a(\theta)} \right.$$
$$\left. + U(\theta)^2 - \frac{\pi^2}{3}\sigma_a(\theta)^2\right]. \tag{3.23}$$

Considering F to be of the following form[7]

$$F[\sigma_1(\theta), \sigma_2(\theta)|A]$$
$$= S[\sigma_1(\theta), \sigma_2(\theta)|A]$$
$$- \frac{1}{2}\sum_{a=1}^{2}\int\sigma_a(\theta)d\theta\fint\sigma_a(\theta')\log\left[\sin\left(\frac{\theta-\theta'}{2}\right)\right]d\theta' \tag{3.24}$$

one can recast Eq. (3.23) as

$$\frac{\partial S}{\partial A} = \frac{1}{2}\int\sigma_a(\theta)\left[\left(\frac{\partial}{\partial\theta}\frac{\delta S}{\delta\sigma_a(\theta)}\right)^2 - \frac{\pi^2}{3}\sigma_a(\theta)^2\right]d\theta. \tag{3.25}$$

As discussed in [22] this equation can be thought of as the Hamilton-Jacobi equation with Hamiltonian

$$H[\sigma,\Pi] = \frac{1}{2}\int\sigma(t,\theta)\left[\left(\frac{\partial\Pi(t,\theta)}{\partial\theta}\right)^2 - \frac{\pi^2}{3}\sigma(t,\theta)^2\right]d\theta \tag{3.26}$$

of a $(1+1)$-dimensional field theory with field $\sigma(t,\theta)$ and conjugate momentum $\Pi(t,\theta) = \frac{\delta S}{\delta\sigma(t,\theta)}$. The Hamiltonian equations of motion are given by

$$\frac{\partial\sigma}{\partial t} + \frac{\partial(\sigma v)}{\partial\theta} = 0,$$
$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial\theta} - \pi^2\sigma\frac{\partial\sigma}{\partial\theta} = 0, \tag{3.27}$$

where

$$v(t,\theta) = \frac{\partial\Pi(t,\theta)}{\partial\theta}. \tag{3.28}$$

These equations are similar to the continuity and Navier-Stokes equations of a one-dimensional fluid moving on a circle with density $\sigma(t,\theta)$, velocity $v(t,\theta)$, and a negative pressure. Finding a solution of Eq. (3.25) for $S(\sigma_1, \sigma_2)$ is equivalent to solve these fluid equations with the boundary conditions

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[6]Since the correlation function is an analytic function of p one can Wick rotate the final answer back to the original p.

[7]Here the cut-integral $\fint d\theta'$ represents integration over θ' over the valid range except the point θ. Such integrals sometimes are denoted by $P\int$.

$$\sigma(t=0,\theta) = \sigma_1(\theta),$$
$$\sigma(t=A,\theta) = \sigma_2(\theta). \qquad (3.29)$$

Suppose $(\bar{\sigma}(t,\theta), \bar{v}(t,\theta))$ is a solution of fluid equations with the desired boundary conditions. The quantity $S(\sigma_1, \sigma_2, A)$ evaluated on this solution is therefore given by

$$\bar{S}(\sigma_1,\sigma_2,A) = \frac{1}{2}\int H[\bar{\sigma}(0,\theta),\bar{v}(0,\theta)]dA + \text{const.} \quad (3.30)$$

Using the series of definitions (3.24), (3.22), and (3.18) one can write the knot invariant for Hopf link in $S^3/\mathbb{Z}_p$ as

$$\mathcal{W}_{\mathcal{R}_1\mathcal{R}_2} = \mathcal{S}_{0\mathcal{R}_1}\mathcal{S}_{0\mathcal{R}_2}\tilde{\mathcal{W}}_{\mathcal{R}_1\mathcal{R}_2}$$
$$= \exp\left[N^2\left(\bar{S}(\sigma_1,\sigma_2,A) + \frac{\pi p}{12} - \ln 2\right)\right]. \quad (3.31)$$

Hence from (3.12) we find

$$\mathrm{W}^{(0)}_{\mathcal{R}_1\mathcal{R}_2} = \bar{S}(\sigma_1,\sigma_2,A) + \frac{\pi p}{12} - \ln 2. \qquad (3.32)$$

Thus we see that the invariant for the Hopf link with any arbitrary large representations of $u(N)_k$ is given by the on-shell free collective field theory action with the boundary conditions (3.29). Further, since the fluid equations (3.27) are dispersionless KdV equations (Burger equations), the whole exercise to find the Hopf link invariant boils down to solving a dispersionless KdV equation with a set of boundary conditions. According to our notation if $\mathcal{R}_2$ (or $\mathcal{R}_1$) is zero, then the corresponding two-point function $\mathcal{W}_{\mathcal{R}_1\mathcal{R}_2}$ becomes a one-point function and gives the invariant for unknot. $\mathcal{R}_2 = 0$ corresponds to a σ_2 given by (3.60).

Our next goal is to find out the invariants for other classes of torus knots with the result of the Hopf link invariant in our disposal.

B. Invariants for a class of torus knots in S^3

In this section we discuss how knot invariants for other torus knots in S^3 (a Seifert manifold with $g=0$, $p=1$) can be computed in the large N limit from the result of the Hopf link.

Torus knots are special kinds of knots which can be put on the surface of a torus. A formalism of knot operators was developed in [7] to compute the invariants of torus knots. In [3] the authors developed a different method to obtain the invariants of links made from braids of up to four strands. It was shown in [3] that for special types of links as shown in Figs. 3(a) and 3(b), the knot invariants can be written in terms of the eigenvalues of the half-twist matrix. The half-twist matrix $\mathcal{B}(\mathcal{R}_1, \mathcal{R}_2)$ introduces right-handed half-twists in parallel oriented strands carrying representations $\mathcal{R}_1$ and $\mathcal{R}_2$ whereas, $\hat{\mathcal{B}}(\mathcal{R}_1, \bar{\mathcal{R}}_2)$ introduces right-handed half-twists
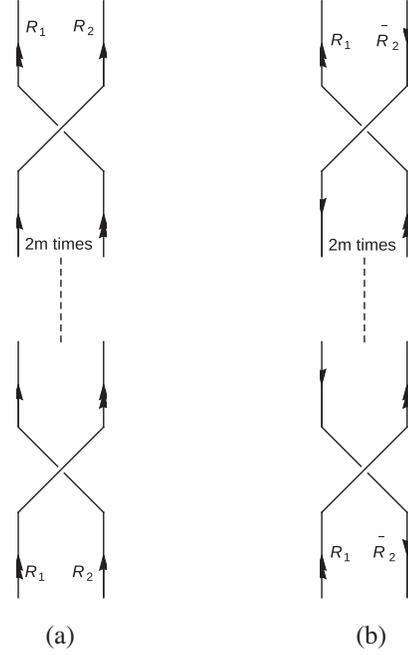

FIG. 3.   Braiding of knots in S^3. (a) For any values of m such braiding gives m links of two unknots in representations R_1 and R_2 with the same orientations. (b) For any values of m such braiding gives $2m$ crossings of two unknots in representations R_1 and R_2 with opposite orientations.

in oppositely oriented strands carrying representations $\mathcal{R}_1$ and $\bar{\mathcal{R}}_2$. The dimensions of $\mathcal{B}$ and $\hat{\mathcal{B}}$ depend on the number of irreducible representations in the product $\mathcal{R}_1$ and $\mathcal{R}_2$ ($\bar{\mathcal{R}}_2$). Denoting the link invariants in Figs. 3(a) and 3(b) by $\mathcal{V}[\mathcal{L}_{2m}(\mathcal{R}_1, \mathcal{R}_2)]$ and $\mathcal{V}[\hat{\mathcal{L}}_{2m}(\mathcal{R}_1, \bar{\mathcal{R}}_2)]$, respectively, it was shown in [3] that they are given by

$$\mathcal{V}[\mathcal{L}_{2m}(\mathcal{R}_1, \mathcal{R}_2)] = \sum_{\mathcal{R}} \dim_q \mathcal{R}(\lambda_{\mathcal{R}}^+(\mathcal{R}_1, \mathcal{R}_2))^{2m} \qquad (3.33)$$

and

$$\mathcal{V}[\hat{\mathcal{L}}_{2m}(\mathcal{R}_1, \bar{\mathcal{R}}_2)] = \sum_{\mathcal{R}} \dim_q \mathcal{R}(\lambda_{\mathcal{R}}^-(\mathcal{R}_1, \bar{\mathcal{R}}_2))^{2m}, \qquad (3.34)$$

where $\lambda_{\mathcal{R}}^{\pm}(\mathcal{R}_1, \mathcal{R}_2)$ are the eigenvalues of $\mathcal{B}(\mathcal{R}_1, \mathcal{R}_2)$ and $\hat{\mathcal{B}}(\mathcal{R}_1, \bar{\mathcal{R}}_2)$, respectively. They are given by

$$\lambda_{\mathcal{R}}^+(\mathcal{R}_1, \mathcal{R}_2) = (-1)^{\epsilon_{\mathcal{R}}}q^{\mathcal{C}_2(\mathcal{R}_1)+\mathcal{C}_2(\mathcal{R}_2)+\frac{|\mathcal{C}_2(\mathcal{R}_1)-\mathcal{C}_2(\mathcal{R}_2)|}{2}-\frac{\mathcal{C}_2(\mathcal{R})}{2}},$$
$$\lambda_{\mathcal{R}}^-(\mathcal{R}_1, \bar{\mathcal{R}}_2) = (-1)^{\epsilon_{\mathcal{R}}}q^{-\frac{|\mathcal{C}_2(\mathcal{R}_1)-\mathcal{C}_2(\mathcal{R}_2)|}{2}+\frac{\mathcal{C}_2(\mathcal{R})}{2}}, \qquad (3.35)$$

where q is given by (3.9). The sum on the right-hand side runs over distinct irreducible representations of $\mathcal{R}_1 \otimes \mathcal{R}_2(\bar{\mathcal{R}}_2)$. The factor $(-1)^{\epsilon_{\mathcal{R}}}$ depends on the symmetric or antisymmetric properties of $\mathcal{R}$ in the product of $\mathcal{R}_1 \otimes \mathcal{R}_2$. For $m=1$, $\mathcal{V}[\mathcal{L}_{2m}(\mathcal{R}_1, \mathcal{R}_2)]$ and $\mathcal{V}[\hat{\mathcal{L}}_{2m}(\mathcal{R}_1, \bar{\mathcal{R}}_2)]$ give the

invariants for the Hopf link in S^3 with the same and opposite orientations respectively.

Denoting

$$\mathcal{G}(\mathcal{R}_1, \mathcal{R}_2) = q^{\mathcal{C}_2(\mathcal{R}_1) + \mathcal{C}_2(\mathcal{R}_2) + \frac{|\mathcal{C}_2(\mathcal{R}_1) - \mathcal{C}_2(\mathcal{R}_2)|}{2}} \qquad (3.36)$$

the knot invariant $\mathcal{V}[\mathcal{L}_{2m}(\mathcal{R}_1, \mathcal{R}_2)]$ can be written as

$$\mathcal{V}[\mathcal{L}_{2m}(\mathcal{R}_1, \mathcal{R}_2)] = \mathcal{G}(m, \mathcal{R}_1, \mathcal{R}_2) \sum_{\mathcal{R}} \dim_q \mathcal{R} q^{-m\mathcal{C}_2(\mathcal{R})}. \qquad (3.37)$$

In the double scaling limit (1.1), using the change of variables (3.3) and denoting the large representations $\mathcal{R}_{1/2}$ by $\sigma_{1/2}$, respectively, the invariants (3.36) can be written as

$$\mathcal{V}[\mathcal{L}_{2m}(\mathcal{R}_1, \mathcal{R}_2)] = \hat{\mathcal{G}}(m, \lambda, \sigma_1, \sigma_2) \mathcal{F}(m, \lambda), \qquad (3.38)$$

where

$$\hat{\mathcal{G}}(m, \lambda, \sigma_1, \sigma_2) = \exp\left[\frac{imN^2}{2\pi\lambda} \left(\int d\theta \theta^2 (\sigma_1 + \sigma_2) + \frac{1}{2} \left| \int d\theta \theta^2 (\sigma_1 - \sigma_2) \right| \right) - \frac{i\pi m \lambda N^2}{3} \right] \qquad (3.39)$$

and

$$\mathcal{F}(m, \lambda) = \int [\mathcal{D}\theta(x)] \exp\left[\frac{N^2}{2} \int dx \fint - dy \log \sin \left| \frac{\theta(x) - \theta(y)}{2} \right| - \frac{imN^2}{2\pi\lambda} \int dx \theta(x)^2 \right]. \qquad (3.40)$$

The function $\mathcal{F}(m, \lambda)$ is difficult to calculate as the functional integration over $\theta(x)$ does not run over all possible configurations [since $\mathcal{R}$ in (3.33) runs over irreducible representations of $\mathcal{R}_1 \otimes \mathcal{R}_2(\bar{\mathcal{R}}_2)$ only]. However in the large N limit one can find the invariant for any m knowing the value of the same for $m = 1$. As we have mentioned that for $m = 1$, $\mathcal{V}[\mathcal{L}_{2m}(\mathcal{R}_1, \mathcal{R}_2)] = \mathcal{W}_{\mathcal{R}_1\mathcal{R}_2}(\lambda)$, the invariant for the Hopf link evaluated in the last section.[8] Now from the expression of the function $\mathcal{F}(m, \lambda)$ we see that

$$\mathcal{F}(m, \lambda) = \mathcal{F}(1, \lambda/m) \qquad (3.41)$$

since the space of functional integration over $\theta(x)$ remains unchanged (for given $\mathcal{R}_1$ and $\mathcal{R}_2$) as we vary m. Therefore we find that in the double scaling limit the link invariants for the class of links shown in Fig. 3(a) are given by

$$\mathcal{V}[\mathcal{L}_{2m}(\mathcal{R}_1, \mathcal{R}_2)] = \left(\frac{\hat{\mathcal{G}}(m, \lambda, \sigma_1, \sigma_2)}{\hat{\mathcal{G}}(1, \lambda/m, \sigma_1, \sigma_2)} \right) \mathcal{W}_{\mathcal{R}_1\mathcal{R}_2}(\lambda/m). \qquad (3.42)$$

In a similar way it is also possible to compute the link invariants $\mathcal{V}[\hat{\mathcal{L}}_{2m}(\mathcal{R}_1, \bar{\mathcal{R}}_2)]$ from $\mathcal{W}_{\mathcal{R}_1\bar{\mathcal{R}}_2}(\lambda)$.

When $\mathcal{R}_2 = \mathcal{R}_1$, one can construct a different braiding as shown in Fig. 4. For even m this is the same as braiding in Fig. 3(a) with $\mathcal{R}_2 = \mathcal{R}_1$. However for odd values of m such braiding gives a single knot. For example, when $m = 3$ we get trefoil in S^3. $m = 1$ gives an unknot. The invariant for this knot is given by

$$\mathcal{V}[\mathcal{L}_m(\mathcal{R}_1, \mathcal{R}_1)] = \sum_{\mathcal{R}} \dim_q \mathcal{R} (\lambda_{\mathcal{R}}^+(\mathcal{R}_1, \mathcal{R}_1))^m. \qquad (3.43)$$

Following the same argument, $\mathcal{V}[\mathcal{L}_m(\mathcal{R}_1, \mathcal{R}_1)]$ for any odd m can be evaluated from $\mathcal{V}[\mathcal{L}_1(\mathcal{R}_1, \mathcal{R}_1)]$ in the double scaling limit. The knot invariant $\mathcal{V}[\mathcal{L}_1(\mathcal{R}_1, \mathcal{R}_1)]$ is given by (3.32) with σ_2 representing a trivial representation (3.60).
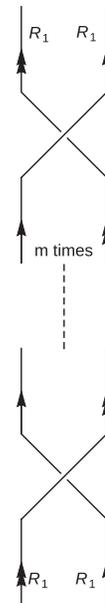

FIG. 4. For any odd values of m, such braiding gives a knot in representations $\mathcal{R}_1$ with m crossings.

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[8]Up to an analytic continuation in $\lambda : \lambda \to i\lambda$. This is due to the fact that in deriving $\mathcal{W}_{\mathcal{R}_1\mathcal{R}_2}$ we did an analytic continuation in p.

C. Large N phases of two-point correlators in $U(N)$ Chern-Simons theory

Large N phase transition in $U(N)$ CS theory has been discussed in [30]. Similar phase transition is also observed in the correlation functions. In this section we give qualitative arguments for such phase transitions.

It can be shown that if the fluid equations (3.27) admit a solution such that there exists a *time* $0 \le t^* \le A$ when the velocity of the fluid is zero, i.e.,

$$v(t^*, \theta) = 0 \qquad (3.44)$$

then the fluid velocity and density satisfy an identity

$$i\pi\sigma^*[\theta - (t - t^*)(v(t,\theta) + i\pi\sigma(t,\theta))] = v(t,\theta) + i\pi\sigma(t,\theta)), \qquad (3.45)$$

where

$$\sigma^*(\theta) = \sigma(t^*, \theta). \qquad (3.46)$$

The maximum value of the density is minimum at $t = t^*$. For such solution one can show that the two-point function (3.14) is dominated by an integrable representation $\rho(y)$ given by (see Appendix D for details)

$$\pi\rho(-\pi\sigma^*(\theta)) = \theta, \qquad (3.47)$$

i.e., inverse of $\sigma^*(\theta)$. If the fluid equations do not admit any such solution then there exists no real saddle points. Needless to mention the existence of a real saddle point depends on the choice of $\sigma_{1/2}$ and λ.

If the fluid equations admit the existence of a real saddle point for a given $\sigma_{1/2}$ then depending on the value of λ the system may undergo a phase transition. In order to discuss this phase transition qualitatively we assume that the initial and final densities $\sigma_1(\theta)$ and $\sigma_2(\theta)$ are even functions of θ and they have gaps (i.e., vanishes for $|\theta|$ greater than some $|\theta_0| \le \pi$). The fluid density $\sigma(t, \theta)$ starting from the configuration $\sigma_1(\theta)$ at $t = 0$ spreads out (i.e., the gap starts decreasing). The absolute value of velocity of the fluid also decreases with time. At some intermediate time t^* when the velocity of the fluid is zero the density has a maximum spread. After that fluid velocity starts increasing (in the opposite direction) and the density starts contracting and reaches the final configuration $\sigma_2(\theta)$ at $t = A$. The system will observe no phase transition if the maximum spread of $\sigma(t, \theta)$ at $t = t^*$ is less than π or at max touches π. Otherwise the system will undergo a phase transition. The Young diagram density $\rho(y)$ have an upper cap $\rho(y) \le 1$. In addition the variable y ranges between $\pm 1/2\lambda$. Since the two functions $\sigma^*(\theta)$ and $\rho(y)$ are functional inverse of each other, σ^* having a gap means $\rho < 1$.

If σ^* is gapless then ρ develops a cap. Depending on the initial conditions and λ we can have four possibilities.

(i) If $\sigma^*(\theta)$ has gap but no cap then $\rho(y)$ has a no cap but a gap.

(ii) If $\sigma^*(\theta)$ is gapless with no cap then the dominant Young diagram $\rho(y)$ will have a cap but it is gapless.

(iii) If $\sigma^*(\theta)$ has a cap with a gap then $\rho(y)$ has no gap but no cap.

(iv) If $\sigma^*(\theta)$ has a cap and no gap then $\rho(y)$ has a cap and no gap.

The last two cases are special in CS theory (unlike 2d YM theory [22]) as all the representations are integrable representations. In the next section we elaborate this qualitative discussion with an example.

D. An example: Explicit computation of two-point function and study of phase structure for Wigner semicircle distributions

Wigner semicircle distribution is a probability distribution on a real line between $\pm R$

$$f_R(x) = \frac{2}{\pi R^2} \Re(\sqrt{R^2 - x^2}). \qquad (3.48)$$

Such distributions appear in different contexts in physics and mathematics. The eigenvalue density of Gaussian Hermitian random matrix theory is given by the semicircle distribution. The transition distribution of the limit shape of asymptotic Young diagrams studied by Vershik-Kerov and Logan-Shepp [32,33] is given by semicircle distribution. In this section we consider the large N distribution $\sigma_{1/2}(\theta)$ corresponding to $\mathcal{R}_1$ and $\mathcal{R}_2$ are given by the semicircle distributions. The reason behind this choice is that the fluid equations can be solved exactly. Our goal is to explicitly calculate the knot invariant for the Hopf link for such representations in the large N limit and study the phase transition of a two-point function.

We take the following semicircular distributions for the Young diagram density corresponds to $\mathcal{R}_{1/2}$

$$\rho_{\mathcal{R}_1/\mathcal{R}_2} = \frac{1}{\pi} \sqrt{L_{1/2} - \frac{L_{1/2}^2 y^2}{4}}. \qquad (3.49)$$

Since $\mathcal{R}_{1/2}$ are integrable representations $L_{1/2}$ satisfies

$$L_1, L_2 \le \pi^2 \quad \text{and} \quad L_1, L_2 \ge 16\lambda^2. \qquad (3.50)$$

Suppose $u(y, t)$ denotes Young diagram distributions that interpolate between $\rho_{\mathcal{R}_1}$ and $\rho_{\mathcal{R}_2}$ from $t = 0$ to $t = A$. We consider

$$u(y, t) = \frac{1}{\pi} \sqrt{\mu(t) - \frac{\mu(t)^2 y^2}{4}} \qquad (3.51)$$
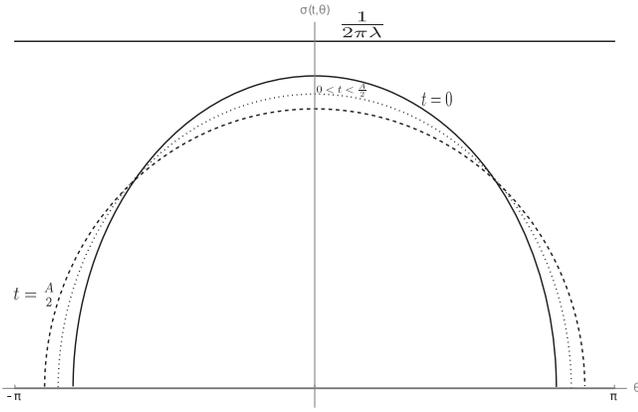
FIG. 5. Plot of $\sigma(\theta, t)$ as a function of θ at $t = 0$ (solid), $t = t^* = A/2$ (dashed), and any arbitrary $0 < t < t^*$ (dotted).

such that

$$\mu(0) = L_1 \quad \text{and} \quad \mu(A) = L_2. \tag{3.52}$$

Considering the relation $\theta = 2\pi\lambda y$, the eigenvalue distribution $\sigma(\theta, t)$ corresponding to $u(y, t)$ is given by

$$\sigma(\theta, t) = \frac{1}{2\pi\lambda} u\left(\frac{\theta}{2\pi\lambda}, t\right) = \frac{1}{2\pi^2\lambda}\sqrt{\mu(t) - \frac{\mu(t)^2\theta^2}{16\pi^2\lambda^2}}. \tag{3.53}$$

For the choice of fluid density (3.53) the fluid velocity can be solved exactly from the continuity equation and is given by

$$v(t, \theta) = -\theta\left(\frac{\dot{\mu}(t)}{2\mu(t)}\right), \tag{3.54}$$

where $\dot{\mu}(t)$ is a derivative of $\mu(t)$ with respect to t. Plugging the ansatz for $\sigma(t, \theta)$ and the solution for

$v(t, \theta)$ in the Navier-Stokes equation we find a differential equation for $\mu(t)$

$$\ddot{\mu}(t) - \frac{3\dot{\mu}(t)^2}{2\mu(t)} - \frac{1}{32\pi^4\lambda^4}\mu(t)^3 = 0. \tag{3.55}$$

This is a second-order nonlinear ordinary differential equation and has a simple solution up to two integration constants. These two constants can be fixed from the boundary conditions (3.52).

First we consider a special case $L_1 = L_2 = L$. Solving Eq. (3.55) for symmetric boundary conditions we find

$$\mu(t) = \frac{4L\pi^2(8\pi^2\lambda^4 + \sqrt{A^2L^2\lambda^4 + 64\pi^4\lambda^8})}{L^2(A - t)t + 4\pi^2(8\pi^2\lambda^4 + \sqrt{A^2L^2\lambda^4 + 64\pi^4\lambda^8})}. \tag{3.56}$$

Since $v(t, \theta)$ is proportional to $\dot{\mu}(t)$ it is easy to see that the velocity is proportional to $A - 2t$ and hence for $L_1 = L_2 = L$ the fluid equations always admit a solution such that $v(t, \theta) = 0$ at $t = A/2$. The density $\sigma(t, \theta)$ coincides with $\sigma_1(\theta)$ at $t = 0$ and then starts spreading out as t increases. The spreading is maximum at $t = A/2$. Finally the density again starts contracting and finally becomes σ_2 at $t = A$. See Fig. 5.

If the maximum spread of $\sigma(t, \theta)$ at $t = A/2$ is less than 2π then $\mathcal{W}_{\mathcal{R}_1\mathcal{R}_2}$ does not observe any phase transition. For

$$L \geq \frac{16\pi^2\lambda^2}{\pi^2 - 4p^2\lambda^2} \tag{3.57}$$

$\sigma(A/2, \theta)$ always has a gap. Since there exists a point $t^* = A/2$ when velocity is zero, there exists a real dominant Young diagram which maximizes $\mathcal{W}_{\mathcal{R}_1\mathcal{R}_2}$. It is given by the (3.47) and has the form

$$\rho(y) = \frac{\sqrt{(8\pi^2\lambda^2 + \sqrt{A^2L^2 + 64\pi^4\lambda^4})(4L - 16\pi^2\lambda^2 y^2) - A^2L^2 y^2}}{2L\pi}. \tag{3.58}$$

Since $\sigma^*(\theta)$ has gap, the dominant Young diagram does not saturate the upper bound. However at $L = \frac{16\pi^2\lambda^2}{\pi^2 - 4p^2\lambda^2}$ the gap in $\sigma^*(\theta)$ vanishes and the corresponding dominant Young diagram touches the upper cap. Therefore $\mathcal{W}_{\mathcal{R}_1\mathcal{R}_2}$ undergoes a Douglas-Kazakov type phase transition at this critical value of L.

The value of the knot invariant can be obtained by calculating the Hamiltonian on this solution and integrating over A. After an explicit calculation we obtain the following result (substituting $A = 2\pi p\lambda$):

This is the HOMFLY-PT polynomial in the double scaling limit for Hopf link where two representations associated with two unknots are the same and given by Wigner semicircle distributions. In general HOMFLY-PT is a

$$W_{LL}^{(0)}(\lambda) = -\frac{2\pi\lambda}{pL} + \frac{1}{2}\sqrt{1 + \frac{16\pi^2\lambda^2}{p^2L^2}} + \frac{\pi p}{12} - \ln 2$$
$$- \frac{1}{2}\sinh^{-1}\left(\frac{pL}{4\pi\lambda}\right). \tag{3.59}$$

polynomial of two variables $q = e^{2\pi i/(k+N)}$ and $s = q^N$. However in the double scaling limit the variable $q = 1$ and $s = e^{2\pi i \lambda}$, hence $W_{LL}^{(0)}(\lambda)$ is a function of λ only.

Surprisingly, the final expression for the action $S(\sigma_1, \sigma_2 | A)$ with $\sigma_{1/2}$, given by the same semicircle distribution of size L, is similar to the limit shape of asymptotic Young diagrams [32–34] up to an analytic continuation in p. In particular

$$\frac{4}{\pi}(S(u) + 1)_{u \to iu} = \hat{\Omega}(u), \quad \text{where } u = \frac{pL}{2\pi\lambda}$$

and $\hat{\Omega}(u)$ is the limit shape. This is nothing but an observation, we think.

One can also choose two different representations by considering the boundary conditions $\mu(0) = L_1$ and $\mu(A) = L_2$. The expression for $\mu(t)$ can also be computed exactly. From these expression we see that the existence of a dominant Young diagram depends on the values of L_1, L_2, and λ. For a given choice of the set (L_1, L_2, λ) if the fluid velocity never reaches zero then there exists no real dominant Young diagram to maximize (3.13). However one can calculate $S(\sigma_1, \sigma_2 | A)$ from (3.25) and hence $W_{L_1 L_2}^0$. In this case $W_{L_1 L_2}^0$ does not observe any phase transition. We see that to get a real dominant Young diagram for $L_1 > L_2$ the parameters must satisfy $A \geq \frac{8\sqrt{L_1 - L_2}\pi^2 \lambda^2}{L_2 \sqrt{L_1}}$ and for $L_2 \geq L_1$ the relation is given by $A \geq \frac{8\sqrt{L_2 - L_1}\pi^2 \lambda^2}{L_1 \sqrt{L_2}}$. In this case there exists a critical relation between L_1, L_2, and A which determines whether the two-point function $W_{L_1 L_2}$ will undergo a phase transition or not. However such condition does not have any handy expression but can be found numerically. One can also compute the knot polynomial in this case.

E. Calculation of partition function

From (2.6) we see that we get back the partition function of CS theory on the Seifert manifold when $\mathcal{R}_a = 0 \; \forall \; a$ (no box in the Young diagram). The phase structure of this theory in the aforementioned double scaling limit was discussed in [30] by directly solving the saddle-point equation. As a consistency check, in this subsection we reproduce the same result from the solution of fluid equations.

When $\mathcal{R}_a = 0$ the corresponding $\sigma_a(\theta)$ is given by

$$\sigma_a(\theta) = \begin{cases} \frac{1}{2\pi\lambda} & \text{for } -\pi\lambda \leq \theta \leq \pi\lambda \\ 0 & \text{for otherwise.} \end{cases} \quad (3.60)$$

We expect that the solutions of the fluid equations (3.27) with the boundary conditions that $\sigma(t, \theta)$ merges with (3.60) at $t = 0$ and $t = A$ will admit an intermediate time $t^* = A/2$ (follows from symmetry) when fluid velocity is zero and the functional inverse of the fluid density at t^* will

give the dominant Young diagram representation obtained in [30].

In order to check our expectation we use the dominant Young diagram found in [30]

$$\rho(y) = \frac{p}{\pi} \tanh^{-1}\left[\sqrt{1 - \frac{e^{-\frac{2\pi\lambda}{p}}}{\cosh(\pi\lambda y)}}\right], \quad (3.61)$$

for $0 \leq \lambda \leq \frac{p}{\pi} \log[\cosh(\pi/p)]$. Inverse of this function gives $\sigma^*(\theta)$. We now use the identity (3.45) to find $v(t, \theta)$ and $\sigma(t, \theta)$. Since both velocity and density are real functions of t and θ, we solve the real and imaginary parts of this equation and find that $\sigma(t, \theta)$ matches with (3.60) at $t = 0$ and $t = A$. For $\lambda > \frac{p}{\pi} \log[\cosh(\pi/p)]$ the dominant Young diagram distribution has a cap. It is difficult to invert that distribution to find $\sigma^*(\theta)$. However, one can numerically check that other phase also solve the fluid equations.

IV. $Sp(N)$ CHERN-SIMONS THEORY AND KNOT INVARIANTS

A symplectic group $Sp(n, F)$ is a group of $2n \times 2n$-dimensional symplectic matrices over a field F under matrix multiplication. A $2n \times 2n$-dimensional matrix A is a symplectic matrix if it satisfies the relation

$$A^T \Omega_{2n} A = \Omega_{2n}, \quad (4.1)$$

where Ω_{2n} is a $2n \times 2n$-dimensional skew-symmetric matrix: $\Omega_{2n}^T + \Omega_{2n} = 0$. A standard choice of Ω_{2n} is

$$\Omega_{2n} = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}. \quad (4.2)$$

Irreducible representations of $Sp(n, F)$ are characterized by Young diagrams with maximum n number of rows.

CS theory for the $Sp(N)$ gauge group is well studied. The large N limit of these theories and their connections with dual string theories were studied in [35]. In this section we shall discuss the phase structure of the theory in the aforementioned double scaling limit. Then we show that the analysis, given in Sec. III A, can be extended for $Sp(N)$ gauge group to obtain the invariant for the Hopf link in terms of the collective field theory action. The analysis is a little different than that of a $U(N)$ theory since the modular transformation matrices $\mathcal{S}$ and $\mathcal{T}$ for $sp(N)_k$ affine algebra have different forms.

A. The partition function

The partition function of CS theory with gauge group $Sp(N)$ and level k on a Seifert manifold is given by

$$\mathcal{Z}[S^3/\mathbb{Z}_p, Sp(N), k] = \sum_{\mathcal{R}} \mathcal{T}_{\mathcal{R}\mathcal{R}}^{-p} \mathcal{S}_{0\mathcal{R}}^2. \quad (4.3)$$

Here $\mathcal{S}$ and $\mathcal{T}$ for $sp(N)_k$ affine lie algebra are given by

$$\mathcal{S}_{\mathcal{R}\mathcal{R}'} = (-i)^{\frac{N(N-1)}{2}} \left(\frac{2}{N+k+1}\right)^{\frac{N}{2}}$$
$$\times \det\left|\sin\left(\frac{\pi f_i(R) f_j(R')}{N+k+1}\right)\right|_{i,j=1}^{N}, \quad (4.4)$$

where

$$f_i(\mathcal{R}) = n_i(\mathcal{R}) - i + N + 1 \equiv h_i(\mathcal{R}) \qquad (4.5)$$

and

$$\mathcal{T}_{\mathcal{R}\mathcal{R}'} = \exp\left[-\frac{i\pi N(N+1)}{12} + \frac{i\pi}{2(N+k+1)}\right.$$
$$\left.\times \sum_{i=1}^{N} h_i(\mathcal{R})^2\right]\delta_{\mathcal{R},\mathcal{R}'}. \qquad (4.6)$$

We define new variables θ_i

$$\theta_i(\mathcal{R}) = \frac{\pi h_i(\mathcal{R}')}{N+K+1} = \pi\lambda\frac{h_i(\mathcal{R})}{N}, \quad \text{where } \lambda = \frac{N}{N+k+1}. \qquad (4.7)$$

The double scaling limit is given by, as before, $N \to \infty$, $k \to \infty$ keeping λ fixed. For an integrable representation $\mathcal{R}$ we have $0 \le h_i(\mathcal{R}) \le N+k+1$; hence the new variables $\theta_i(\mathcal{R})$ satisfies

$$0 \le \theta_i(\mathcal{R}) \le \pi. \qquad (4.8)$$

Although $\theta_i \ge 0$ for $sp(N)$ representations, we introduce a distribution function $\sigma^{sp}(\theta)$ which defines a symmetric distribution of eigenvalues between $-\pi$ and π,

$$\sigma^{sp}(\theta) = \frac{1}{2N}\sum_{i=1}^{N}\delta(\theta-\theta_i) + \frac{1}{2N}\sum_{i=1}^{N}\delta(\theta+\theta_i). \qquad (4.9)$$

Introducing mirror images of the eigenvalues

$$\theta_{-i} = -\theta_i \qquad (4.10)$$

$\sigma^{sp}(\theta)$ can be written as

$$\sigma^{sp}(\theta) = \frac{1}{2N}\sum_{\substack{i=-N \\ i\neq 0}}^{N}\delta(\theta-\theta_i). \qquad (4.11)$$

Hence $\sigma^{sp}(\theta)$ is a distribution of $2N$ eigenvalues: N θ_is and their mirror images θ_{-i}s. We should remember the relation

(4.10) while taking derivative with respect to θ_i. We also note that $\sigma^{sp}(\theta)$ has an upper cap given by

$$\sigma^{sp}(\theta) \le \frac{1}{2\pi\lambda} \qquad (4.12)$$

similar to the $U(N)$ case (3.6).

The partition function in the continuum limit is therefore given by

$$\mathcal{Z}[S^3/\mathbb{Z}_p, Sp(N), k] = \int[d\theta]\exp\left[-\frac{N^2}{\lambda^2}S_{\text{eff}}[\sigma^{sp}]\right], \quad (4.13)$$

where

$$S_{\text{eff}}[\sigma^{sp}] = -\lambda^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sigma^{sp}(\theta)\sigma^{sp}(\theta')$$
$$\times \log\left[4\sin^2\left(\frac{\theta-\theta'}{2}\right)\right]d\theta d\theta'$$
$$+ \frac{2p\lambda}{\pi}\int_{-\pi}^{\pi}\sigma^{sp}(\theta)\left(\frac{\theta^2}{4}-\frac{\pi^2}{12}\right)$$
$$+ \frac{2p\pi\lambda(1-\lambda)}{12}. \qquad (4.14)$$

The saddle-point equation is given by

$$\fint_{-\pi}^{\pi}\sigma^{sp}(\theta')\cot\left(\frac{\theta-\theta'}{2}\right)d\theta' = \frac{p}{2\pi\lambda}\theta. \qquad (4.15)$$

Thus we see that the saddle-point equation for $Sp(N)$ CS theory on $S^3/\mathbb{Z}_p$ is the same as that of $U(N)$ CS theory (B10). Hence in the large N limit the phase structure of these two theories are identical. Therefore, in the double scaling limit, $Sp(N)$ CS theory on $S^3/\mathbb{Z}_p$ admits a third-order phase transition at $\lambda = p\log(\cosh\frac{\pi}{p})/\pi$.

B. Two-point correlator and Hopf link invariants

Though the structure of the modular $\mathcal{S}$ and $\mathcal{T}$ matrices for $sp(N)_k$ affine algebra is different than that for $u(N)_k$, the two-point correlator for the Hopf link admits a description in the language of incompressible fluid and hence can be written in terms of the on-shell action of a free collective field theory. The calculation follows the similar line as what we did for $U(N)$ theory but the intermediate steps are different since hook numbers of the Young diagrams of $sp(N)$ representations are always positive. The bulk of the calculations are given in Appendix C 2; here we outline the main steps.

Using the expression of modular $\mathcal{S}$ and $\mathcal{T}$ matrices we write down the modified two-point correlation function (3.1) as

$$\tilde{\mathcal{W}}_{\mathcal{R}_1\mathcal{R}_2}(S^3/\mathbb{Z}_p, Sp(N), N, k) = \frac{e^{\frac{ip\pi N(2N+1)}{12}}}{2^{2N(N-1)}} \sum_{y_i(\mathcal{R})} \left(\frac{\det[\sin(N\theta_j(\mathcal{R}_1)y_i(\mathcal{R}))]}{D[\theta(\mathcal{R}_1)]} \right)$$

$$\times \left(\frac{\det[\sin(N\theta_j(\mathcal{R}_2)y_i(\mathcal{R}))]}{D[\theta(\mathcal{R}_2)]} \right) e^{-\frac{i\pi p\lambda N}{2}\sum_{i=1}^{N} y_i(\mathcal{R})^2}, \qquad (4.16)$$

where

$$y_i(\mathcal{R}) = \frac{h_i(\mathcal{R})}{N} \qquad (4.17)$$

and

$$D[\theta(\mathcal{R}_1)] = \prod_{i=1}^{N} \sin(\theta_i(\mathcal{R}_1)) \prod_{i<j}^{N} \sin\left(\frac{\theta_i(\mathcal{R}_1) + \theta_j(\mathcal{R}_1)}{2}\right)$$

$$\times \sin\left(\frac{\theta_i(\mathcal{R}_1) - \theta_j(\mathcal{R}_1)}{2}\right). \qquad (4.18)$$

We follow the same procedure, what we did for $U(N)$ CS theory, and define (after a wick rotation $p \to -ip$) $\tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2}(S^3/\mathbb{Z}_p, Sp(N), N, k)$,

$$\tilde{\mathcal{W}}_{\mathcal{R}_1\mathcal{R}_2}(S^3/\mathbb{Z}_p, Sp(2N), N, k)$$
$$= \frac{1}{2^{2N(N-1)}} e^{\frac{p\pi N(2N+1)}{12}} \tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2}(S^3/\mathbb{Z}_p, Sp(2N), N, k). \quad (4.19)$$

Following (3.22), in the large N limit we define a similar function $F(\sigma_{1/2}^{sp}, A)$ where $\sigma_{1/2}^{sp}(\theta)$ are eigenvalue distributions corresponding to $\mathcal{R}_{1/2}$, respectively. Segregating the pure $\sigma_{1/2}^{sp}$ dependent part from $F(\sigma_{1/2}^{sp}, A)$, we define a function $S(\sigma_{1/2}^{sp}, A)$ given in (C13) and show that $S[\sigma_{1/2}^{sp}, A]$ satisfies

$$\frac{\partial S}{\partial A} = \frac{1}{2} \int \sigma_a^{sp}\theta) \left[\left(\frac{1}{2} \frac{\partial}{\partial \theta} \frac{\delta S}{\delta \sigma_a^{sp}(\theta)} \right)^2 - 4 \frac{\pi^2}{3} \sigma_a^{sp}(\theta)^2 \right] d\theta, \qquad (4.20)$$

where

$$A = \pi p \lambda. \qquad (4.21)$$

Thus we see that $S[\sigma_a^{sp}, A]/4$ [Hopf link invariant in $Sp(N)$ CS theory] satisfies the same Hamilton-Jacobi equation and hence the saddle point is governed by $Sp(N)$ free collective field theory equations.

The real dominant Young diagram in the large N limit can also be obtained by studying the Hamiltonian equations. The real dominant representation, if it exists, is given by the inverse of $Sp(N)$ fluid density $\sigma^{sp}(\theta, t)$ at some intermediate time when the fluid velocity is zero. The

detailed calculation for dominant representation is given in Appendix D 1.

V. DISCUSSION

In this paper, we find that the computation of a two-point correlation function in $U(N)$ and $Sp(N)$ CS theory in $S^3/\mathbb{Z}_p$ (which renders invariant for the Hopf link) in the large N limit boils down to finding solutions of continuity and Navier-Stokes equations of an incompressible one-dimensional fluid evolving from $t = 0$ to $t = A$ with the initial and final densities corresponding to the representations $\mathcal{R}_1$ and $\mathcal{R}_2$. The Hopf link invariant $W_{\mathcal{R}_1\mathcal{R}_2}^{(0)}(\lambda)$ in the large N limit satisfies the Hamilton-Jacobi equation where $p\lambda$ plays the role of time and the Hamiltonian is given by a $U(N)$ [or $Sp(N)$] free collective field theory Hamiltonian. The invariant for the Hopf link turns out to be equal to the on-shell action. Using the method developed in [3] we finally show that invariants for other torus knots $(2, k = \text{odd})$ and links $(2, k = \text{even})$ can be obtained from the invariants for the Hopf link and unknot.

We further discuss the large N phase structure of two-point correlators in CS theory. Whether the two-point function undergoes a phase transition or not depends on the evolution of the fluid. The absolute value of the fluid velocity at a given point θ decreases with time as the fluid starts evolving from $t = 0$. The existence of a real dominant representation depends on whether the absolute value of velocity can reach zero at some intermediate time for all $-\pi \leq \theta \leq \pi$. If one starts with a class of $\mathcal{R}_1$ and $\mathcal{R}_2$ such that corresponding $\sigma_1(\theta)$ and $\sigma_2(\theta)$ are gapped, then the density of the fluid starts spreading from its initial distribution $\sigma_1(\theta)$. The spreading is maximum when the velocity is zero (if such solution exists) and then starts shrinking and goes to its final distribution $\sigma_2(\theta)$ at $t = A$. The maximum spreading of fluid density (at $t = t^*$) depends on the choice of $\mathcal{R}_{1/2}$ and λ. If the maximum spread lies between $-\pi$ and π then there is no phase transition in $W_{\mathcal{R}_1\mathcal{R}_2}^{(0)}$. However, $W_{\mathcal{R}_1\mathcal{R}_2}^{(0)}$ observes a third-order phase transition otherwise. The CS theory enjoys *level-rank* duality. A theory with rank N and level k is dual to a theory with level and rank exchanged. The two-point correlation function in the dual theory also admits the similar fluid structure. It would be interesting to understand the relation between the fluid and its dual fluid in the large N limit.

The study of large N correlation functions is important in many aspects. It sets up the platform to check the generalized volume conjecture [18–21]. Further, based on the Gopakumar-Vafa conjecture [36] it was observed in [37] that the CS invariants are mapped to topological string amplitudes on Riemann surfaces with boundaries in the topological string theory side. The knot invariants were reformulated in terms of new invariants (integer invariants) capturing the spectrum of M2 branes ending on M5 branes. The results were checked explicitly for unknot. Large N analysis of this observation and its generalization to other links were considered in [9]. Our analysis to compute invariants for a class of knots and links will be useful to further investigate the connection between CS theory and topological string theory beyond partition function.

As we mentioned in Sec. II, in S^3 there exists a canonical framing $\mathcal{K} = \mathcal{S}$. In this framing the two-point correlation function for the Hopf link is given by $\mathcal{S}_{\mathcal{R}_1 \mathcal{R}_2}$ and does not show phase transition for any $\mathcal{R}_{1/2}$. In Seifert framing the same quantity is given by (3.11). The sum is over the integrable representations. Using the properties of $\mathcal{S}$ and $\mathcal{T}$ matrices one can show that these two expressions are related to each other up to a phase factor. The question is why we see a phase transition in the two-point function in the double scaling limit. If one takes the $N, k \to \infty$ limit without any restriction, then the sum in (3.11) runs over all possible Young diagrams with any number of rows and any number of columns. However, here we are considering a particular limit $N, k \to \infty$ keeping N/k fixed. Under this condition the sum becomes restricted—one does not sum over all possible Young diagrams. Therefore we do not expect that in the double scaling limit the above equality holds. This was also the reason behind the phase transition in CS theory in S^3 studied in [30]. The framing dependence of correlation functions in CS theory has a mining in the topological string theory side. Framing is related to inherent ambiguity in the open topological string amplitude related to the IR geometry of the D-brane [38,39]. It would be interesting to study the topological string amplitudes in the same double scaling limit and understand the connection better. We keep the problem for future.

ACKNOWLEDGMENTS

APPENDIX A: FRAMING DEPENDENCE OF PARTITION FUNCTION

Partition function of CS theory in three dimensions depends on choice of framing [1]. In this appendix we discuss the framing dependence in details.

Canonical quantization of CS theory on a three manifold $\mathcal{M}$ with a boundary Σ produces the "physical Hilbert space" $\mathcal{H}(\Sigma)$. In [1] Witten constructed $\mathcal{H}(\Sigma)$ in terms of conformal blocks of the WZW model on Σ with a gauge group G and level k. One can explicitly construct the Hilbert space for $\Sigma = S^2$ and $\Sigma = \mathbb{T}^2$. For $\Sigma = S^2$ the Hilbert space is trivial (dimension one). However for $\Sigma = \mathbb{T}^2$ the space of conformal blocks has one-to-one correspondence with integrable representations of $\mathbf{g}_k$. There is a natural choice of basis for $\mathcal{H}(\Sigma = \mathbb{T}^2)$. The basis vectors are given by integrable representations $|\mathcal{R}_a\rangle$ of $\mathbf{g}_k$: these are the states associated with the partition function of CS theory in a solid torus $\mathbf{T}^2$ with a WL in representation $\mathcal{R}_a$ along the noncontractible cycle. Therefore the Hilbert space is finite dimensional and spanned by these integrable representations.

In order to write the partition function of CS theory on a generic three manifold $\mathcal{M}$, we split the manifold into two parts $\mathcal{X}_L$ and $\mathcal{X}_R$ sharing a common boundary Σ. The path integral of CS theory on $\mathcal{X}_R$ corresponds to a vector $|\phi\rangle$ in $\mathcal{H}(\Sigma)$. Since the boundary Σ of $\mathcal{X}_L$ has an opposite orientation of that of $\mathcal{X}_R$ the path integral on $\mathcal{X}_L$ is mapped to a vector $\langle\psi| \in \mathcal{H}^*(\Sigma)$ where $\mathcal{H}^*(\Sigma)$ is dual of $\mathcal{H}(\Sigma)$. Since the manifold $\mathcal{M}$ can be obtained by gluing $\mathcal{X}_L$ and $\mathcal{X}_R$ along Σ, the partition function of CS theory on $\mathcal{M}$ is therefore given by [1]

$$\mathcal{Z}(\mathcal{M}) = \langle\psi|\phi\rangle. \tag{A1}$$

Using this result Witten showed that one can write the partition function and correlation functions of CS theory in S^3 and other generic three manifolds from the partition function and WLs in $S^2 \times S^1$.

In order to understand the prescription in detail let us start with CS theory on a three manifold $\mathcal{M}$. We consider a WL $\mathcal{W}_{\mathcal{R}_a} = \text{Tr}_{\mathcal{R}_a} U_K$ in $\mathcal{M}$ along a knot K, where $U_K = P \exp(\int_K A)$ and $\mathcal{R}_a$ is an integrable representation of $\mathbf{g}_k$. We take a tubular neighborhood of the knot K which is a solid torus $\mathbf{T}^2$ such that $\partial \mathbf{T}^2 = \mathbb{T}^2$ is a torus. We take the $\mathbf{T}^2$ out of the manifold and hence the three manifold $\mathcal{M}$ is now a connected sum of $\mathcal{X}_R = \mathbf{T}^2$ with a WL inserted and the reminder $\mathcal{X}_L$. Note both $\mathcal{X}_L$ and $\mathcal{X}_R$ have common boundary $\mathbb{T}^2$. Following the work of Verlinde [40] Witten showed that the path integral over $\mathbf{T}^2$ with a WL $\mathcal{W}_{\mathcal{R}_a}$ along the noncontractible cycle of $\mathbf{T}_2$ is mapped to a state $|\mathcal{R}_a\rangle$ in $\mathcal{H}(\mathbb{T}^2)$. Thus, following (A1) we see that the expectation value of WL $\mathcal{W}_{\mathcal{R}_a}$ in representation R_a in $\mathcal{M}$ can be written as

$$\langle\mathcal{W}_{\mathcal{R}_a}\rangle_\mathcal{M} = \langle\psi|\mathcal{R}_a\rangle, \tag{A2}$$

where $\langle\psi| \in \mathcal{H}^*(T^2)$ is the path integral over $\mathcal{X}_L$. For $\mathcal{R}_a$ to be a trivial representation, the WL is equal to 1. Therefore $\langle\psi|0\rangle$ is the partition function on $\mathcal{M}$.

Now, before gluing the solid torus with $\mathcal{X}_L$ one can also make a diffeomorphism on the boundary of $\mathbf{T}^2$. Such operation (scooping out $\mathbf{T}^2$ from $\mathcal{M} \to$ apply diffeomorphism on the boundary $\to$ gluing back with $\mathcal{X}_L$) generates a new manifold $\widetilde{\mathcal{M}}$. Let us first understand this with the help of a simple example. Suppose our $\mathcal{M} = S^2 \times S^1$. The manifold $S^2 \times S^1$ can be written as a connected sum of two solid tori. To understand this one can think that a solid torus is a *disc* times a *circle*: $\mathbf{T}^2 = D \times S^1$. When we glue two discs at the boundary we get the two manifold S^2. Therefore, when we glue two solid tori along their boundaries without any diffeomorphism we get $S^2 \times S^1$. Thus we see that when we scoop out a solid torus from $S^2 \times S^1$ the reminder is also a solid torus. Consider now a solid torus $\mathbf{T}^2$ embedded in S^3 which is $R^3 \cup \infty$. $\mathbf{T}^2$ is invariant under inversion. The exterior of this $\mathbf{T}^2$ is another solid torus $\mathbf{T}'^2$ (as we have identified the points at infinity). However, there is a difference. The contractible cycle in $\mathbf{T}^2$ is mapped to the noncontractible cycle in $\mathbf{T}'^2$ and vice versa. Therefore these two tori are related to each other by S modular transformation on the boundary $\mathbb{T}^2$. Thus when we glue $\mathbf{T}^2$ and $\mathbf{T}'^2$ we get S^3. Now starting from $\mathcal{M} = S^2 \times S^1$ we split the manifold in two solid tori. Then we perform an S modular transformation (i.e., inverting the torus) on the boundary of one of them (say the right one) and then glue them again. This surgery produces S^3 from $S^2 \times S^1$.

Consider now a CS theory in a solid torus with a WL in representation $\mathcal{R}_a$ along the noncontractible circle. The path integral maps to $|\mathcal{R}_a\rangle \in \mathcal{H}(\mathbb{T}^2)$. If K is the diffeomorphism that acts on the boundary of the solid torus, the path integral on $\mathbf{T}^2$ changes and hence is mapped to a different state in $\mathcal{H}(\mathbb{T}^2)$. The new state can be written as

$$|\chi\rangle = \sum_{\mathcal{R}_b} \mathcal{K}_a^b |\mathcal{R}_b\rangle. \tag{A3}$$

The matrix $\mathcal{K} \in \mathrm{Hom}(\mathcal{H}(\mathbb{T}^2), \mathcal{H}(\mathbb{T}^2))$ depends on the diffeomorphism K that acts on the torus. If we now glue this solid torus with $\mathcal{X}_L$ we get a new manifold $\widetilde{\mathcal{M}}$ with a WL in $\mathcal{R}_a$. The expectation value of the WL in representation $\mathcal{R}_a$ in $\widetilde{\mathcal{M}}$ can be written in terms of expectation value of WLs in $\mathcal{M}$

$$\langle W_{R_a}\rangle_{\widetilde{\mathcal{M}}} = \sum_{\mathcal{R}_b} \mathcal{K}_a^b \langle W_{\mathcal{R}_b}\rangle_{\mathcal{M}}. \tag{A4}$$

Hence the partition function on $\widetilde{\mathcal{M}}$ can be obtained by considering R_a to be a trivial representation

$$Z(\widetilde{\mathcal{M}}) = \sum_{\mathcal{R}_b} \mathcal{K}_0^b \langle W_{R_b}\rangle_{\mathcal{M}}. \tag{A5}$$

This is a very powerful relation.

The partition function of Chern-Simons theory on $S^2 \times S^1$ can be calculated using (A5). As explained above the manifold $S^2 \times S^1$ can be written as a connected sum of two solid tori. As mentioned earlier, a partition function in $\mathbf{T}^2$ without any WL is mapped to $|0\rangle$ in $\mathcal{H}(\mathbb{T}^2)$. Thus we get a partition function of CS theory on $S^2 \times S^1$ is given by

$$Z(S^2 \times S^1) = \langle 0|0\rangle = 1. \tag{A6}$$

Similarly when we have a nontrivial WL in $S^2 \times S^1$ we can split the manifold into two solid tori with one torus containing the WL along the noncontractible cycle. Hence the expectation value of a WL in $S^2 \times S^1$ is given by

$$\langle W_{\mathcal{R}_a}\rangle_{S^2 \times S^1} = \langle 0|\mathcal{R}_a\rangle = \delta_{0\mathcal{R}_a}. \tag{A7}$$

Our goal is to generate S^3 from $S^2 \times S^1$ by surgery. Starting from $S^2 \times S^1$ we split the manifold into two solid tori. Then we perform an S modular transformation on the boundary $\mathbb{T}^2$ and then glue them again. This surgery produces S^3. The diffeomorphism K is the modular transformation S and hence the corresponding $\mathcal{K}$ matrix is the modular transform matrix $\mathcal{S}$ in $\mathcal{H}(\mathbb{T}^2)$. Therefore following (A5) and (A6) we find the CS partition function on S^3 is given by

$$Z(S^3) = \sum_{\mathcal{R}_b} \mathcal{S}_{0\mathcal{R}_b} \delta_{0\mathcal{R}_b} = \mathcal{S}_{00}. \tag{A8}$$

However, instead of choosing $\mathcal{K} = \mathcal{S}$ if we choose $\mathcal{K} = \mathcal{TST} = \mathcal{ST}^{-1}\mathcal{S}$, this also produces S^3 but in a different framing called *Seifert framing*. In this framing the partition function is given by

$$Z_{SF}(S^3) = \sum_{\mathcal{R}} \mathcal{S}_{0\mathcal{R}}^2 \mathcal{T}_{\mathcal{R}\mathcal{R}}^{-1} \tag{A9}$$

which is the same as $Z(S^3)$ up to a phase.

Partition function for CS theory on a generic lens space $S^3/\mathbb{Z}_p$ (a Seifert manifold with $g = 0$) can be obtained from $S^2 \times S^1$ by choosing $\mathcal{K} = (\mathcal{TST})^p = (\mathcal{ST}^{-1}\mathcal{S})^p$. The partition function is given by (B1) with $g = 0$.

In order to get the partition function of Hopf link in $S^3/\mathbb{Z}_p$ we can start with CS theory on $S^2 \times S^1$ with two WLs. We consider a solid torus along one WL and split the manifold with two parts scooping out the solid torus. The remainder is also a solid torus with the other loop. Partition functions on these two tori are given by $|\mathcal{R}_a\rangle$ and $\langle \mathcal{R}_b|$. Hence the CS partition function with two WLs is given by

$$Z(S^2 \times S^1, \mathcal{R}_a \mathcal{R}_b) = \langle \mathcal{R}_b|\mathcal{R}_a\rangle = \delta_{\mathcal{R}_a \mathcal{R}_b}. \tag{A10}$$

Before gluing if we give an inversion on the right torus by choosing $\mathcal{K} = (\mathcal{ST}^{-1}\mathcal{S})^p$ we get Hopf link in $S^3/\mathbb{Z}_p$ given by (3.11).

We now want to find the correlation of three WLs in representations $\mathcal{R}1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ in $S^2 \times S^1$. We can split $S^2 \times S^1$ with three WLs into two solid tori: one contains a WL in $\mathcal{R}_1$ and the other torus contains two WLs in $\mathcal{R}_2$ and $\mathcal{R}_3$. Suppose the CS partition function in $\mathbf{T}^2$ with two WLs along the noncontractible circle is mapped to a state $|\mathcal{R}_2 \mathcal{R}_3\rangle$ in $\mathcal{H}(\mathbb{T}^2)$. Since $\mathcal{R}_i$'s correspond to primary fields in WZW we have

$$|\mathcal{R}_2 \mathcal{R}_3\rangle = \sum_{\mathcal{R}} \mathcal{N}^{\mathcal{R}}_{\mathcal{R}_1 \mathcal{R}_2} |\mathcal{R}\rangle, \qquad (A11)$$

where $\mathcal{N}^{\mathcal{R}}_{\mathcal{R}_1 \mathcal{R}_2}$ is the Verlinde numbers. Hence

$$Z(S^2 \times S^1, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3) = \mathcal{N}^{\mathcal{R}_1}_{\mathcal{R}_2 \mathcal{R}_3}. \qquad (A12)$$

One can now use the result (A11) to write the n-point correlation function in $S^2 \times S^1$ given by (2.3) for $g = 0$ with the help of the Verlinde formula,

$$\mathcal{N}^{\mathcal{R}_1}_{\mathcal{R}_2 \mathcal{R}_3} = \sum_{\mathcal{R}} \frac{\mathcal{S}_{\mathcal{R} \mathcal{R}_2} \mathcal{S}_{\mathcal{R} \mathcal{R}_3} \mathcal{S}^*_{\mathcal{R} \mathcal{R}_1}}{\mathcal{S}_{0 \mathcal{R}}}. \qquad (A13)$$

Starting from (2.3) we take a solid torus out from $S^2 \times S^1$ with any of the n WLs inside, apply an inversion on the torus, and then put that back inside. In this process we generate an S^3 with the link, shown in Fig. 2 inside and the invariant is given by (2.2).

APPENDIX B: REVIEW OF $U(N)_k$ CHERN-SIMONS THEORY ON SEIFERT MANIFOLD

The partition function for $U(N)$ CS theory on a Seifert manifold is given by

$$\mathcal{Z}_{N,k} = \sum_{\mathcal{R}} \mathcal{S}^{2-2g}_{0\mathcal{R}} \mathcal{T}^{-p}_{\mathcal{R}\mathcal{R}}. \qquad (B1)$$

Since the affine Lie algebra $u(N)_k$ is the quotient of $su(N)_k \times u(1)_{N(k+N)}$ by $\mathbb{Z}_N$, the $u(N)$ representations can be expressed in terms of $su(N)$ representations (denoted by R) and eigenvalues of $u(1)$ generator Q: $\mathcal{R} = (R, Q)$, where $Q = r(R)$ mod N and $r(R)$ is the number of boxes in R. The trivial representation $\mathcal{R} = 0$ means both $R = 0$ and $Q = 0$. The modular transform matrix $\mathcal{S}_{\mathcal{R}\mathcal{R}'}$ for $u(N)_k$ in terms of $su(N)$ representations and the $u(1)$ charges is given by [26,29,41]

$$\mathcal{S}_{\mathcal{R}\mathcal{R}'} = \frac{(-i)^{\frac{N(N-1)}{2}}}{(k+N)^{\frac{N}{2}}} e^{-\frac{2\pi i Q Q'}{N(N+k)}} \det M(R, R'), \qquad (B2)$$

where $M(R, R')$ is a $N \times N$ matrix with elements,

$$M_{ij}(R, R') = \exp\left[\frac{2\pi i}{k+N} \phi_i(R) \phi_j(R')\right], \qquad (B3)$$

$$\phi_i(R) = l_i - \frac{r(R)}{N} - i + \frac{1}{2}(N+1) \qquad (B4)$$

and l_i's are the number of boxes in the ith row in R. The modular transformation matrix $\mathcal{T}_{\mathcal{R}\mathcal{R}'}$ is given by

$$\mathcal{T}_{\mathcal{R}\mathcal{R}'} = e^{2\pi i (h_{\mathcal{R}} - \frac{c}{24})} \delta_{\mathcal{R}\mathcal{R}'}, \quad h_{\mathcal{R}} = \frac{1}{2} \frac{C_2(\mathcal{R})}{k+N}, \quad c = \frac{N(Nk+1)}{k+N}, \qquad (B5)$$

where $C_2(\mathcal{R})$ is the quadratic Casimir of $u(N)_k$. The $u(N)$ representations $\mathcal{R}$ can be characterized by extended YDs by introducing the number of boxes in the ith row $n_i = l_i + s$, for $1 \le i \le N - 1$ and $n_N = s$ where $s \in \mathbb{Z}$. n_is can be negative and the corresponding YDs will have *antiboxes* [42]. In terms of these n_i's the quadratic Casimir $C_2(\mathcal{R})$ is given by

$$C_2(\mathcal{R}) = \sum_{i=1}^{N} n_i(n_i - 2i + N + 1). \qquad (B6)$$

For an integrable representation $\mathcal{R}$ of $u(N)_k$

$$-\frac{k}{2} \le n_N \le \cdots \le n_1 \le \frac{k}{2}, \qquad (B7)$$

and hence

$$-\frac{k}{2} < h_N < \cdots < h_1 \le \frac{k}{2} + N - 1. \qquad (B8)$$

In terms of the variables θ_i introduced in (3.3), the CS partition function (B1) in $S^3/\mathbb{Z}_p$ in the continuum limit is given by

$$\mathcal{Z}^p_{N,k} = \int [d\theta] e^{-(N+K)^2 S_{\text{eff}}[\sigma]}, \quad \text{where}$$

$$S_{\text{eff}}[\sigma] = \frac{p\lambda}{\pi} \int \sigma(\theta) \left(\frac{\theta^2}{4} - \frac{\pi^2}{12}\right) d\theta + \frac{\pi p \lambda (1-\lambda)}{12}$$
$$- \frac{\lambda^2}{2} \int \!\!\!\!\!\!\!\diagup \sigma(\theta) \sigma(\theta') \log\left[4 \sin^2\left(\frac{\theta - \theta'}{2}\right)\right] d\theta d\theta'. \qquad (B9)$$

The saddle-point equation for $\sigma(\theta)$, obtained from the effective action is given by

$$\int \!\!\!\!\!\!\!\diagup \sigma(\theta') \cot\left(\frac{\theta - \theta'}{2}\right) d\theta' = \frac{p}{2\pi\lambda} \theta. \qquad (B10)$$

In the large N limit we have to solve this equation for $\sigma(\theta)$ in the presence of this constraint (3.6).

The unitary matrix model (B9) has a gapped phase in the large k, N limit and the eigenvalue distribution is given by [43,44]

$$\sigma(\theta) = \frac{p}{2\pi^2\lambda} \tanh^{-1}\left[\sqrt{1 - \frac{e^{-\frac{2\pi\lambda}{p}}}{\cos^2\frac{\theta}{2}}}\right], \quad \text{for}$$

$$-2\cos^{-1}e^{-\frac{\pi\lambda}{p}} < \theta < 2\cos^{-1}e^{-\frac{\pi\lambda}{p}}. \tag{B11}$$

It turns out that the solution saturates the upper bound (3.6) at $\lambda = p/\pi \log\cosh(\pi/p) \equiv \lambda^*$ [44] and is not valid beyond that.

Cap-gap phase.—It was observed in [30] that for $\lambda > \lambda^*$ the saddle-point equation admits a *cap-gap* solution. The eigenvalue density $\sigma(\theta)$ for $\lambda > p/\pi \log\cosh(\pi/p)$ is given by

$$\sigma(\theta) = \begin{cases} \frac{1}{2\pi\lambda} & \text{for } -\theta_2 < \theta < \theta_2 \\ \hat{\sigma}(\theta) & \text{for } -\theta_1 < \theta < -\theta_2 \text{ and } \theta_2 < \theta < \theta_1, \end{cases} \tag{B12}$$

where

$$\hat{\sigma}(\theta) = \frac{|\sin\phi|}{\pi^2\lambda} \frac{\sqrt{(\sin^2\frac{\phi}{2} - \sin^2\frac{\theta_2}{2})(\sin^2\frac{\theta_1}{2} - \sin^2\frac{\phi}{2})}}{\sqrt{(1 + \cos\theta_2)(1 - \cos\theta_1)}} \left[\frac{4(\Pi(n_2, m_2) - \sin^2\frac{\phi}{2}K(m_2))}{\sin^2\phi}\right.$$
$$\left. \times \frac{2p(\cos^2\frac{\theta_1}{2}\Pi(\psi, n_1, m_1) - \cos^2\frac{\phi}{2}F(\psi, m_1))}{(1 + \cos\phi)(\cos\phi - \cos\theta_1)}\right]. \tag{B13}$$

The constants m_1, m_2, n_1, n_2, ψ are given in [30].

APPENDIX C: EXPLICIT DERIVATION OF HAMILTON-JACOBI EQUATION

1. $U(N)$ Chern-Simons theory

Replacing $\tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2} = e^{N^2 F_N}$ in (3.20) and using the relation

$$\frac{1}{D[\theta^{(a)}]} \frac{\partial^2 D[\theta^{(a)}]}{\partial\theta_k^{(a)2}} = N\frac{\partial U_k}{\partial\theta_k^{(a)}} + N^2 U_k^2 \tag{C1}$$

for

$$U_k = \frac{1}{N} \frac{\partial\log[D[\theta^{(a)}]]}{\partial\theta_k^{(a)}} = \frac{1}{2N}\sum_{\substack{j=1\\j\neq k}}^{N}\cot\left(\frac{\theta_k^{(a)} - \theta_j^{(a)}}{2}\right) \tag{C2}$$

we get

$$2\frac{\partial F_N}{\partial A} = \frac{1}{N}\sum_{k=1}^{N}\frac{\partial^2 F_N}{\partial\theta_k^{(a)2}} + \frac{1}{N}\sum_{K=1}^{N}\left(N\frac{\partial F_N}{\partial\theta_k^{(a)}}\right)^2$$
$$+ \frac{2}{N}\sum_{k=1}^{N}U_k N\frac{\partial F_N}{\partial\theta_k^{(a)}} + \frac{1}{N^2}\sum_{k=1}^{N}\frac{\partial U_k}{\partial\theta_k^{(a)}} + \frac{1}{N}\sum_{k=1}^{N}U_k^2. \tag{C3}$$

Calculating

$$\frac{1}{N^2}\sum_{k=1}^{N}\frac{\partial U_k}{\partial\theta_k^{(a)}} = -\frac{1}{4N^3}\sum_{k=1}^{N}\sum_{\substack{j=1\\j\neq k}}^{N}\frac{1}{\sin^2\left(\frac{\theta_k^{(a)} - \theta_j^{(a)}}{2}\right)} \tag{C4}$$

we note that the right-hand side is *zero* in the large N limit when $\theta_k^{(a)} \neq \theta_j^{(a)}$. It only gives a nonzero contribution for $\theta_k^{(a)} \approx \theta_j^{(a)}$. In the large N limit we define continuous distribution functions $\sigma_a(\theta)$ for $\{\theta_i^{(a)}\}$ given by (3.5)

$$\theta_i^{(a)} = \theta^{(a)}(x). \tag{C5}$$

Thus $\theta_k^{(a)} - \theta_j^{(a)} \approx \frac{|k-j|}{N\sigma_{(a)}(\theta_k)}$, also in this limit the *sum* will be replaced by integration and all the partial derivatives by corresponding functional derivatives

$$\frac{1}{N}\sum_{k=1}^{N} \rightarrow \int\sigma_a(\theta)d\theta;$$
$$N\frac{\partial}{\partial\theta_k^{(a)}} \rightarrow \frac{\partial}{\partial\theta}\frac{\delta}{\delta\sigma_a(\theta)}. \tag{C6}$$

Finally using the identity

$$\sum_{\substack{j=1\\j\neq k}}^{N\to\infty}\frac{1}{(j-k)^2} = \frac{\pi^2}{3}$$

the equation (C3) reduced to (3.23). Here we have neglected the term $\frac{\partial^2 F_N}{\partial\theta_k^{(a)2}}$ as it is $\mathcal{O}(\frac{1}{N})$ in the large N limit.

2. $Sp(N)$ Chern-Simons theory

The derivation of the Hamilton-Jacobi equation for $Sp(N)$ CS theory falls in the same line as that of a

$U(N)$ theory once we define the eigenvalue density for the eigenvalues and their mirror images (4.11).

The function $\tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2}(S^3/\mathbb{Z}_p, Sp(N), N, k)$ is given by

$$\tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2}(S^3/\mathbb{Z}_p, Sp(N), N, k)$$
$$= \sum_{y_i(\mathcal{R})} \frac{\det[\sin(N\theta_j(\mathcal{R}_1)y_i(\mathcal{R}))]}{D[\theta(\mathcal{R}_1)]}$$
$$\times \frac{\det[\sin(N\theta_j(\mathcal{R}_2)y_i(\mathcal{R}))]}{D[\theta(\mathcal{R}_2)]} e^{-\frac{AN}{2}\sum_{i=1}^{N} y_i(\mathcal{R})^2}. \quad (C7)$$

The quantity $\tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2}$ in (C7) will satisfy (3.20) with $A = \pi p\lambda$. The quantity $\log[D[\theta]]$ for $Sp(N)$ theory has the form

$$\log[D[\theta]] = \frac{1}{4}\sum_{i=-N}^{N} \log[\sin\theta_i]$$
$$+ \frac{1}{4}\sum_{k=-N}^{N}\sum_{\substack{j=-N\\j\neq k}}^{N} \log\left[\sin\left(\frac{\theta_k-\theta_j}{2}\right)\right]. \quad (C8)$$

For $Sp(N)$ theory we define the quantity U_k as follows:

$$U_k = \frac{1}{N}\frac{\partial\log[D[\theta^{(a)}]]}{\partial\theta_k^{(a)}}$$
$$= \frac{1}{2N}\cot\theta_k^{(a)} + \frac{1}{2N}\sum_{\substack{j=-N\\j\neq k}}^{N}\cot\left(\frac{\theta_k^{(a)}-\theta_j^{(a)}}{2}\right) \quad (C9)$$

and find that it satisfies

$$\frac{1}{N^2}\frac{\partial U_k}{\partial\theta_k^{(a)}} = -\frac{1}{4N^3}\sum_{\substack{j=-N\\j\neq k}}^{N}\frac{1}{\sin^2\left(\frac{\theta_k^{(a)}-\theta_j^{(a)}}{2}\right)} - \frac{3}{4N^3}\frac{1}{\sin^2\theta_k}. \quad (C10)$$

In the large N limit the only contribution will come from those θ_j which are close to θ_k. Thus $\theta_k^{(a)} - \theta_j^{(a)} \approx \frac{|k-j|}{2N\hat{\sigma}_{(a)}(\theta_k)}$ and using the identity $\lim_{N\to\infty}\sum_{\substack{j=-N\\j\neq k}}^{N}\frac{1}{(j-k)^2} = \frac{\pi^2}{3}$ we find that

$$\frac{1}{N}\sum_{k=1}^{N}\frac{\partial U_k}{\partial\theta_k^{(a)}} \approx -\frac{1}{N^2}\sum_{k=1}^{N} 4N^2\frac{\pi^2}{3}\hat{\sigma}_{(a)}(\theta_k). \quad (C11)$$

Assuming $\tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2}$ is dominated by a single representation in the large N limit, we use the ansatz $\tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2} = e^{N^2 F_N}$, where F_N can be written

$$F_N = S_N - \frac{1}{N^2}\sum_{a=1}^{2}\left[\frac{1}{4}\sum_{i=-N}^{N}\log[\sin\theta_i^{(a)}]\right.$$
$$\left. + \frac{1}{4}\sum_{k=-N}^{N}\sum_{\substack{j=-N\\j\neq k}}^{N}\log\left[\sin\left(\frac{\theta_k^{(a)}-\theta_j^{(a)}}{2}\right)\right]\right]. \quad (C12)$$

From the above equation we can find two relations,

$$\frac{\partial F_N}{\partial\theta_k^{(a)}} = \frac{\partial S_N}{\partial\theta_k^{(a)}} - \frac{1}{N}U_k,$$
$$\frac{\partial F_N}{\partial A} = \frac{\partial S_N}{\partial A}. \quad (C13)$$

Using the above two relations we can reduce the differential equation of $\tilde{\mathcal{Z}}_{\mathcal{R}_1\mathcal{R}_2}$ to differential equation of S_N

$$\frac{\partial S_N}{\partial A} = \frac{1}{2N}\sum_{K=1}^{N}\left[\left(N\frac{\partial S_N}{\partial\theta_k^a}\right)^2 + \frac{1}{N}\frac{\partial U_k}{\partial\theta_k}\right]. \quad (C14)$$

In the continuum we assume that the density converges to a smooth function and the sum and partial derivative replaced by

$$\lim_{N\to\infty}\frac{1}{2N}\sum_{j=-N}^{N} \to \int \sigma_{(a)}^{sp}(\theta)d\theta \quad 2N\frac{\partial}{\partial\theta_k^{(a)}} \to \frac{\partial}{\partial\theta}\frac{\delta}{\delta\sigma_{(a)}^{sp}(\theta)}\Big|_{\theta=\theta_k^{(a)}}. \quad (C15)$$

Replacing the sum by integral and using (C11), we get the continuum version of (C14) as given by (4.20). With redefinition of $S = 4S'$ the above equation is the same as (3.25) which can be mapped to a Hamilton-Jacobi equation with Hamiltonian

$$H[\sigma, \Pi] = \frac{1}{2}\int \sigma(\theta)^2\left[\left(\frac{\partial\Pi}{\partial\theta}\right)^2 - \frac{\pi^2}{3}\sigma(\theta)^2\right]d\theta. \quad (C16)$$

The Hamiltonian equation of motion is the same as (3.27) with the boundary condition $\sigma(t=0,\theta) = \hat{\sigma}_1(\theta)$ and $\sigma(t=A,\theta) = \hat{\sigma}_2(\theta)$.

APPENDIX D: FROM EIGENVALUE DENSITY TO YOUNG TABLEAU DENSITY

To find the dominant Young tableau density, we can use the large N properties of the Itzykson-Zuber integral

$$I_N(A,B) \equiv \int [dU]e^{N\text{Tr}(AUBU^\dagger)} = \frac{\det||e^{Na_k b_j}||}{\Delta(a)\Delta(b)}. \quad (D1)$$

Here A and B are arbitrary Hermitian matrices, a_k and b_j are their eigenvalues, and $\Delta(a) = \prod_{i<j}(a_i - a_j)$. One nice property of the integral is

$$\frac{\det ||e^{Na_k b_j}||}{\Delta(a)\Delta(b)} = e^{\frac{N}{2}\left[\sum_{k=1}^{N} a_k^2 + \sum_{j=1}^{N} b_j^2\right]} \frac{\det ||e^{-\frac{N}{2}(a_k - b_j)^2}||}{\Delta(a)\Delta(b)}. \quad (D2)$$

Then one can define the quantity

$$J_N(t, A, B) = \frac{1}{t^{\frac{N}{2}}} \frac{\det ||e^{-\frac{N}{2t}(a_k - b_j)^2}||}{\Delta(a)\Delta(b)} \quad (D3)$$

which satisfies the partial differential equation

$$2N\frac{\partial J_N}{\partial t} = \frac{1}{\Delta(a)} \sum_{k=1}^{N} \frac{\partial^2}{\partial a_k^2}[\Delta(a)J_N]. \quad (D4)$$

Equation (D4) is similar to (3.20) with

$$U_k = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq k}}^{N} \frac{1}{a_k - a_j} \quad (D5)$$

and like (3.24) we can also write

$$\lim_{N\to\infty} \frac{1}{N^2} \log[J_N] = S_J[\alpha, \beta, t]$$

$$-\frac{1}{2}\int\!\!\!\!\int \alpha(a)\alpha(a')\log[a - a']$$

$$-\frac{1}{2}\int\!\!\!\!\int \beta(b)\beta(b')\log[b - b'] \quad (D6)$$

with $\alpha(a)$ and $\beta(b)$ being the densities of a and b. $S_J[\alpha, \beta, t]$ satisfies the differential equation

$$\frac{\partial S_J}{\partial t} = \frac{1}{2}\int \alpha(a)\left[\left(\frac{\partial}{\partial a}\frac{\delta S_J}{\delta\alpha(a)}\right)^2 - \frac{\pi^2}{3}\alpha(a)^2\right]da. \quad (D7)$$

One can think of this as the Hamilton-Jacobi equation for the dynamical system with the Hamiltonian

$$H[\rho, \Pi] = \frac{1}{2}\int \rho(a)\left[\left(\frac{\partial}{\partial a}\frac{\delta S_J}{\delta\rho(a)}\right)^2 - \frac{\pi^2}{3}\rho(a)^2\right]da. \quad (D8)$$

We are interested to the solution which connects $\rho(t = 0, a) = \alpha(a)$ and $\rho(t = 1, b) = \beta(b)$ within time $t = 1$. The variational derivative at the end of the trajectory is

$$\frac{\delta S_J}{\delta\alpha(a)} = \Pi(t = 0, a), \qquad \frac{\delta S_J}{\delta\beta(b)} = -\Pi(t = 1, b). \quad (D9)$$

The equation of motion which follows from the above Hamiltonian can be transformed to a single equation (Hopf equation) of the function $f_J(t, a) = \frac{\partial \Pi(a)}{\partial a} + i\pi\rho(a)$,

$$\frac{\partial f_J}{\partial t} + f_J\frac{\partial f_J}{\partial a} = 0. \quad (D10)$$

The general solution of (D10) can be written in parametric form

$$x = R(\xi) + F(\xi)t, \qquad f_J(t, x) = F(\xi).$$

If we introduce two analytic functions,

$$G_+(x) = x + f_J(t = 0, x), \qquad G_-(x) = x - f_J(t = 1, x),$$

then one can show that

$$G_+(G_-(x)) = G_-(G_+(x)) = x. \quad (D11)$$

After replacing $a_k = y_k' = \tau y_k$ and $b_j = \theta_j$ followed by $\tau \to i$ we can transform the Itzykson-Zuber integral to a character of the U_N group

$$\det ||e^{Na_k b_j}|| \to \det ||e^{iN y_k \theta_j}|| = \chi_R(U)\Delta(e^{i\theta}). \quad (D12)$$

Also, the densities of y' and y are related by

$$\rho_\tau(y) = -\frac{\partial x}{\partial y'} = \frac{1}{\tau}\rho\left(\frac{y}{\tau}\right)\Big|_{\tau=i}. \quad (D13)$$

With this we can define two types of velocity

$$U(y) = \frac{\partial}{\partial y}\frac{\delta S_J[\rho_\tau, \sigma]}{\delta\rho(y)}, \qquad U_\tau(y) = \frac{\partial}{\partial y}\frac{\delta S_J[\rho_\tau, \sigma]}{\delta\rho_\tau(y)},$$

$$V(\theta) = -\frac{\partial}{\partial\theta}\frac{\delta S_J[\rho_\tau, \sigma]}{\delta\sigma(\theta)}, \qquad v_2(\theta) = -\frac{\partial}{\partial y}\frac{\delta S[\sigma_1, \sigma_2]}{\delta\sigma_2(\theta)}, \quad (D14)$$

where the quantity $S[\sigma_1, \sigma_2]$ is from (3.24). The above four velocities are related to each other by

$$U(y) = \tau U_\tau(\tau y)|_{\tau=i}, \qquad -V(\theta) + \theta = -v_2(\theta) \quad (D15)$$

from the large N limit of $\chi_R(U) = e^{N^2 \Xi[\rho,\sigma]}$ and using on-shell condition one can show

$$\frac{\partial}{\partial y}\frac{\delta\Xi[\rho, \sigma_1]}{\delta\rho(y)} = \tau U_\tau(\tau y)|_{\tau=i} - y = \frac{A}{2}y. \quad (D16)$$

We can also define two analytic functions with these variables:

$$G_+(x) = x + U_\tau(x) + i\pi\rho_\tau(x) = -\frac{A}{2}x + -\pi\rho_\tau(x),$$

$$G_-(x) = x - V(x) - i\pi\sigma_1(x) = -v_2(x) - i\pi\sigma_1(x). \quad (D17)$$

Hence from (3.45) with $\sigma_1 = \sigma_2$, in this case $t^* = \frac{A}{2}$ and at $t = A$ we have

$$i\pi\sigma^*\left[\theta - \frac{A}{2}(v_2(\theta) + i\pi\sigma_1(\theta))\right] = v_2 + i\pi\sigma_1(\theta). \quad (D18)$$

By using (D13) and (D17) we can show that

$$\pi\rho(-\pi\sigma^*(\theta)) = \theta. \tag{D19}$$

1. $Sp(N)$

To find the Young-Tableau density for the $Sp(N)$ group we can extend the above procedure of $U(N)$ by a simple observation:

$$\det\|\sinh Na_k b_j\| = \frac{1}{2^N} e^{\frac{N}{2}[\sum_{k=1}^{N} a_k^2 + \sum_{j=1}^{N} b_j^2]} \det\|e^{-\frac{N}{2}(a_k - b_j)^2}$$
$$- e^{-\frac{N}{2}(a_k + b_j)^2}\|. \tag{D20}$$

This helps us to define

$$J_N(t, a, b) = \frac{1}{t^{\frac{N}{2}}} \frac{\det\|e^{-\frac{N}{2t}(a_k - b_j)^2} - e^{-\frac{N}{2t}(a_k + b_j)^2}\|}{\Delta(a)\Delta(b)} \tag{D21}$$

which satisfies the same differential equation as (D4), with $\Delta(a) = \prod_{i=1}^{N} a_i \prod_{j<j}^{N} (a_i + a_j)(a_i - a_j)$. Following the same procedure as depicted in (IV B), i.e., identifying $a_{-k} = -a_k$ and the same for b_k also, we can write everything in terms of this new a_k as

$$\log[\Delta(a)] = \frac{1}{4} \sum_{i=-N}^{N} \sum_{\substack{j=-N \\ j \neq k}}^{N} \log[a_i - a_j] + \frac{1}{4} \sum_{i=-N}^{N} \log[2a_i] \tag{D22}$$

with

$$U_k = \frac{1}{N} \log[\Delta(a)] = \frac{1}{N} \sum_{\substack{j=-N \\ j \neq k}}^{N} \frac{1}{a_k - a_j} + \frac{1}{2a_k} \tag{D23}$$

and

$$\frac{1}{N} \frac{\partial U_k}{\partial a_k} \approx -4\frac{\pi^2}{3} \alpha(a_k)^2. \tag{D24}$$

Assuming that J_N at the large N is given by $J_N = e^{N^2 F_N^J}$ and

$$F_N^J = S_N^J - \frac{1}{N^2} (\log[\Delta(a)] + \log[\Delta(b)]), \tag{D25}$$

then $S^J = \lim_{N\to\infty} S_N^J$ will satisfy the differential equation

$$\frac{\partial S^J}{\partial t} = \frac{1}{2} \int \alpha(a) \left[\left(\frac{1}{2} \frac{\partial}{\partial a} \frac{\delta S^J}{\delta \alpha(a)}\right)^2 - 4\frac{\pi^2}{3} \alpha(a)^2 \right] da. \tag{D26}$$

After redefinition of $S^J =\to 4\tilde{S}^J$, the above equation reduces to the Hamilton-Jacobi-like Eq. (D7). Thus the function $f_J(t, a) = v(t, a) + i\pi\rho(t, a)$ follows the Burger's equation with the boundary condition:

$$\text{Im}[f_J(t=0, a)] = \pi\alpha(a), \quad \text{Re}[f_J(t=0, a)] = \frac{\partial}{\partial a} \frac{\delta S^J}{\delta \alpha(a)},$$
$$\text{Im}[f_J(t=1, b)] = \pi\beta(b), \quad \text{Re}[f_J(t=1, b)] = -\frac{\partial}{\partial b} \frac{\delta S^J}{\delta \beta(b)}. \tag{D27}$$

Similar to the $U(N)$ case, after replacing $a_k = y'_k = \tau y_k$ and $b_j = \theta_j$ followed by $\tau \to i\tau$ we get

$$\det\|\sinh Ny'_k \theta_j\| \to \frac{\det\|\sin Ny_k \theta_j\|}{D[\theta]} (i)^N D[\theta]. \tag{D28}$$

We can follow the same procedure as above and prove that when $\sigma_1(\theta) = \sigma_2(\theta)$ the dominant young tableau density satisfies

$$\pi\rho[-\pi\sigma^*(\theta)] = \theta. \tag{D29}$$

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