Comparison of two theories of Type-IIa minimally modified gravity

Antonio De Felice,[1,*] Kei-ichi Maeda,[2,1,†] Shinji Mukohyama,[1,3,‡] and Masroor C. Pookkillath[1,§]

[1]*Center for Gravitational Physics and Quantum Information, Yukawa Institute for Theoretical Physics, Kyoto University, 606-8502, Kyoto, Japan*

[2]*Department of Physics, Waseda University, Shinjuku, Tokyo 169-8555, Japan*

[3]*Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, Kashiwa, Chiba 277-8583, Japan*

(Received 25 April 2022; accepted 15 June 2022; published 18 July 2022)

We investigate two Type-IIa minimally modified gravity theories, namely V (φ) Cold Dark Matter (VCDM) and Cuscuton theories. We confirm that all acceptable Cuscuton solutions are always solutions for VCDM theory. However, the inverse does not hold. We find that VCDM allows for the existence of exact general relativity (GR) solutions with or without the presence of matter fields and a cosmological constant. We determine the conditions of existence for such GR-VCDM solutions in terms of the trace of the extrinsic curvature and on the fields which define the VCDM theory. On the other hand, for the Cuscuton theory, we find that the same set of exact GR solutions (such as Schwarzschild and Kerr spacetimes) is not compatible with timelike configurations of the Cuscuton field and therefore cannot be considered as acceptable solutions. Nonetheless, in Cuscuton theory, there could exist solutions which are not the same but close enough to GR solutions. We also show the conditions to determine intrinsic-VCDM solutions, i.e., solutions which differ from GR and do not belong to the Cuscuton model. We finally show that in cosmology a mapping between VCDM and the Cuscuton is possible, for a generic form of the VCDM potential. In particular, we find that for a quadratic potential in VCDM theory, this mapping is well defined giving an effective redefinition of the Planck mass for the cosmological background solutions of both theories.

DOI: 10.1103/PhysRevD.106.024028

I. INTRODUCTION

Even though general relativity (GR) is a successful theory of gravity, it still needs to explain the dark sector of our universe at large scales in a way the theory and experiments and observations can agree with each other. Hence, exploring modified gravity theories at the cosmological scales has been showing a constantly growing interest [1,2]. In most cases, modified gravity theories introduce some additional degrees of freedom, which are not present in GR. For example, in the scalar-tensor theories of gravity, in addition to the two polarizations of the gravitational waves, we typically have an additional propagating scalar mode [3]. Whereas in vector-tensor theories of gravity, one expects to find five propagating degrees of freedom, in general [4–6] (in addition to the standard model fields). Since all the modifications are amending the Einstein-Hilbert action, it is natural to study the existence and validity of solutions of these modified gravity models also beyond cosmology, describing, e.g., other gravitational systems like black holes, stars, etc. To pass the astrophysical constraints for these new theories, one typically needs some kind of screening mechanisms at least at solar system scales to hide the otherwise additional propagating modes [7–9].

On the other hand, there has been a recent development in a class of modified gravity theories, generally called minimally modified gravity (MMG) [10–14]. These theories do not contain any additional local degrees of freedom other than those that are present in GR. This minimalist's approach is aimed at avoiding the constraints connected to the existence of extra degrees of freedom. The MMG theories are classified into Type-I and Type-II, where Type-I theories are endowed with an Einstein frame and Type-II not [15]. If an MMG has the same propagation speed for both electromagnetic and gravitational waves, then this model is classified as Type-Ia or Type-IIa. On the other hand, if the propagation speed is different between electromagnetic waves and gravitational waves, it is classified as Type-Ib or Type-IIb [16]. Several investigations have been performed for these theories both in the context of astrophysics and cosmology [17–24].

One example of such MMG theory was introduced very recently [13]. It is a Type-IIa theory [16] and it is dubbed

*antonio.defelice@yukawa.kyoto-u.ac.jp
†maeda@waseda.jp
‡shinji.mukohyama@yukawa.kyoto-u.ac.jp
§masroor.cp@yukawa.kyoto-u.ac.jp

V (φ) Cold Dark Matter (VCDM) theory.[1] The construction of the VCDM theory is the following: (1) perform a canonical transformation of GR Hamiltonian; (2) add a cosmological constant in the new canonical frame; (3) add a gauge fixing term which works as a constraint as to have only two degrees of freedom in the gravity sector; (4) perform an inverse canonical transformation as to have a resulting Hamiltonian which differs from GR; (5) make a Legendre transformation in order to obtain the VCDM Lagrangian; (6) add standard matter fields.

Some exact solutions of VCDM theory have been found and studied. In particular, black-holes/vacuum solutions have been explored, see, e.g., [27]. Although the theory, by construction, does not possess any extra degrees of freedom, still it breaks the Birkhoff theorem, and one needs to find the most general solutions compatible with some symmetry and set the free parameters of the solutions either by imposing appropriate boundary conditions, or by matching with observations. This is due to the presence of a shadowy mode, which leads to the presence of additional free parameters other than mass and the cosmological constant. The spherically symmetric static star solutions were also studied in the context of the VCDM theory.[2] It was shown that once we fix the physical boundary for the Misner-Sharp mass of the system the solution exactly matches those of GR [29]. The cosmology of the VCDM theory was also explored and it was shown that the H_0 tension can be reduced and addressed within this theory [30], since the theory allows for general dynamics for $H(z)$ (with $H(z) > 0$) without introducing unstable or ghost degrees of freedom.

Another Type-IIa theory that is discussed in the literature is the Cuscuton theory [31]. If one starts from a scalar tensor theory which, to the standard Einstein Hilbert term, adds a term in the form $P(X,\varphi) = \mu^2\sqrt{-X} - U(\varphi)$, where $X = (\partial\varphi)^2$, provided that the scalar field φ is timelike (and this proves to be a crucial assumption), then in the unitary gauge ($\varphi = t$), it is straightforward to show that the theory has only two degrees of freedom coming from the gravity sector. This theory, for a timelike field φ, defines the Cuscuton theory, which can be regarded, *a posteriori*, as being a Type-IIa MMG theory [16]. Many aspects of the Cuscuton theory have already been explored, see, e.g., the following references [32–41].

In both these theories, VCDM and Cuscuton, there exists a scalar field which is not propagating, leaving only two gravitational degrees of freedom in the gravity sector. This scalar field is associated to the existence of a shadowy mode, which, by definition, obeys an elliptic equation of motion instead of a hyperbolic one. As mentioned above, at least for known solutions within the VCDM theory, the influence of the shadowy mode on background solutions can be removed if an appropriate physical boundary condition is imposed. In other words, the behavior of the shadowy mode is controlled by the physical boundary conditions provided by the environment.

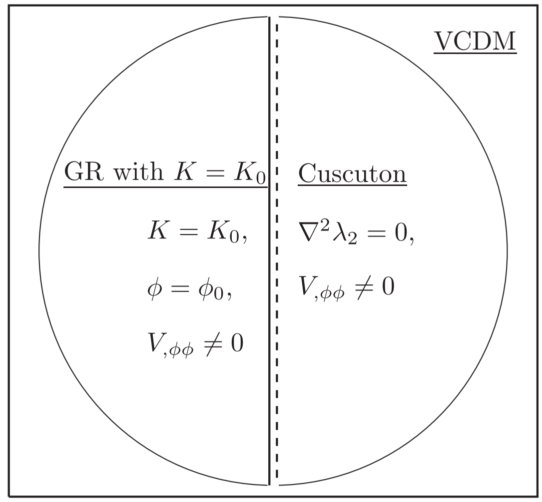

FIG. 1. Summary of the results. We classify solutions (with or without matter) for VCDM and the Cuscuton model. The dashed line corresponds to the solutions in Cuscuton theory that are close enough to (but not exactly equal to) the GR solutions. In the present paper we mainly focus on the cases with $V_{,\phi\phi} \neq 0$ since the VCDM cosmology with a linear potential is indistinguishable from the standard ΛCDM both at the background and at the linear perturbation level and thus is less motivated.

Since both these theories are Type-IIa MMG theories, it is interesting to explore the differences between these two theories. To address this question, it is a good idea to study the known nonperturbative solutions allowed for these theories. On top of that, it is natural to ask if the allowed solutions for both these theories can coincide with solutions of GR or not, and if not, explore their difference. In this work, we address all these issues. We find that under certain conditions there exists a set of solutions in the VCDM which can be exactly matched with those of GR. On the other hand, not all solutions of VCDM are also GR solutions or Cuscuton solutions. Instead, the solutions in the Cuscuton theory cannot exactly coincide with the ones of GR otherwise the Cuscuton field would stop being timelike. This last property of the Cuscuton field does not necessarily exclude the phenomenology of this theory, since, after all, the solutions do not need to be exactly equal to the ones of GR but only close enough to them, compatibly with known experimental and observational constraints.

Figure 1 summarizes the results of this paper. There exists a set of vacuum solutions of VCDM (with generic potential V, i.e., satisfying $V_{,\phi\phi} \neq 0$) which are also solutions of GR (i.e., GR solutions in the presence of minimally coupled matter fields and a cosmological constant) once we impose

[1] This theory should not be confused with other "VCDM" theories, such as those introduced in [25,26].

[2] The same spherically symmetric static star solution valid in VCDM is also valid for another Type-II MMG theory named VCCDM [28].

that both the extrinsic curvature K and the field ϕ to be constants (in space and time). The left semicircle in the figure 1 represents GR solutions in VCDM theory.

In the unitary gauge,[3] solutions of the Cuscuton Lagrangian are also solutions of the VCDM theory when we impose $\nabla^2\lambda_2 = 0$ (λ_2 can be interpreted as being the shadowy mode in the VCDM theory), provided that $V_{,\phi\phi} \neq 0$ and φ remains timelike, as first shown in [16]. As we will see later on, GR solutions cannot be exact solutions of the Cuscuton theory (being φ forced to remain timelike), however there are cases (at least known examples in cosmology exist) for which Cuscuton solutions may be close to GR, provided a well-behaved limit $(\partial\varphi)^2 \to 0$ exists. The dashed line in Figure 1 shows the Cuscuton solutions that are close enough to (but not exactly equal to) GR. Finally, solutions for which $\nabla^2\lambda_2 \neq 0$, and at the same time K is not a constant (in space or time) are VCDM-intrinsic solutions, i.e., solutions which differ from GR and which do not belong to the Cuscuton theory.

The rest of this paper is organized as follows. In Sec. II we investigate the condition under which the VCDM admits solutions of ΛGR, i.e., solutions of GR in the presence of a cosmological constant. We find these conditions by comparing the VCDM Hamilton equations of motion for a general background to those of ΛGR. We show in particular in this section that, e.g., the Schwarzschild-de Sitter and the Kerr-de Sitter are valid vacuum solutions in VCDM theory. Then in Sec. III we investigate VCDM solutions with matter. In particular we study the weak field limit of this theory and confirm that it reduces to that of GR, compatibly with previous studies [27]. We also show that VCDM admits solutions of GR in the presence of minimally coupled matter fields. For this purpose we introduce a four dimensional covariant action which reduces to the one of VCDM after choosing the unitary gauge for one of the fields. Furthermore, we discuss the cosmological not-necessarily-flat background and show the reconstruction of a given $H(z)$ for VCDM. Subsequently, in Sec. IV we briefly discuss Cuscuton theory and discuss various backgrounds (including Schwarzschild and Kerr ones) which are perfectly valid in VCDM, but which are, on the other hand, not acceptable in the Cuscuton theory. We also investigate an exact mapping on a cosmological background between VCDM and the Cuscuton theory. Finally we give our concluding remarks in Sec. V.

Notation: the Latin letters are used for the three dimensional spatial indices for example $a, b, c, \ldots = 1, 2, 3$, while the Greek letter are used to indicate four dimensional spacetime indices $\alpha, \beta, \gamma, \ldots = 0, 1, 2, 3$. We work in the units for which $c = 1$. Also we have the space time metric signature convention $(-,+,+,+)$. Finally, by $V(\phi)$ we will denote the potential term of VCDM theory and ϕ denotes the scalar field of VCDM theory. Instead $U(\varphi)$ denotes the potential in the Cuscuton theory and φ denotes the Cuscuton scalar field.

[3]Since the Cuscuton field is bound to be timelike, it is always possible for acceptable solutions in Cuscuton theory to pick up the φ-unitary gauge.

II. VACUUM GR SOLUTIONS IN VCDM

From the previous investigations of different spherically symmetric solutions of VCDM theory (see, e.g., [27,29]), we know that there exist solutions inside VCDM which are the same as those of GR, provided that we set appropriate physical boundary conditions for the shadowy mode, e.g., the finiteness of the (generalized) Misner-Sharp mass for a spherically symmetric isolated compact gravitational body/system. Nevertheless, even though these GR/VCDM solutions do exist, VCDM theory, by construction, is different from GR. In particular, the presence of the shadowy mode, by construction, implies the existence of a mode whose spatial dependence is determined by an elliptic equation of motion, which requires a preferred slicing where to set boundary conditions. Therefore, by construction, the theory, since it requires fixing boundary conditions on this field, is bound to pick up a natural slicing for the theory which on the other hand breaks the general 4-D diffeomorphism invariance. At the same time, the constraints which define the theory are such that VCDM has the same number of gravitational propagating modes of GR, namely the two standard tensor polarization of GR. In summary VCDM differs from GR although it shares the same physical degrees of freedom. Hence, it is natural to ask whether there exist (or not) VCDM background solutions which exactly match GR solutions, and if so, to determine the conditions of existence of such solutions. In this section, we study these conditions of equality of the solutions in VCDM theory compared to GR solutions in the presence, at most, of a cosmological constant. We will extend this discussion in the presence of matter fields in Sec. III B.

A. Vacuum VCDM equations of motion

In the following we will work in the VCDM-natural slicing, the one which sets the shadowy mode to fulfill a Laplacian equation of motion. After having chosen this slicing, we will make use of the standard ADM splitting for the metric. Since we are looking for solutions which are required, by assumption, to reduce to the same solutions of GR for a generic background/slicing, then we have that the VCDM three dimensional metric γ_{ab}, lapse N, and shift N_a fields are the same as those of GR.

As a consequence, after finding the equations of motion for VCDM in the unitary gauge for a generic background, we will consider the variables γ_{ab}, N, and N_a as satisfying also the equations of motion of GR. This, in turn, will lead to imposing some nontrivial conditions in the VCDM theory, that we want to determine.

On using the $1+3$ ADM splitting for a generic background, we find it convenient to determine the equations of motion by using the Hamiltonian approach. Hence, first of all, we write down the Hamiltonian for VCDM in vacuum as follows

$$H = \int dt d^3x \sqrt{\gamma}\Big\{\lambda_C(M_P^2\phi - \gamma_{ab}\tilde{\pi}^{ab}) - 2N^a D_b \tilde{\pi}_a{}^b - N\left[\frac{1}{2}M_P^2 R - \frac{2}{M_P^2}\left(\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \frac{1}{2}\tilde{\pi}^a{}_a\tilde{\pi}^b{}_b\right) - M_P^2 V(\phi)\right] + M_P^2\lambda_{\rm gf}^a D_a\phi + \lambda_\phi\tilde{\pi}_\phi\Big\}, \tag{1}$$

whereas the presence of other matter fields will be discussed in Sec. III B. In the expression of Eq. (1) we have that λ_C, λ_ϕ, $\lambda_{\rm gf}^a$, N^a, N are to be considered as Lagrange multipliers which set all the constraints of the theory. Also, we have defined $\tilde{\pi}^{ab} \equiv \pi^{ab}/\sqrt{\gamma}$ and $\tilde{\pi}_\phi \equiv \pi_\phi/\sqrt{\gamma}$, where π^{ab} and π_ϕ are the momenta conjugate to the metric variables γ_{ab} and the VCDM field ϕ respectively. Here and in the following R represents the 3-D Ricci scalar.

From the above VCDM Hamiltonian (1), for a generic background, we have to fulfill all the following constraints, which are set by the above mentioned Lagrangian multipliers:

$$0 \approx \mathcal{C}_1 \equiv \sqrt{\gamma}\left[\frac{2}{M_P^2}\left(\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \frac{1}{2}\tilde{\pi}^a{}_a\tilde{\pi}^b{}_b\right) - \frac{1}{2}M_P^2 R + M_P^2 V(\phi)\right], \tag{2}$$

$$0 \approx \mathcal{C}_2 \equiv \sqrt{\gamma}\tilde{\pi}_\phi, \tag{3}$$

$$0 \approx \mathcal{C}_3 \equiv \sqrt{\gamma}(M_P^2\phi - \gamma_{ab}\tilde{\pi}^{ab}), \tag{4}$$

$$0 \approx \mathcal{C}_{4a} \equiv -2\sqrt{\gamma}D_b\tilde{\pi}_a{}^b + \sqrt{\gamma}\tilde{\pi}_\phi D_a\phi, \tag{5}$$

$$0 \approx \mathcal{C}_{5a} \equiv \sqrt{\gamma}M_P^2 D_a\phi. \tag{6}$$

As a consequence of the constraints $\mathcal{C}_3$ and $\mathcal{C}_{5a}$, we find that $D_a\tilde{\pi}^b{}_b \approx 0$ on the surface constraint. Now, setting the time evolution of these constraints to vanish generically leads to equations which set the value for the Lagrange multipliers on the solutions. However, for $\mathcal{C}_{4a}$, this does not happen because they represent first class constraints for the system[4] and they show that the three dimensional diffeomorphism invariance holds for this theory. In particular we find that

$$\{\mathcal{C}_2, H\} \approx 0 \to \quad D_a\lambda_{\rm gf}^a = \lambda_C + NV_{,\phi}, \tag{7}$$

where $\{f,g\}$ denotes the Poisson bracket of f and g.[5] Also we have

$$\{\mathcal{C}_3, H\} \approx 0 \to$$
$$D_aD^aN + N\left[V + \frac{1}{M_P^4}(\tilde{\pi}^a{}_a\tilde{\pi}^b{}_b - 4\tilde{\pi}^{ab}\tilde{\pi}_{ab})\right] + \lambda_\phi = 0, \tag{8}$$

which can be used to fix the lapse N. Then

$$\{\mathcal{C}_{5a}, H\} \approx 0: \quad D_a\lambda_\phi = 0, \tag{9}$$

which sets the field λ_ϕ. The equation of motion which instead fixes λ_C is found as follows

$$\{\mathcal{C}_1, H\} \approx 0:$$
$$D_aD^a\lambda_C + \lambda_C\left[V + \frac{1}{M_P^4}(\tilde{\pi}^a{}_a\tilde{\pi}^b{}_b - 4\tilde{\pi}^{ab}\tilde{\pi}_{ab})\right] - \lambda_\phi V_{,\phi} \approx 0. \tag{10}$$

So far, the treatment was fully general. We can now proceed to find the general dynamical equations of motion for VCDM. Let us start by writing the following ones

$$\dot{\gamma}_{ab} = \{\gamma_{ab}, H\} = \frac{2N}{M_P^2}(2\tilde{\pi}_{ab} - \gamma_{ab}\tilde{\pi}^c{}_c) + D_aN_b + D_bN_a - \lambda_C\gamma_{ab}, \tag{11}$$

$$\dot{\phi} = \{\phi, H\} = \lambda_\phi, \tag{12}$$

$$\dot{\pi}_\phi = \{\pi_\phi, H\} = 0. \tag{13}$$

From the Eq. (11) we can find

$$\tilde{\pi}^{ab} = \frac{M_P^2}{2}(K^{ab} - K\gamma^{ab}) - \frac{M_P^2}{2}\frac{\lambda_C}{N}\gamma^{ab}, \tag{14}$$

where we have used the definition of the extrinsic curvature

$$K_{ab} \equiv \frac{1}{2N}(\dot{\gamma}_{ab} - D_aN_b - D_bN_b), \tag{15}$$

[4]The term $\mathcal{C}_{4a}$ is the redefinition of the momentum constraint $\tilde{\mathcal{C}}_{4a} \equiv -2\sqrt{\gamma}D_b\tilde{\pi}_a{}^b \approx 0$, which is given as $\mathcal{C}_{4a} \equiv \tilde{\mathcal{C}}_{4a} + \sqrt{\gamma}\tilde{\pi}_\phi D_a\phi \approx 0$, and which is just a linear combination of constraints. However, with this redefinition the momentum constraint is now a first class constraint. In other words the Poisson bracket of $\mathcal{C}_{4a}$ with any other constraint vanishes. That is, there is a internal gauge freedom in the 3-D space, which ensures 3-D diffeomorphism invariance.

[5]More in detail $\{f,g\} = \sum_i \int d^3z (\frac{\delta f}{\delta q_i}\frac{\delta g}{\delta \pi^i} - \frac{\delta f}{\delta \pi^i}\frac{\delta g}{\delta q_i})$, and the sum is over all the dynamical fields, γ_{ab} and ϕ in this section.

which can be used anywhere in the equations of motion as to write them for the variable K^{ab}. Now we want to write the following dynamical equations

$$\begin{aligned}\dot{\tilde{\pi}} &= \{\tilde{\pi}^a{}_a, H\}\\ &\approx -M_{\rm P}^2 N\left[V + \frac{1}{M_{\rm P}^4}(\tilde{\pi}^a{}_a\tilde{\pi}^b{}_b - 4\tilde{\pi}^{ab}\tilde{\pi}_{ab})\right] - M_{\rm P}^2 D_a D^a N\\ &= M_{\rm P}^2\lambda_\phi = M_{\rm P}^2\dot{\phi}, \end{aligned}\tag{16}$$

where $\tilde{\pi} \equiv \tilde{\pi}^a{}_a$ and we have used the constraint for N Eq. (8) as well as $D_a\mathcal{C}_3 = 0$ from Eq. (4), (6), and (13).

Also we find, by taking the trace of Eq. (14), that

$$\tilde{\pi} = -M_{\rm P}^2 K - \frac{3}{2}M_{\rm P}^2\frac{\lambda_C}{N},\tag{17}$$

which, on using also Eq. (4), reduces to

$$\frac{\lambda_C}{N} = -\frac{2}{3}(K+\phi).\tag{18}$$

On replacing the above equation back into Eq. (14), we get

$$\tilde{\pi}^{ab} = \frac{M_{\rm P}^2}{2}(K^{ab} - K\gamma^{ab}) + \frac{1}{3}M_{\rm P}^2(K+\phi)\gamma^{ab}.\tag{19}$$

Then we write the full Hamilton equations for $\dot{\tilde{\pi}}^{ab} = \{\tilde{\pi}^{ab}, H\}$, as

$$\begin{aligned}\dot{\tilde{\pi}}^{ab} = &-\frac{1}{2}M_{\rm P}^2 N R^{ab} + \frac{1}{4}M_{\rm P}^2\gamma^{ab}NR - \frac{1}{2}M_{\rm P}^2\gamma^{ab}NV + \frac{5}{2}\lambda_C\tilde{\pi}^{ab} - \frac{4N\tilde{\pi}^{ac}\tilde{\pi}^b{}_c}{M_{\rm P}^2} + \frac{3N\tilde{\pi}^{ab}\tilde{\pi}^c{}_c}{M_{\rm P}^2} + \frac{\gamma^{ab}N\tilde{\pi}_{cd}\tilde{\pi}^{cd}}{M_{\rm P}^2}\\ &-\frac{\gamma^{ab}N\tilde{\pi}^c{}_c\tilde{\pi}^d{}_d}{2M_{\rm P}^2} - \frac{1}{2}M_{\rm P}^2\gamma^{ab}\lambda_C\phi + \frac{1}{2}M_{\rm P}^2 D^b D^a N + N^c D_c\tilde{\pi}^{ab} - \frac{1}{2}M_{\rm P}^2\gamma^{ab}D_c D^c N - \tilde{\pi}^b{}_c D^c N^a - \tilde{\pi}^a{}_c D^c N^b,\end{aligned}\tag{20}$$

where the left-hand side can be replaced with the time derivative of Eq. (19). So far, we have considered the Hamilton equations of motion for a general background in VCDM. We can now proceed to find the conditions for them to be satisfied also by ΛGR solutions.

B. Solutions in VCDM which reduce to ΛGR solutions

Let us now consider ΛGR solutions. This means we consider the same solution for the lapse N, the shift N_a, and the spatial 3-D metric γ_{ab} which exist for GR on a given slicing. This also implies that the expressions of K_{ab} in VCDM and ΛGR will coincide on this slicing. It should be noticed that, in the case of ΛGR, we have only first class constraints, so that all the Lagrange multipliers N and N^a cannot be determined by the Hamiltonian procedure. By calling the GR-momentum conjugate to γ_{ab} as Π^{ab}, then in ΛGR, we find that the equations of motion $\dot{\gamma}_{ab} = \{\gamma_{ab}, H\}$ lead to

$$\tilde{\Pi}^{ab} = \frac{M_{\rm P}^2}{2}(K^{ab} - K\gamma^{ab}),\tag{21}$$

where we have $\tilde{\Pi}^{ab} \equiv \Pi^{ab}/\sqrt{\gamma}$. Taking the trace of the above relation we also get

$$\tilde{\Pi} \equiv \tilde{\Pi}^a{}_a = -M_{\rm P}^2 K.\tag{22}$$

The ΛGR constraints can be written as

$$\frac{\mathcal{C}_1^{\rm GR}}{\sqrt{\gamma}} = M_{\rm P}^2\Lambda - \frac{1}{2}M_{\rm P}^2 R + \frac{2(\tilde{\Pi}_{ab}\tilde{\Pi}^{ab} - \frac{1}{2}\tilde{\Pi}^a{}_a\tilde{\Pi}^b{}_b)}{M_{\rm P}^2} \approx 0,\tag{23}$$

$$\frac{\mathcal{C}_{4a}^{\rm GR}}{\sqrt{\gamma}} = -2D_b\tilde{\Pi}_a{}^b \approx 0.\tag{24}$$

On using the Hamilton equations of motion, we can also find the time evolution of conjugate momenta. For instance, we have

$$\begin{aligned}\dot{\tilde{\Pi}} &= -\frac{3}{2}M_{\rm P}^2\Lambda N + \frac{1}{4}M_{\rm P}^2 NR + \frac{3N\tilde{\Pi}_{ab}\tilde{\Pi}^{ab}}{M_{\rm P}^2} - \frac{N\tilde{\Pi}^a{}_a\tilde{\Pi}^b{}_b}{2M_{\rm P}^2} + N^a D_a\tilde{\Pi}^b{}_b - M_{\rm P}^2 D_a D^a N\\ &\approx -M_{\rm P}^2\Lambda N + \frac{4N\tilde{\Pi}_{ab}\tilde{\Pi}^{ab}}{M_{\rm P}^2} - \frac{N\tilde{\Pi}^a{}_a\tilde{\Pi}^b{}_b}{M_{\rm P}^2} + N^a D_a\tilde{\Pi}^b{}_b - M_{\rm P}^2 D_a D^a N,\end{aligned}\tag{25}$$

where we have used the Hamiltonian constraint, Eq. (23). Furthermore we have that

$$\dot{\tilde{\Pi}}^{ab} = -\frac{1}{2}M_{\rm P}^2\Lambda\gamma^{ab}N - \frac{1}{2}M_{\rm P}^2 NR^{ab} + \frac{1}{4}M_{\rm P}^2\gamma^{ab}NR - \frac{4N\tilde{\Pi}^{ac}\tilde{\Pi}^b{}_c}{M_{\rm P}^2} + \frac{3N\tilde{\Pi}^{ab}\tilde{\Pi}^c{}_c}{M_{\rm P}^2} + \frac{\gamma^{ab}N\tilde{\Pi}_{cd}\tilde{\Pi}^{cd}}{M_{\rm P}^2} - \frac{\gamma^{ab}N\tilde{\Pi}^c{}_c\tilde{\Pi}^d{}_d}{2M_{\rm P}^2} + \frac{1}{2}M_{\rm P}^2 D^bD^aN + N^cD_c\tilde{\Pi}^{ab} - \frac{1}{2}M_{\rm P}^2\gamma^{ab}D_cD^cN - \tilde{\Pi}^b{}_cD^cN^a - \tilde{\Pi}^a{}_cD^cN^b. \quad (26)$$

We are now ready to study the conditions under which the solutions of VCDM and GR coincide with each other, at least locally. Here the logic is to apply the above GR-solutions as to constrain the VCDM Hamiltonian constraints/equations of motion.

The first thing we notice by comparing Eq. (14) with Eq. (21) is that

$$\tilde{\pi}^{ab} = \tilde{\Pi}^{ab} - \frac{M_{\rm P}^2}{2}\frac{\lambda_C}{N}\gamma^{ab}. \quad (27)$$

On applying the operator D_b on both sides of this equation and on using the momentum constraint in VCDM Eq. (5) together with the gauge fixing constraint Eq. (6) and Eq. (24), we find

$$D_a(\lambda_C/N) = 0. \quad (28)$$

On using Eq. (27) the Hamiltonian constraint in VCDM written in terms of $\tilde{\Pi}^{ab}$ is

$$\frac{\mathcal{C}_1}{\sqrt{\gamma}} = M_{\rm P}^2\left[-\frac{3\lambda_C^2}{4N^2} - \frac{1}{2}R + V\right] + \frac{\lambda_C\tilde{\Pi}^a{}_a}{N} + \frac{2\tilde{\Pi}_{ab}\tilde{\Pi}^{ab} - \tilde{\Pi}^a{}_a\tilde{\Pi}^b{}_b}{M_{\rm P}^2} = 0. \quad (29)$$

On comparing the above expression with the Hamiltonian constraint Eq. (23) of ΛGR, and using Eq. (22), we require a second condition to hold, namely

$$\Lambda = V(\phi) - \frac{3}{4}\frac{\lambda_C^2}{N^2} - \frac{\lambda_C}{N}K. \quad (30)$$

By taking a spatial covariant derivative of the above expression, we reach another condition, namely

$$D_aK = 0, \quad \text{or} \quad D_a\tilde{\Pi}^b{}_b = 0, \quad (31)$$

where we have used the gauge constraint Eq. (6) and Eq. (28). For the special case $\lambda_C = 0$, from Eq. (18) we know that $K = -\phi$, which, after taking a covariant derivative and using Eq. (6), again leads to the condition Eq. (31).

In VCDM the time derivative of $\tilde{\pi}$, using also Eq. (17), leads to

$$\begin{aligned}\dot{\tilde{\pi}} &= \dot{\tilde{\Pi}} - \frac{3}{2}M_{\rm P}^2\frac{d}{dt}\left(\frac{\lambda_C}{N}\right) \\ &= \frac{3M_{\rm P}^2{\lambda_C}^2}{4N} - M_{\rm P}^2NV(\phi) - \lambda_C\tilde{\Pi}^a{}_a + \frac{4N\tilde{\Pi}_{ab}\tilde{\Pi}^{ab}}{M_{\rm P}^2} \\ &\quad - \frac{N\tilde{\Pi}^a{}_a\tilde{\Pi}^b{}_b}{M_{\rm P}^2} - \frac{3M_{\rm P}^2N^aD_a\lambda_C}{2N} + \frac{3M_{\rm P}^2\lambda_CN^aD_aN}{2N^2} \\ &\quad + N^aD_a\tilde{\Pi}^b{}_b - M_{\rm P}^2D_aD^aN \\ &= \frac{3M_{\rm P}^2{\lambda_C}^2}{4N} - M_{\rm P}^2NV(\phi) - \lambda_C\tilde{\Pi}^a{}_a + \dot{\tilde{\Pi}} + M_{\rm P}^2\Lambda N, \end{aligned} \quad (32)$$

where we have used Eqs. (27), (25) and (31). Then this result together with Eq. (22) lead to

$$-\Lambda - \frac{3}{2}\frac{1}{N}\frac{d}{dt}\left(\frac{\lambda_C}{N}\right) = \frac{\lambda_C}{N}K + \frac{3\lambda_C^2}{4N^2} - V(\phi). \quad (33)$$

Comparing the condition Eq. (30) with the above, for consistency we reach the condition

$$\frac{d}{dt}\left(\frac{\lambda_C}{N}\right) = 0, \quad \text{or} \quad \lambda_C = \lambda_0 N, \qquad \lambda_0 = \text{constant}. \quad (34)$$

The above relation, used in Eq. (18), leads to

$$K = -\phi - \frac{3}{2}\lambda_0, \quad (35)$$

which also gives $\dot{K} = -\dot{\phi}$.

Substituting this last relation for K together with Eq. (34) into Eq. (33) and taking a time derivative we obtain

$$\dot{\phi}(\lambda_0 + V_{,\phi}) = 0, \quad (36)$$

which is solved in general only for a constant ϕ, and we will not consider the case of a special linear form for the potential in detail, as giving trivial results in cosmology.[6] Therefore for a general VCDM potential, we have

[6]In fact, for the case of a linear potential $V = \beta_0 + \beta_1\phi$, we would have, as a possible solution of (36), that $\lambda_0 = -\beta_1$. This would not set ϕ to be necessarily constant, leaving $\phi = \phi(t)$, as well as $K = K(t)$. This is what actually happens in cosmology, as a linear potential makes VCDM solutions exactly reduce to ΛCDM, see, e.g., [13]. However, even for a linear potential V, there could be nontrivial, non-GR, VCDM-solutions when K becomes space-and-time dependent.

$$\phi=\phi_0,\quad K=K_0=-\phi_0-\frac{3}{2}\lambda_0,\quad \Lambda=\lambda_0\phi_0+\frac{3}{4}\lambda_0^2+V(\phi_0),\tag{37}$$

or in other words when both ϕ and K are constants VCDM is equivalent to ΛGR with an effective cosmological constant Λ given by the expression in Eq. (37). On considering the other equation of motion for $\dot{\tilde{\pi}}^{ab}$, we find

$$\frac{1}{2}M_{\rm P}^2\Lambda\gamma^{ab}N-\frac{3}{8}M_{\rm P}^2{\lambda_0}^2\gamma^{ab}N-M_{\rm P}^2\lambda_0\gamma^{ab}NK$$
$$-\frac{1}{2}M_{\rm P}^2\gamma^{ab}NV(\phi)-\lambda_0\gamma^{ab}N\tilde{\Pi}^c{}_c-\frac{1}{2}M_{\rm P}^2\lambda_0\gamma^{ab}N\phi=0,\tag{38}$$

which can be shown to lead to

$$\frac{1}{2}M_{\rm P}^2\gamma^{ab}N[\Lambda-V(\phi_0)-\frac{1}{4}\lambda_0(3\lambda_0+4\phi_0)]=0,\tag{39}$$

which is automatically satisfied.

As for the other Lagrange multipliers of VCDM, we find that

$$\lambda_\phi=\dot{\phi}=0,\tag{40}$$

whereas the equation of motion defining N is automatically satisfied as well as the one defining $\lambda_C=\lambda_0 N$. The only leftover nontrivial equation of motion is then

$$D_a\lambda_{\rm gf}^a=N[V_{,\phi_0}-\frac{2}{3}(K_0+\phi_0)],\tag{41}$$

which can be used in order to solve for $\lambda_{\rm gf}^a$. It should be noticed that VCDM-ΛGR solutions do not necessarily have a vanishing $D_a\lambda_{\rm gf}^a$. These VCDM solutions were first found, for the particular case of a static, spherically symmetric background, in [27], which were shown to correspond to the Schwarzschild-de Sitter solutions of GR in the constant-K slicing.

In summary, any ΛGR solution in a constant-K slicing can be embedded in the VCDM theory as a solution. We have all relevant equations that determine the VCDM fields once a ΛGR solution and a constant-K slicing are specified.

C. Example: Kerr-de Sitter solutions

As a lemma based on the previous discussion, for the special case of $K_0=0$, i.e., in the maximal slicing, we have an effective cosmological constant given by $\Lambda=V(\phi_0)-\frac{1}{3}\phi_0^2$. Here we are assuming that any nontrivial cosmological time dependence for ϕ can be set to be negligible at astrophysical scales. Beside the aforementioned case of the Schwarzschild-de Sitter solutions of ΛGR first found in [27], we want to add here as a nontrivial case, the Kerr-de Sitter solutions in Boyer–Lindquist coordinates, which describe the empty spacetime around an axisymmetric distribution of matter. We are now going to show that they are solutions not only for ΛGR, but also for the VCDM theory. In fact, one has that the three dimensional line element for this background in this slicing can be written as

$$ds_{(3)}^2=\frac{a^2z^2+r^2}{\Delta}dr^2+\frac{a^2z^2+r^2}{(1+\frac{\Lambda a^2z^2}{3})(1-z^2)}dz^2$$
$$+\left[(a^2+r^2)^2\left(1+\frac{\Lambda a^2z^2}{3}\right)-\Delta a^2(1-z^2)\right]$$
$$\times\frac{(1-z^2)d{\theta_2}^2}{(a^2z^2+r^2)(1+\frac{\Lambda a^2}{3})^2},\tag{42}$$

where $\Delta=(a^2+r^2)(1-\frac{\Lambda r^2}{3})-2mr$, $z=\cos\theta_1$, and θ_2 is the angle which defines the axis of symmetry. Here a is standard Kerr spin parameter and m is the mass parameter. Then for the same background solution, the lapse and shift vector can be written as

$$\frac{1}{N^2}=\frac{[3(z^2-1)a^2\Delta+(a^2+r^2)^2(\Lambda a^2z^2+3)](\Lambda a^2+3)^2}{9\Delta(a^2z^2+r^2)(\Lambda a^2z^2+3)},\tag{43}$$

$$N_a\,dx^a=\left[\Delta-\left(1+\frac{\Lambda a^2z^2}{3}\right)(a^2+r^2)\right]$$
$$\times\frac{a(1-z^2)}{(a^2z^2+r^2)(1+\frac{\Lambda a^2}{3})^2}d\theta_2,\tag{44}$$

which lead to

$$K=\gamma^{ab}K_{ab}=\frac{\gamma^{ab}}{2N}(\dot{\gamma}_{ab}-D_aN_b-D_bN_a)$$
$$=-\frac{1}{N}\gamma^{ab}D_{(a}N_{b)}=0,\tag{45}$$

and confirms that this GR-solution is also a solution for VCDM. We also discuss the existence of the McVittie solution in VCDM theory later on.

III. VCDM SOLUTIONS WITH MATTER

A. Weak field solutions

Let us consider the weak field limit, namely a situation in which the matter fields are supposed to source small perturbations around the Minkowski background. The 3D metric, the lapse and the shift can be written as

$$ds_3^2=(1+2\zeta)\delta_{ij}dx^idx^j.\tag{46}$$

$$N = 1 + \alpha, \tag{47}$$

$$N_i = \partial_i \chi, \tag{48}$$

whereas the VCDM fields are instead given by

$$\phi = \phi(t) + \delta\phi, \tag{49}$$

$$\lambda^i_{\rm gf} = \delta^{ij}\partial_j \delta\lambda_2, \tag{50}$$

$$\lambda = \lambda(t) + \delta\lambda. \tag{51}$$

Here λ is a Lagrangian multiplier related to the field λ_C introduced in the VCDM Hamiltonian (1) by $\lambda = \lambda_C/N$, and $\delta\lambda_2$ corresponds to the perturbation of the shadowy mode present in the VCDM theory as will be explained in the discussion after Eq. (69).

For the Minkowski background, the VCDM equations of motion lead to

$$\phi(t) = \phi_0, \tag{52}$$

$$V = \frac{1}{3}\phi_0^3, \tag{53}$$

$$V_{,\phi} = \frac{2}{3}\phi_0, \tag{54}$$

$$\lambda(t) = -\frac{2}{3}\phi_0. \tag{55}$$

These are compatible with our previous finding connecting GR solutions to VCDM solutions. Looking for the first nontrivial corrections, we find the effective Einstein tensor elements and set them equal to the stress-energy tensor elements of a fluid, whereas the equations of motion in VCDM which are not sourced by the matter fields are then solved by themselves. For example we have

$$\nabla^2 \delta\phi = 0, \tag{56}$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ on this background. Along the same lines at leading order (assuming no shear and the fluid velocity to be nonzero, but of subleading order):

$$\nabla^2 \chi = \delta\phi, \tag{57}$$

$$\delta\lambda = -2\dot{\zeta}, \tag{58}$$

$$\nabla^2 \zeta = -\nabla^2 \alpha - \nabla^2 \dot{\chi}, \tag{59}$$

$$2M_{\rm P}^2 \nabla^2 \zeta = -\rho, \tag{60}$$

with ρ satisfying the standard continuity equation as expected. We find that $\nabla^4\zeta = -\nabla^4\alpha$, which on imposing appropriate boundary conditions at infinity, leads to the same results of GR, namely $\zeta = -\alpha$, and the standard Poisson equation for the Newtonian potential.

For the tensor mode, this theory does not modify the dispersion relation from that of GR. Hence this theory is called as Type-IIa MMG theory [16].

B. Covariant action and GR solutions with matter fields

In this subsection we show that under a certain condition, a solution of GR in the presence of a cosmological constant and minimally coupled matter fields can be embedded in VCDM as a consistent solution. For this purpose it is convenient to use a covariant theory which reduces to VCDM in the unitary gauge for the time coordinate. In the following ϕ, α, T, are 4D scalar fields, and their connection with other geometrical objects is determined by the Lagrange multipliers λ, λ_2 and λ_T. Let us start by writing the following gravitational action

$$S_g = M_{\rm P}^2 \int d^4x \sqrt{-g}\bigg\{\frac{1}{2}R^{(4)} - V(\phi) - \frac{3}{4}\lambda^2 - \lambda(\nabla^\sigma n_\sigma + \phi) + \frac{\lambda_2}{\alpha}[\gamma^{\tau\rho}\nabla_\tau\nabla_\rho\phi + n^\rho(\nabla_\rho\phi)\nabla^\sigma n_\sigma] + \lambda_T(1 + g^{\mu\nu}n_\mu n_\nu)\bigg\}, \tag{61}$$

$$n_\mu \equiv -\alpha \nabla_\mu T, \tag{62}$$

$$\gamma^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu. \tag{63}$$

After integrating out the field α by using the equation of motion for λ_T, we find

$$S_g = M_{\rm P}^2 \int d^4x \sqrt{-g}\bigg\{\frac{1}{2}R^{(4)} - V(\phi) - \frac{3}{4}\lambda^2 - \lambda(\nabla^\sigma n_\sigma + \phi) + (-g^{\mu\nu}\nabla_\mu T \nabla_\nu T)^{1/2}\lambda_2[\gamma^{\tau\rho}\nabla_\tau\nabla_\rho\phi + n^\rho(\nabla_\rho\phi)\nabla^\sigma n_\sigma]\bigg\}, \tag{64}$$

$$n_\mu = -(-g^{\mu\nu}\nabla_\mu T \nabla_\nu T)^{-1/2}\nabla_\mu T, \tag{65}$$

$$\gamma^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu, \tag{66}$$

so that $\nabla_\mu T$ is, by construction, timelike. Then on choosing $T = t$, not as the solution of some equations of motion, but rather as a free choice of the time coordinate, we find the following action

$$S_g = \int d^4x N\sqrt{\gamma}\bigg[\frac{M_{\rm P}^2}{2}[R + K_{ij}K^{ij} - K^2 - 2V(\phi)] + \frac{1}{N}\lambda_2 M_{\rm P}^2 \gamma^{ij} D_i D_j \phi - \frac{3M_{\rm P}^2\lambda^2}{4} - M_{\rm P}^2\lambda(K + \phi)\bigg]. \tag{67}$$

This agrees with the action of VCDM. Notice that λ_2 imposes an elliptic equation on ϕ, and vice versa ϕ imposes a Laplacian operator on λ_2. Therefore the original VCDM action can be thought of being the action of Eq. (64) written in T-unitary-gauge.

It should be noted that we can integrate out the field λ by using its own equation of motion[7]

$$\lambda = -\frac{2}{3}(K + \phi). \tag{68}$$

Furthermore, for a generic potential V, we can also integrate out the field ϕ by using its own algebraic equation of motion[8]

$$\phi = F\left[(\gamma^{ij} D_i D_j \lambda_2)/N + \frac{2}{3}K\right]. \tag{69}$$

Finally the VCDM Lagrangian can be written only in terms of the metric variables and $\gamma^{ij} D_i D_j \lambda_2$, the shadowy mode, whose equation of motion is clearly elliptical. This latter field cannot be further integrated out, unless we introduce nonlocal terms into the action, avoiding in this way the Lovelock theorem. Although we have found a covariant theory which reduces to VCDM, the choice of the slicing $T = t$ is precisely chosen because of the presence of the shadowy mode. In fact, the shadowy mode, by its own equation of motion, sets a preferred frame on which its elliptic differential operator is defined. Then the T-unitary gauge is the natural choice for the time coordinate for the above VCDM-covariant Lagrangian. Although this covariant action may seem a redundant knowledge, nonetheless, in same cases, one can use it in a proficient way, for example when the T-equation of motion is needed (which is written in Eq. (A8) of Appendix A, and which, in unitary gauge, can be found only after an appropriate manipulation the other equations of motion) or when it is helpful to have an explicit expression for $T_{\mu\nu}$ (even when it is evaluated, after finding it, in unitary gauge).

Let us now use the covariant action of VCDM, introduced in Eq. (61), in order to show that VCDM indeed admits GR solutions with minimally coupled matter. The modified Einstein equations in covariant VCDM can be written as

$$M_{\rm P}^2 G^\mu{}_\nu = T^\mu{}_\nu + \mathcal{T}_{\rm v}{}^\mu{}_\nu, \tag{70}$$

where $T^\mu{}_\nu$ stands for the total matter field stress energy tensor (i.e., excluding the VCDM contribution). Let us try to find the condition under which we can embed GR solutions in VCDM. In this case we require that $\mathcal{T}_{\rm v}{}^\mu{}_\nu$ should give a cosmological constant contribution. Therefore, as we have also seen in the vacuum case, let us consider the case of $\phi = \phi_0 =$ constant. Furthermore, let us assume that the solution admits $K = K_0 =$ constant, where $K = \nabla^\sigma n_\sigma$ is the trace of the extrinsic curvature induced by the T-coordinate choice.[9] In this case, the equation of motion for λ sets also λ itself to be a constant, i.e., $\lambda = \lambda_0$, on this background, independently of the presence of matter fields since

$$\lambda = -\frac{2}{3}(\nabla^\sigma n_\sigma + \phi) = -\frac{2}{3}(K_0 + \phi_0) = \lambda_0. \tag{71}$$

Now, the equation of motion for α, corresponding to Eq. (A6) of Appendix A, evaluated for a constant λ and ϕ, sets the following constraint on the solution

$$\frac{2M_{\rm P}^2 \lambda_T}{\alpha} = 0, \tag{72}$$

which makes λ_T vanish. Then in this case, we find that the stress energy tensor of VCDM, given in Eq. (A12) of Appendix A, can be rewritten as

$$\mathcal{T}_{\rm v}{}^\mu{}_\nu = -\frac{1}{4}M_{\rm P}^2[4V(\phi_0) + \lambda_0(3\lambda_0 + 4\phi_0)]\delta^\mu{}_\nu = -M_{\rm P}^2 \Lambda \delta^\mu{}_\nu, \tag{73}$$

where the effective cosmological constant on this background is given by

$$\Lambda = \frac{3}{4}\lambda_0^2 + \lambda_0\phi_0 + V(\phi_0), \tag{74}$$

which agrees with Eq. (37).

In summary this shows that all GR solutions, written in the constant-K slicing (whenever this choice of slicing is allowed), are also solutions of VCDM. An example of this case is given in [29], where the extrinsic curvature of the solutions is vanishing, finding indeed that the static profile

[7]Here we can integrate out the Lagrange multiplier λ because its equation of motion is purely algebraic, getting a Lagrangian equivalent to the VCDM Lagrangian.

[8]We should avoid the temptation of integrating out ϕ, by solving the differential equation $D^2\phi = 0$ imposed by the field λ_2 at the level of the Lagrangian, not being an algebraic equation. In fact, this in general leads to a different theory. For instance, on considering an analogue case, i.e., having a similar structure, in classical mechanics, take the following simple model $\mathcal{L} = \lambda_2 \dot{q} - m^2 q^2$. On integrating out q by solving the differential equation imposed by λ_2 as $q = q_0$ would lead to a nonequivalent Lagrangian $\mathcal{L} = -m^2 q_0^2$, which gives no more dynamics for any variable. Instead, one should first integrate by parts $\dot{q}$ giving $\mathcal{L} = -q\dot{\lambda}_2 - m^2 q^2$, and then integrating out q, which has become now a Lagrange multiplier, by using its own algebraic equation of motion, $q = -\dot{\lambda}_2/(2m^2)$, leads to a reduced Lagrangian $\mathcal{L} = \dot{\lambda}_2^2/(4m^2)$, out of which one finds equivalent equations of motion.

[9]This corresponds to a constant-K slicing.

of spherically symmetric stars solutions are also solutions of VCDM.

Motivated from Sec. III A, a more general PPN treatment, which holds at higher order in the post Newtonian expansion, can be performed by looking at the effective $\mathcal{T}_{\rm v}{}^{\mu}{}_{\nu}$ of VCDM found by using the covariant Lagrangian of the previous section, and which is written in Eq. (A12) of Appendix A.

C. Cosmological solutions

Here we look at the dynamics of the cosmological background endowed with Friedmann-Lemaître-Robertson-Walker (FLRW) metric and nonzero spatial curvature. The three dimensional spatial metric is given as

$$\mathrm{d}s_3^2 = \left[a^2\frac{\mathrm{d}r^2}{1-\kappa r^2}+\Phi\right]\mathrm{d}r^2 + a^2r^2(1+\zeta)\left\{\frac{\mathrm{d}z^2}{(1-z^2)}+(1-z^2)\mathrm{d}\theta_2{}^2\right\}, \quad (75)$$

where κ is the curvature constant and the terms in the curly bracket define the two dimensional line element of a unit-radius sphere, being $z=\cos\theta_1$, namely

$$\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta_1{}^2+\sin^2\theta_1\mathrm{d}\theta_2{}^2 = \frac{\mathrm{d}z^2}{(1-z^2)}+(1-z^2)\mathrm{d}\theta_2{}^2, \quad (76)$$

and the lapse is instead defined as

$$N = [-{}^{(4)}g^{00}]^{-1/2} = \bar{N}(t)(1+\alpha), \quad (77)$$

whereas the shift contributes, on a homogeneous and isotropic background, only perturbatively as follows

$$N^i\partial_i = N^r\partial_r \equiv \chi\partial_r. \quad (78)$$

The field variables $\alpha, \chi, \Phi, \zeta$, are linear perturbations and have been introduced in order to derive the background equations of motion.

We also define the scalar and vector fields in the VCDM Lagrangian as in

$$\phi = \bar{\phi}(t)+\delta\phi, \quad \lambda=\bar{\lambda}(t)+\delta\lambda, \quad \lambda^i_{\rm gf}\partial_i = [\bar{\lambda}_2(t,r)+\delta\lambda_2]\partial_r, \quad (79)$$

where a bar stands for background quantities.

Now we include the matter fields using a Schutz-Sorkin Lagrangian [42,43] for each matter component, namely

$$S_I = -\int \mathrm{d}^4x N\sqrt{\gamma}[\rho_I(n_I)+J_I\partial_t l_I + J^i_I\partial_i l_I], \quad (80)$$

where we have named, for each matter component labeled by I, the 0th component of the vector field J^μ_I as $J^0_I = J_I$. In the above Lagrangian, ρ_I is energy density of each matter component, J^μ_I is conserved number current density of the matter, i.e., $J^\mu_I = n_I u^\mu_I$ and $\nabla_\mu J^\mu_I = 0$, whereas l_I is the field variable related to the scalar part of the velocity of the matter component. We have also defined the number density of the fluid as

$$n_I \equiv \sqrt{-g_{\mu\nu}J^\mu_I J^\nu_I} = \sqrt{-[(N^iN_i-N^2)J_I^2+2N_iJ^i_IJ_I+\gamma_{ij}J^i_IJ^j_I]}. \quad (81)$$

We find it useful to introduce on the FLRW background a decomposition for the matter fields given as follows

$$J_I = \bar{J}_I(t)/\bar{N}(t)+\delta J_I, \qquad l_I \equiv \bar{l}_I(t)+\delta l_I, \qquad J^i_I\partial_i = \delta J^r_I\partial_r. \quad (82)$$

Here we have imposed homogeneity and isotropy in order to set the three dimensional fluid velocity to vanish, i.e., $\bar{u}^i = 0$, or $\bar{J}^i = \bar{u}^i/\bar{n} = 0$, where n the fluid number density on the background is only a function of time, i.e., $n=\bar{n}(t)$ on the background. This leads to also $\rho_I = \rho_I(n_I)$ and $P_I = P_I[\rho_I(n_I)]$ to be only functions of time on the background, where P_I is pressure of the matter component.

For the matter sector we have the following background equations of motion

$$\bar{n}_I = \bar{J}_I = \frac{N_{I,\rm tot}}{a^3}, \qquad \frac{\dot{\rho}_I}{\bar{N}}+3H(\rho_I+P_I) = 0, \qquad \bar{l}_I = -\int_0^t \bar{N}(t')\rho_{I,n_I}[\bar{n}(t')]\mathrm{d}t', \quad (83)$$

where $N_{I,\rm tot}$ is a constant, corresponding to the constant number of I-fluid particles for each matter component, whereas $P_I = n_I\rho_{I,n_I}-\rho_I$ is the pressure of the I-fluid component, and, finally, $H=\dot{a}/(aN)$ is the Hubble parameter. For the gravity sector we have instead the following equations of motion

$$E_1 \equiv -\phi^2+3V(\phi)+\frac{3\rho}{M_{\rm P}^2} = 0, \quad (84)$$

$$E_2 \equiv -\frac{\dot{\phi}}{\bar{N}}+\frac{3}{2M_{\rm P}^2}(\rho+P) = 0, \quad (85)$$

and the total conservation equation

$$E_3 \equiv \frac{\dot{\rho}}{\bar{N}}+3H(\rho+P) = 0, \quad (86)$$

where we have defined ρ as the total effective energy density, namely

$$\rho \equiv \sum_I \rho_I + \rho_\kappa, \tag{87}$$

$$\rho_\kappa \equiv -\frac{3M_{\rm P}^2\kappa}{a^2}, \tag{88}$$

$$P \equiv \sum_I P_I + P_\kappa, \tag{89}$$

$$P_\kappa \equiv \frac{M_{\rm P}^2\kappa}{a^2}, \tag{90}$$

which also implies that $\dot{\rho}_\kappa/\bar{N} + 3H(\rho_\kappa + P_\kappa) = 0$. Note that for these background equations, the role of the curvature amounts to giving an extra effective component for the term $\rho + P$. This implies that, even in the absence of standard matter fields components, we still have a nontrivial dynamics for $\phi(t)$ in nonflat FLRW solutions.

Now we also have for the background that

$$K_{ij} = H\gamma_{ij}, \quad \text{and} \quad K = 3H, \tag{91}$$

which, together with the equation of motion for λ gives

$$\frac{2}{3}\phi + 2H + \lambda = 0. \tag{92}$$

By considering a combination of E_1, E_2, and E_3 we obtain the following equation,

$$\left(3H - \frac{3}{2}V_{,\phi} + \phi\right)(\rho + P) = 0. \tag{93}$$

Since in general $\rho + P \neq 0$, this leads to

$$3H + \phi = \frac{3}{2}V_{,\phi}. \tag{94}$$

The last equation of motion, the one for $\delta\phi$, is

$$D_i\lambda^i_{\rm gf} = 0, \tag{95}$$

which implies that $\lambda^i_{\rm gf}$ vanishes, otherwise $\bar{\lambda}_2$ would be singular at $r = 0$. The fact that in general both $\dot{\phi}$ and K do not vanish leads to the consequence that on the cosmological background VCDM solutions are different from ΛCDM, except for the special case of a linear potential[10] $V(\phi) = \beta_0 + \beta_1\phi$.

We now prove that for any given or desired dynamics $H(z)$, with $H > 0$, the VCDM potential is in general always re-constructable, even in the presence of a nonzero spatial curvature in the 3-D metric, generalizing the result previously found in [13]. Let us rewrite then the second Friedmann equation and the matter equation of motion with the e-fold number $\mathcal{N} \equiv \ln(a/a_0)$, by assuming a known matter sector and on imposing a given dynamics for the Hubble factor, i.e., $H = H(\mathcal{N})$.

$$\frac{{\rm d}\phi}{{\rm d}\mathcal{N}} = \frac{3}{2}\frac{\rho + P}{HM_{\rm P}^2}, \tag{96}$$

Integrating Eq. (96) with respect to $\mathcal{N}$ we get

$$\phi(\mathcal{N}) = \phi_0 + \frac{3}{2}\frac{1}{M_{\rm P}^2}\int_0^{\mathcal{N}} \frac{\rho(\mathcal{N}') + P(\mathcal{N}')}{H(\mathcal{N}')}{\rm d}\mathcal{N}'. \tag{97}$$

Now, on assuming that

$$\rho + P > 0, \quad H > 0, \tag{98}$$

i.e., H is positive definite as well as the total matter-curvature contribution for $\rho + P$, the found function $\phi(\mathcal{N})$ is an increasing function of $\mathcal{N}$. Hence, there exists a unique inverse function

$$\mathcal{N} = \mathcal{N}(\phi). \tag{99}$$

Then, on using the first Friedmann equation Eq. (84), we can finally write

$$V = \frac{\phi^2}{2} + \frac{\rho(\mathcal{N}(\phi))}{M_{\rm P}^2}. \tag{100}$$

Notice that the potential is not uniquely defined, since there is a free choice for the constant ϕ_0. In the spatially flat case we have to impose the null energy condition as already mentioned in [13]. On the other hand, in the spatially curved case ρ and P include contributions from the curvature term [see (87)–(90)] and thus (98) is either stronger or weaker than the null energy condition, depending on the sign of the spatial curvature.

We also discuss the existence of the McVittie solution in VCDM theory in the Appendix B.

IV. COMPARISON BETWEEN VCDM AND CUSCUTON

VCDM theory and Cuscuton theory are sharing similar properties: in both theories there are no additional degrees of freedom other than that of GR, so that it is natural to ask if the solution of theses theories share the same solutions or not. In a more mathematical language, we ask if there exist a well-defined mapping from solutions of VCDM to Cuscuton theory and vice versa.

At first we discuss the Cuscuton theory itself. The covariant action for the Cuscuton theory is given by

[10]Instead a quadratic potential, namely $V(\phi) = \beta_0 + \beta_1\phi + \frac{1}{2}\beta_2\phi^2$ would instead lead, for $\beta_2 < 2/3$, to a ΛCDM background with an effective redefined cosmological-background-Planck mass $M_v^2 = 2M_{\rm P}^2/(2 - 3\beta_2)$, but still with $G_{\rm eff} = G_N$ for dust perturbations.

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}{}^{(4)}R + \mu^2\sqrt{-X} - U(\varphi)\right] + S_m, \tag{101}$$

where S_m represents the contribution from standard matter fields, and we consider only the case of a timelike field φ, so that $X < 0$,[11] where

$$X \equiv g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi. \tag{102}$$

Now, from the above Cuscuton action, we have the covariant equations of motion

$$M_P^2 G_{\mu\nu} = T_{\mu\nu} - g_{\mu\nu}U + \frac{\mu^2}{\sqrt{-X}}[\partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}X], \tag{103}$$

$$U_{,\varphi} = \frac{\mu^2}{\sqrt{-X}}\left[g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi - \frac{1}{2X}\nabla_\mu X\nabla^\mu\varphi\right], \tag{104}$$

where $T_{\mu\nu}$ represents the total stress energy tensor for the matter fields, which satisfies the usual conservation equations $\nabla_\mu T^\mu{}_\nu = 0$.

A. Cuscuton cosmology with quadratic potential

Considering an homogeneous and isotropic FLRW metric with nonzero spatial curvature we have the following equations of motion

$$U_{,\varphi} = -3\mu^2 H \operatorname{sign}(\dot\varphi), \tag{105}$$

$$H^2 = \frac{\rho_m + U}{3M_P^2} - \frac{\kappa}{a^2}, \tag{106}$$

$$\dot H = -\frac{\rho_m + P_m}{2M_P^2} - \frac{\mu^2|\dot\varphi|}{2M_P^2} + \frac{\kappa}{a^2}. \tag{107}$$

Here we will assume that $\dot\varphi \neq 0$, so that $\dot\varphi$ does not change its sign during the evolution of the universe. However, we will also discuss the limiting case, namely $\dot\varphi \to 0$, and determine the conditions for which this limit can be taken while the theory remains a valid effective field theory.

Using Eqs. (105) and (106), we obtain Eq. (107), after assuming the standard energy conservation in the matter sector, namely $\dot\rho_m + 3H(\rho_m + P_m) = 0$. We then have to solve only two independent equations, for instance Eqs. (105) and (106). Indeed, from Eqs. (105) and (106), we find that the following equation always holds

$$U - \frac{M_P^2 U_{,\varphi}^2}{3\mu^4} = \frac{3M_P^2\kappa}{a^2} - \rho_m. \tag{108}$$

On assuming the following form for the potential

$$U = U_0 + \frac{1}{2}m^2\varphi^2, \tag{109}$$

we find that Eq. (108) leads to

$$\frac{1}{2}\left[1 - \frac{2M_P^2 m^2}{3\mu^4}\right]m^2\varphi^2 = \frac{3M_P^2\kappa}{a^2} - \rho_m - U_0. \tag{110}$$

(We shall study cosmology with a general potential in Sec. IV C.) Using this equation for φ, we rewrite the Friedmann equation (106) as

$$3M_c^2 H^2 = \rho_m + U_0 - \frac{3M_P^2\kappa}{a^2}, \tag{111}$$

where

$$M_c^2 \equiv M_P^2 - \frac{3\mu^4}{2m^2},$$

provided that

$$m^2 < 0, \quad \text{or} \quad m^2 > \frac{3}{2}\frac{\mu^4}{M_P^2}. \tag{112}$$

Notice that we have found an equation of motion, Eq. (111), which on the background, up to a redefinition of the background effective gravitational constant, is identical to the Friedmann equation in ΛCDM. However, it can be shown that the growth of structure for this theory will still feel the standard Newtonian gravitational constant, G_N. Hence, both the background and the perturbations overall differ from ΛCDM. We can further perform a time redefinition as $t = (M_c/M_P)\tilde t$, as to make the Friedmann equation take the same form as in GR, namely

$$3M_P^2\left(\frac{1}{a}\frac{da}{d\tilde t}\right)^2 = \rho_m + U_0 - \frac{3M_P^2\kappa}{a^2}, \tag{113}$$

out of which one can deduce the known GR solutions in terms of $a(\tilde t)$. For instance, in vacuum, on calling $U_0 \equiv 3M_P^2\tilde H_0^2$, we find

$$a(\tilde t) \propto \begin{cases} \cosh[\tilde H_0\tilde t] & \text{for} \quad \kappa = 1, \\ \exp[\tilde H_0\tilde t] & \text{for} \quad \kappa = 0, \\ \sinh[\tilde H_0\tilde t] & \text{for} \quad \kappa = -1, \end{cases} \tag{114}$$

[11] In fact, we can consider an opposite sign convention, but here we follow the $(-,+,+,+)$ convention for the metric and demand the Cuscuton field φ to be timelike.

as expected.[12] Here the $\kappa = 0$ solution should be discarded, as leading to a constant φ. However, if $\kappa = \pm 1$, then H becomes time dependent, as well as φ, and these solutions can be accepted for the Cuscuton theory.

Let us recast the effective Friedmann equation, Eq. (111), in another way which is more suitable for phenomenology. Indeed let us write

$$3M_{\rm P}^2 H^2 = \rho_m + \rho_\Lambda - \frac{3M_{\rm P}^2\kappa}{a^2} + \rho_{\rm cusc}, \tag{115}$$

where

$$\rho_{\rm cusc} \equiv 3(M_{\rm P}^2 - M_c^2)H^2, \tag{116}$$

$$\rho_\Lambda \equiv U_0. \tag{117}$$

Then we have that

$$1 = \Omega_m + \Omega_\Lambda + \Omega_\kappa + \Omega_{\rm cusc}, \tag{118}$$

where $\Omega_m = \rho_m/(3M_{\rm P}^2H^2)$, $\Omega_\Lambda = \rho_\Lambda/(3M_{\rm P}^2H^2)$, $\Omega_\kappa = -\kappa/(a^2H^2)$, and $\Omega_{\rm cusc} = 1 - M_c^2/M_{\rm P}^2$. This shows that $\Omega_{\rm cusc} =$ constant, which will prevent in general the other components' Ω to become unity when they dominate the dynamics. The parameter $\Omega_{\rm cusc}$ corresponds to an additional free parameter of the Cuscuton theory (with a quadratic potential), on which one can set in general constraints.

B. Unacceptable solutions of Cuscuton theory

In this section we discuss the ΛGR solutions which are not acceptable solutions of Cuscuton theory, but, as previously shown, acceptable in the VCDM theory.

1. Static spherically symmetric solutions of VCDM

Here we consider, for simplicity, spherically symmetric static solutions of VCDM found in [27]. The Cuscuton theory does not allow for such a solutions, even outside the unitary gauge choice, as φ must be timelike and the presence of the potential does not allow staticity for the spherically symmetric solutions of the theory. In particular for such existing VCDM solutions we have $\phi =$ constant, and

$$D_i\lambda^i_{\rm gf} = \frac{3V_{,\phi} + 6b_0 - 2\phi}{F(r)} \neq 0, \tag{119}$$

where $b_0 = -K/3$, K being the extrinsic curvature and, $F(r)$ is the rr component of the spherically symmetric static metric. The expression (119) does not vanish in general. Therefore these solutions have constant K and ϕ but in general $D_i\lambda^i_{\rm gf} \neq 0$. As shown in [16], all Cuscuton solutions are also solutions of VCDM provided that $D_i\lambda^i_{\rm gf} = 0$ (as well as imposing that $V_{\phi\phi} \neq 0$ while φ remains timelike). Here, in addition to the fact that ϕ is constant, these solutions have in general a nonvanishing $D_i\lambda^i_{\rm gf}$, which makes them outside the reach of Cuscuton theory. Nonetheless, these solutions still belong to the class where VCDM admits ΛGR solutions (because both ϕ and K are constant in time and space). Indeed, the static solutions found in [27] are nothing but the Schwarzschild-de Sitter solutions only written in a K-constant slicing coordinate system. The time-dependent spherically symmetric solutions found in [27] have both $\dot K \neq 0$ and $D_i\lambda^i_{\rm gf} \neq 0$, so that they represent intrinsic-VCDM solutions, i.e., solutions which are outside both GR and the Cuscuton theory.

2. GR vacuum solutions

Let us consider now ΛGR vacuum solutions, that is four dimensional solutions for the metric $g_{\mu\nu}$ which satisfy the following tensorial equations of motion

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}, \tag{120}$$

and we seek the condition for these solutions to hold also in the Cuscuton theory. Before we look into the answer of this problem, let us rewrite the Cuscuton action as proposed in [44], namely

$$S_g = \int {\rm d}^4x\sqrt{-g}\left[\frac{M_{\rm P}^2}{2}{}^{(4)}R + \mu^2 u^\mu\nabla_\mu\varphi - U(\varphi) + \frac{M_{\rm P}^2}{2}\sigma(g_{\mu\nu}u^\mu u^\nu + 1)\right], \tag{121}$$

out of which we can find covariant equations of motion for the metric as

$$G^\mu{}_\nu = \frac{1}{M_{\rm P}^2}\mathcal{T}_{\bf c}{}^\mu{}_\nu, \tag{122}$$

where

$$\mathcal{T}_{\bf c}{}^\mu{}_\nu = [\mu^2 u^\alpha\nabla_\alpha\varphi - U(\varphi)]\delta^\mu{}_\nu + M_{\rm P}^2\sigma u^\mu u_\nu, \tag{123}$$

$$\mu^2\nabla_\mu\varphi + M_{\rm P}^2\sigma u_\mu = 0, \tag{124}$$

$$g_{\mu\nu}u^\mu u^\nu = -1. \tag{125}$$

On using $u_\mu = -\mu^2/(M_{\rm P}^2\sigma)\nabla_\mu\varphi$, and multiplying Eq. (124) by u^μ we find

$$-g_{\mu\nu}\nabla^\mu\varphi\nabla^\nu\varphi = \frac{M_{\rm P}^4}{\mu^4}\sigma^2, \tag{126}$$

[12] In the case of $U_0 = 0$ and $\rho_m = 0$ we the find for $\kappa = -1$ a Milne-like universe ($a = \tilde t$), which differs from the GR's one ($a = t$) because of the different time rescaling.

or

$$\sigma = \frac{\mu^2}{M_{\rm P}^2}\sqrt{-X}, \qquad (127)$$

where we have chosen the positive square root for σ. Then in this case $u_\mu = -\frac{\nabla_\mu \varphi}{\sqrt{-X}}$, as expected. In this case, for ΛGR vacuum solutions which are also solution for the Cuscuton theory, we need to set

$$\mathcal{T}_{\bf c}{}^{\mu}{}_{\nu} = -M_{\rm P}^2 \Lambda \delta^\mu{}_\nu, \qquad (128)$$

which leads to imposing

$$Z_{\mu\nu} \equiv [\mu^2 u^\alpha \nabla_\alpha \varphi + M_{\rm P}^2 \Lambda - U(\varphi)] g_{\mu\nu} + M_{\rm P}^2 \sigma u_\mu u_\nu = 0. \qquad (129)$$

Then $g^{\mu\nu} Z_{\mu\nu} = 0$ and $Z_{\mu\nu} u^\mu u^\nu = 0$, upon using (125), imply that

$$4[\mu^2 u^\alpha \nabla_\alpha \varphi + M_{\rm P}^2 \Lambda - U(\varphi)] - M_{\rm P}^2 \sigma = 0, \qquad (130)$$

$$-[\mu^2 u^\alpha \nabla_\alpha \varphi + M_{\rm P}^2 \Lambda - U(\varphi)] + M_{\rm P}^2 \sigma = 0, \qquad (131)$$

which lead in particular to

$$3M_{\rm P}^2 \sigma = 0, \qquad X = 0. \qquad (132)$$

This clearly contradicts the basic requirement of timelike $\partial_\mu \varphi$, and thus cannot be accepted in the Cuscuton theory. So these ΛGR solutions do not exist in Cuscuton. In particular, this result excludes exact Minkowski, de Sitter or Schwarzschild-de Sitter solutions, as the solution $X = 0$ cannot be accepted. The same solution would be instead accepted for quintessence models for which the configuration $X = 0$ is allowed.

3. GR solutions in the presence of matter fields

Let us consider also exact GR, in the presence of matter fields, that is solutions of the following Einstein equations

$$G^\mu{}_\nu = -\Lambda \delta^\mu{}_\nu + \frac{1}{M_{\rm P}^2} T^\mu{}_\nu, \qquad (133)$$

where $T_{\mu\nu}$ represent the total stress-energy tensor for matter fields (which, by construction, we suppose to be minimally coupled with gravity). On the other hand, a similar environment, in the Cuscuton theory, would lead to the following equations of motion

$$G^\mu{}_\nu = \frac{1}{M_{\rm P}^2} \mathcal{T}_{\bf c}{}^{\mu}{}_{\nu} + \frac{1}{M_{\rm P}^2} T^\mu{}_\nu \qquad (134)$$

and once again we end up with the following necessary condition for the ΛGR solutions to be solutions of the Cuscuton theory.

$$\mathcal{T}_{\bf c}{}^{\mu}{}_{\nu} = -M_{\rm P}^2 \Lambda \delta^\mu{}_\nu. \qquad (135)$$

Again this condition implies $X = 0$, which is not acceptable for the Cuscuton model. This results still holds even if in the Cuscuton theory there is an explicit cosmological constant contribution Λ_c, as this merely leads to a shift in the effective cosmological constant, as in $\Lambda \to \Lambda - \Lambda_c$.

4. Possible acceptable solutions close to GR solutions

The Cuscuton field, by definition, is required to have timelike derivative. This prevents the Cuscuton from admitting exact GR solutions. However, it is possible in some situations that the field may be timelike but may also be reaching an attractor for which $\dot\varphi \to 0$. Then we have a Cuscuton solution which is not exactly GR but very close to it. In this case, it is necessary to understand whether or not the Cuscuton theory still stands as a good effective low energy theory. As to understand this point better we study the quantity $\delta X/X$ in linear perturbation theory in cosmology, adopting the ansatz (75)–(78), and then determine which dynamics can give an acceptable behavior for the perturbations fields. We achieve this goal by undoing the unitary gauge, and using, instead the $\zeta = 0$ gauge, which is always well defined, as long as $H \neq 0$. We also introduce a perfect fluid as a matter field. Then we find, that on defining $\tilde k = k/(aH)$, $w = P/\rho$, $\Omega = \rho/(3M_{\rm P}^2 H^2)$, $c_s^2 = \dot P/\dot\rho$, we have after removing all the auxiliary fields that

$$\frac{\delta X}{X} = \left[\frac{2\tilde k^2 (1 + 3c_s^2)}{2\tilde k^2 + 9(1+w)\Omega} - \frac{\ddot\varphi}{H\dot\varphi}\right] \frac{6\Omega}{2\tilde k^2 + 9(1+w)\Omega} \delta_{\rm FG}, \qquad (136)$$

where for simplicity we have fixed the background lapse function to unity ($\bar N = 1$), and have also assumed $\dot\varphi > 0$. Here we have also introduced the gauge invariant variable $\delta_{\rm FG} = \delta\rho/\rho - [\dot\rho/(H\rho)]\zeta$. So in the limit $X \to 0$, whether or not $\delta X/X$ blows up, hence going out of the EFT validity, depends on the ratio $\frac{\ddot\varphi}{H\dot\varphi}$. So even approaching $X = 0$ does not necessarily mean that the theory loses predictability. Indeed, we can choose dynamics, i.e., suitable Cuscuton potentials, for which this ratio is always of order one, leading to a consistent evolution of both the background and perturbations [40]. Otherwise, the EFT breaks down as the configuration approaches a GR solution with or without matter fields.

It seems Cuscuton is doomed to be away from exact ΛCDM solutions, but this does not necessarily mean that the theory is ruled out, as we have already discussed above. Solutions might not be the same as GR but close enough to them, in fact we could be even thinking of cases for which, on the background, $T^c_{\mu\nu} \propto G_{\mu\nu}$, giving a non-ΛCDM solution, which on the other hand could be different from it only up to a redefinition of the effective Planck mass for that particular background. This was indeed the case when $U(\varphi) = U_0 + \frac{1}{2}m^2\varphi^2$, as we have seen in Sec. IV A. In this case though, we should be seeing a difference between the cosmological effective gravitational constant and the gravitational constant which determines the evolution of dust perturbation, which is still G_N.

C. Cosmology: VCDM vs Cuscuton

As we have stated before, both VCDM and Cuscuton theories are MMG Type-IIa theories, with only two propagating degrees of freedom, but still both theories are different from GR, in general. Hence, it is natural to check if the cosmology of these theories are related with each other. In fact, since $D_i\lambda^i_{\rm gf}$ vanishes and in general $\phi = \phi(t)$ with $\dot\phi \propto \rho + P \neq 0$ (excluding an exact de Sitter case), one should expect to find a correspondence between VCDM and the Cuscuton theory (see [16]). Let us stress that this equivalence is accidental, and holds only in particular cases, such as on a homogeneous and isotropic background. As discussed so far, the two theories have different solutions and as such the equivalence in general breaks.

In the following we will always consider both the conditions $H \neq 0$ (standard cosmological background) and $\dot\varphi \neq 0$ (always holding at any finite time as to avoid EFT-breaking). Giving a FLRW ansatz to the Cuscuton action Eq. (101) we can obtain the Cuscuton Friedmann equation

$$H^2 = \frac{1}{3M_{\rm P}^2}[U(\varphi) + \rho], \tag{137}$$

on replacing H by means of Eq. (94), and ρ, the total matter energy density, by means of the VCDM Friedmann equation, namely Eq. (84), we find

$$\left(\frac{1}{2}V_{,\phi} - \frac{1}{3}\phi\right)^2 = \frac{1}{3M_{\rm P}^2}\left[U(\varphi) + \frac{M_{\rm P}^2}{3}(\phi^2 - 3V)\right]. \tag{138}$$

Imposing that this equation must hold at all times, we obtain

$$\begin{aligned} U(\varphi) &= 3M_{\rm P}^2\left(\frac{1}{2}V_{,\phi} - \frac{1}{3}\phi\right)^2 - \frac{M_{\rm P}^2}{3}(\phi^2 - 3V) \\ &= \frac{3M_{\rm P}^2}{4}V_{,\phi}^2 + M_{\rm P}^2(V - V_{,\phi}\phi). \end{aligned} \tag{139}$$

So that

$$\begin{aligned} U_{,\varphi}d\varphi &= \left[\frac{3}{2}V_{,\phi} - \phi\right]M_{\rm P}^2 V_{,\phi\phi}d\phi \\ &= 3HM_{\rm P}^2 V_{,\phi\phi}d\phi. \end{aligned} \tag{140}$$

On the other hand, the timelike Cuscuton satisfies also the following condition

$$U_{,\varphi} = -3\mu^2 H \operatorname{sign}(\dot\varphi). \tag{141}$$

Since now on we impose that during the known history of the universe $H \neq 0$, this implies that $U_{,\varphi} \neq 0$. Then Eq. (140) becomes

$$\operatorname{sign}(\dot\varphi)d\varphi = -\frac{M_{\rm P}^2}{\mu^2}V_{,\phi\phi}d\phi. \tag{142}$$

Therefore, we also require that $V_{,\phi\phi} \neq 0$, for the mapping to exist. As expected, this condition makes VCDM dynamics different from ΛCDM.

Let us give an example for a well defined behavior of such a mapping. Let us consider the case of a quadratic potential for the VCDM field, namely

$$V = \beta_0 + \beta_1\phi + \frac{1}{2}\beta_2\phi^2. \tag{143}$$

Then Eq. (142) leads to

$$\operatorname{sign}(\dot\varphi)d\varphi = -\frac{\beta_2 M_{\rm P}^2}{\mu^2}d\phi, \tag{144}$$

which can be integrated to give

$$\operatorname{sign}(\dot\varphi)\varphi = -\frac{\beta_2 M_{\rm P}^2}{\mu^2}\phi + \beta_3, \tag{145}$$

and β_3 is a free constant of integration. Then on using Eq. (139), we find that on fixing the free parameter β_3 as in

$$\beta_3 = \frac{M_{\rm P}^2}{\mu^2}\frac{3\beta_1\beta_2}{2 - 3\beta_2}, \tag{146}$$

the potential for the Cuscuton field can be written as

$$U(\varphi) = U_0 + \frac{1}{2}m^2\varphi^2, \tag{147}$$

where

$$m^2 = -\frac{\mu^4(2 - 3\beta_2)}{2M_{\rm P}^2\beta_2}, \tag{148}$$

$$\frac{U_0}{M_{\rm P}^2} = \beta_0 + \frac{3\beta_1^2}{4 - 6\beta_2}. \tag{149}$$

This Cuscuton potential agrees with the one in (109) and thus admits a ΛCDM background with an effective cosmological gravitational constant M_c^2 which differs from M_P^2. Concretely, we have

$$M_c^2 = M_\mathrm{P}^2 - \frac{3\mu^4}{2m^2} = M_v^2, \tag{150}$$

where $M_v^2 = 2M_\mathrm{P}^2/(2 - 3\beta_2)$. This is a working example for which finding cosmological solutions in VCDM leads to knowing mirror solutions in the Cuscuton theory and vice versa.

V. SUMMARY AND DISCUSSIONS

In the present era, when some cosmological data seem to be either inconsistent with each other or with general relativity (GR), it is of special interest to investigate the possibilities of modifying gravity in several possible ways. In particular, since at solar system scales no evidence has been found so far as to motivate the existence of any new degree of freedom connected to the gravity sector, it makes sense to look for those theories which do not add, by construction, any new degree of freedom beside the two polarizations of gravitational waves in the gravity sector. This possibility is now known to exist in the framework of the so called "minimally modified gravity" (MMG). In particular, those theories which do not allow the existence of an Einstein frame are called of Type II. Among these, we name of Type-IIa those theories in which gravitational waves propagate, on any background, at the speed of light.

Both VCDM and Cuscuton theories are Type-IIa MMG theories and we have discussed here the relation between these two theories. In fact, both theories on a cosmological background lead to an effective time dependent extra energy-density component, which, however, does not lead to new propagating degrees of freedom. This feature make them appealing as to provide possibilities at solving e.g., the so called H_0-tension.

We have shown that the two theories in general are not equivalent. We have in fact, explicitly shown this statement by mainly comparing solutions which exist in VCDM but not in Cuscuton, as demonstrated in Fig. 1. The following two facts clearly show the nonequivalence of the theories. First, the derivative of the Cuscuton scalar needs to be always timelike on any background. As a consequence, backgrounds which require the Cuscuton field φ to be constant (in time and space) are not acceptable solutions for this theory. This situation takes place when we consider exact GR solutions (in the presence of minimally coupled matter, including possibly a cosmological constant). Although the Cuscuton field can lead to solutions which are close to the GR counterparts, it does not allow for exact GR solutions to be also solutions of the theory.

Second, on the other hand, we have shown that when a GR-solution (with or without matter fields, in the presence of a cosmological constant) allows for a foliation which is endowed with a constant trace of the extrinsic curvature (both in time and space) then these same solutions also are solutions in the VCDM theory. For instance, this result holds true in VCDM for both the static Schwarzschild-de Sitter metric (for any slicing admitting $K = K_0$) and the vacuum Kerr-de Sitter solutions (in Boyer–Lindquist coordinates), since both solutions have a constant trace for the extrinsic curvature K.

As a consequence, we also worked out various limits of the VCDM theory, say weak field limit and the de Sitter limit of the VCDM theory and show that all these limits are well defined, e.g., no strong coupling is present, and exactly match the GR solutions.

We also find that in the context of cosmology these two theories are always related in general to each other, since $D_i\lambda^i_\mathrm{gf} = \nabla^2\lambda_2 = 0$. For a special form of potential, i.e., for a quadratic potential for VCDM and Cuscuton, this mapping is well defined. However, the effective Planck mass for cosmological backgrounds is modified to be $M_v^2 = 2M_\mathrm{P}^2/(2 - 3\beta_2)$.

In summary, we have confirmed that all acceptable Cuscuton solutions are also solutions of VCDM (see, e.g., [16], and in particular, cosmological solutions belong to this case). However, in addition to these solutions, VCDM has other solutions which, as mentioned above, are exact solutions of GR (with our without matter fields and a cosmological constant) which are not, on the other hand, acceptable solutions in Cuscuton. Finally, besides these, VCDM has a third category of solutions, which consists of solutions which are intrinsic only to VCDM, which are neither GR (K is not a constant in time or space) nor Cuscuton (no mapping in this case exists).

This study opens up several possible future directions. One direction is to look for possible signatures coming from the properties of gravitational waves propagating on intrinsic-VCDM background solutions. Another direction worth investigating is, whether the VCDM theory can be recast as an IR limit of some Lorentz breaking UV theory. It was shown in [45], that using braneworld model with k-essence we can have a self-tuning of the cosmological constant. Interestingly, the self-tuning mechanism constraints the Lagrangian to spacelike Cuscuton. Hence, it is interesting to explore the braneworld scenario with spacelike VCDM in five dimension to see if a self-tuning mechanism is possible with VCDM theory.

ACKNOWLEDGMENTS

The work of A. D. F. was supported by Japan Society for the Promotion of Science Grants-in-Aid for Scientific Research No. 20K03969. K. M. would like to acknowledges the Yukawa Institute for Theoretical Physics at Kyoto University, where the present work was begun during the Visitors Program of FY2021. The work of K. M. was supported by JSPS KAKENHI Grants No. JP17H06359 and No. JP19K03857. The work of S. M. is supported in part by Japan Society for the Promotion of Science Grants-in-Aid for Scientific Research No. 17H02890, and No. 17H06359 and by World Premier International Research Center Initiative, The Ministry of Education, Culture, Sports, Science and Technology, Japan. The work of M. C. P. was supported by the Japan Society for the Promotion of Science Grant-in-Aid for Scientific Research No. 17H06359.

APPENDIX A: COVARIANT VCDM EQUATIONS OF MOTION

In this appendix, we explicitly write down all the equations of motion for the VCDM covariant action introduced in Eq. (61), and evaluate them, as an example, on a FLRW background. In the remaining part of this section we find it convenient to perform the following field redefinition

$$\lambda_T \equiv \frac{\lambda_N}{\alpha^2}. \tag{A1}$$

The equation of motion for λ_T (or, equivalently, for λ_N) leads to

$$\frac{E_{\lambda_T}}{M_{\rm P}^2} \equiv 1 + \alpha^2 \nabla_\alpha T \nabla^\alpha T = 0, \tag{A2}$$

which, on a FLRW manifold on which $T = t$, gives

$$\alpha(t) = N(t), \tag{A3}$$

as expected (discarding the other solution $\alpha = -N$). The equation of motion for λ instead gives, on a general background,

$$\frac{E_\lambda}{M_{\rm P}^2} = \frac{3}{2}\lambda + \phi - \alpha \nabla_\mu \nabla^\mu T - \nabla_\mu \alpha \nabla^\mu T = 0, \tag{A4}$$

which evaluated on FLRW returns

$$\lambda = -\frac{2}{3}\phi - 2H, \quad \text{with} \quad H \equiv \frac{\dot{a}}{Na}. \tag{A5}$$

Next, let us consider the equation of motion for α. It can be written as

$$\begin{aligned} \frac{E_\alpha}{M_{\rm P}^2} = {} & \frac{2\lambda_N}{\alpha^3} + \frac{\lambda_2}{\alpha^2}\nabla_\mu \nabla^\mu \phi + \nabla_\mu \lambda \nabla^\mu T \\ & + (\nabla_\mu \lambda_2 \nabla^\mu T)(\nabla_\nu \phi \nabla^\nu T) \\ & + \lambda_2 (\nabla^\mu T)(\nabla^\nu \phi)\nabla_\mu \nabla_\nu T = 0, \end{aligned} \tag{A6}$$

which can be used to fix λ_N in terms of the other fields. On doing this on FLRW we do find

$$\lambda_N(t) = -\frac{1}{2}\frac{\dot{\phi}}{N}\dot{\lambda}_2 + \frac{3\dot{a}\,\dot{\phi} + a\ddot{\phi}}{2aN}\lambda_2 + \frac{1}{2}N\dot{\lambda}. \tag{A7}$$

On setting this constraint on λ_N, we have that the covariant equation of motion for the T field,

$$\begin{aligned} \frac{E_T}{M_{\rm P}^2} = {} & 2\lambda_N \nabla_\nu \nabla^\nu T - \alpha \nabla_\nu \nabla^\nu \lambda + 2\nabla_\mu \lambda_N \nabla^\mu T - \nabla_\mu \lambda \nabla^\mu \alpha + \lambda_2 (\nabla_\mu \phi \nabla^\mu \alpha)(\nabla_\nu \nabla^\nu T) \\ & + \alpha(\nabla_\mu \phi \nabla^\mu \lambda_2)(\nabla_\nu \nabla^\nu T) - \alpha(\nabla_\mu \phi \nabla^\mu T)(\nabla_\nu \nabla^\nu \lambda_2) + \alpha\lambda_2 (\nabla_\nu \nabla^\nu T)(\nabla_\mu \nabla^\mu \phi) \\ & + \lambda_2 (\nabla_\mu T \nabla^\mu \alpha)(\nabla_\nu \nabla^\nu \phi) + \alpha\lambda_2 \nabla^\mu T (\nabla_\nu \nabla^\nu \nabla_\mu \phi) - (\nabla_\mu \lambda_2 \nabla^\mu \alpha)(\nabla_\nu \phi \nabla^\nu T) \\ & + \lambda_2 (\nabla^\mu T)(\nabla^\nu \alpha)\nabla_\mu \nabla_\nu \phi + (\nabla_\mu \lambda_2 \nabla^\mu \phi)(\nabla_\nu \alpha \nabla^\nu T) - \alpha\lambda_2 R_{\mu\nu}(\nabla^\mu T)(\nabla^\nu \phi) \\ & - 2\alpha(\nabla^\mu \lambda_2)(\nabla^\nu \phi)\nabla_\mu \nabla_\nu T + \lambda_2 (\nabla^\mu T)(\nabla^\nu \phi)\nabla_\mu \nabla_\nu \alpha = 0, \end{aligned} \tag{A8}$$

is automatically satisfied on FLRW. Let us now consider the equation of motion for ϕ. This can be written as

$$\frac{E_{\lambda_2}}{M_{\rm P}^2} = -\frac{1}{\alpha}\nabla_\mu \nabla^\mu \phi - \alpha(\nabla_\mu \phi \nabla^\mu T)(\nabla_\nu \nabla^\nu T) - (\nabla_\mu \alpha \nabla^\mu T)(\nabla_\nu \phi \nabla^\nu T) - \alpha(\nabla^\mu T)(\nabla^\nu T)(\nabla_\mu \nabla_\nu \phi) = 0, \tag{A9}$$

which also identically vanishes on FLRW, as expected. We also need to evaluate the equation of motion for λ_2, which reads

$$\begin{aligned} \frac{E_\phi}{M_{\rm P}^2} = {} & \lambda + V_{,\phi} + \frac{\lambda_2}{\alpha^2}(\nabla_\nu \nabla^\nu \alpha) - \frac{1}{\alpha}(\nabla_\nu \nabla^\nu \lambda_2) - \frac{2\lambda_2}{\alpha^3}(\nabla_\mu \alpha \nabla^\mu \alpha) + \frac{2}{\alpha^2}(\nabla_\mu \lambda_2 \nabla^\mu \alpha) \\ & - \alpha(\nabla_\mu T \nabla^\mu \lambda_2)(\nabla_\nu \nabla^\nu T) - \alpha\lambda_2 \nabla^\mu T(\nabla_\nu \nabla^\nu \nabla_\mu T) - (\nabla_\mu \lambda_2 \nabla^\mu T)(\nabla_\nu \alpha \nabla^\nu T) \\ & - \alpha(\nabla^\mu T)(\nabla^\nu T)\nabla_\mu \nabla_\nu \lambda_2 - \lambda_2 (\nabla^\mu \alpha)(\nabla^\nu T)\nabla_\mu \nabla_\nu T - 2\alpha(\nabla^\mu \lambda_2)(\nabla^\nu T)\nabla_\mu \nabla_\nu T \\ & - \alpha\lambda_2 (\nabla_\mu \nabla_\nu T)(\nabla^\mu \nabla^\nu T) = 0, \end{aligned} \tag{A10}$$

and gives on the homogeneous background

$$\phi - \frac{3}{2}V_{,\phi} + 3H = 0, \tag{A11}$$

but on another generic background it would be used as to fix λ_2. Finally, let us evaluate the stress-energy tensor as

$$\begin{aligned}\frac{\mathcal{T}_{\mathrm{v}}{}^{\mu}{}_{\nu}}{M_{\mathrm{P}}^2} = \delta^{\mu}{}_{\nu}\Big\{&-V - \frac{3}{4}\lambda^2 - \lambda\phi + \lambda_2\alpha^{-2}(\nabla_\beta\phi\nabla^\beta\alpha) - \alpha^{-1}(\nabla_\beta\phi\nabla^\beta\lambda_2) - \alpha(\nabla_\beta\lambda\nabla^\beta T) - \alpha(\nabla^\gamma\lambda_2\nabla_\gamma T)(\nabla_\beta\phi\nabla^\beta T)\\ &- \alpha\lambda_2\nabla^\beta\phi(\nabla_\beta\nabla_\gamma T)\nabla^\gamma T\Big\} - \nabla^\mu T\nabla_\nu T\{2\lambda_N + \alpha\lambda_2(\nabla_\beta\nabla^\beta\phi) + \lambda_2(\nabla_\beta\phi\nabla^\beta\alpha) + \alpha(\nabla_\beta\phi\nabla^\beta\lambda_2)\}\\ &- \frac{\lambda_2}{\alpha^2}(\nabla^\mu\phi\nabla_\nu\alpha + \nabla^\mu\alpha\nabla_\nu\phi) + \frac{1}{\alpha}(\nabla^\mu\phi\nabla_\nu\lambda_2 + \nabla^\mu\lambda_2\nabla_\nu\phi) + \alpha(\nabla^\mu T\nabla_\nu\lambda + \nabla^\mu\lambda\nabla_\nu T)\\ &+ \alpha(\nabla^\mu T\nabla_\nu\phi + \nabla^\mu\phi\nabla_\nu T)(\nabla_\beta\lambda_2\nabla^\beta T) + \alpha(\nabla^\mu T\nabla_\nu\lambda_2 + \nabla^\mu\lambda_2\nabla_\nu T)(\nabla_\beta\phi\nabla^\beta T)\\ &+ \alpha\lambda_2\nabla^\beta T[\nabla^\mu\phi(\nabla_\beta\nabla_\nu T) + (\nabla_\beta\nabla^\mu T)\nabla_\nu\phi].\end{aligned} \tag{A12}$$

Then on constructing

$$M_{\mathrm{P}}^2 G^{\mu}{}_{\nu} = \mathcal{T}_{\mathrm{v}}{}^{\mu}{}_{\nu} + \sum_I T^{\mu}_{(I)}{}_{\nu}, \tag{A13}$$

we find on a FLRW background

$$\frac{1}{3}\phi^2 - V + \frac{3\kappa}{a^2} = \frac{1}{M_{\mathrm{P}}^2}\sum_I \rho_I, \tag{A14}$$

$$\frac{\dot\phi}{N} - \frac{1}{2}\phi^2 + \frac{3}{2}V - \frac{3\kappa}{2a^2} = \frac{3}{2M_{\mathrm{P}}^2}\sum_I P_I, \tag{A15}$$

as expected.

APPENDIX B: MCVITTIE SOLUTION IN VCDM

In the following we consider the McVittie solution in VCDM. Let us assume we have the following metric ansatz

$$\begin{aligned}ds^2 = &-N(t)^2\left(\frac{2ar-m}{2ar+m}\right)^2 dt^2\\ &+ a^2\left(1+\frac{m}{2ar}\right)^4\left[dr^2 + r^2\left(\frac{dz^2}{1-z^2} + (1-z^2)d\theta_2{}^2\right)\right],\end{aligned} \tag{B1}$$

with m being a constant. We will also consider the matter content only consists of a bare cosmological constant. In this case, by looking for example at $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, we can see the spacetime is not homogeneous. Therefore in this case, besides the choice of coordinate

$$T(t) = t, \text{as a choice of coordinates,} \tag{B2}$$

we suppose a spherically symmetric profile for all the fields in the theory, namely

$$\begin{aligned}&\phi = \phi(t,r), \qquad \alpha = \alpha(t,r), \qquad \lambda = \lambda(t,r),\\ &\lambda_2 = \lambda_2(t,r), \qquad \lambda_T = \lambda_T(t,r).\end{aligned} \tag{B3}$$

The equation of motion for λ_T sets

$$\alpha(t,r) = \frac{2ar-m}{2ar+m}N(t), \tag{B4}$$

while, the equation of motion for λ gives

$$\lambda(t,r) = -\frac{2}{3}\phi(t,r) - 2H, \tag{B5}$$

where we have defined

$$H \equiv \frac{\dot a}{Na}, \tag{B6}$$

out of which the trace of the extrinsic curvature is given by $K = 3H$. Notice that at this level, we cannot impose homogeneity on ϕ or λ. Next solving the equation of motion for α, we find λ_T as

$$\lambda_T = \lambda_T(\partial_t\lambda_2, \lambda_2, \partial_r^2\phi, \partial_t\phi, \partial_r\phi, \dot H, H, N, a, r). \tag{B7}$$

The equation of motion for λ_2 sets the following constraint

$$(2ar+m)\partial_r^2\phi + 4a\partial_r\phi = 0, \tag{B8}$$

which can be solved for

$$\phi(t,r) = \phi_h(t) + \frac{\phi_n(t)}{r+\frac{m}{2a}}, \tag{B9}$$

so ϕ in general might have an inhomogeneous contribution. In principle, on matching the field ϕ with cosmological boundary conditions would set $\lim_{r\to\infty}\phi(t,r)=\phi_h(t)$ giving an homogeneous profile (so we cannot use the boundary conditions to set $\phi_n(t)$ to vanish in this case). Therefore, we will keep this solution as it is, and see whether the equations of motion set the values of ϕ_h or ϕ_n. In fact, since the Einstein equations are

$$M_{\rm P}^2 G^\mu{}_\nu = \mathcal{T}_{\rm v}{}^\mu{}_\nu, \tag{B10}$$

we find that the (0,1) component of these equations lead to

$$\frac{8\phi_n a^2}{N(3m^2-12a^2r^2)}=0, \tag{B11}$$

which requires

$$\phi_n(t)=0, \tag{B12}$$

leading to

$$\phi(t,r)=\phi_h(t)=\phi(t), \tag{B13}$$

or the field is homogeneous. At this level, looking at the (0,0) component of the Einstein equations we find

$$E_1\equiv\frac{1}{3}\phi(t)^2-V(\phi(t))=0. \tag{B14}$$

Therefore, for a generic potential[13] we have

$$\phi(t)=\phi_0=\text{constant}. \tag{B15}$$

All the nondiagonal components of the Einstein equations now vanish whereas the (i,i) components lead once again to the condition $E_1=0$. Now all the Einstein equations are satisfied. At this level also the equation of motion for T is automatically satisfied. There is one last equation of motion we need solve, the equation of motion for ϕ, which gives

$$\partial_r^2\lambda_2+\frac{4a}{2ar+m}\partial_r\lambda_2=-N(t)\frac{(2ar-m)(2ar+m)^3}{48a^2r^4}[2\phi_0-3V_{,\phi}(\phi_0)+6H(t)]. \tag{B16}$$

The equation can be solved as

$$\lambda_2=\lambda_{2,h}(t)+\frac{\lambda_{2,n}(t)}{(2ar+m)}+\frac{N[\frac{2}{3}\phi_0-3V_{,\phi}(\phi_0)+2H(t)]}{96a^2r^2(2ar+m)}\left[-32a^5r^5-144ma^4r^4-96m^2a^3r^3\left(\ln r+\frac{2}{3}\right)\right.$$
$$\left.-48m^3a^2r^2\left(\ln r+\frac{8}{3}\right)+18m^4ra+m^5\right]. \tag{B17}$$

Notice that λ_2 grows as $r\to\infty$, although the source and the Riemann tensor tends to vanish for large r's. On fixing boundary conditions so that λ_2 does not diverge at infinity (as otherwise $\lambda_2\propto r^2$), we require

$$\frac{2}{3}\phi_0-3V_{,\phi}(\phi_0)+2H(t)=0, \tag{B18}$$

which states that

$$H=H_0=\text{constant}. \tag{B19}$$

In this case the solution automatically reduces to the MacVittie's solution obtained in GR, since K becomes a constant. With these boundary conditions, λ_2 reduces to the flat-de Sitter solution, $\lambda_2=\lambda_2(t)$, in the limit $\frac{m}{ar}\ll 1$. Therefore the chosen boundary conditions for λ_2 make λ_2 match an homogeneous profile at infinity.

[13]We will not consider here the possibility of a special form of a quadratic potential such that $V=V_0+\frac{1}{3}\phi^2$.

[1] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Phys. Rep. **513**, 1 (2012).
[2] S. Tsujikawa, Lect. Notes Phys. **800**, 99 (2010).
[3] Y. Fujii and K. Maeda, *The Scalar-Tensor Theory of Gravitation*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2007).
[4] L. Heisenberg, in *Proceedings of the 52nd Rencontres de Moriond on Gravitation* (2017), pp. 233–241, arXiv:1705.05387.

[5] A. De Felice, L. Heisenberg, R. Kase, S. Mukohyama, S. Tsujikawa, and Y.-l. Zhang, J. Cosmol. Astropart. Phys. 06 (2016) 048.
[6] A. De Felice, C.-Q. Geng, M. C. Pookkillath, and L. Yin, J. Cosmol. Astropart. Phys. 08 (2020) 038.
[7] A. Joyce, B. Jain, J. Khoury, and M. Trodden, Phys. Rep. **568**, 1 (2015).
[8] P. Brax and A.-C. Davis, J. Cosmol. Astropart. Phys. 10 (2015) 042.
[9] K. Koyama, Rep. Prog. Phys. **79**, 046902 (2016).
[10] A. De Felice and S. Mukohyama, Phys. Lett. B **752**, 302 (2016).
[11] C. Lin and S. Mukohyama, J. Cosmol. Astropart. Phys. 10 (2017) 033.
[12] S. Mukohyama and K. Noui, J. Cosmol. Astropart. Phys. 07 (2019) 049.
[13] A. De Felice, A. Doll, and S. Mukohyama, J. Cosmol. Astropart. Phys. 09 (2020) 034.
[14] K. Aoki, M. A. Gorji, and S. Mukohyama, Phys. Lett. B **810**, 135843 (2020).
[15] K. Aoki, A. De Felice, C. Lin, S. Mukohyama, and M. Oliosi, J. Cosmol. Astropart. Phys. 01 (2019) 017.
[16] K. Aoki, F. Di Filippo, and S. Mukohyama, J. Cosmol. Astropart. Phys. 05 (2021) 071.
[17] A. De Felice and S. Mukohyama, J. Cosmol. Astropart. Phys. 04 (2016) 028.
[18] A. De Felice, F. Larrouturou, S. Mukohyama, and M. Oliosi, Phys. Rev. D **98**, 104031 (2018).
[19] K. Aoki, A. De Felice, S. Mukohyama, K. Noui, M. Oliosi, and M. C. Pookkillath, Eur. Phys. J. C **80**, 708 (2020).
[20] A. De Felice, S. Mukohyama, and M. C. Pookkillath, J. Cosmol. Astropart. Phys. 12 (2021) 011.
[21] K. Aoki, M. A. Gorji, and S. Mukohyama, J. Cosmol. Astropart. Phys. 09 (2020) 014; 05 (2021) E01.
[22] K. Aoki, M. A. Gorji, S. Mizuno, and S. Mukohyama, J. Cosmol. Astropart. Phys. 01 (2021) 054.
[23] J. C. N. de Araujo, A. De Felice, S. Kumar, and R. C. Nunes, Phys. Rev. D **104**, 104057 (2021).
[24] M. C. Pookkillath, Astronomy Reports **65**, 1021 (2021).
[25] L. Parker and A. Raval, Phys. Rev. D **60**, 063512 (1999); **67**, 029901(E) (2003).
[26] J. Beltran Jimenez and A. L. Maroto, Phys. Rev. D **78**, 063005 (2008).
[27] A. De Felice, A. Doll, F. Larrouturou, and S. Mukohyama, J. Cosmol. Astropart. Phys. 03 (2021) 004.
[28] A. De Felice and S. Mukohyama, J. Cosmol. Astropart. Phys. 04 (2021) 018.
[29] A. De Felice, S. Mukohyama, and M. C. Pookkillath, Phys. Rev. D **105**, 104013 (2022).
[30] A. De Felice, S. Mukohyama, and M. C. Pookkillath, Phys. Lett. B **816**, 136201 (2021).
[31] N. Afshordi, D. J. H. Chung, and G. Geshnizjani, Phys. Rev. D **75**, 083513 (2007).
[32] N. Afshordi, D. J. H. Chung, M. Doran, and G. Geshnizjani, Phys. Rev. D **75**, 123509 (2007).
[33] N. Afshordi, Phys. Rev. D **80**, 081502 (2009).
[34] S. S. Boruah, H. J. Kim, and G. Geshnizjani, J. Cosmol. Astropart. Phys. 07 (2017) 022.
[35] S. S. Boruah, H. J. Kim, M. Rouben, and G. Geshnizjani, J. Cosmol. Astropart. Phys. 08 (2018) 031.
[36] A. Iyonaga, K. Takahashi, and T. Kobayashi, J. Cosmol. Astropart. Phys. 12 (2018) 002.
[37] J. Quintin and D. Yoshida, J. Cosmol. Astropart. Phys. 02 (2020) 016.
[38] A. Iyonaga, K. Takahashi, and T. Kobayashi, J. Cosmol. Astropart. Phys. 07 (2020) 004.
[39] S. Panpanich and K.-i. Maeda, arXiv:2109.12288 [Phys. Rev. D (to be published)].
[40] K.-i. Maeda and S. Panpanich, Phys. Rev. D **105**, 104022 (2022).
[41] N. Bartolo, A. Ganz, and S. Matarrese, J. Cosmol. Astropart. Phys. 05 (2022) 008.
[42] B. F. Schutz and R. Sorkin, Ann. Phys. (N.Y.) **107**, 1 (1977).
[43] M. C. Pookkillath, A. De Felice, and S. Mukohyama, Universe **6**, 6 (2019).
[44] J. Bhattacharyya, A. Coates, M. Colombo, A. E. Gumrukcuoglu, and T. P. Sotiriou, Phys. Rev. D **97**, 064020 (2018).
[45] O. Lacombe and S. Mukohyama, arXiv:2203.16322.