Rotating black holes at large D in Einstein-Gauss-Bonnet theory

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Applying the large D approach to the Einstein-Gauss-Bonnet theory, we construct equally rotating black hole solutions in odd dimensions. This provides the first example of the analytic solutions that describe not-slowly rotating black holes. For the next-to-leading-order solutions in the $1/D$ expansion, we discuss the physical aspects such as thermodynamics and the phase diagram.

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The Einstein-Gauss-Bonnet (EGB) theory is a simplest extension of the Einstein theory to the theory with higher curvature terms, which describes string theory inspired ultraviolet corrections to the Einstein gravity [1]. In particular, the EGB theory in $D = 5$ can be regarded as the low energy limit of string theory when the theory is dimensionally reduced from $D = 11$ to $D = 5$ by compactifying 6 of the 11 dimensions in compact Calabi-Yau threefold [2,3]. Furthermore, such quadratic terms of curvatures appears as a one-loop correction of heterotic string theory [1]. Thus, the physics of black holes in the $D = 5$ EGB theory has been the subject of increased attention for the reason that it provides us some insight on a quantum aspect of black holes.

The first exact solutions of black holes in the EGB theory were found by Boulware and Deser for a spherically symmetric and static case in Ref. [4]. The static solutions were also generalized to an electrically charged case [5,6]. However, so far, finding rotating black hole solutions in the EGB theory has been considered to be a hard and unsolved problem, since the Kerr-Schild formalism, which is a powerful tool for finding rotating black hole solutions, cannot work at all in this EGB theory. In spite of the technical difficulty, there have been some attempts to construct rotating EGB black hole solutions. Equally rotating black hole solutions in $D = 5$ were obtained as numerical solutions [7], and slowly rotating charged anti–de Sitter black hole solutions in $D \geq 5$ were obtained as perturbative and analytic solutions [8].

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The large dimension limit, or large D limit [9–11], is a useful approximation, which largely simplifies the black hole analysis in higher dimensions. Because of the localization of the gravity at large D, the dynamical degrees of freedom of the horizon are confined within a thin layer of the near-horizon region, which form an effective theory insensitive to the global structure of the spacetime [12–15].

So far the large D effective theory approach has been a viable tool to study the black hole dynamics, not only in general relativity (GR), but also in EGB theory. The (in)stabilities of the static EGB black holes [16] and black strings [17] were studied by using the large D approach, in which the black string instability is weakened by the Gauss-Bonnet (GB) term for the small GB coupling, whereas it is enhanced for the large GB coupling. Moreover, black ring solutions at large D in the EGB theory were also studied [18], where they obtained the quasinormal modes of the EGB black ring and showed that the thin EGB black ring becomes unstable against nonaxisymmetric perturbation.

In this paper, we construct new rotating black hole solutions with equal angular momenta in an odd-dimensional EGB theory by using the $1/D$ expansion up to the next-to-leading order (NLO). The assumption of equal angular momenta in odd dimensions enhances a spacetime symmetry to a class of cohomogeneity one. The further key assumption is that the metric of a rotating black hole at $D \to \infty$ is locally similar to that of the boosted black string, which was first noticed in the studies of rotating black holes in GR [19,20]. By imposing this assumption, the leading-order equations are decoupled to be simply solvable. The thermodynamic property is also studied up to the relevant order in $1/D$.

The action of the EGB theory is given by

$$S_{\rm EGB} = \frac{1}{16\pi G} \int \sqrt{-g}(R + \alpha_{\rm GB}\mathcal{L}_{\rm GB}) d^D x, \qquad (1)$$

where the GB Lagrangian is

$$\mathcal{L}_{\rm GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{2}$$

The equations of motion become

$$R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu} + \alpha_{\rm GB}H_{\mu\nu} = 0, \tag{3}$$

where

$$H_{\mu\nu} = -\frac{1}{2}\mathcal{L}_{\rm GB}g_{\mu\nu} + 2RR_{\mu\nu} - 4R_{\mu\alpha}R^{\alpha}{}_{\nu} - 4R_{\mu\alpha\nu\beta}R^{\alpha\beta} + 2R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma}. \tag{4}$$

The outcome of the large D limit depends on which scales are fixed in the limit, i.e., which scale of physics we are going to focus on. To obtain the black hole horizon, we must fix the length scale of the horizon radius r_0 at $\mathcal{O}(1)$. With the fixed horizon scale r_0, the scalar curvature around the horizon has the magnitude of $\mathcal{O}(D^2/r_0^2)$. We are interested in the intermediate regime in which the Einstein-Hilbert and GB terms become comparable $R \sim \alpha_{\rm GB}\mathcal{L}_{\rm GB} \sim \alpha_{\rm GB}R^2$, otherwise the equation of motion reduces to that of the Einstein or pure GB theory. Thus, we assume the GB coupling scales as $\alpha_{\rm GB} = \mathcal{O}(r_0^2/D^2)$ at large D.

Even for the large D limit, it is not so easy to solve the Einstein equations under the general rotating ansatz, since the metric functions are nonlinearly coupled already at the leading order. Instead, we assume that the EGB rotating black holes have the same property as GR rotating black holes, i.e., the large D limit of the Myers-Perry metric reduces to that of the boosted black brane [19,20]. For instance, in the Einstein-Maxwell theory, the same strategy has been successful in constructing charged rotating black holes in the large D limit both with a single angular momentum [21] and equal angular momenta [22].

We thus start from the following metric ansatz of equally rotating black holes in $D = 2n + 3$ dimensions with the Eddington-Finkelstein gauge:

$$ds^2 = -A(r)(e^{(0)})^2 + 2U(r)e^{(0)}e^{(1)} + 2C(r)e^{(0)}e^{(2)} + H(r)(e^{(2)})^2 + r^2 d\Sigma^2, \tag{5}$$

where $d\Sigma^2$ is the Fubini-Study metric on CP^n and other tetrad bases are defined by

$$e^{(0)} = \frac{dt - \Omega r(d\phi + \mathcal{A})}{\sqrt{1-\Omega^2}}, \quad e^{(2)} = \frac{r(d\phi + \mathcal{A}) - \Omega dt}{\sqrt{1-\Omega^2}}, \quad e^{(1)} = dr, \tag{6}$$

with the dimensionless spin parameter Ω, which produces the local Lorentz boost in the subspace $(dt, r(d\phi + \mathcal{A}))$. Here $\mathcal{A}$ is the Kähler potential of CP^n. In what follows, we use $1/n$ as the expansion parameter rather than $1/D$ itself, since the large D owes to the large dimension of CP^n. We impose that the metric reduces to that of the boosted black brane at $n \to \infty$,[1]

$$C = \mathcal{O}(n^{-1}), \qquad H = 1 + \mathcal{O}(n^{-1}). \tag{7}$$

As the asymptotic boundary condition, we impose

$$A \to 1, \qquad U \to 1, \qquad C \to 0, \qquad H \to 1 \tag{8}$$

at $r \to \infty$, so that the ansatz (5) is asymptotically flat. To resolve the thin near-horizon region at the large n limit, we introduce the following often-used radial coordinate:

$$\mathsf{R} := r^{2n}. \tag{9}$$

Here we set the horizon scale $r_0 = 1$ using the scaling degree of freedom. The metric components are expanded by $1/n$ as a function of R,

$$A = \sum_{i=0}^{\infty}\frac{1}{n^i}A_i(\mathsf{R}), \qquad U = \sum_{i=0}^{\infty}\frac{1}{n^i}U_i(\mathsf{R}),$$
$$C = \sum_{i=0}^{\infty}\frac{1}{n^i}C_i(\mathsf{R}), \qquad H = \sum_{i=0}^{\infty}\frac{1}{n^i}H_i(\mathsf{R}). \tag{10}$$

To keep the Einstein-Hilbert and GB terms comparable at the large n limit in Eq. (3), we introduce the rescaled GB coupling parameter, which remains finite at $n \to \infty$,

$$\alpha := (2n)^2\alpha_{\rm GB}. \tag{11}$$

With the assumption (7), we can decouple the leading-order equation, which yields

$$A_0 = 1 + \frac{1}{2\alpha} - \frac{1}{2\alpha}\sqrt{1 + \frac{4\alpha(\alpha+1)m}{\mathsf{R}}},$$
$$U_0 = 1, \qquad C_0 = 0, \qquad H_0 = 1, \tag{12}$$

where the integration constant m introduces the horizon at $\mathsf{R} = m$. As one can see in the form of A_0, the leading-order metric, therefore, reduces to the boosted black string metric at large D as in GR [17]. Note that, for the existence of the horizon, we only consider the parameter region $\alpha > -1/2$.

To obtain the information for $D < \infty$, we need to solve the $1/n$ correction to the above leading-order metric. In the higher-order analysis, A_i and C_i get extra integration constants, which are not determined by the boundary condition. They actually correspond to the parameter shift in the mass parameter m and horizon velocity Ω_H in each

[1] The assumption $C = \mathcal{O}(n^{-1})$ alone gives $H, U = \text{const} + \mathcal{O}(n^{-1})$ in GR. However, we could not decouple the leading-order equation only with the assumption for C in the EGB theory.

order of n^{-i}. Here Ω_H is determined so that $k = \partial_t + \Omega_H \partial_\phi$ becomes the null generator of the horizon. To fix the above integration constants, we simply set

$$A_i(\mathsf{R} = m) = 0, \qquad C_i(\mathsf{R} = m) = 0. \tag{13}$$

This sets the horizon at $\mathsf{R} = m$ and angular velocity as

$$\Omega_H = \Omega m^{-\frac{1}{2n}} \tag{14}$$

in all order of $1/n$. In the original coordinate, the horizon radius is given by

$$r_H \coloneqq m^{\frac{1}{2n}}. \tag{15}$$

For the other metric functions, we simply impose the regularity at $\mathsf{R} = m$ and asymptotic boundary condition. In the derivation, it is convenient to introduce an auxiliary variable [18]

$$X \coloneqq \sqrt{1 + \frac{4\alpha(\alpha+1)m}{\mathsf{R}}}, \tag{16}$$

which takes $X = 1$ at $\mathsf{R} = \infty$ and $X = 1 + 2\alpha$ on the horizon.

Having these in mind, the next-to-leading-order solution is determined as

$$C_1 = \frac{\Omega(X-1)}{4\alpha(1-\Omega^2)} \log\left(\frac{4\alpha(1+\alpha)}{X^2-1}\right), \tag{17}$$

$$U_1 = \frac{(X-1)\Omega^2(\alpha(X-1)-1)}{2(\alpha+1)(2\alpha+1)(X^2+1)(\Omega^2-1)}, \tag{18}$$

$$H_1 = \frac{\Omega^2}{(\alpha+1)(1-\Omega^2)}\left[\log\left(\frac{X+1}{2}\right) - \arctan X + \frac{\pi}{4} - \frac{1}{2(2\alpha+1)}\log\left(\frac{X^2+1}{2}\right)\right], \tag{19}$$

and

$$A_1 = \frac{(X^2-1)\Omega^2 \log(X^2+1)}{16\alpha(2\alpha^2+3\alpha+1)X(\Omega^2-1)} + \frac{(X^2-1)\Omega^2(\arctan X - \arctan(1+2\alpha))}{8\alpha(\alpha+1)X(\Omega^2-1)} - \frac{(X-1)(X+2\Omega^2-1)\log(X-1)}{4\alpha X(\Omega^2-1)} - \frac{(X-1)\log(X+1)(\alpha(4\Omega^2-2)+X(2\alpha+\Omega^2+2)+5\Omega^2-2)}{8\alpha(\alpha+1)X(\Omega^2-1)} + a_0 + a_1 X + \frac{a_2}{X}, \tag{20}$$

where the coefficients a_0, a_1, and a_2 are given by

$$a_0 = \frac{\log(4m\alpha(\alpha+1))}{2\alpha} + \frac{1}{4\alpha(1-\Omega^2)}, \tag{21}$$

$$a_1 = \frac{2(\alpha+1)\log(4\alpha(1+\alpha)) + 2\log m + 1}{8\alpha(1+\alpha)(\Omega^2-1)} + \Omega^2 \frac{2(1+2\alpha)\log(2(1+\alpha)/m^2) - \log(1+(1+2\alpha)^2)}{16\alpha(1+\alpha)(1+2\alpha)(\Omega^2-1)}, \tag{22}$$

$$a_2 = \frac{(1+2\alpha)(1+2\log m)}{8\alpha(1+\alpha)(\Omega^2-1)} + \frac{\log(4\alpha(1+\alpha))}{4\alpha(\Omega^2-1)} - \frac{\Omega^2(4(\alpha+1)\log(2\alpha) + (4\alpha+5)\log(2(\alpha+1)))}{8\alpha(\alpha+1)(\Omega^2-1)} + \frac{\Omega^2(\log(1+(1+2\alpha)^2) - 4(2\alpha+1)^2\log m)}{16\alpha(1+\alpha)(1+2\alpha)(\Omega^2-1)}. \tag{23}$$

One can easily check that the GR limit $\alpha \to 0$ reproduces the equally rotating Myers-Perry solutions up to NLO in the corresponding gauge. The large α limit gives another simplification

$$A \to 1 - \sqrt{\frac{m}{\mathsf{R}}}\left(1 + \frac{m\log(\mathsf{R}/m)}{2n\mathsf{R}(1-\Omega^2)}\right), \tag{24}$$

$$C \to \sqrt{\frac{m}{\mathsf{R}}}\frac{\Omega\log(\mathsf{R}/m)}{2n(1-\Omega^2)}, \tag{25}$$

and

$$U \to 1 + \mathcal{O}(n^{-2}), \qquad H \to 1 + \mathcal{O}(n^{-2}), \tag{26}$$

which could imply the existence of the analytic form in the pure GB theory.

The ergosurface of the leading-order metric (12) is given by the same condition as in GR,

$$0 = g_{tt} = (1-\Omega^2)^{-1}(-A - 2\Omega C + \Omega^2 H), \tag{27}$$

which is solved as

$$\mathsf{R}_{\rm ergo} = \frac{(1+\alpha)m}{(1-\Omega^2)(1+\alpha(1-\Omega^2))} + \mathcal{O}(n^{-1}). \quad (28)$$

This is a monotonically increasing function of α, and hence the ergoregion is extended by the GB correction. For $\alpha \to \infty$, $\mathsf{R}_{\rm ergo}$ approaches to a finite value.

In the EGB theory, the thermodynamic variables are obtained as in GR, except the entropy defined by the Iyer-Wald formula [23,24]

$$\mathcal{S} = \frac{1}{4G}\int_H (1+2\alpha_{\rm GB}\mathcal{R})\sqrt{h}d^{D-2}x, \quad (29)$$

where h and $\mathcal{R}$ are the spacial metric and curvature of the horizon cross section, respectively. Note that the angular velocity is already given in Eq. (14). Up to NLO, the Arnowitt-Deser-Misner (ADM) mass and angular momentum, temperature, and entropy are given by

$$M = \frac{n\Omega_{2n+1}}{8\pi G}\frac{(1+\alpha_H)m}{1-\Omega^2}\left[1 - \frac{1}{8n(1-\Omega^2)(\alpha_H+1)(2\alpha_H+1)}(4 - 8\Omega^2\alpha_H^2 + (-8\Omega^4 + 2(\pi-6)\Omega^2 + 8)\alpha_H - 2\Omega^2\log(2\alpha_H^2 + 2\alpha_H + 1) + 4\Omega^2(2\alpha_H+1)(\log(\alpha_H+1) - \arctan(2\alpha_H+1)) + \Omega^2(\pi-4))\right], \quad (30)$$

$$J = \frac{n\Omega_{2n+1}}{8\pi G}\frac{(1+\alpha_H)m^{\frac{2n+1}{2n}}\Omega}{1-\Omega^2}\left[1 - \frac{1}{8n(1-\Omega^2)(\alpha_H+1)(2\alpha_H+1)}(8(2\Omega^2-1)\alpha_H^2 + 2\Omega^2\log(2\alpha_H^2+2\alpha_H+1) + 4(1+2\alpha_H)\Omega^2(\arctan(2\alpha_H+1) - \log(\alpha_H+1)) - 2\alpha_H((\pi-16)\Omega^2+10) - (\pi-8)\Omega^2 - 8)\right], \quad (31)$$

$$T = \frac{n}{\pi}\frac{1+\alpha_H}{1+2\alpha_H}m^{-\frac{1}{2n}}\sqrt{1-\Omega^2}\left[1 - \frac{(4\Omega^2+1)\alpha_H + 4\alpha_H^2 + 2\Omega^2}{2n(1-\Omega^2)(\alpha_H+1)(2\alpha_H+1)}\right], \quad (32)$$

$$S = \frac{\Omega_{2n+1}}{4G}\frac{(1+2\alpha_H)m^{\frac{2n+1}{2n}}}{\sqrt{1-\Omega^2}}\left[1 + \frac{1}{8n(1-\Omega^2)(\alpha_H+1)(2\alpha_H+1)}(8(1-2\Omega^2)\alpha_H^2 + 8\alpha_H(1-2\Omega^2) + 4(1+2\alpha_H)\Omega^2(\log(1+\alpha_H) - \arctan(2\alpha_H+1) + \pi/4) - 2\Omega^2\log(2\alpha_H^2+2\alpha_H+1))\right], \quad (33)$$

where the GB coupling is written in the scale invariant form $\alpha_H \coloneqq \alpha/r_H^2 = \alpha/m^{\frac{1}{n}}$. The first law $dM = TdS + \Omega_H dJ$ is easily checked by differentiating with m and Ω up to NLO with α fixed.

From Eq. (32), one can expect the extremal limit would exist approximately at

$$\Omega = 1 - \frac{2+5\alpha+4\alpha^2}{4n(1+2\alpha)(1+\alpha)}. \quad (34)$$

Unfortunately, we will see that T includes $(1-\Omega^2)^{-2}$ in next-to-next-to-leading order [25], which invalidates the $1/n$ expansion around the extremal limit. This fact should not be so remarkable, since as pointed out already in the Einstein gravity [20,26], the large D limit is incompatible to the extremal limit, so that we need some remedy to eliminate the apparent breakdown of the $1/n$ expansion near the extremal limit, as actually performed for charged squashed black holes [27]. Finding the analytic solution of equally rotating black holes in the pure GB theory could shed some light on the extremal limit in the EGB theory. Interestingly, the extremal limit of the equally rotating black holes was examined for small α in $D = 5$ [28], where the inner horizon only appears close to the extremality.

Below, we present the phase diagram in the terms of dimensionless variables related by

$$s = \frac{\sqrt{1-j^2}(2\alpha_H+1)}{\alpha_H+1}\left[1 + \frac{1}{2n(1-j^2)}\left(\log\left(\frac{1-j^2}{1+\alpha_H}\right) + \frac{\alpha_H(4(1-j^2)\alpha_H - 4j^2 + 3)}{(\alpha_H+1)(2\alpha_H+1)}\right)\right], \quad (35)$$

where the angular momentum and entropy are normalized by the mass scale

$$j \coloneqq \frac{8\pi GJ}{(n+1)\Omega_{2n+1}}\left(\frac{8\pi GM}{(n+1/2)\Omega_{2n+1}}\right)^{-\frac{2n+1}{2n}}, \quad (36)$$

$$s \coloneqq \frac{4GS}{(n+1)\Omega_{2n+1}}\left(\frac{8\pi GM}{(n+1/2)\Omega_{2n+1}}\right)^{-\frac{2n+1}{2n}}. \quad (37)$$

Here the spin parameter is expressed as the function of j,

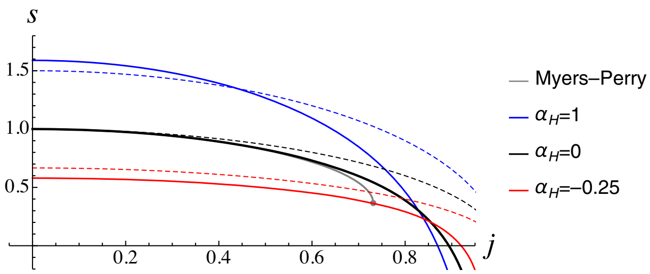

FIG. 1. The phase diagram in the space of the entropy and angular momentum normalized by the mass for $n = 4(D = 11)$. The thick and dashed curves represents the NLO and LO results, respectively. The exact Myers-Perry solutions for $n = 4$ are also plotted by the gray curve.

$$\Omega = j - \frac{j}{2n}\left[\log\left(\frac{1-j^2}{1+\alpha_H}\right) - \frac{2\alpha_H j^2}{(1+\alpha_H)(1+2\alpha_H)}\right]. \tag{38}$$

In Fig. 1, the phase diagram shows that the positive (negative) value of α gives larger (smaller) entropy than GR solutions for each j, succeeding the property of the static solutions. Near the extremality, the convergence of $1/n$ expansion becomes bad.

In this work, using the large D approach, we have obtained the first analytic solutions of not-slowly rotating black holes to the EGB theory in odd dimensions. For larger α, the size of the ergoregion becomes larger, and then for $\alpha \to \infty$, it saturates. We have also determined the first phase diagram of equally rotating EGB black holes. More technical details and higher-order corrections will be presented in a forthcoming paper [25].

By introducing the dependence of the metric on the time and angular coordinates, one can obtain the large D effective theory, which enables the stability analysis of the horizon. We expect the same strategy also applies to the singly rotating case. It would be also interesting to explore the large D rotating black holes in the more general Lovelock theory [29] with the same ansatz.

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