Dark matter production in Weyl R^2 inflation

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Dark matter and inflation are two key elements to understand the origin of cosmic structures in modern cosmology, and yet their exact physical models remain largely uncertain. The Weyl scaling invariant theory of gravity may provide a feasible scheme to solve these two puzzles jointly, which contains a massive gauge boson playing the role of dark matter candidate and allows the quadratic scalar curvature term, namely R^2, to realize a viable inflationary mechanism in agreement with current observations. We ponder on the production of dark matter in the Weyl R^2 model, including the contribution from the nonperturbative production due to the quantum fluctuations from inflationary vacuum and perturbative ones from scattering. We demonstrate that there are generally three parameter regions for viable dark matter production: (1) If the reheating temperature is larger than 10^3 GeV, then the Weyl gauge boson as dark matter can be produced abundantly with mass larger than the inflation scale $\sim 10^{13}$ GeV. (2) Small mass region with 3×10^{-13} GeV for a higher reheating temperature. (3) Annihilation channel becomes important in the case of higher reheating temperature, which enables the Weyl gauge boson with a mass up to 4×10^{16} GeV to be produced through freeze-in.

I. INTRODUCTION

Dark matter is an effective paradigm to explain the various observations in galactic rotation curve, cosmic large scale structures, and the gravitational lensing of clusters [1–4]. Studies of the primordial nucleosynthesis [5] and the power spectrum of cosmic microwave background (CMB) [6] also provide strong evidence that there is abundant missing matter whose present total amount is about five times that of visible matter described by the standard model (SM) of particle physics. As for the physical essence of dark matter, however, it still remains obscure. All experiments so far have not identified whether dark matter interacts with SM particles except gravity [7–9].

In this paper, we ponder on a new kind of vector dark matter [10,11] from the Weyl scaling invariant theory of gravity [12,13]. Weyl symmetry with various forms has wide

applications in cosmological physics [14–18]. It can naturally introduce a local scaling symmetry associated with a gauge boson (called a Weyl gauge boson or WGB), which acquires a mass from the symmetry breaking. The consideration of Weyl gravity is also motivated by the reasonableness of introducing a quadratic scalar curvature term R^2 into theory [19–21]. It can provide a successful inflationary mechanism with observational conformity, albeit it has a different form under the Weyl geometry [22–25]. Our previous study has discussed Weyl R^2 inflation [21] and briefly the production of dark matter. However, the intrinsic connection between the inflaton and WGBs has not been fully explored. The purposes of this paper are twofold. First, we use the latest cosmological observations to update the preferred parameter space for Weyl R^2 inflation. Second, we identify the viable dark matter mass range by considering all the production channels.

There are several contributions for the production of WGBs. If dark matter is already massive during inflation, there will be an important nonperturbative production mechanism [26–37] that is due to the quantum fluctuations in rapidly expanding background [38–45]. Other important contributions are also present after inflation, such as producing from inflaton decay or the annihilation from particles in thermal bath after reheating [46–59]. The previous study

[21] has only estimated the annihilation channel through scalars, but not taken into account the gravitational annihilation, nonperturbative production and the decay process. Our goal here is to figure out how these production mechanisms determine the relic abundance of WGBs as dark matter, and identify the possible mass ranges.

The paper is organized as follows. In Sec. II, we introduce the formalism of Weyl R^2 gravity with inflaton and massive WGBs. Then in Sec. III, we discuss the constraints on the Weyl R^2 inflation from the latest observations. Later in Sec. IV, we focus on the WGB as dark matter. We first discuss the equation of motion and quantization of the Weyl gauge field in a cosmological background, and elaborate the method to calculate the energy density of WGBs produced from the inflationary fluctuations. Then based on the inflation and reheating process, we compute the production rate and final relic abundance of WGBs from all possible channels, and show the constraints on the mass and reheating temperature in detail. Finally, we summarize and give our conclusion.

II. FORMALISM AND MODEL

We use the following conventions in this paper: the metric $\eta_{\mu\nu} = (-1, +1, +1, +1)$, and natural unit $\hbar = c = 1$, $M_P \equiv 1/\sqrt{8\pi G} = 2.435 \times 10^{18}$ GeV $= 1$.

The complete Lagrangian that is motivated by gauge theory of gravity can be found in Refs. [12,13,60,61]. For the interest of dark matter production and inflation, we can use the metric formalism and illustrate by starting with the following simplest global scaling-invariant Lagrangian $\mathcal{L}$ with linear and quadratic scalar curvature

$$\frac{\mathcal{L}}{\sqrt{-g}} = \frac{1}{2}\phi^2 R + \frac{\beta}{12}R^2 - \frac{\lambda}{2}\partial^\mu\phi\partial_\mu\phi, \qquad (1)$$

where g is the determinant of metric tensor $g_{\mu\nu}$, Ricci scalar R are defined as usual and λ is a nonzero real number. A scalar field ϕ is introduced here to maintain the global scaling symmetry with the following transformation rules:

$$\text{metric:} \quad g_{\mu\nu} \to g'_{\mu\nu} = f^2 g_{\mu\nu},$$
$$\text{scalar:} \quad \phi \to \phi' = f^{-1}\phi,$$
$$\text{Ricci scalar:} \quad R \to R' = f^{-2}R, \qquad (2)$$

where f is a constant scale factor. Note that one can introduce a scalar potential ϕ^4 without spoiling the symmetry, but for our purpose in this paper, such a term is not necessary and we neglect here.

For the global scaling transformation, the Christoffel connection

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}), \qquad (3)$$

is invariant. When we extend the global symmetry into the local one, namely $f \to f(x)$, the connection and $\mathcal{L}$ would not be invariant. To make the theory locally scaling invariant, we construct the following connection

$$\hat{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + (W_\mu g^\rho_\nu + W_\nu g^\rho_\mu - W^\rho g_{\mu\nu}), \qquad (4)$$

by introducing a vector $W_\mu \equiv g_W w_\mu$, called Weyl gauge field (g_W is the gauge coupling), with the following transformation rule:

$$W_\mu \to W'_\mu = W_\mu - \partial_\mu \ln f(x). \qquad (5)$$

$\hat{\Gamma}^\rho_{\mu\nu}$ can be also obtained by replacing $\partial_\mu g_{\rho\sigma} \to (\partial_\mu + 2W_\mu)g_{\rho\sigma}$ in Eq. (3). Then the corresponding Ricci scalar $\hat{R}$ should be defined through $\hat{\Gamma}^\rho_{\mu\nu}$ and Lagrangian in Eq. (1) is modified to a new form

$$\frac{\mathcal{L}}{\sqrt{-g}} = \frac{1}{2}\phi^2\hat{R} + \frac{\beta}{12}\hat{R}^2 - \frac{\lambda}{2}D^\mu\phi D_\mu\phi - \frac{1}{4g_W^2}F_{\mu\nu}F^{\mu\nu}, \qquad (6)$$

where the derivative in the kinetic term of ϕ has been replaced to the covariant form, $D_\mu = \partial_\mu - W_\mu$, and the invariant field strength of W_μ is defined as $F_{\mu\nu} \equiv \partial_\mu W_\nu - \partial_\nu W_\mu$. This model is also discussed in [19–21].

To make the theoretical formalism more transparent for analyzing inflation and dark matter, we shall utilize the equivalence between $f(R)$ theory and the scalar-tensor theory [62,63]. We define

$$F(\hat{R}) = \frac{1}{2}\phi^2\hat{R} + \frac{\beta}{12}\hat{R}^2, \qquad (7)$$

and introduce an auxiliary scalar field χ. Then Eq. (6) can be written as an equivalent form

$$\frac{\mathcal{L}}{\sqrt{-g}} = \frac{1}{2}[F'(\chi^2)(\hat{R} - \chi^2) + F(\chi^2)]$$
$$- \frac{\lambda}{2}D^\mu\phi D_\mu\phi - \frac{1}{4g_W^2}F_{\mu\nu}F^{\mu\nu},$$
$$= \frac{1}{2}\left(\phi^2 + \frac{\beta}{3}\chi^2\right)\hat{R} - \frac{1}{12}\beta\chi^4$$
$$- \frac{\lambda}{2}D^\mu\phi D_\mu\phi - \frac{1}{4g_W^2}F_{\mu\nu}F^{\mu\nu}, \qquad (8)$$

where $F'(\chi^2) \equiv |\frac{\partial F(\hat{R})}{\partial\hat{R}}|_{\hat{R}=\chi^2}$. The equivalence relation $\chi^2 = \hat{R}$ can be easily derived by using the equation of motion from $\delta\mathcal{L}/\delta\chi = 0$.

Note that the local scaling symmetry of Eq. (8) allows us to choose convenient $f(x)$ in Eqs. (2) and (5) to fix the gauge and simplify the Lagrangian. To compare with the

Einstein gravity, we adopt $f(x) = \sqrt{\phi^2 + \frac{\beta}{3}\chi^2}$ and get $\frac{1}{2}(\phi^2 + \frac{\beta}{3}\chi^2)\hat{R} = \frac{1}{2}\hat{R}$, or, equivalently, $\phi^2 + \frac{\beta}{3}\chi^2 = 1$. Putting things together, we can obtain the following Lagrangian

$$\frac{\mathcal{L}}{\sqrt{-g}} = \frac{1}{2}R - \frac{\lambda}{2}D^\mu\phi D_\mu\phi - \frac{3}{4\beta}(1-\phi^2)^2$$

$$- \frac{1}{4g_W^2}F_{\mu\nu}F^{\mu\nu} - 3W^\mu W_\mu,$$

$$= \frac{1}{2}R - \frac{3\lambda}{6+\lambda\phi^2}\partial^\mu\phi\partial_\mu\phi - \frac{3}{4\beta}(1-\phi^2)^2 - \frac{1}{4g_W^2}F_{\mu\nu}F^{\mu\nu}$$

$$- \frac{1}{2}(6+\lambda\phi^2)\left[W_\mu - \frac{1}{2}\partial_\mu\ln|6+\lambda\phi^2|\right]^2. \quad (9)$$

where we have used the relation between $\hat{R}$ and R,

$$\hat{R} = R - 6W_\mu W^\mu - \frac{6}{\sqrt{-g}}\partial_\mu(\sqrt{-g}W^\mu), \quad (10)$$

and neglect the surface term at infinite due to the total derivative term above.

We define a new field variable Φ,

$$\Phi \equiv \pm \begin{cases} \sqrt{6}\ln\left[\sqrt{\frac{+\lambda\phi^2}{6}} + \sqrt{\frac{+\lambda\phi^2+6}{6}}\right] & \text{for } \lambda > 0 \\ \sqrt{6}\ln\left[\sqrt{\frac{-\lambda\phi^2}{6}} + \sqrt{\frac{-\lambda\phi^2-6}{6}}\right] & \text{for } \lambda < 0 \end{cases}, \quad (11)$$

to get the canonical kinetic term, and make a transformation for the Weyl gauge field

$$\tilde{W}_\mu \equiv W_\mu - \frac{1}{2}\partial_\mu\ln|6+\lambda\phi^2| \equiv g_W\tilde{w}_\mu. \quad (12)$$

Then the Lagrangian of Weyl R^2 model can be rewritten in a simple form as

$$\frac{\mathcal{L}}{\sqrt{-g}} = \frac{1}{2}R - \frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi - \frac{1}{4g_W^2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$$

$$- \frac{1}{2}m^2(\Phi)\tilde{W}^\mu\tilde{W}_\mu - V(\Phi), \quad (13)$$

where we have defined the mass term

$$m^2(\Phi) = \begin{cases} +6\cosh^2\left(\frac{\Phi}{\sqrt{6}}\right) & \text{for } \lambda > 0 \\ -6\sinh^2\left(\frac{\Phi}{\sqrt{6}}\right) & \text{for } \lambda < 0 \end{cases} \quad (14)$$

and scalar potential

$$V(\Phi) = \begin{cases} \frac{3}{4\beta}\left[1 - \frac{6}{\lambda}\sinh^2\left(\frac{\Phi}{\sqrt{6}}\right)\right]^2 & \text{for } \lambda > 0 \\ \frac{3}{4\beta}\left[1 + \frac{6}{\lambda}\cosh^2\left(\frac{\Phi}{\sqrt{6}}\right)\right]^2 & \text{for } \lambda < 0 \end{cases}. \quad (15)$$

Now it is clear that this theory can be reformulated as the Einstein gravity with a massive WGB $\tilde{W}_\mu$ and a self-interacting scalar Φ. In the following sections, we shall show that Φ can be the inflaton field while $\tilde{W}_\mu$ is a dark matter candidate due to the Z_2 symmetry, $\tilde{W}_\mu \to -\tilde{W}_\mu$.

III. INFLATION IN WEYL R^2 GRAVITY

Now we concentrate on the inflationary mechanism in this model. The physics of R^2 term has been extensively studied as the well-known Starobinsky model [64–71]. It was proposed originally to solve the cosmological initial singularity problem, and later demonstrated to be an elegant mechanism to give rise to an inflationary universe. In this section, we shall show the predictions in our model are different from those in Starobinsky model.

The potential $V(\Phi)$ in Fig. 1 for a hilltoplike shape regardless of the sign of λ. The parameter β decides the height of the hilltop, and λ determines the position of valley. Figure 1 shows the potentials with $\beta = 1.56 \times 10^{10}$ in Planck units and several values of $\lambda > 0$ ($\lambda < 0$ is similar because $\cosh^2 x = 1 + \sinh^2 x$).

At the potential minimum Φ can outstrip the Planck scale when $|\lambda|$ is not too small. It is a general picture that for large $|\lambda|$ the potential can be flat and lead to an inflationary universe when Φ slow rolls on the hilltop, until it falls into the valley.

To compare with the latest observational constraints by BICEP/*Keck* collaboration [72], we calculate the spectral
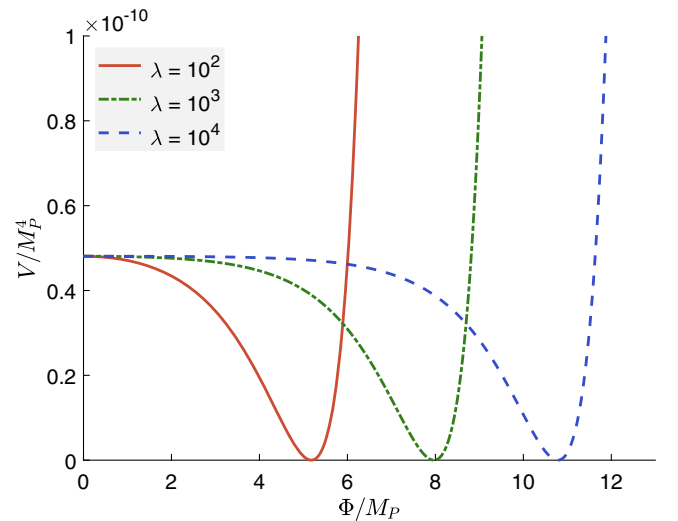

FIG. 1. The inflationary potentials of the Weyl R^2 model with $\lambda = 10^2, 10^3, 10^4$. It is a hilltoplike potential whose typical vacuum expectation value is larger than the Planck scale.
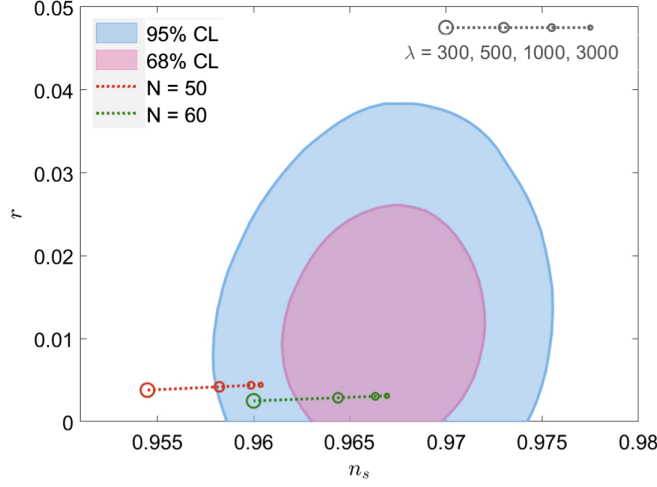
FIG. 2. The predictions of spectral index n_s and tensor-to-scalar ratio r for the Weyl R^2 model with λ from 300 to 3000, and e-folding number $N = 50, 60$, in comparison with the recent constraints shown by the BICEP/*Keck* collaboration [72].

index n_s and tensor-to-scalar ratio r for the primordial perturbations. These values are related to the slow-roll parameters ϵ and η through

$$\epsilon = \frac{1}{2}\left[\frac{V'(\Phi)}{V}\right]^2, \quad \eta = \frac{V''(\Phi)}{V}, \quad n_s = 1 - 6\epsilon + 2\eta, \quad r = 16\epsilon. \tag{16}$$

We numerically compute n_s and r for this model with λ from 300 to 3000, and show the results in Fig. 2, where N is defined as the e-folding number of the inflationary multiples from the horizon scale as large as the present one to the end of inflation [73], $N \equiv \ln(a_e/a_i) \sim (50, 60)$. The figure indicates that as λ increases, n_s and r will approach fixed values predicted in Starobinsky model for every e-folds number N. The physical reasons has been explained in detail in Ref. [21] in which the authors show the equivalence between this model and Starobinsky inflation as $|\lambda| \to +\infty$. There is another noteworthy point in Fig. 2, the typical tensor-to-scalar ratio in this model is greater than $\mathcal{O}(10^{-3})$, which implies that the Weyl R^2 model may be tested by the next generation experiment of CMB B-mode polarization [74]. Current observation can give constraints on $\lambda \gtrsim 500$ for $N = 50$ and $\lambda \gtrsim 250$ for $N = 60$.

On the other hand, the observation of CMB perturbation also provide a constraint on the Hubble parameter during inflation (denoted by $H_{\rm inf}$) and the mass of inflaton m_Φ in this model. It is known that the amplitude of scalar spectra $\Delta_s^2 \sim V/24\pi^2\epsilon$, and the observation of CMB has confirmed that $\Delta_s^2 \sim 2.1 \times 10^{-9}$. Thus this result constrains the parameter β in Eq. (15), which determines the height of the potential V at the hilltop. In this way, we find the Hubble parameter during inflation can be limited to $H_{\rm inf} \sim \sqrt{V/3} \sim 10^{13}$ GeV. Similarly, the mass of inflaton

is also constrained to $m_\Phi \sim 2 \times 10^{13}$ GeV defined at the minimum of the scalar potential.

IV. WEYL GAUGE BOSON AS DARK MATTER

Now we discuss the physics of Weyl gauge boson. As we have shown, the Lagrangian has Z_2 symmetry, $\tilde{W}_\mu \to -\tilde{W}_\mu$, which enables the stability of $\tilde{W}_\mu$. The mass term $-\frac{1}{2}m^2\tilde{W}^\mu\tilde{W}_\mu$ in Eq. (13), however, shows that the negative sign would appear for $\lambda < 0$. This does not necessarily mean an inconsistent theory because one could introduce another scalar field ψ with $D_\mu\psi D^\mu\psi$ that can contribute to the mass of $\tilde{W}_\mu$ when $\psi \neq 0$ at its potential minimum. For illustration, we shall only focus on the $\lambda > 0$ case in the rest of this paper.

Since the magnitude of Φ is evolving in the inflation and subsequent reheating process, the mass term in Eq. (14) is also changing in this period, until Φ completely stops at its potential minimum Φ_0. We can expand the mass term at $\Phi = \Phi_0$ as

$$\begin{aligned} m^2 &= 6\cosh^2\left(\frac{\varphi + \Phi_0}{\sqrt{6}}\right), \\ &= 6\cosh^2\left(\frac{\Phi_0}{\sqrt{6}}\right) + \sqrt{6}\sinh\left(\frac{2\Phi_0}{\sqrt{6}}\right)\varphi \\ &\quad + \cosh^2\left(\frac{\Phi_0}{\sqrt{6}}\right)\varphi^2 + \mathcal{O}(\varphi^3). \end{aligned} \tag{17}$$

It implies that the term $-\frac{1}{2}m^2\tilde{W}^\mu\tilde{W}_\mu$ not only describes the mass of WGBs, but also $\varphi - \tilde{W}$ interactions that may lead to the decay or annihilation of inflaton into WGBs after inflation. The eventual mass of a WGB is determined by the parameters λ and g_W

$$m_W = \sqrt{6}|g_W|\cosh[\Phi_0(\lambda)/\sqrt{6}]. \tag{18}$$

The couplings between φ and WGBs also depend on these two parameters and can be read from above. In the rest of this section, we shall discuss the production mechanism for the WGB as a dark matter candidate in detail by including all of the above couplings and interactions.

A. Weyl gauge field in a cosmological background

The production of vector dark matter includes both contributions from perturbative scattering and nonperturbative mechanism from vacuum fluctuations due to the quantum effect in curved background [32–35]. Let us concentrate on the part of WGBs in Eq. (13)

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{4g_W^2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{2}m^2\tilde{W}^\mu\tilde{W}_\mu\right]. \tag{19}$$

The metric is set to the Friedmann-Lemaitre-Robertson-Walker form

$$g_{\mu\nu}dx^\mu dx^\mu = a^2(\tau)(-d\tau^2 + d\vec{x}^2), \tag{20}$$

where a is the scale factor of the Universe, and τ is the conformal time, which has the following relation with the physical time $d\tau = dt/a$. We can rewrite Eq. (19) in components

$$S = \int d^3x d\tau \frac{1}{2}(|\vec{\tilde{w}}' - \nabla\tilde{w}_0|^2 - |\nabla \times \vec{\tilde{w}}|^2 + a^2\tilde{m}^2\tilde{w}_0^2 - a^2\tilde{m}^2|\vec{\tilde{w}}|^2), \tag{21}$$

where $' \equiv \frac{\partial}{\partial\tau}$, $\tilde{m} \equiv |g_W|m(\Phi)$ and $\vec{\tilde{w}}$ denotes the spatial components.

It is obvious that $\tilde{w}_0$ in Eq. (21) has no kinetic term, which is commonly known as that only three of the four components of massive vectors are independent and physical modes. Hence, we can decouple the part of W_0 from Eq. (21) with a Fourier transform and only retain the physical modes, including two transverse modes and a longitudinal mode

$$\vec{\tilde{w}}(\tau, \vec{k}) \equiv \sum_{j=\pm,L} \vec{\epsilon}_j(\vec{k})W_j(\tau, \vec{k}), \tag{22}$$

where $\vec{\epsilon}_j(\vec{k})$ is the orthonormal polarization vector. Their actions are derived as follows

$$S_\pm = \int \frac{d^3k d\tau}{(2\pi)^3}\frac{1}{2}(|W'_\pm|^2 - (k^2 + a^2\tilde{m}^2)|W_\pm|^2), \tag{23}$$

$$S_L = \int \frac{d^3k d\tau}{(2\pi)^3}\frac{1}{2}\left(\frac{a^2\tilde{m}^2}{k^2 + a^2\tilde{m}^2}|W'_L|^2 - a^2\tilde{m}^2|W_L|^2\right), \tag{24}$$

where k is the magnitude of comoving momentum $\vec{k}$, and $\vec{k}\cdot\vec{\tilde{w}} = kW_L$ is used in the derivation.

For the sake of simplicity, we redefine a new field $\mathcal{W}_L$ for the longitudinal mode

$$\mathcal{W}_L \equiv \frac{a\tilde{m}}{\sqrt{k^2 + a^2\tilde{m}^2}}W_L. \tag{25}$$

Then the equation of motion for each physical mode can be written as oscillator equations

$$W''_\pm + \omega_\pm^2 W_\pm = 0, \tag{26}$$

$$\mathcal{W}''_L + \omega_L^2 \mathcal{W}_L = 0, \tag{27}$$

with frequency variables

$$\omega_\pm^2 = k^2 + a^2\tilde{m}^2, \tag{28}$$

$$\omega_L^2 = k^2 + a^2\tilde{m}^2 - \frac{k^2}{k^2 + a^2\tilde{m}^2}\frac{a''}{a} + 3\frac{k^2\tilde{m}^2 a'^2}{(k^2 + a^2\tilde{m}^2)^2}$$
$$- \frac{k^2}{k^2 + a^2\tilde{m}^2}\frac{\tilde{m}''}{\tilde{m}} + 3\frac{k^2a^2\tilde{m}'^2}{(k^2 + a^2\tilde{m}^2)^2}$$
$$- \frac{a'}{a}\frac{\tilde{m}'}{\tilde{m}}\frac{2k^2(k^2 - 2a^2\tilde{m}^2)}{(k^2 + a^2\tilde{m}^2)^2}. \tag{29}$$

The mode functions of WGBs can be determined by solving Eqs. (26) and (27).

Next, we show the relation between the energy density and the mode functions. The energy-momentum tensor of field in a curved background is

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}S)}{\delta g^{\mu\nu}}, \tag{30}$$

and the energy density is defined as the vacuum expectation of T_{00}

$$\rho = \langle 0|T_{00}|0\rangle. \tag{31}$$

Therefore, to calculate the energy density of Weyl gauge field, we need to define the vacuum state $|0\rangle$ of the theory. In other words, a quantized theory is required. The first step of quantization is to give the Heisenberg representation for Weyl gauge field

$$\hat{W}_\pm(\tau, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}[\hat{a}_{\vec{k}}\vec{\epsilon}_\pm(\vec{k})W_\pm(\tau, \vec{k})e^{i\vec{k}\cdot\vec{x}} + \hat{a}_{\vec{k}}^\dagger\vec{\epsilon}_\pm^*(\vec{k})W_\pm^*(\tau, \vec{k})e^{-i\vec{k}\cdot\vec{x}}],$$

$$\hat{\mathcal{W}}_L(\tau, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}[\hat{b}_{\vec{k}}\vec{\epsilon}_L(\vec{k})\mathcal{W}_L(\tau, \vec{k})e^{i\vec{k}\cdot\vec{x}} + \hat{b}_{\vec{k}}^\dagger\vec{\epsilon}_L^*(\vec{k})\mathcal{W}_L^*(\tau, \vec{k})e^{-i\vec{k}\cdot\vec{x}}]. \tag{32}$$

Then the quantization demands that the ladder operators should satisfy the following commutative relations

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}}^\dagger] = \delta_{jj'}\delta(\vec{k} - \vec{k}'), \quad [\hat{b}_{\vec{k}}, \hat{b}_{\vec{k}}^\dagger] = \delta(\vec{k} - \vec{k}'), \tag{33}$$

where j is the index that refers to the two components of transverse mode. The commutative relations can also be represented as a canonical form

$$[\hat{W}_\pm(\tau, \vec{x}), \hat{W}'_\pm(\tau, \vec{y})] = i\delta(\vec{x} - \vec{y}),$$
$$[\hat{\mathcal{W}}_L(\tau, \vec{x}), \hat{\mathcal{W}}_L'(\tau, \vec{y})] = i\delta(\vec{x} - \vec{y}), \tag{34}$$

which imply the following normalization condition:

$$W_\pm W_\pm^{*\prime} - W_\pm^* W'_\pm = \mathcal{W}_L\mathcal{W}_L^{*\prime} - \mathcal{W}_L^*\mathcal{W}'_L = i. \tag{35}$$

Now we can define the vacuum $|0\rangle$ as a state that complies with $\hat{a}_{\vec{k}}|0\rangle = \hat{b}_{\vec{k}}|0\rangle = 0$. Then the nonperturbative production of particles from vacuum can be calculated by $\rho = \langle 0|T_{00}|0\rangle$, described by the time-dependent mode functions. Finally, with Eqs. (23), (24), and (30)–(33), the energy density of Weyl gauge field in a cosmological background is obtained as

$$\rho_{\pm} = \int \frac{k^2 dk}{4\pi^2 a^4} [|W'_{\pm}|^2 + (k^2 + a^2\tilde{m}^2)|W_{\pm}|^2 - \omega_{\pm}], \quad (36)$$

$$\rho_L = \int \frac{k^2 dk}{4\pi^2 a^4} \left[|\mathcal{W}'_L|^2 + (k^2 + a^2\tilde{m}^2)|\mathcal{W}_L|^2 \right.$$
$$+ \left(\frac{(\frac{a'}{a} + \frac{\tilde{m}'}{\tilde{m}})k^2}{k^2 + a^2\tilde{m}^2} \right)^2 |\mathcal{W}_L|^2$$
$$\left. - \frac{(\frac{a'}{a} + \frac{\tilde{m}'}{\tilde{m}})k^2}{k^2 + a^2\tilde{m}^2}(\mathcal{W}'_L\mathcal{W}^*_L + \mathcal{W}'^*_L\mathcal{W}_L) - \omega_L \right], \quad (37)$$

where the zero-point energy at flat spacetime has been subtracted by the term $-\omega$. The total energy density is the sum of two transverse modes and one longitudinal mode, $\rho_{\rm tot} = 2\rho_{\pm} + \rho_L$.

Next we numerically solve the equations of motion (26), (27) in the inflationary background and calculate the energy density of the WGB. Directly solving these equations might be inaccurate in some cases. For instance, when ρ is many orders of magnitude smaller than the zero-point energy, the numerical calculation of ρ will meet the situation that subtracting two very close numbers, which will make the numerical result unstable. Fortunately, this intractable case usually has an adiabatic vacuum, which means the vacuum evolves slowly. Then, a strategy called adiabatic approximation can be used to deal with the problems [75]. In the adiabatic approximation, the mode functions can be written as the following form

$$W_{\pm}(\tau) = \mathcal{A}^k_{\pm}(\tau) v^k_{\pm}(\tau) + \mathcal{B}^k_{\pm}(\tau) v^{k*}_{\pm}(\tau),$$
$$\mathcal{W}_L(\tau) = \mathcal{A}^k_L(\tau) v^k_L(\tau) + \mathcal{B}^k_L(\tau) v^{k*}_L(\tau),$$
$$v^k_{\pm(L)} = \frac{1}{\sqrt{2\omega_{\pm(L)}}} \exp\left[-i \int^{\tau} \omega_{\pm(L)} d\tau \right], \quad (38)$$

where $\mathcal{A}^k(\tau)$ and $\mathcal{B}^k(\tau)$ are some functions for the k mode and satisfy the normalization condition

$$|\mathcal{A}^k|^2 - |\mathcal{B}^k|^2 = 1. \quad (39)$$

Substituting Eq. (38) into Eq. (32), we can redefine a new set of ladder operators by the Bogolyubov transformation

$$\tilde{a}_{\vec{k}} = \hat{a}_{\vec{k}}\mathcal{A}^k + \hat{a}^{\dagger}_{\vec{k}}\mathcal{B}^{k*}, \qquad \tilde{b}_{\vec{k}} = \hat{b}_{\vec{k}}\mathcal{A}^k + \hat{b}^{\dagger}_{\vec{k}}\mathcal{B}^{k*}, \quad (40)$$

which satisfy the similar commutative relations as Eq. (33)

$$[\tilde{a}_{\vec{k}}, \tilde{a}^{\dagger}_{\vec{k}}] = \delta_{jj'}\delta(\vec{k} - \vec{k}'), \qquad [\tilde{b}_{\vec{k}}, \tilde{b}^{\dagger}_{\vec{k}}] = \delta(\vec{k} - \vec{k}'). \quad (41)$$

Then the energy density of the Weyl gauge field can be written as

$$\rho_{\pm/L} = \int \frac{k^2 dk}{2\pi^2 a^4} |\mathcal{B}^k_{\pm/L}|^2 \omega_{\pm/L}. \quad (42)$$

Comparing to Eqs. (36) and (37), it can be seen the zero-point energy has disappeared in this expression, which means the aforementioned numerical problem is alleviated with the adiabatic approximation. In addition, the function $\mathcal{B}^k$ obeys

$$\mathcal{A}'v = \frac{\omega'_k}{2\omega_k} v^*\mathcal{B}, \qquad \mathcal{B}'v^* = \frac{\omega'_k}{2\omega_k} v\mathcal{A}, \quad (43)$$

which are identical to Eqs. (26) and (27).

Finally, it is essential to note that the adiabatic approximation can only be used to the case with positive ω^2. This is the prerequisite for the existence of adiabatic vacuum. For the transverse mode Eq. (28), this condition is completely satisfied. But for the longitudinal mode Eq. (29), the tachyonic situation ($\omega_L^2 < 0$) is possible, and it is common in the case of $\tilde{m} \ll H_{\rm inf}$. Consequently, if the tachyonic situation appears in the longitudinal mode, it would violate the adiabatic condition. Then the mode functions and energy density can only be solved with original formulas [Eqs. (27) and (37)].

B. Production of Weyl gauge boson

Now we are in a position to apply the above formalism in the cosmological background of inflation. It can be seen from Eqs. (26)–(29) that to solve the mode functions we also need the evolution of scale factor a over conformal time τ, which is determined by the inflation and reheating in our model.

1. Inflation and reheating

The evolution of cosmic scale a is described by the Friedmann equation

$$\mathcal{H}^2 \equiv \left(\frac{a'}{a} \right)^2 = \frac{a^2\rho}{3M_P^2}, \quad (44)$$

where $' \equiv \frac{\partial}{\partial\tau}$. To solve this equation, we need to discuss how the matter energy density evolves in our model. To be more specific, we need reheating process to transfer the energy of the inflaton Φ into other relativistic particles, for example through decay into SM particles [75–79]. Here, instead we first consider a new scalar ψ that couples to the inflaton and SM particles simultaneously. Then in this way, the inflaton can first decay into ψ before eventually transferring its energy to the SM

particles. To maintain the scaling symmetry, we illustrate with the following toy model:

$$\mathcal{L}_{\text{reh}} = -g_\psi^2 \phi^2 \psi^2 - \frac{1}{2} D^\mu \psi D_\mu \psi + \mathcal{L}(\psi, SM),$$

$$= -\frac{6g_\psi^2}{\lambda} \sinh^2\left(\frac{\Phi}{\sqrt{6}}\right) \psi^2 - \frac{1}{2} D^\mu \psi D_\mu \psi + \mathcal{L}(\psi, SM),$$

$$\sim -\frac{g_\psi^2}{\lambda} \Phi^2 \psi^2 - \frac{1}{2} D^\mu \psi D_\mu \psi + \mathcal{L}(\psi, SM), \quad (45)$$

where $\mathcal{L}(\psi, SM)$ denotes the coupling between ψ and SM particles, which we do not specify here. Here we have neglected higher order interactions in the last step. We reiterate that the definition $\Phi \equiv \varphi + \Phi_0$ has been introduced in the previous section. Then the first term in Eq. (45) can be expanded as

$$-\frac{g_\psi^2}{\lambda} \Phi^2 \psi^2 = -\frac{g_\psi^2}{\lambda} (\varphi + \Phi_0)^2 \psi^2,$$

$$= -\frac{g_\psi^2}{\lambda} \Phi_0^2 \psi^2 - 2\frac{g_\psi^2}{\lambda} \Phi_0 \varphi \psi^2 - \frac{g_\psi^2}{\lambda} \varphi^2 \psi^2. \quad (46)$$

From the left are the mass term of ψ, the three-scalar vertex, and the four-scalar vertex. The feasibility of φ's decay into ψ requires that the mass of ψ must be less than the half of φ. Note that the mass of inflaton is limited to $m_\Phi \sim 2 \times 10^{13}$ GeV in our model, thus it is clear that $m_\psi \equiv g_\psi \Phi_0/\sqrt{\lambda} < 10^{13}$ GeV. With a typical vacuum expectation value $\Phi_0 \sim 10 M_P$, the coupling is limited to $g_\psi/\sqrt{\lambda} \lesssim 4 \times 10^{-7}$, which gives an upper limit to the reheating temperature T_r. Note that the major factor determining T_r is the $\varphi \to \psi\psi$ decay process caused by the three-scalar vertex in Eq. (46), then the probability of inflaton decay is easily derived as

$$\Gamma(\varphi \to \psi\psi) = \frac{4g_\psi^4 \Phi_0^2}{8\pi\lambda^2} \frac{\sqrt{(m_\Phi/2)^2 - m_\psi^2}}{m_\Phi^2}$$

$$= \frac{g_\psi^2 m_\psi^2}{2\pi\lambda} \frac{\sqrt{(m_\Phi/2)^2 - m_\psi^2}}{m_\Phi^2} \lesssim 10^{-20} M_P. \quad (47)$$

As a consequence, the reheating temperature is estimated as $T_r \sim 0.4\sqrt{\Gamma M_P} \lesssim 10^8$ GeV, which is far less than the inflation energy scale.

With a specific reheating mechanism, the evolution of inflaton field obeys the following equation

$$\Phi'' + 2\mathcal{H}\Phi' + a^2 \frac{\partial V}{\partial \Phi} + a\Gamma\Phi' = 0. \quad (48)$$

Then we can write down the energy density of the inflaton

$$\rho_\Phi = \frac{\Phi'^2}{2a^2} + V(\Phi), \quad (49)$$

and the evolving equation for ψ and SM particles as radiation

$$\rho_r' + 4\mathcal{H}\rho_r - \frac{\Gamma\Phi'^2}{a} = 0. \quad (50)$$

Note that the total energy density in the Friedmann equation is the sum of ρ_Φ and ρ_r. So combining Eq. (44) with (48)–(50) and (15), we can obtain a numerical solution of $a(\tau)$, which describes the evolution of cosmic scale within the period of inflation, reheating, and the ensuing radiation-dominated epoch.

2. Nonperturbative production

We take $\Gamma(\varphi \to \psi\psi) = 10^{-20} M_P$ as an example in this subsection, which is the maximum rate allowed in the toy model we constructed above. We shall keep in mind that generally the reheating temperature is free as long as it satisfies the bound from Big Bang Nucleosynthesis and energy density limit after inflation.

We solve the numerical result of $a(\tau)$, and substitute it into Eqs. (26)–(29), the mode functions for each eventual mass m_W [defined as Eq. (18) or $\tilde{m}|_{\Phi=\Phi_0}$] and comoving momentum k can be derived with an initial condition. Because all the k modes we are interested in once had the wavelength much smaller than the horizon scale in the far past of the inflation, it is then expected that the term k^2 is dominant in Eqs. (28) and (29) under the limit of $\tau \to -\infty$. Therefore, the solution to the equation of harmonic oscillator is suggested to be an initial condition for both the transverse mode and the longitudinal mode, which is the well-known Bunch-Davies vacuum state [80]

$$\lim_{\tau \to -\infty} W_\pm (\text{or } \mathcal{W}_L) = \frac{1}{\sqrt{2k}} e^{-ik\tau}. \quad (51)$$

With this initial condition, we illustrate with the eventual mass $m_W = 10^{10}$ GeV in Fig. 3, where f_k is defined as a function that is proportional to the energy density Eqs. (36) and (37)

$$f_{k,\pm} \equiv [|W_\pm'|^2 + (k^2 + a^2\tilde{m}^2)|W_\pm|^2]/2\omega_\pm - \frac{1}{2} = |\mathcal{B}_\pm^k|^2,$$

$$f_{k,L} \equiv [|\mathcal{W}_L'|^2 + (k^2 + a^2\tilde{m}^2)|\mathcal{W}_L|^2 + \left(\frac{(\frac{a'}{a} + \frac{\tilde{m}'}{\tilde{m}})k^2}{k^2 + a^2\tilde{m}^2}\right)^2 |\mathcal{W}_L|^2$$

$$- \frac{(\frac{a'}{a} + \frac{\tilde{m}'}{\tilde{m}})k^2}{k^2 + a^2\tilde{m}^2}(\mathcal{W}_L'\mathcal{W}_L^* + \mathcal{W}_L'^*\mathcal{W}_L)]/2\omega_L - \frac{1}{2} = |\mathcal{B}_L^k|^2,$$

$$(52)$$

and a_{end} denotes the scale factor when the inflation ends (defined as the moment that the slow-roll parameter $\epsilon = 1$).
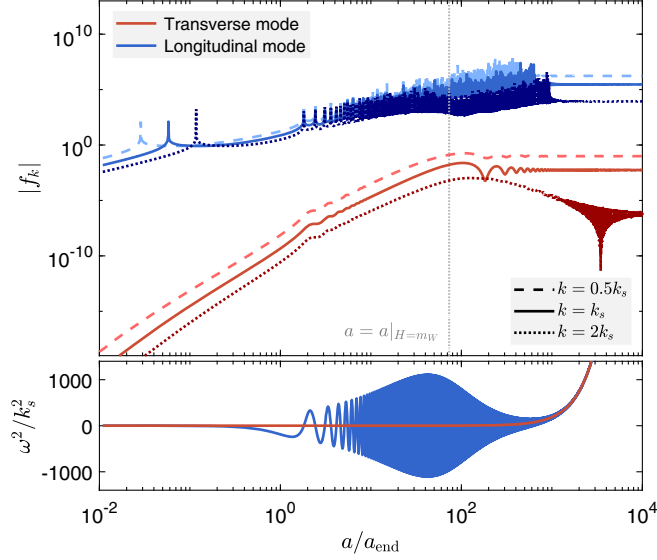
FIG. 3. Evolution of $|f_k|$ and ω^2 for the transverse mode (red line, solved by the adiabatic approximation) and longitudinal mode [blue line, solved directly in Eq. (27)] with $m_W = 10^{10}$ GeV, $\Gamma = 10^{-20}M_P$, and $k = 0.5, 1, 2k_s$ (k_s denotes the dominate mode of production). It depicts an increasing abundance of dark matters in the comoving volume until the Hubble parameter becomes lower than the mass of WGBs. Moreover, the existence of tachyonic enhancement makes the creation of the longitudinal mode far more productive than the transverse mode.

Moreover, k_s denotes the dominate mode of production, which is approximately equal to the product of m_W and the scale factor a for the moment that the Hubble parameter $H = m_W$ (if $m_W > H_{\rm inf}$, the dominant mode is about $k \sim a_{\rm end}H_{\rm inf}$).

There are two interesting aspects in Fig. 3. First, the production of the longitudinal mode is more abundantly than the transverse one, which has been shown by [32,34]. This behavior can be understood as follows. When $a \ll a_{\rm end} \lesssim k/m_W$, the frequency $\omega_\pm$ of the transverse mode is almost constant, which leads to a nonincreasing mode function as plane wave. However, for the longitudinal mode, we can show that $\omega_L^2 \sim k^2 - a''/a$ with $a \ll a_{\rm end} \lesssim k/m_W$, here we ignore the impact of changing mass for convenience. Since the Universe is approximately regarded as the de Sitter space during the inflationary epoch, that is $a \sim -1/H_{\rm inf}\tau$, we have $\omega_L^2 \sim k^2 - 2a^2H_{\rm inf}^2$, which decreases as the Universe expands. Substituting it into Eq. (27), then we obtain an increasing mode function

$$\mathcal{W}_L|_{a\ll k/m_w} \sim \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 + \frac{iaH_{\rm inf}}{k}\right). \qquad (53)$$

It indicates that when $a \ll k/H_{\rm inf}$, the mode function of the longitudinal mode is also a plane wave like the transverse one. But once the scale factor becomes large enough to

make the negative ω_L^2 emerge, the imaginary part of Eq. (53) will surge exponentially. This situation is usually called the tachyonic enhancement, which leads to the high yield of the longitudinal mode compared to the transverse one.

The second noticeable aspect is that the function f_k for the dominant mode k_s gradually approaches a stationary value after the Hubble parameter H becoming smaller than m_W. This is because the term $a^2m_W^2$ in Eqs. (28) and (29) turns into a leading term after that moment, which leads to a steady solution with $f_k \propto \rho a^3 \sim$ constant (see [35]). And the particles become nonrelativistic due to the expansion of the Universe. We can choose an arbitrary moment that satisfies $H \ll m_W$ to evaluate the spectrum of $f_k(k)$ around the dominant mode k_s, and calculate the integral (36), (37), or (42) to obtain the energy density ρ of a WGB at that moment. In practice, we consider a moment that is later than the end of reheating as the conserved quantity ρ/s can be defined for the WGB, where $s = \frac{2\pi^2}{45}g_{*S}(T)T^3$ is the total entropy density for the reheated particles, and $g_{*S}(T)$ denotes the effective number of degrees of freedom in entropy with temperature T. This quantity measures the relic abundance of nonperturbative produced particles.

Figure 4 shows how relic abundance varies with m_W and decay constant Γ, where m_W is determined by the parameters λ and g_W, and the parameter β has been fixed according to the constraint of observation on $H_{\rm inf}$ discussed in Sec. II. We can see that there are three cases in the relationship between ρ/s and m_W, which are summarized as
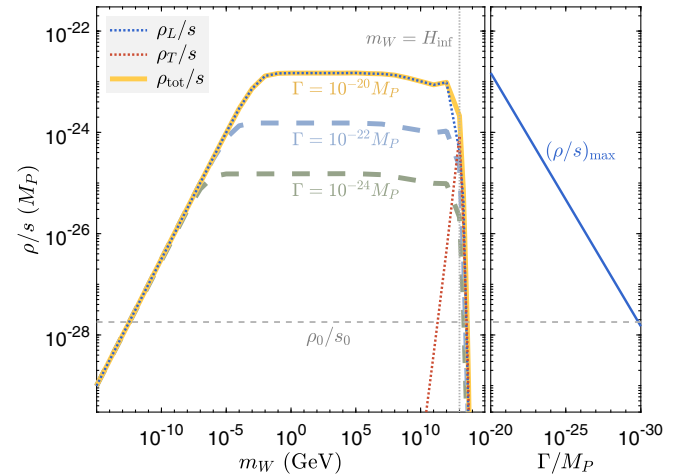

FIG. 4. Left: the relation between the relic abundance and the mass of nonperturbative produced WGBs. The feasible mass of WGBs should be restricted to the range that $\rho/s \leq \rho_0/s_0$. Right: the maximal abundance decreases proportionately with the total decay rate of inflaton. It shows that if $\Gamma < 10^{-30}M_P$ or $T_r < 10^3$ GeV, then the nonperturbative produced WGBs would only be part of dark matter.

$$\rho/s \propto \begin{cases} \sqrt{m_W}, & \text{for } m_W < H_{\text{reh}} \\ \text{almost constant}, & \text{for } H_{\text{reh}} < m_W < H_{\text{inf}}, \\ \text{exponential decrease}, & \text{for } m_W > H_{\text{inf}} \end{cases}$$

$$\tag{54}$$

where H_{reh} is the Hubble parameter at the end of reheating. For the current Universe, $T_0 = 2.73$ K $= 9.65 \times 10^{-32} M_P$, $g_{*S0} = 3.91$, and the present energy density for the dark matters is $\rho_0 = 2.73 \times 10^{-121} M_P^4$, then we can derive $\rho_0/s_0 = 1.8 \times 10^{-28} M_P$. This observational result gives a constraint to the mass of WGBs. It can be clearly identified in Fig. 4 that the relic abundance for the WGBs produced in a nonperturbative pattern will be larger than that of dark matters in the current Universe if the mass of WGBs is in the interval of 3×10^{-13} GeV to $\sim 5 \times 10^{13}$ GeV (the upper limit is in the range of 10^{13}–10^{14} GeV, which is slightly dependent on the decay rate Γ). Therefore, the observation constrains two possible ranges of mass, $m_W \lesssim 3 \times 10^{-13}$ GeV or $m_W \gtrsim 5 \times 10^{13}$ GeV, such that the relic density is not larger than the observation value.

When $m_W < 3 \times 10^{-13}$ GeV, according to the $\varphi - \tilde{W}$ interaction discussed in Sec. II, there will be other production channels for WGBs through the perturbative decay $\varphi \to \tilde{W}\tilde{W}$ and the annihilation $\varphi\varphi \to \tilde{W}\tilde{W}$. Since the contribution of decay is much larger than that of annihilation in our case, we only consider the decay. Based on Eqs. (17) and (18), the decay rate of $\varphi \to \tilde{W}\tilde{W}$ can be easily calculated as

$$\Gamma(\varphi \to \tilde{W}\tilde{W}) = \frac{3\sinh^2(2\Phi_0/\sqrt{6})g_W^4}{\pi}$$
$$\times \frac{\sqrt{(m_\Phi/2)^2 - m_W^2}}{m_\Phi^2}\left(3 + \frac{m_\Phi^4}{4m_W^4} - \frac{m_\Phi^2}{m_W^2}\right),$$
$$\sim \frac{3m_\Phi^3}{288\pi}\frac{\sinh^2(2\Phi_0/\sqrt{6})}{\cosh^4(\Phi_0/\sqrt{6})}, \quad \text{when } m_W \ll m_\Phi.$$

$$\tag{55}$$

Note that the mass of inflaton is around $m_\Phi \sim 2 \times 10^{13}$ GeV $\sim 10^{-5} M_P$ in this model, and the typical field value $\Phi_0 \sim 10 M_P$, thus the decay rate can be estimated as $\Gamma(\varphi \to \tilde{W}\tilde{W}) \sim 10^{-17} M_P$. Comparing to Eq. (47), the decay rate of $\varphi \to \tilde{W}\tilde{W}$ is far larger than that of $\varphi \to \psi\psi$. This situation will lead to an inadequate reheating to SM, and result in additional dark radiation in our Universe. In such a case, we would need other reheating mechanisms that can transfer most of energy in inflaton field into other particles, and subsequently SM particles.

For the large mass range $m_W \gtrsim 5 \times 10^{13}$ GeV, there is a $-\psi^2 W^\mu W_\mu$ term due to the covariant derivative $D_\mu = \partial_\mu - W_\mu$ in Eq. (45). This implies that the annihilation $\psi\psi \to \tilde{W}\tilde{W}$ is allowed in the large mass case. However, if

the reheating temperature is too low, then the annihilation rate will be negligible, because the number density n of ψ with enough energy to participate in the production of WGBs is exponentially reduced with decreasing temperature T, $n \propto \exp(-E/T)$. Therefore, with a low-temperature reheating mechanism, the nonperturbative production can reasonably provide enough abundance for WGBs as dark matter with the mass $m_W \sim 5 \times 10^{13}$ GeV.

Another visible aspect in Fig. 4 is that the decay rate Γ strongly affects the nonperturbative production. Note that Γ denotes the total rate of all decay channels of inflaton, including decay to SM particles and to dark matters. Hence if $m_W < m_\Phi/2$, then the $\varphi \to \tilde{W}\tilde{W}$ channel is open, and the total decay rate is never lower than $\Gamma(\varphi \to \tilde{W}\tilde{W})$. Then if $m_W > m_\Phi/2$, the total decay rate is equivalent to $\Gamma(\varphi \to \psi\psi)$ as the reheating rate. It is apparent on the right of Fig. 4 that the maximal abundance of nonperturbative produced WGBs decreases proportionately with Γ, and if $\Gamma < 10^{-30} M_P$, then the nonperturbative pattern is unable to produce sufficient WGBs as dark matter. Since we have shown $\Gamma(\varphi \to \tilde{W}\tilde{W}) \sim 10^{-17} M_P$ in this model (for $m_W < m_\Phi/2$), the $\Gamma < 10^{-30} M_P$ case only appears when $m_W > m_\Phi/2$, which corresponds to the reheating temperature $T_r \sim 0.4\sqrt{\Gamma M_P} < 10^3$ GeV. As a result, for this range of T_r, only heavy WGBs can be a dark matter candidate but with insufficient abundance.

3. High-temperature reheating

The preceding discussions consider a low-temperature reheating mechanism that is constructed for simplicity. In this section, we investigate the production of WGBs with a high temperature of reheating.

Let us focus on the small mass range first. The perturbative pattern of production is primarily the decay of inflaton $\varphi \to \tilde{W}\tilde{W}$. The annihilation $\psi\psi \to \tilde{W}\tilde{W}$ is also present (ψ denotes an arbitrary scalar in the thermal equilibrium after reheating), but it is negligible, because the coupling of this process is too small according to the relation $g_W \propto m_W$ under our definition. Hence we only need to consider the decay. In previous sections, we have derived that the decay rate of $\varphi \to \tilde{W}\tilde{W}$ is $\Gamma_W \sim 10^{-17} M_P$. If this decay rate is far larger than that to the SM particles, $\Gamma_W \gg \Gamma_{\text{SM}}$, WGBs can contribute as dark radiation. Therefore, the upper limit of abundance of dark radiation constrains the lower limit of Γ_{SM} or T_r. Substituting $\Gamma = \Gamma_W + \Gamma_{\text{SM}}$ into Eq. (48), and combining it with the equation

$$\rho'_{\text{SM}/W} + 4\mathcal{H}\rho_{\text{SM}/W} - \frac{\Gamma_{\text{SM}/W}\Phi'^2}{a} = 0, \tag{56}$$

we can derive the energy density of dark radiation ρ_W after reheating. Then the amount of dark radiation at present (denoted as Ω_{DR}) can be estimated. The result is,

if $\Omega_{\mathrm{DR}} < \Omega_{\mathrm{SM}}$ is expected at matter-radiation equality, then the magnitude of Γ_{SM} cannot be lower than the level of Γ_W, which restricts the reheating temperature to be $T_r > 4 \times 10^9$ GeV. In conclusion, under this constraint, the small mass WGBs can be the dark matter candidate, which embodies a tiny amount of dark radiation produced in the inflaton decay and the majority of cold component originates from the vacuum fluctuation during inflation. Meanwhile, if it contributes to the whole abundance of dark matters, then the mass should be around $m_W \sim 3 \times 10^{-13}$ GeV.

For the heavy WGB case, the decay process is forbidden kinematically. However, the annihilation channel becomes significant, which is a kind of "freeze-in" mechanism of production [81–83]. There are two main channels for annihilation, direct annihilation owing to the $\psi^2 W^\mu W_\mu$ coupling and the other by gravitational interaction [46–49]. The amplitude $\mathcal{A} \equiv \sum |\mathcal{M}|^2$ of them can be calculated as follows

$$\mathcal{A}(\psi\psi \to \tilde{W}\tilde{W}) = 16 g_W^4 \left(3 + \frac{\bar{s}(\bar{s}/4 - m_W^2)}{m_W^4} \right), \quad (57)$$

$$\mathcal{A}(\psi\psi \overset{\mathrm{grav}}{\to} \tilde{W}\tilde{W}) = \kappa^4 \left(\frac{101 m_W^4 m_\psi^4}{60\bar{s}^2} - \frac{m_W^2 m_\psi^2}{20\bar{s}} (11 m_\psi^2 + m_W^2) \right.$$
$$+ \frac{1}{240} (19 m_\psi^4 + 76 m_W^2 m_\psi^2)$$
$$\left. - \frac{7\bar{s}}{240} (m_W^2 + m_\psi^2) + \frac{\bar{s}^2}{160} \right), \quad (58)$$

where $\bar{s} \equiv (p_{\psi 1} + p_{\psi 2})^2$, $\kappa \equiv \sqrt{32\pi G} = 2$. Then the total cross section can be easily derived as

$$\sigma_{\mathrm{tot}} = \frac{1}{16\pi \bar{s} t g_i^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} \mathcal{A}_{\mathrm{tot}}, \quad (59)$$

where $t = 2$ is the symmetric factor for the identical final states and g_i is the degree of freedom for initial state. Finally, the relic abundance of dark matters produced in the annihilation process can be calculated with the following formula (see also in [48]):

$$Y = \int_{\sim 0}^{\sim T_r} \frac{dT}{HTs} \left[\frac{Tg_i^2}{32\pi^4} \int d\bar{s} \sigma_{\mathrm{tot}} \sqrt{\bar{s}} (\bar{s} - 4m_\psi^2) K_1 \left(\frac{\sqrt{\bar{s}}}{T} \right) \right], \quad (60)$$

where H is the Hubble parameter, s is the total entropy density mentioned earlier, and K_1 is the modified Bessel function of the second kind with order one. It is seen that, for heavy particles, larger relic abundance of dark matter would result from higher reheating temperature.

With supposing $m_\psi \ll m_W$, we can calculate the relic abundance Y for each set of T_r and m_W. Then comparing it
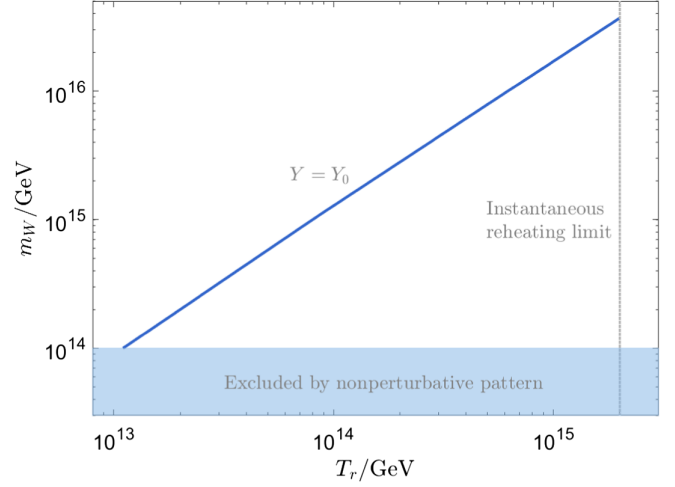

FIG. 5. The proper mass m_W for the "freeze-in" production (annihilation channel) that derives an observation-consistent relic abundance is approximately linear with the reheating temperature T_r, where the $m_W < 10^{14}$ GeV range is overproduced due to nonperturbative production. Under the instantaneous reheating limit, the WGB can possess an enormous mass of up to 4×10^{16} GeV.

with present abundance of dark matter $Y_0 \sim 5 \times 10^{-9}/m_W$, we can get the relation between m_W and T_r for $Y = Y_0$, as shown in Fig. 5. It shows that WGBs in the large mass range is feasible to be the dark matter candidate with an abundance that is consistent with observation as long as the reheating temperature $T_r > 10^{13}$ GeV, and the m_W corresponding to the proper abundance is approximately linear with T_r. Moreover, since the instantaneous reheating limit is estimated as $T_{\mathrm{max}} \sim (3H_{\mathrm{inf}}^2/g_{*S}(T_r))^{1/4} \sim 2 \times 10^{15}$ GeV, the WGB can reach an enormous mass of up to 4×10^{16} GeV.

Now we can summarize the main results we have demonstrated in the preceding discussions. First, if the reheating temperature is too small, $T_r < 10^3$ GeV, then the WGB can only serve as part of dark matter with insufficient abundance. Because the nonperturbative production is inefficient for all mass ranges and the perturbative contribution is also not productive or forbidden by the observations, then, for 10^3 GeV $< T_r \lesssim 4 \times 10^9$ GeV, only the nonperturbative production can reasonably provide enough abundance for WGBs as dark matter, and only the large mass is allowed, $m_W \sim 10^{13}$–10^{14} GeV. Moreover, for 4×10^9 GeV $\lesssim T_r < 10^{13}$ GeV, there are two feasible values of mass for WGBs, the small mass $m_W \sim 3 \times 10^{-13}$ GeV and the large mass $m_W \sim 10^{14}$ GeV. The production channel is also mainly the nonperturbative pattern. Finally, for 10^{13} GeV $< T_r < T_{\mathrm{max}}$, the small mass case is similar to the foregoing one, which is restricted to $m_W \sim 3 \times 10^{-13}$ GeV, and produced mainly in the nonperturbative channel. But the permissible large mass case is different from the above, because the perturbative

annihilation becomes dominant under extremely high T_r, which enables WGBs with mass from 10^{14} to 4×10^{16} GeV to be a dark matter candidate.

V. CONCLUSIONS

We work on a scaling invariant theory of gravity with a quadratic scalar curvature, namely the Weyl R^2 gravity. The model contains a viable inflationary scenario that is different with the conventional Starobinsky R^2 model. We calculate the spectral index and tensor-to-scalar ratio, and confront them with the latest cosmological observations. Our results indicate that this model's predictions agree with the observations and its differences from Starobinsky model can be tested with future CMB experiment.

The model also contains a stable gauge boson, WGB, which can serve as a dark matter candidate. The breaking of local scaling symmetry makes WGBs massive in the inflationary period, which causes an inevitable nonperturbative production for WGBs, namely emerging from the inflationary quantum fluctuations. Our investigations demonstrate that the relic abundance of nonperturbatively produced WGBs is related to the mass of WGB m_W and the decay rate Γ of inflaton. By comparing with the observation, we give the possible viable mass range of WGBs, $m_W \sim 3 \times 10^{-13}$ GeV or $m_W > H_{\text{inf}}$. Moreover, we also give the lower limit of decay width of the inflaton Γ in this model, which corresponds to the reheating temperature $T_r > 10^3$ GeV.

We also demonstrate that there is a specific coupling between a WGB and inflaton Φ, which can lead to inflaton's decay into WGBs if $m_W < m_\Phi/2$. This provides a stringent constraint on the allowable reheating temperature in the case of small m_W. Finally, for the case of extremely high reheating temperature T_r, the perturbative freeze-in channel becomes important and the dominant contribution, namely the annihilation of particles in the thermal bath into WGBs through direct coupling or gravitational interaction. It opens new mass range of WGBs up to 4×10^{16} GeV as dark matter candidate.

[1] V. C. Rubin and W. K. Ford, Jr., Astrophys. J. **159,** 379 (1970).

[2] S. W. Allen, A. C. Fabian, R. W. Schmidt, and H. Ebeling, Mon. Not. R. Astron. Soc. **342,** 287 (2003).

[3] A. Refregier, Annu. Rev. Astron. Astrophys. **41,** 645 (2003).

[4] J. A. Tyson, G. P. Kochanski, and I. P. Dell'Antonio, Astrophys. J. Lett. **498,** L107 (1998).

[5] B. Fields, P. Molaro, and S. Sarkar (Particle Data Group Collaboration), Prog. Theor. Exp. Phys. **2020,** 083C01 (2020).

[6] N. Aghanim et al. (Planck Collaboration), Astron. Astrophys. **641,** A6 (2020); **652,** C4(E) (2021).

[7] T. Lin, Proc. Sci., 333 (**2019**) 009 [arXiv:1904.07915].

[8] M. Schumann, J. Phys. G **46,** 103003 (2019).

[9] J. Billard et al., Rep. Prog. Phys. **85,** 056201 (2022).

[10] Y. Tang and Y.-L. Wu, Phys. Lett. B **803,** 135320 (2020).

[11] Y. Tang and Y.-L. Wu, J. Cosmol. Astropart. Phys. 03 (2020) 067.

[12] Y.-L. Wu, Phys. Rev. D **93,** 024012 (2016).

[13] Y.-L. Wu, Eur. Phys. J. C **78,** 28 (2018).

[14] H. Weyl, Z. Phys. **56,** 330 (1929).

[15] P. D. Mannheim and D. Kazanas, Astrophys. J. **342,** 635 (1989).

[16] C. Romero, J. B. Fonseca-Neto, and M. L. Pucheu, Classical Quantum Gravity **29,** 155015 (2012).

[17] I. Bars, P. Steinhardt, and N. Turok, Phys. Rev. D **89,** 043515 (2014).

[18] E. Scholz, Einstein Stud. **14,** 261 (2018).

[19] D. M. Ghilencea, J. High Energy Phys. 10 (2019) 209.

[20] P. G. Ferreira, C. T. Hill, J. Noller, and G. G. Ross, Phys. Rev. D **100,** 123516 (2019).

[21] Y. Tang and Y.-L. Wu, Phys. Lett. B **809,** 135716 (2020).

[22] J. Beltran Jimenez, L. Heisenberg, and T. S. Koivisto, J. Cosmol. Astropart. Phys. 04 (2016) 046.

[23] I. Oda, Mod. Phys. Lett. A **35,** 2050304 (2020).

[24] D. M. Ghilencea, Eur. Phys. J. C **81,** 510 (2021).

[25] R.-G. Cai, Y.-S. Hao, and S.-J. Wang, Commun. Theor. Phys. (2022).

[26] L. Parker, Phys. Rev. **183,** 1057 (1969).

[27] L. Parker, Phys. Rev. D **3,** 346 (1971); **3,** 2546(E) (1971).

[28] L. H. Ford, Phys. Rev. D **35,** 2955 (1987).

[29] D. H. Lyth and D. Roberts, Phys. Rev. D **57,** 7120 (1998).

[30] D. J. H. Chung, P. Crotty, E. W. Kolb, and A. Riotto, Phys. Rev. D **64**, 043503 (2001).

[31] H. Baer, K.-Y. Choi, J. E. Kim, and L. Roszkowski, Phys. Rep. **555**, 1 (2015).

[32] P. W. Graham, J. Mardon, and S. Rajendran, Phys. Rev. D **93**, 103520 (2016).

[33] Y. Ema, K. Nakayama, and Y. Tang, J. High Energy Phys. 09 (2018) 135.

[34] Y. Ema, K. Nakayama, and Y. Tang, J. High Energy Phys. 07 (2019) 060.

[35] A. Ahmed, B. Grzadkowski, and A. Socha, J. High Energy Phys. 08 (2020) 059.

[36] D. J. H. Chung, E. W. Kolb, and A. J. Long, J. High Energy Phys. 01 (2019) 189.

[37] E. W. Kolb and A. J. Long, J. High Energy Phys. 03 (2021) 283.

[38] Y. Ema, R. Jinno, K. Mukaida, and K. Nakayama, J. Cosmol. Astropart. Phys. 02 (2017) 045.

[39] Y. Ema, R. Jinno, K. Mukaida, and K. Nakayama, Phys. Rev. D **94**, 063517 (2016).

[40] L. Li, S. Lu, Y. Wang, and S. Zhou, J. High Energy Phys. 07 (2020) 231.

[41] L. Li, T. Nakama, C. M. Sou, Y. Wang, and S. Zhou, J. High Energy Phys. 07 (2019) 067.

[42] N. Herring and D. Boyanovsky, Phys. Rev. D **101**, 123522 (2020).

[43] E. Babichev, D. Gorbunov, and S. Ramazanov, Phys. Lett. B **794**, 69 (2019).

[44] S. Hashiba and J. Yokoyama, Phys. Rev. D **99**, 043008 (2019).

[45] S. Ling and A. J. Long, Phys. Rev. D **103**, 103532 (2021).

[46] M. Garny, M. Sandora, and M. S. Sloth, Phys. Rev. Lett. **116**, 101302 (2016).

[47] Y. Tang and Y.-L. Wu, Phys. Lett. B **758**, 402 (2016).

[48] Y. Tang and Y.-L. Wu, Phys. Lett. B **774**, 676 (2017).

[49] M. Garny, A. Palessandro, M. Sandora, and M. S. Sloth, J. Cosmol. Astropart. Phys. 02 (2018) 027.

[50] S.-L. Chen and Z. Kang, J. Cosmol. Astropart. Phys. 05 (2018) 036.

[51] N. Bernal, M. Dutra, Y. Mambrini, K. Olive, M. Peloso, and M. Pierre, Phys. Rev. D **97**, 115020 (2018).

[52] S. Aoki, H. M. Lee, A. G. Menkara, and K. Yamashita, J. High Energy Phys. 05 (2022) 121.

[53] S. Clery, Y. Mambrini, K. A. Olive, and S. Verner, Phys. Rev. D **105**, 075005 (2022).

[54] E. Frangipane, S. Gori, and B. Shakya, arXiv:2110.10711.

[55] M. Redi and A. Tesi, J. High Energy Phys. 12 (2021) 060.

[56] B. Barman, N. Bernal, A. Das, and R. Roshan, J. Cosmol. Astropart. Phys. 01 (2022) 047.

[57] Y. Mambrini and K. A. Olive, Phys. Rev. D **103**, 115009 (2021).

[58] M. R. Haque and D. Maity, arXiv:2112.14668.

[59] S. Clery, Y. Mambrini, K. A. Olive, A. Shkerin, and S. Verner, Phys. Rev. D **105**, 095042 (2022).

[60] Y.-L. Wu, Int. J. Mod. Phys. A **36**, 2143001 (2021).

[61] Y.-L. Wu, Int. J. Mod. Phys. A **36**, 2143002 (2021).

[62] J. D. Barrow and S. Cotsakis, Phys. Lett. B **214**, 515 (1988).

[63] A. De Felice and S. Tsujikawa, Living Rev. Relativity **13**, 3 (2010).

[64] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980).

[65] K.-i. Maeda, Phys. Rev. D **37**, 858 (1988).

[66] J. A. R. Cembranos, Phys. Rev. Lett. **102**, 141301 (2009).

[67] F. L. Bezrukov and D. S. Gorbunov, Phys. Lett. B **713**, 365 (2012).

[68] M. He, A. A. Starobinsky, and J. Yokoyama, J. Cosmol. Astropart. Phys. 05 (2018) 064.

[69] V.-M. Enckell, K. Enqvist, S. Rasanen, and L.-P. Wahlman, J. Cosmol. Astropart. Phys. 02 (2019) 022.

[70] S. Pi, Y.-l. Zhang, Q.-G. Huang, and M. Sasaki, J. Cosmol. Astropart. Phys. 05 (2018) 042.

[71] I. Antoniadis, A. Karam, A. Lykkas, and K. Tamvakis, J. Cosmol. Astropart. Phys. 11 (2018) 028.

[72] P. A. R. Ade et al. (BICEP, Keck Collaborations), Phys. Rev. Lett. **127**, 151301 (2021).

[73] A. R. Liddle and S. M. Leach, Phys. Rev. D **68**, 103503 (2003).

[74] K. Abazajian et al. (CMB-S4 Collaboration), Astrophys. J. **926**, 54 (2022).

[75] L. Kofman, A. D. Linde, and A. A. Starobinsky, Phys. Rev. D **56**, 3258 (1997).

[76] A. Albrecht, P. J. Steinhardt, M. S. Turner, and F. Wilczek, Phys. Rev. Lett. **48**, 1437 (1982).

[77] L. Kofman, A. D. Linde, and A. A. Starobinsky, Phys. Rev. Lett. **73**, 3195 (1994).

[78] B. A. Bassett, S. Tsujikawa, and D. Wands, Rev. Mod. Phys. **78**, 537 (2006).

[79] K. D. Lozanov, arXiv:1907.04402.

[80] T. S. Bunch and P. C. W. Davies, Proc. R. Soc. A **360**, 117 (1978).

[81] L. J. Hall, K. Jedamzik, J. March-Russell, and S. M. West, J. High Energy Phys. 03 (2010) 080.

[82] M. Blennow, E. Fernandez-Martinez, and B. Zaldivar, J. Cosmol. Astropart. Phys. 01 (2014) 003.

[83] N. Bernal, M. Heikinheimo, T. Tenkanen, K. Tuominen, and V. Vaskonen, Int. J. Mod. Phys. A **32**, 1730023 (2017).