Quantum electrodynamics in the null-plane causal perturbation theory

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We study quantum electrodynamics (QED) in the light-front dynamical form by using null-plane causal perturbation theory. We establish the equivalence with instant dynamics for the scattering processes, whose normalization allows one to construct the instantaneous terms of the usual null-plane QED Lagrangian density. Then we study vacuum polarization and normalize it by studying its insertions into Møller's scattering process, obtaining the complete photon propagator, which turns out to be equivalent to the one of instant dynamics only when gauge invariance is taken into account.

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I. INTRODUCTION

Quantum electrodynamics (QED) is the theory of the interaction between leptons and photons. It was the first quantum field theory to be constructed and, by virtue of its accessibility to experimental testing as well as its great success in predicting physical quantities such as the gyromagnetic ratio of the electron and Lamb's shift, it becomes the paradigm of this entire area of physics and one of the most studied theories. The most famous formulation of QED is the one developed by Tomonaga [1–5], Schwinger [6–8], Feynman [9], and Dyson [10] between 1946 and 1949. It was also Dyson, in 1949, who invented the techniques of regularization of the QED integrals and renormalization of the theory by the absorption of the infinities into the mass and charge terms [11].

In 1949, Dirac [12] started the study of the relativistic dynamical forms. He discovered three possibilities: (a) Instant dynamics: The one in which the isochronic surfaces are the planes of constant x^0. (b) Point-form dynamics: The isochronic surface is the superior branch of the hyperboloid $a^2 = x^2$, the parameter a^2 being the time. (c) Light-front dynamics: The isochronic surfaces are null planes of constant x^+. This list was completed by Leutwyler and Stern [13] in 1978; they encountered two more dynamical forms [14], with the following isochronic surfaces: (d) the superior branch of the hyperboloid $(x^0)^2 - (x^1)^2 - (x^2)^2 = a^2$, and (e) the superior branch of the hyperboloid $(x^0)^2 - (x^3)^2 = a^2$; in both cases the parameter a^2 being the time. Among these dynamical forms the light-front one is special: in it the number of Poincaré generators independent of the interaction is maximum [12]; also, the null planes are the characteristic surfaces of Klein-Gordon-Fock's equation [15,16]. These theoretical advantages of light-front dynamics translate into its success in treating a variety of practical problems, for example, in the context of current algebra [17–19], in the study of laser fields [16,20,21], for treating deep-inelastic scattering [22–25], or in QCD for the study of hadron physics [26].

However, being, as there are, many ways to describe the relativistic dynamics, there will also be various quantum field theories. The obvious question to ask in this context is which one of them is the correct theory, or if they are physically equivalent. To develop such theories is necessary in order to answer that question. Focusing on light-front dynamics, the quantization of fields on the null plane and the corresponding formulation of null-plane QED were done by Chang and Ma [27], Kogut and Soper [28], Rohrlich and Neville [16,29], and Leutwyler, Klauder, and Streit [30]. The equivalence between null-plane QED and the conventional instant one was then considered by Ten Eyck and Rohrlich [31,32] and by Yan [33,34]. In all these calculations, Feynman's amplitudes at one-loop level exhibited double-pole singularities as a consequence of an inconsistent treatment of the poles of the gauge field propagator in the null-plane gauge $A^+ = 0$; this problem was solved by Pimentel and Suzuki [35,36], who proposed a prescription to treat those poles in a causal way. Perturbative renormalization of null-plane QED was studied by Brodsky, Roskies, and Suaya [37] and by Mustaki, Pinsky, Shigemitsu, and Wilson [38]. Additionally, the constraint structure of classical QED and scalar QED in light-front dynamics and in the null-plane gauge was studied by Casana, Pimentel, and Zambrano [39].

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However, the equivalence between null-plane QED and the instant one is still under discussion; particularly, the importance of the instantaneous terms in the fermion and gauge fields propagators is not clear; recent reviews on the *status quo* of the gauge field propagator can be found in Refs. [40,41]. Very recently, the equivalence problem for one-loop radiative corrections was studied in Refs. [42,43], and the fulfilment of Ward-Takahashi's identity at one-loop order in Ref. [44].

It is in this context that the axiomatic approaches could offer a new insight into the subtleties of null-plane QED. Particularly, we will adopt the "S-matrix program" point of view, initiated by Heisenberg [45] in 1943 as an attempt to go beyond the Lagrangian theory, and which was axiomatized in instant dynamics in the works by Stückelberg and Rivier [46,47] and Bogoliubov, Medvedev, and Polivanov [48–50]. The detailed perturbative solution to Bogoliubov-Madvedev-Polivanov's axioms was carried out in 1973 by Epstein and Glaser [51], in a method in which the causality axiom plays an essential role, and its first application to QED was done by Scharf in 1989—in a monograph which is the first edition of Ref. [52]. This theory is now called causal perturbation theory. Within this framework, QED in $2+1$ dimensions was considered by Scharf, Wreszinski, Pimentel, and Tomazelli [53], while Dütsch, Krahe, and Scharf [54] used this theory to study scalar QED (SQED), showing that in the causal approach it suffices to start with the first-order coupling and the second-order vertex of the usual formulation is automatically generated as a normalization term in the second-order step. SQED was also investigated by Lunardi, Pimentel, Valverde, Manzoni, Beltrán, and Soto [55,56], who have used causal perturbation theory (CPT) to study the equivalence between Klein-Gordon-Fock's and Duffin-Kemmer-Petiau's scalar quantum electrodynamics. Also Podolsky's second-order electrodynamics was considered from the causal point of view by Bufalo, Pimentel, and Soto [57,58]. So, QED in instant dynamics CPT is well established. It is the purpose of this paper, which is the first in a series, to start the study of null-plane QED in the causal framework. The formulation of CPT on light-front dynamics was done in Refs. [59,60], in which the causality axiom is referred to the null-plane time coordinate x^+, and has been successfully applied to obtain the radiative corrections for Yukawa's model [61], directly showing the equivalence with the instant dynamics formulation [62] for that model. In view of the mentioned successes of CPT, we hope that this framework would lead to a very clear formulation of null-plane QED.

This paper is organized as follows. Section II is devoted to the definition of the field operators of the electron and photon. In Sec. III we offer a short review of null-plane CPT. The construction of the second-order causal distribution for null-plane QED is done in Sec. IV. Møller's and Compton's scattering processes are studied in Sec. V, in which also a comparison with the Lagrangian approach is discussed. Then, in Sec. VI we turn to vacuum polarization. Section VII contains our conclusions and perspectives of future work.

II. QUANTIZED FIELD OPERATORS OF NULL-PLANE QED

QED deals with fermion and photon fields; the quantization of them was done in Appendix A of Ref. [60] by the method of direct construction of Fock's space and a careful choice of the basis functions in the one-particle Hilbert space, exploiting the relation with the classical Goursat's problem. Then it was obtained that the fermion quantized field operator has the following expression:

$$\psi(x) = (2\pi)^{-3/2}\sum_s \int d\mu(\boldsymbol{p})\sqrt{2p_-}(u_s(\boldsymbol{p})b_s(\boldsymbol{p})e^{-ipx} + v_s(\boldsymbol{p})d_s^\dagger(\boldsymbol{p})e^{ipx}), \tag{1}$$

with the four-component functions $u, \bar{u}$ and $v, \bar{v}$ normalized so as to satisfy the sum rules:

$$\sum_s u_s(\boldsymbol{p})\bar{u}_s(\boldsymbol{p}) = \frac{E\gamma^+ + |p_-|\gamma^- + p_\perp\gamma^\perp + m}{|2p_-|}, \tag{2}$$

$$\sum_s v_s(\boldsymbol{p})\bar{v}_s(\boldsymbol{p}) = \frac{E\gamma^+ + |p_-|\gamma^- + p_\perp\gamma^\perp - m}{|2p_-|}, \tag{3}$$

and the emission and absorption field operators satisfying the following anticommutation relations:

$$\{b_s(\boldsymbol{p}); b_r^\dagger(\boldsymbol{q})\} = 2p_-\delta_{sr}\delta(\boldsymbol{p}-\boldsymbol{q}) = \{d_s(\boldsymbol{p}); d_r^\dagger(\boldsymbol{q})\}. \tag{4}$$

The anticommutator of Dirac's field with its Dirac's adjoint is

$$\{\psi(x); \bar{\psi}(y)\} = -iS(x-y), \tag{5}$$

with the distribution $S(x)$ the one which in the classical case solves Goursat's problem for Dirac's equation:

$$S(x) = i(2\pi)^{-3}\int d^4p(\not{p}+m)\mathrm{sgn}(p_-)\delta(p^2-m^2)e^{-ipx} = (i\not{\partial}+m)D(x). \tag{6}$$

Particularly, the equal-time anticommutation relation is

$$\{\psi(x^+;\boldsymbol{x}); \bar{\psi}(x^+;\boldsymbol{y})\} = (\not{\partial} - im)D(x-y)|_{y^+=x^+}. \tag{7}$$

In the right-hand side of this equation, the derivation with respect to the variables x^- and $x^\perp$ can be directly done, with the following result:

$$D(0;\boldsymbol{x}-\boldsymbol{y}) = (2\pi)^{-1}\delta(x^\perp - y^\perp)\int_0^{+\infty} dp_- \frac{\sin[p_-(x^- - y^-)]}{p_-}$$
$$= \frac{1}{4}\mathrm{sgn}(x^- - y^-)\delta(x^\perp - y^\perp). \tag{8}$$

The derivation with respect to x^+, instead, must be done before the evaluation at $y^+ = x^+$. It leads to

$$\begin{aligned}
&\partial_+ D(x-y)|_{y^+=x^+} \\
&= (2\pi)^{-3}\int d^4p\,\mathrm{sgn}(p_-)p_+\delta(p^2-m^2)e^{-ip(x-y)}\Big|_{y^+=x^+} \\
&= (2\pi)^{-3}\int d^3\boldsymbol{p}\,\frac{1}{4p_-^2}(p_\perp^2+m^2)e^{-ip_-(x^- - y^-) - ip_\perp(x^\perp - y^\perp)} \\
&= -\frac{1}{4\partial_-^2}\delta(x^- - y^-)(-\partial_\perp^2 + m^2)\delta(x^\perp - y^\perp) \\
&= -\frac{1}{8}|x^- - y^-|(-\partial_\perp^2 + m^2)\delta(x^\perp - y^\perp).
\end{aligned} \tag{9}$$

Therefore,

$$\begin{aligned}
&\{\psi(x^+;\boldsymbol{x});\bar{\psi}(x^+;\boldsymbol{y})\} \\
&= \frac{1}{2}\Big\{\gamma^-\delta(x^- - y^-) + \frac{1}{2}\mathrm{sgn}(x^- - y^-)(\gamma^\perp\partial_\perp - im) \\
&\quad - \frac{1}{4}\gamma^+|x^- - y^-|(-\partial_\perp^2 + m^2)\Big\}\delta(x^\perp - y^\perp).
\end{aligned} \tag{10}$$

This result coincides with the one obtained by Kogut and Soper [28], by Rohrlich and Neville [16,29], and by the use of Dirac-Bergmann's method via the correspondence principle in Ref. [39].

The photon quantized field operator in the null-plane gauge, on the other hand, is

$$A^a(x) = (2\pi)^{-3/2}\sum_{\lambda=1,2}\int d\mu(\boldsymbol{p})\varepsilon_\lambda(\boldsymbol{p})^a$$
$$\times (a_\lambda(\boldsymbol{p})e^{-ipx} + a_\lambda^\dagger(\boldsymbol{p})e^{ipx}). \tag{11}$$

As we can see, only the physical degrees of freedom—the transversal polarization vectors—appear in it. For completeness we give the expression of the four polarization vectors, which can be found in a classical analysis [63]:

$$\varepsilon_1(\boldsymbol{p})^a = \left(0;1;0;-\frac{p_1}{p_-}\right), \qquad \varepsilon_2(\boldsymbol{p})^a = \left(0;0;1;-\frac{p_2}{p_-}\right),$$
$$\varepsilon_+(\boldsymbol{p})^a = \left(1;-\frac{p_1}{p_-};-\frac{p_2}{p_-};\frac{p_\perp^2}{2p_-^2}\right), \qquad \varepsilon_-(\boldsymbol{p})^a = (0;0;0;1). \tag{12}$$

The photon emission and absorption field operators satisfy the following commutation relations:

$$[a_\lambda(\boldsymbol{p}); a_\sigma^\dagger(\boldsymbol{q})] = 2p_-\delta_{\lambda\sigma}\delta(\boldsymbol{p}-\boldsymbol{q}). \tag{13}$$

The commutation distribution for this field operator is

$$[A^a(x); A^b(y)] =: iD^{ab}(x-y), \tag{14}$$

with

$$D^{ab}(x) = i(2\pi)^{-3}\int d^4p\,e^{-ipx}$$
$$\times\,\mathrm{sgn}(p_-)\delta(p^2)\left(g^{ab} - \frac{p^a\eta^b + \eta^a p^b}{p_-}\right). \tag{15}$$

Here, the vector η has components $(\eta^a) = (0;0_\perp;1)$. We can also obtain the restriction of these commutation relations to the null plane $y^+ = x^+$. The equal-time commutators between the transversal components of the quantized field operators are

$$[A_\alpha(x^+;\boldsymbol{x}); A_\beta(x^+;\boldsymbol{y})]$$
$$= -\frac{i}{4}\delta_\beta^\alpha\mathrm{sgn}(x^- - y^-)\delta(x^\perp - y^\perp). \tag{16}$$

Again, these results agree with the ones in Refs. [16,29,39].

III. NULL-PLANE CAUSAL PERTURBATION THEORY

In the causal theory one uses the operation of "adiabatic switching" [49], by means of which the coupling constant of the interaction theory is multiplied by a "switching function" $g \in \mathscr{S}(\mathbb{R}^4)\colon \mathbb{R}^4 \to \mathbb{R}$, in order to isolate the problem of infrared divergences, and with it, the problem of the confinement in the real (physical) asymptotic states; it is through the adiabatic limit $g \to 1$ that the real interaction is recovered. This operation allows the usage of the free fields for the construction of the $S(g)$ scattering operator, which is subjected to Bogoliubov-Medvedev-Polivanov's axioms [48–50]: (i) translation invariance, (ii) causality—now referred to the x^+ null-plane time, (iii) unitarity, (iv) Lorentz invariance, and (v) vacuum stability. For the construction of CPT only the axioms (i) and (ii) are necessary, while (iii), (iv), and (v) are physical conditions imposed for the normalization of the scattering operator. The details of the formulation of this theory on light-front dynamics can be found in Ref. [60].

Being a perturbation theory, in CPT the $S(g)$ operator is written as the following formal series:

$$S(g) = 1 + \sum_{n=1}^{+\infty}\frac{1}{n!}\int dX T_n(X)g(X), \tag{17}$$

with the notations $T_n(X) \equiv T_n(x_1;\ldots;x_n)$, $g(X) \equiv g(x_1)\ldots g(x_n)$, and $dX \equiv d^4x_1\ldots d^4x_n$. Equation (17) is also the definition of the transition distributions of order n or

n-point distributions $T_n(x_1;\ldots;x_n) \in \mathscr{S}'(\mathbb{R}^{4n})$, which are symmetrical in the coordinates $x_1,\ldots,x_n$ as the products of functions $g(x_1)\ldots g(x_n)$ are.

The inverse operator $S(g)^{-1}$ is given as a perturbation series as well:

$$S(g)^{-1} = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} \int dX \tilde{T}_n(X) g(X),$$

$$\tilde{T}_n(X) = \sum_{r=1}^{n} (-1)^r \sum_{\substack{X_1,\ldots,X_r \neq \varnothing \\ X_1 \cup \ldots \cup X_r = X \\ X_j \cap X_k = \varnothing,\, \forall j \neq k}} T_{n_1}(X_1)\ldots T_{n_r}(X_r). \tag{18}$$

The causality axiom implies that the transition distributions are "chronologically ordered"—in the sense of the x^+ time:

$$T_n(X) = T_m(X_2) T_{n-m}(X_1) \quad \text{for } X_1 < X_2;$$
$$[T_n(X); T_m(Y)] = 0 \quad \text{for } X \sim Y. \tag{19}$$

Because of this, we can define the advanced distribution of order n as the following distribution:

$$A_n(Y; x_n) = \sum_{\substack{X \cup X' = Y \\ X \cap X' = \varnothing}} \tilde{T}_m(X) T_{n-m}(X' \cup \{x_n\}), \tag{20}$$

and the retarded distribution of order n as

$$R_n(Y; x_n) = \sum_{\substack{X \cup X' = Y \\ X \cap X' = \varnothing}} T_{n-m}(X' \cup \{x_n\}) \tilde{T}_m(X). \tag{21}$$

In these distributions the n-point distribution appears once. Separating it from the other terms:

$$A_n(Y; x_n) = T_n(Y \cup \{x_n\}) + A'_n(Y; x_n),$$
$$R_n(Y; x_n) = T_n(Y \cup \{x_n\}) + R'_n(Y; x_n), \tag{22}$$

with the following definitions of the advanced subsidiary distribution and of the retarded subsidiary distribution, respectively, which do not contain T_n, but only the transition distributions T_m with $m \leq n-1$:

$$A'_n(Y; x_n) \coloneqq \sum_{\substack{X \cup X' = Y \\ X \cap X' = \varnothing \\ X \neq \varnothing}} \tilde{T}_m(X) T_{n-m}(X' \cup \{x_n\}), \tag{23}$$

$$R'_n(Y; x_n) \coloneqq \sum_{\substack{X \cup X' = Y \\ X \cap X' = \varnothing \\ X \neq \varnothing}} T_{n-m}(X' \cup \{x_n\}) \tilde{T}_m(X). \tag{24}$$

The transition distribution of order n is then equal to (a similar formula holds with the advanced distribution)

$$T_n(Y \cup \{x_n\}) = R_n(Y; x_n) - R'_n(Y; x_n). \tag{25}$$

Therefore, the n-point distribution can be found by encountering the retarded distribution of order n, which can be done by splitting [64–66] the causal distribution of order n:

$$D_n(Y; x_n) \coloneqq R_n(Y; x_n) - A_n(Y; x_n)$$
$$= R'_n(Y; x_n) - A'_n(Y; x_n). \tag{26}$$

It must be done as follows: Suppose that the causal distribution of order n was already constructed by means of the inductive procedure; it has, in general, the following form:

$$D_n(x_1;\ldots;x_n) = \sum_k d_n^k(x_1;\ldots;x_n) \colon C_k(u^A) \colon, \tag{27}$$

with $d_n^k(x_1;\ldots;x_n)$ a numerical distribution and $\colon C_k(u^A)\colon$ a Wick's monomial of the different quantized free field operators u^A of the theory. Since these field operators do not restrict the support of the complete distribution, it is sufficient to consider the splitting of the numerical distribution d_n^k, whose support, then, is causal by hypothesis. Also the advanced and retarded distributions will maintain the operator fields structure of the causal distribution:

$$A_n(x_1;\ldots;x_n) = \sum_k a_n^k(x_1;\ldots;x_n) \colon C_k(u^A) \colon, \tag{28}$$

$$R_n(x_1;\ldots;x_n) = \sum_k r_n^k(x_1;\ldots;x_n) \colon C_k(u^A) \colon, \tag{29}$$

with a_n^k and r_n^k the advanced and retarded parts, respectively, of the numerical distribution d_n^k. Using the translational invariance, define the numerical distribution $d \in \mathscr{S}'(\mathbb{R}^{4n-4})$ as

$$d(x) \coloneqq d_n^k(x_1 - x_n;\ldots;x_{n-1} - x_n; 0), \tag{30}$$

with $\mathrm{supp}(d) \subseteq \Gamma^+_{n-1}(0) \cup \Gamma^-_{n-1}(0)$, and which will be split as

$$d = r - a; \quad \mathrm{supp}(r) \subseteq \Gamma^+_{n-1}(0), \quad \mathrm{supp}(a) \subseteq \Gamma^-_{n-1}(0). \tag{31}$$

Here we are denoting

$$\Gamma^+_n(0) \coloneqq \{(x_1;\cdots;x_n) \in \mathbb{M}^n |\ \forall j \in \{1,\ldots,n\} \colon,$$
$$x_j^+ \geq 0 \wedge (\exists x_k \in \overline{V^+}(0)(k \neq j) \colon x_j \in \tilde{V}^+(x_k))\},$$

with $V^\pm(x)$ the interior of the future or past, respectively, light-cone with vertex at the point x, $\overline{V^\pm}(x)$ its closure, and $\tilde{V}^\pm(x)$ the union of its closure and the x^- axis. An analogous definition holds for $\Gamma^-_n(0)$. Additionally, in Eq. (30) we have written $d(x)$; x means $(x_1 - x_n;\ldots;x_{n-1} - x_n)$. In the following we will use Schwartz's multi-index notation [67]. We will also use the notation $x^a \equiv (x_1^a - x_n^a; \ldots; x_{n-1}^a - x_n^a)$.

To perform the splitting, it is crucial to remember that the product of a distribution with a discontinuous function can be ill defined if the distribution has a singularity precisely on the discontinuity surface of the function [69]. In our case we then need to control the behavior of the causal distribution near the splitting region. In instant dynamics, in which the splitting region is the vertex of the light cone, the concept of Vladimirov-Drozzinov-Zavialov's quasiasymptotics [70] was introduced to cast that behavior [71]. In null-plane dynamics, the splitting region is the intersection of the null plane $x^+ = 0$ with the light cone, which is the entire x^- axis, hence the concept of quasiasymptotics by selected variable [72] is most adequate for this purpose:

Definition.—Let $d \in \mathscr{S}'(\mathbb{R}^m)$ be a distribution, and let ρ be a continuous positive function. If the (distributional) limit

$$\lim_{s\to 0^+} \rho(s) s^{3m/4} d(sx^+; sx^\perp; x^-) = d_-(x) \qquad (32)$$

exists in $\mathscr{S}'(\mathbb{R}^m)$ and is non-null, then the distribution d_- is called the quasiasymptotics of d at the x^- axis, with regard to the function ρ.

The function $\rho(s)$ can be shown to be a regularly varying at zero function, also called an automodel function [52,70], which means that for every $a > 0$: $\lim_{s\to 0^+} \rho(as)/\rho(s) = a^\alpha$ for some $\alpha \in \mathbb{R}$, called the order of automodelity of the function ρ. This number serves as a characterizing parameter of the distribution, which is called its singular order at the x^- axis and is denoted by ω_-.

In momentum space the following splitting formulas are found: For negative singular order, $\omega_- < 0$,

$$\hat{r}(p) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\hat{d}(p_+ - k; \boldsymbol{p})}{k + i0^+} dk. \qquad (33)$$

For non-negative singular order, $\omega_- \geq 0$, the retarded distribution with normalization line $(q_+; q_\perp; p_-)$ is

$$\hat{r}_q(p) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{dk}{k + i0^+} \Bigg\{ \hat{d}(p_+ - k; \boldsymbol{p}) - \sum_{|c|=0}^{\lfloor \omega_- \rfloor} \frac{1}{c!} (p_{+,\alpha} - q_{+,\alpha})^c D^c_{+,\alpha} \hat{d}(q_+ - k; q_\perp; p_-) \Bigg\}. \qquad (34)$$

A particular case of normalization line is $(0; 0_\perp; p_-)$; the solution normalized at it is called the central solution.

Finally, if r_1 and r_2 are two solutions of the splitting problem, then they could be different only by normalization terms which are distributions with support on the x^- axis. In momentum space,

$$\hat{r}_1(p) - \hat{r}_2(p) = \sum_{|b|=0}^{M} \hat{C}_b(p_-) p^b_{+,\perp}, \qquad (35)$$

with $\hat{C}_b(p_-)$ some distributions of the variable p_-. The singular order of each one of these terms is $|b|$, independently of which is the distribution $\hat{C}_b(p_-)$ is, because the variable p_- is not scaled in the singular order calculus in light-front dynamics; this leads to a richer variety of possible normalization terms when compared to instant dynamics; particularly, instantaneous normalization terms are now allowed. The procedure of determining these unknown distributions by the imposition of physical requirements is called the normalization process.

IV. CAUSAL DISTRIBUTION OF THE SECOND-ORDER QED

For QED the first-order term of the $S(g)$ operator is given by the one-point distribution:

$$T_1(x) = i{:}j^a(x){:}A_a(x) \equiv ie{:}\bar{\psi}(x)\gamma^a\psi(x){:}A_a(x). \qquad (36)$$

The construction of the second-order causal distribution starts with the definition of the subsidiary ones:

$$A'_2(x_1; x_2) = \tilde{T}_1(x_1)T_1(x_2) = -T_1(x_1)T_1(x_2), \qquad (37)$$

$$R'_1(x_1; x_2) = T_1(x_2)\tilde{T}_1(x_1) = -T_1(x_2)T_1(x_1), \qquad (38)$$

with which the causal distribution $D_2 = R'_2 - A'_2$ is equal to

$$D_2(x_1; x_2) = [T_1(x_1); T_1(x_2)]. \qquad (39)$$

The explicit expression of this distribution is obtained by replacing Eq. (36) into Eq. (39), and by using Wick's theorem with the contractions:

$$\overline{\psi_a(x)\bar{\psi}_b}(y) = \frac{1}{i} S_{ab+}(x - y), \qquad (40)$$

$$\overline{\bar{\psi}_a(x)\psi_b}(y) = \frac{1}{i} S_{ba-}(y - x), \qquad (41)$$

$$\overline{A_a(x)A_b}(y) = iD_{ab+}(x - y). \qquad (42)$$

We will need the subsidiary retarded distribution [see Eq. (25)] given by

$$\begin{aligned} R'_2(x_1; x_2) = e^2\gamma^a_{ab}\gamma^b_{cd}[&{:}\bar{\psi}_a(x_1)\psi_b(x_1)\bar{\psi}_c(x_2)\psi_d(x_2){:} \\ &+ iS_{bc-}(x_1 - x_2){:}\bar{\psi}_a(x_1)\psi_d(x_2){:} \\ &+ iS_{da+}(x_2 - x_1){:}\psi_b(x_1)\bar{\psi}_c(x_2){:} \\ &- S_{bc-}(x_1 - x_2)S_{da+}(x_2 - x_1)] \\ &\times [{:}A_a(x_1)A_b(x_2){:} + iD_{ab+}(x_2 - x_1)], \end{aligned} \qquad (43)$$

and the causal distribution, whose final form is

$$D_2 = D_2^{(M)} + D_2^{(C)} + D_2^{(VP)} + D_2^{(SE)} + D_2^{(VG)}, \tag{44}$$

with (we use the relative coordinate $y \equiv x_1 - x_2$)

$$D_2^{(M)}(x_1;x_2) = -ie^2 D_{ab}(y) \times :\bar{\psi}(x_1)\gamma^a\psi(x_1)\bar{\psi}(x_2)\gamma^b\psi(x_2):, \tag{45}$$

$$D_2^{(C)}(x_1;x_2) = ie^2 :A_a(x_1)A_b(x_2): \times [:\bar{\psi}(x_1)\gamma^a S(y)\gamma^b\psi(x_2): - :\bar{\psi}(x_2)\gamma^b S(-y)\gamma^a\psi(x_1):], \tag{46}$$

$$D_2^{(VP)}(x_1;x_2) = -e^2 :A_a(x_1)A_b(x_2): \times \mathrm{Tr}[\gamma^a S_-(y)\gamma^b S_+(-y) - \gamma^a S_+(y)\gamma^b S_-(-y)], \tag{47}$$

$$D_2^{(SE)}(x_1;x_2) = -e^2 :\bar{\psi}(x_1)\gamma^a[S_-(y)D_{ab+}(-y) + S_+(y)D_{ab+}(y)]\gamma^b\psi(x_2): + e^2 :\bar{\psi}(x_2)\gamma^a \times [S_+(-y)D_{ab+}(-y) + S_-(-y)D_{ab+}(y)] \times \gamma^b\psi(x_1):, \tag{48}$$

$$D_2^{(VG)}(x_1;x_2) = -ie^2 D_{ab+}(-y) \times \mathrm{Tr}[\gamma^a S_-(y)\gamma^b S_+(-y) - \gamma^a S_-(-y)\gamma^b S_+(y)]. \tag{49}$$

In this form, Eqs. (45)–(49) allow one to directly identify the terms which will contribute to each process: the noncontracted quantized field operators determine the initial and final states which will give a non-null contribution to the amplitude $|\langle\Psi; S(g)\Phi\rangle|^2$. Hence, the distribution $D_2^{(M)}$ describes the scattering of two fermions, $D_2^{(C)}$, the scattering of a fermion by a photon; the distributions $D_2^{(VP)}$ and $D_2^{(SE)}$ represent the vacuum polarization and fermion's self-energy, respectively; finally, the distribution $D_2^{(VG)}$ does not describe any physical process.

V. SCATTERING PROCESSES

In this section we will show in a very direct manner that the equivalence with instant dynamics for the scattering processes at second order can be obtained by a suitable choice of the normalization terms.

A. Møller's scattering

The scattering of two leptons, called Møller's scattering, is described by the causal distribution in Eq. (45). The numerical distribution contained in it is the commutation distribution of the radiation field, whose expression in momentum space and in the null-plane gauge is

$$\hat{D}_{ab}(p) = \frac{i}{2\pi}\mathrm{sgn}(p_-)\delta(p^2)\left(g_{ab} - \frac{p_a\eta_b + \eta_a p_b}{p_-}\right). \tag{50}$$

Some of the components of this causal distribution have singular order $\omega_- = -2$, while others have $\omega_- = -1$. In any case, the singular order is negative, so the retarded part is found by application of Eq. (33). In order to do that, it will be convenient to define the following distributions:

$$\hat{d}_1(p) := \frac{i}{2\pi}\mathrm{sgn}(p_-)\delta(p^2), \quad \hat{d}_{2a}(p) := \hat{d}_1(p)\frac{p_a}{p_-}; \tag{51}$$

as a function of which the commutation distribution is

$$\hat{D}_{ab}(p) = g_{ab}\hat{d}_1(p) - [\hat{d}_{2a}(p)\eta_b + \eta_a\hat{d}_{2b}(p)]. \tag{52}$$

To find the retarded part of $\hat{D}_{ab}$ is then equivalent to finding the retarded parts of $\hat{d}_1$ and $\hat{d}_{2a}$. We find, by using the variable $s = -2kp_-$,

$$\begin{aligned}\hat{r}_1(p) &= -(2\pi)^{-2}\int_{-\infty}^{+\infty}\frac{\mathrm{sgn}(p_-)\delta(p^2 - 2kp_-)}{k + i0^+}dk \\ &= (2\pi)^{-2}\int_{-\infty}^{+\infty}\frac{\delta(s + p^2)}{s - ip_-0^+}ds \\ &= -(2\pi)^{-2}\frac{1}{p^2 + ip_-0^+};\end{aligned} \tag{53}$$

since the variables $p_{\alpha,-}$ do not change in the splitting formula of Eq. (33):

$$\hat{r}_{2\alpha,-}(p) = \frac{p_{\alpha,-}}{p_-}\hat{r}_1(p) = -(2\pi)^{-2}\frac{1}{p^2 + ip_-0^+}\frac{p_{\alpha,-}}{p_-}; \tag{54}$$

and finally,

$$\begin{aligned}\hat{r}_{2+}(p) &= -(2\pi)^{-2}\int_{-\infty}^{+\infty}\frac{\mathrm{sgn}(p_-)\delta(p^2 - 2kp_-)(p_+ - k)}{p_-(k + i0^+)}dk \\ &= (2\pi)^{-2}\int_{-\infty}^{+\infty}\frac{\delta(s + p^2)(p_+ + \frac{s}{2p_-})}{p_-(s - ip_-0^+)}ds \\ &= -(2\pi)^{-2}\left(\frac{1}{p^2 + ip_-0^+}\frac{p_+}{p_-} - \frac{1}{2p_-}\right).\end{aligned} \tag{55}$$

Equations (53)–(55) imply that the retarded part of the commutation distribution of the massless vector field is [see Eq. (52)]

$$\hat{D}^{\rm ret}_{ab}(p) = -\frac{(2\pi)^{-2}}{p^2 + ip_- 0^+} \times \left\{ g_{ab} - \frac{p_a \eta_b + \eta_a p_b}{p_-} + \frac{p^2}{2p_-^2}[\delta_{a+}\eta_b + \eta_a \delta_{b+}] \right\}. \tag{56}$$

But, since δ_{a+} are precisely the components of η_a, the above equation simplifies to

$$\hat{D}^{\rm ret}_{ab}(p) = \frac{-(2\pi)^{-2}}{p^2 + ip_- 0^+} \left\{ g_{ab} - \frac{p_a \eta_b + \eta_a p_b}{p_-} + \frac{p^2}{p_-^2}\eta_a \eta_b \right\}. \tag{57}$$

Subtracting the subsidiary retarded distribution, which corresponds to the negative frequency part of the commutation distribution,

$$\hat{D}_{ab-}(p) = -\frac{i}{2\pi}\Theta(-p_-)\delta(p^2)\left\{ g_{ab} - \frac{p_a \eta_b + \eta_a p_b}{p_-} \right\}, \tag{58}$$

we obtain Feynman's propagator of this quantized field:

$$\hat{D}^F_{ab}(p) := \hat{D}^{\rm ret}_{ab}(p) - \hat{D}_{ab-}(p) = -\frac{(2\pi)^{-2}}{p^2 + i0^+}\left\{ g_{ab} - \frac{p_a \eta_b + \eta_a p_b}{p_-} + \frac{p^2}{p_-^2}\eta_a \eta_b \right\}, \tag{59}$$

which is the one that enters into the transition distribution for Møller's scattering:

$$T_2^{(M)}(x_1; x_2) = -ie^2 D^F_{ab}(y) \times :\bar{\psi}(x_1)\gamma^a \psi(x_1)\bar{\psi}(x_2)\gamma^b \psi(x_2):. \tag{60}$$

As we see in Eq. (57), we have obtained an instantaneous term in the splitting process of the causal distribution. This has led to the so-called doubly transverse gauge propagator, shown in Eq. (59), which means that $\hat{D}^F_{ab}(p)$ is transverse both to p^a and η^a [31,32,73]; here we have obtained it in a very natural way. Now, the singular order of this distribution is $\omega_-[D^F_{ab}] = 0$, so it is allowed a normalization term of the form $\hat{C}(p_-)$. Choosing

$$\hat{C}(p_-) = -ie^2(2\pi)^{-2}\frac{\eta_a \eta_b}{p_-^2}, \tag{61}$$

the instantaneous term which arose in the splitting procedure cancels out and we are left with

$$\hat{d}_{ab}(p) \equiv \hat{D}^F_{ab}(p) + \hat{C}(p_-) = -\frac{(2\pi)^{-2}}{p^2 + i0^+}\left\{ g_{ab} - \frac{p_a \eta_b + \eta_a p_b}{p_-} \right\}. \tag{62}$$

Consider now the following initial and final states, respectively, with definite momenta:

$$\frac{b^\dagger_{s_1}(\boldsymbol{p}_1)}{\sqrt{2p_{1-}}}\frac{b^\dagger_{r_1}(\boldsymbol{q}_1)}{\sqrt{2q_{1-}}}\Omega, \qquad \frac{b^\dagger_{s_2}(\boldsymbol{p}_2)}{\sqrt{2p_{2-}}}\frac{b^\dagger_{r_2}(\boldsymbol{q}_2)}{\sqrt{2q_{2-}}}\Omega. \tag{63}$$

For these states, the S operator to second order in the adiabatic limit $g \to 1$ is

$$\begin{aligned} S_{12}^{(M)} &= \frac{1}{\sqrt{2p_{1-}2q_{1-}2p_{2-}2q_{2-}}}(\Omega; b_{r_2}(\boldsymbol{q}_2) b_{s_2}(\boldsymbol{p}_2) S_2^{(M)} b_{s_1}(\boldsymbol{p}_1)^\dagger b_{r_1}(\boldsymbol{q}_1)^\dagger \Omega) \\ &= -\frac{1}{\sqrt{2p_{1-}2q_{1-}2p_{2-}2q_{2-}}}\frac{ie^2}{2(2\pi)^2}\int d^4k d^4x_1 d^4x_2 e^{-iky}\hat{d}_{ab}(k)(\Omega; b_{r_2}(\boldsymbol{q}_2) b_{s_2}(\boldsymbol{p}_2) \\ &\quad \times :\bar{\psi}(x_1)\gamma^a\psi(x_1)\bar{\psi}(x_2)\gamma^b\psi(x_2): b_{s_1}(\boldsymbol{p}_1)^\dagger b_{r_1}(\boldsymbol{q}_1)^\dagger\Omega). \end{aligned} \tag{64}$$

Inside the parentheses, the non-null contributions are found by using Wick's theorem. There are four contributions, which we obtain by using the contractions

$$\begin{aligned} \overline{b_s(\boldsymbol{p})\bar{\psi}}(x) &= (2\pi)^{-3/2}\Theta(p_-)\sqrt{2p_-}\bar{u}_s(\boldsymbol{p})e^{ipx}, \\ \overline{\psi(x) b_s}(\boldsymbol{p})^\dagger &= (2\pi)^{-3/2}\Theta(p_-)\sqrt{2p_-}u_s(\boldsymbol{p})e^{-ipx}, \end{aligned} \tag{65}$$

Then,

$$\begin{aligned} S_{12}^{(M)} = \frac{ie^2}{2(2\pi)^8}\int & d^4k d^4x_1 d^4x_2 e^{-iky}\hat{d}_{ab}(k)\Theta(p_{1-})\Theta(q_{1-})\Theta(p_{2-})\Theta(q_{2-}) \\ & \times \{\bar{u}_{s_2}(\boldsymbol{p}_2)\gamma^a u_{r_1}(\boldsymbol{q}_1)\bar{u}_{r_2}(\boldsymbol{q}_2)\gamma^b u_{s_1}(\boldsymbol{p}_1)e^{i(p_2-q_1)x_1 + i(q_2-p_1)x_2} \\ & - \bar{u}_{s_2}(\boldsymbol{p}_2)\gamma^a u_{s_1}(\boldsymbol{p}_1)\bar{u}_{r_2}(\boldsymbol{q}_2)\gamma^b u_{r_1}(\boldsymbol{q}_1)e^{i(p_2-p_1)x_1 + i(q_2-q_1)x_2} \\ & - \bar{u}_{r_2}(\boldsymbol{q}_2)\gamma^a u_{r_1}(\boldsymbol{q}_1)\bar{u}_{s_2}(\boldsymbol{p}_2)\gamma^b u_{s_1}(\boldsymbol{p}_1)e^{i(q_2-q_1)x_1 + i(p_2-p_1)x_2} \\ & + \bar{u}_{r_2}(\boldsymbol{q}_2)\gamma^a u_{s_1}(\boldsymbol{p}_1)\bar{u}_{s_2}(\boldsymbol{p}_2)\gamma^b u_{r_1}(\boldsymbol{q}_1)e^{i(q_2-p_1)x_1 + i(p_2-q_1)x_2}\}. \end{aligned} \tag{66}$$

In all of the terms the following integral appears:

$$\int d^4x_1 d^4x_2 e^{-iky} e^{iPx_1+iQx_2} = (2\pi)^8 \delta(k-P)\delta(P+Q), \quad (67)$$

so that, integrating in the variable k and using the symmetries of the $\hat{d}_{ab}$ distribution [see Eq. (62)],

$$\hat{d}_{ab}(k) = \hat{d}_{ab}(-k) \quad \text{and} \quad \hat{d}_{ab}(k) = \hat{d}_{ba}(k), \quad (68)$$

we finally find

$$\begin{aligned} S_{12}^{(M)} = {} & ie^2 \delta(p_2 + q_2 - p_1 - q_1) \\ & \times \Theta(p_{1-})\Theta(q_{1-})\Theta(p_{2-})\Theta(q_{2-}) \\ & \times \{\bar{u}_{s_2}(\boldsymbol{p}_2)\gamma^a u_{r_1}(\boldsymbol{q}_1)\bar{u}_{r_2}(\boldsymbol{q}_2)\gamma^b u_{s_1}(\boldsymbol{p}_1)\hat{d}_{ab}(p_2 - q_1) \\ & - \bar{u}_{s_2}(\boldsymbol{p}_2)\gamma^a u_{s_1}(\boldsymbol{p}_1)\bar{u}_{r_2}(\boldsymbol{q}_2)\gamma^b u_{r_1}(\boldsymbol{q}_1)\hat{d}_{ab}(p_2 - p_1)\}. \end{aligned} \quad (69)$$

The wave-functions $u(p)$ and $\bar{u}(p)$ satisfy Dirac's equation in momentum space:

$$\not{p} u(p) = m u(p), \qquad \bar{u}(p)\not{p} = m\bar{u}(p). \quad (70)$$

Therefore,

$$\begin{aligned} \bar{u}_{s_2}(\boldsymbol{p}_2)\gamma^a u_{r_1}(\boldsymbol{q}_1)(p_{2a} - q_{1a}) &= \bar{u}_{s_2}(\boldsymbol{p}_2)(\not{p}_2 - \not{q}_1)u_{r_1}(\boldsymbol{q}_1) \\ &= 0, \end{aligned} \quad (71)$$

and the noncovariant terms in the $\hat{d}_{ab}$ distribution do not contribute to $S_{12}^{(M)}$ [see Eqs. (62) and (69)]. We conclude that all of the nonlocal terms cancel out, and the result is the same as if we would consider the covariant part of the radiation field commutation distribution only:

$$-\frac{(2\pi)^{-2}}{k^2 + i0^+} g_{ab}, \quad (72)$$

establishing the equivalence with instant dynamics.

B. Compton's scattering

Now we turn to the study of Compton's scattering, this is to say, the scattering of a fermion by a photon, whose causal distribution at second order is the one in Eq. (46). Defining the numerical distribution,

$$d^{ab}(y) = ie^2 \gamma^a S(y) \gamma^b, \quad (73)$$

we will have

$$\begin{aligned} D_2^{(C)}(x_1; x_2) = {} & :A_a(x_1)A_b(x_2):(:\bar{\psi}(x_1)d^{ab}(y)\psi(x_2): \\ & - :\bar{\psi}(x_2)d^{ba}(-y)\psi(x_1):). \end{aligned} \quad (74)$$

The distribution $d^{ab}(y)$ has singular order $\omega_- = -1$, and its retarded part is

$$\hat{r}^{ab}(y) = ie^2 \gamma^a S^{\text{ret}}(y) \gamma^b. \quad (75)$$

Hence we need to obtain the retarded part of the anti-commutation distribution of the fermion field. In momentum space it is

$$\hat{S}(p) = \frac{i}{2\pi}(\not{p} + m)\text{sgn}(p_-)\delta(p^2 - m^2). \quad (76)$$

As it was said, its singular order at the x^- axis is $\omega_- = -1 < 0$, so its retarded part is given by Eq. (33):

$$\begin{aligned} \hat{S}^{\text{ret}}(p) = {} & -(2\pi)^{-2} \int \frac{dk}{k + i0^+} \text{sgn}(p_-)[(p_+ - k)\gamma^+ \\ & + p_\perp \gamma^\perp + p_- \gamma^- + m]\delta(2p_+p_- - 2kp_- - \omega_p^2). \end{aligned} \quad (77)$$

Using the variable $s = -2kp_-$, the above integral is equal to

$$\begin{aligned} \hat{S}^{\text{ret}}(p) &= (2\pi)^{-2} \int \frac{ds}{s - iq0^+} \delta(s + 2p_+p_- - \omega_p^2) \\ & \quad \times \left(p_+\gamma^+ + p_\perp\gamma^\perp + p_-\gamma^- + m + \frac{s}{2p_-}\gamma^+\right) \\ &= -(2\pi)^{-2} \frac{\not{p} + m - \frac{2p_+p_- - \omega_p^2}{2p_-}\gamma^+}{2p_+p_- - \omega_p^2 + ip_-0^+} \\ &= -(2\pi)^{-2} \left(\frac{\not{p} + m}{p^2 - m^2 + ip_-0^+} - \frac{\gamma^+}{2p_-}\right). \end{aligned} \quad (78)$$

Subtracting the corresponding $\hat{r}'^{ab}(y)$ subsidiary distribution, which corresponds to the negative frequency part of the anticommutation distribution,

$$\hat{S}_-(p) = -\frac{i}{2\pi}\Theta(-p_-)(\not{p} + m)\delta(p^2 - m^2), \quad (79)$$

we obtain for the numerical part of the transition distribution

$$t^{ab}(y) = ie^2 \gamma^a S^F(y) \gamma^b, \quad (80)$$

with Feynman's propagator being

$$\begin{aligned} \hat{S}^F(p) &\coloneqq \hat{S}_-(p) - \hat{S}^{\text{ret}}(p) \\ &= (2\pi)^{-2} \left(\frac{\not{p} + m}{p^2 - m^2 + i0^+} - \frac{\gamma^+}{2p_-}\right). \end{aligned} \quad (81)$$

As we see, in the splitting process of the causal distribution an instantaneous term arises. Writing the normalization term that is allowed for $\omega_- = 0$, which is the singular order of the distribution t^{ab},

$$\hat{t}^{ab}(p) = \frac{ie^2}{(2\pi)^2}\gamma^a\left(\frac{\not{p}+m}{p^2-m^2+i0^+} - \frac{\gamma^+}{2p_-}\right)\gamma^b + \hat{C}(p_-). \tag{82}$$

Choosing

$$\hat{C}(p_-) = \frac{ie^2}{(2\pi)^2}\frac{\gamma^a\gamma^+\gamma^b}{2p_-}, \tag{83}$$

the instantaneous noncovariant term in the fermion Feynman's propagator is canceled out, and we arrive at the final result,

$$\hat{t}^{ab}(p) = \frac{ie^2}{(2\pi)^2}\gamma^a\frac{\not{p}+m}{p^2-m^2+i0^+}\gamma^b, \tag{84}$$

showing, also for this scattering process, the equivalence with instant dynamics.

C. Interaction Lagrangian density

In our study of the scattering processes we have seen that Lorentz's covariance requires the introduction of very specific normalization terms. In the case of Møller's scattering, the contribution of the normalization term in Eq. (61) to the second-order $S(g)$ operator in the adiabatic limit $g \to 1$ is

$$\begin{aligned} &+\frac{i}{2}\int d^4x_1 d^4x_2 :j^a(x_1)\delta(y)\frac{\eta_a\eta_b}{\partial_-^2}j^b(x_2): \\ &= \int d^4x_1 :j^+(x_1)\frac{i}{2\partial_-^2}j^+(x_1):. \end{aligned} \tag{85}$$

This is precisely the instantaneous term which in the usual approach appears in the Lagrangian density by solving the constraint equation for the radiation field in the null-plane gauge in the interacting theory [31,32].

Another normalization term was required to obtain a covariant transition distribution for Compton's scattering—Eq. (83); its contribution to the scattering operator in the adiabatic limit $g \to 1$ is, taking into account the two terms in Eq. (74),

$$\begin{aligned} &+\frac{1}{2}\int d^4x_1 d^4x_2 e^2\bigg\{:(\bar{\psi}(x_1)\gamma^a A_a(x_1)) \\ &\quad\times\delta(y)\frac{\gamma^+}{2\partial_-}(\gamma^b A_b(x_2)\psi(x_2)): \\ &\quad+:(\bar{\psi}(x_2)\gamma^b A_b(x_2))\delta(y)\frac{\gamma^+}{2\partial_-}(\gamma^a A_a(x_1)\psi(x_1)):\bigg\} \\ &= \int d^4x_1 e^2 :\bar{\psi}(x_1)\gamma^a A_a(x_1)\frac{\gamma^+}{2\partial_-}\gamma^b A_b(x_1)\psi(x_1):. \end{aligned} \tag{86}$$

The term so obtained is the one which corresponds, in the Lagrangian approach, to the instantaneous interaction term which arises when solving the constraint equation for the fermion field in the interacting theory [31,32].

Joining Eqs. (36), (85), and (86) we can identify the interaction Lagrangian density, defined as $-i$ times the one-point transition distribution plus $-i$ times the contribution of the normalization terms of the next-order transition distributions to the scattering operator in the adiabatic limit:

$$\begin{aligned} \mathscr{L} &= :j^a(x)A_a(x): - :\frac{1}{2}\left(\frac{1}{\partial_-}j^+(x)\right)^2: \\ &\quad+\frac{e^2}{2}:(\bar{\psi}(x)\gamma^a A_a(x))\frac{\gamma^+}{i\partial_-}(\gamma^b A_b(x)\psi(x)):. \end{aligned} \tag{87}$$

This Lagrangian density was first obtained by Kogut and Soper [28]. We remind the reader that the Lagrangian density of Eq. (87) is of first order in e when written as a function of interacting fields; its second-order structure arises when the constraint equations are solved and reintroduced in it. Therefore, that these terms appear in null-plane CPT at the second order is in accordance with the philosophy of the causal approach, which works with free fields only.

Now, it is a debate question if the instantaneous terms in this Lagrangian density cancel exactly the terms coming from the instantaneous terms in the field propagators. Null-plane CPT answers this question in a direct way: Since the normalization terms cancel the instantaneous terms of the propagators at second order, it will cancel them at all orders in a perturbation series based on $\mathscr{L}$, because the next-order causal distributions are constructed with the normalized transition distributions. Here we see the advantage of working in an inductive framework.

VI. VACUUM POLARIZATION

We consider in this subsection the radiative correction known as vacuum polarization, which will be precisely defined later on, and which comes from the study of the causal distribution in Eq. (47); we write it as

$$D_2^{(VP)}(x_1;x_2) = (P^{ab}(y) - P^{ba}(-y)):A_a(x_1)A_b(x_2):, \tag{88}$$

with

$$P^{ab}(y) = e^2\mathrm{Tr}[\gamma^a S_+(y)\gamma^b S_-(-y)]. \tag{89}$$

Fourier's transform of the P^{ab} distribution is

$$\begin{aligned} \hat{P}^{ab}(k) &= e^2(2\pi)^{-2}\int d^4p\,\mathrm{Tr}[\gamma^a S_+(p)\gamma^b S_-(p-k)] \\ &= e^2(2\pi)^{-2}\int d^4p\,\mathrm{Tr}[\gamma^a(\not{p}+m)\gamma^b(\not{p}-\not{k}+m)] \\ &\quad\times\hat{D}_+(p)\hat{D}_-(p-k). \end{aligned} \tag{90}$$

The trace which appears here is calculated by the usual techniques:

$$\begin{aligned}&\mathrm{Tr}[\gamma^a(\not p+m)\gamma^b(\not p-\not k+m)]\\&\quad=\mathrm{Tr}[\gamma^a\not p\gamma^b(\not p-\not k)]+m^2\mathrm{Tr}[\gamma^a\gamma^b]\\&\quad=4[p^a(p^b-k^b)+p^b(p^a-k^a)-g^{ab}(p(p-k)-m^2)].\end{aligned}\tag{91}$$

Also, since

$$\hat{D}_{\pm}(p)=\pm\frac{i}{2\pi}\Theta(\pm p_-)\delta(p^2-m^2),\tag{92}$$

in the integrand of Eq. (90) the following Dirac's delta distributions will appear: $\delta(p^2-m^2)$ and $\delta((p-k)^2-m^2)$; they imply that $p^2=m^2$ and $k^2=2pk$. Hence,

$$\begin{aligned}\hat{P}^{ab}(k)=&\frac{4e^2}{(2\pi)^4}\int d^4p[2p^ap^b-p^ak^b-k^ap^b+g^{ab}pk]\\&\times\Theta(p_-)\delta(p^2-m^2)\Theta(k_--p_-)\delta(k^2-2pk).\end{aligned}\tag{93}$$

We can see from Eq. (93) that the distribution $\hat{P}^{ab}(k)$ is symmetric in its indices. And, in addition, it is orthogonal to the momentum k, since the multiplication by this vector gives

$$k_a\hat{P}^{ab}(k)\sim(2pk-k^2)p^b,\tag{94}$$

which is null by means of the support of $\delta(k^2-2pk)$. Therefore, $\hat{P}^{ab}(k)$ must be proportional to the projector $k^ak^b-k^2g^{ab}$:

$$\hat{P}^{ab}(k)=(k^ak^b-k^2g^{ab})B(k^2).\tag{95}$$

Taking the trace of this equation and also in Eq. (93), we obtain the following formula for $B(k^2)$:

$$B(k^2)=-\frac{1}{3k^2}\hat{P}^a{}_a(k)=-\frac{4e^2}{3(2\pi)^4}\left(1+\frac{2m^2}{k^2}\right)I(k),\tag{96}$$

with $I(k)$ the following integral:

$$I(k)=\int d^4p\Theta(p_-)\Theta(k_--p_-)\delta(p^2-m^2)\delta(k^2-2pk).\tag{97}$$

This integral is the same which appears in the calculus of the boson's self-energy in Yukawa's model; see Ref. [61]. It is equal to

$$I(k)=\frac{\pi}{2}\Theta(k_-)\Theta(k^2-4m^2)\sqrt{1-\frac{4m^2}{k^2}}.\tag{98}$$

Then,

$$\begin{aligned}\hat{P}^{ab}(k)=&\frac{e^2}{3(2\pi)^3}\left(g^{ab}-\frac{k^ak^b}{k^2}\right)(k^2+2m^2)\Theta(k_-)\\&\times\Theta(k^2-4m^2)\sqrt{1-\frac{4m^2}{k^2}}.\end{aligned}\tag{99}$$

The numerical distribution associated with vacuum polarization is therefore [see Eq. (88)],

$$\begin{aligned}\hat{d}^{ab}(k)=&\frac{e^2}{3(2\pi)^3}\left(g^{ab}-\frac{k^ak^b}{k^2}\right)(k^2+2m^2)\mathrm{sgn}(k_-)\\&\times\Theta(k^2-4m^2)\sqrt{1-\frac{4m^2}{k^2}}.\end{aligned}\tag{100}$$

In order to obtain its retarded part we will factorize a second-order polynomial:

$$\hat{d}^{ab}(k)=\frac{e^2}{3(2\pi)^3}(k^2g^{ab}-k^ak^b)\hat{d}_1(k),\tag{101}$$

with $\hat{d}_1(k)$ the following distribution:

$$\hat{d}_1(k)=\left(1+\frac{2m^2}{k^2}\right)\mathrm{sgn}(k_-)\Theta(k^2-4m^2)\sqrt{1-\frac{4m^2}{k^2}}.\tag{102}$$

Now, a great amount of calculations could be avoided if one uses the following result, which can be shown by using the general splitting formulas given in Eqs. (33) and (34):

For a causal distribution which in momentum space is of the form

$$\hat{d}(p)=P(p)\hat{d}_1(p),\tag{103}$$

with P a polynomial, if $\hat{r}_1(p)$ is a retarded distribution corresponding to $\hat{d}_1(p)$, then $P(p)\hat{r}_1(p)$ is a retarded distribution corresponding to $\hat{d}(p)$. This property is very convenient for practical purposes because it assures that it suffices to split the distribution $\hat{d}_1(p)$, which is less singular than $\hat{d}(p)$.

Therefore, returning to our problem, we only need to obtain the retarded part of $\hat{d}_1(k)$ given in Eq. (102), whose singular order at the x^- axis is

$$\omega_-[\hat{d}_1]=0.\tag{104}$$

Its retarded part is given by (we use the variable $s=-2k_-q$)

$$\hat{r}_1(k) = -\frac{i}{2\pi}\int \frac{ds}{s - ik\_0^+}\left\{\Theta(s+k^2-4m^2) \times \left(1+\frac{2m^2}{s+k^2}\right)\sqrt{1-\frac{4m^2}{s+k^2}} - \Theta(s-4m^2)\left(1+\frac{2m^2}{s}\right)\sqrt{1-\frac{4m^2}{s}}\right\}. \quad (105)$$

Applying Sokhotskiy's formula in the first integral, then changing the variable to $s + k^2 \to s$ in it, we find

$$\hat{r}_1(k) = \frac{i}{2\pi}k^2 E(k) + \frac{1}{2}\mathrm{sgn}(k_-)\Theta(k^2-4m^2) \times \left(1+\frac{2m^2}{k^2}\right)\sqrt{1-\frac{4m^2}{k^2}} \quad (106)$$

with

$$E(k) = \int_{4m^2}^{+\infty} \frac{(s+2m^2)\sqrt{1-\frac{4m^2}{s}}}{s^2(k^2-s)}ds. \quad (107)$$

This is the same integral which appears in instant dynamics [52], and has the value

$$E(k) = \frac{m^2}{k^4}\left[\frac{1+\xi}{1-\xi}\left(\xi - 4 + \frac{1}{\xi}\right)\log(\xi) + \frac{5}{3}\left(\xi+\frac{1}{\xi}\right) - \frac{22}{3}\right], \quad (108)$$

with the parameter ξ defined by the relation

$$\frac{k^2}{m^2} = -\frac{(1-\xi)^2}{\xi}. \quad (109)$$

Therefore, the retarded distribution is

$$\hat{r}_1(k) = \frac{i}{2\pi}\frac{m^2}{k^2}\left\{\left[\frac{1+\xi}{1-\xi}\left(\xi-4+\frac{1}{\xi}\right)\log(\xi) + \frac{5}{3}\left(\xi+\frac{1}{\xi}\right) - \frac{22}{3}\right] - i\pi\mathrm{sgn}(k_-) \times \Theta(k^2-4m^2)(k^2+2m^2)\sqrt{1-\frac{4m^2}{k^2}}\right\}. \quad (110)$$

Note however that by Eq. (109)

$$\xi + \frac{1}{\xi} = 2 - \frac{k^2}{m^2}, \quad (111)$$

so the terms in the first line of Eq. (110) which do not multiply the logarithm have coefficients subjected to normalization. Finally, putting this result into Eq. (101) to obtain the retarded distribution $\hat{r}^{ab}(k)$ and subtracting the subsidiary distribution $\hat{r}'^{ab}(k)$, we are able to define the vacuum polarization tensor $\Pi^{ab}(k)$ as

$$\hat{t}^{ab}(k) =: -i\hat{\Pi}^{ab}(k),$$

$$T_2^{(VP)}(x_1;x_2) = -i:A_a(x_1)\Pi^{ab}(x_1-x_2)A_b(x_2):, \quad (112)$$

so that

$$\hat{\Pi}^{ab}(k) =: (2\pi)^{-4}\left(\frac{k^a k^b}{k^2} - g^{ab}\right)\hat{\Pi}(k), \quad (113)$$

with

$$\hat{\Pi}(k) = \frac{e^2m^2}{3}\left\{\left[\frac{1+\xi}{1-\xi}\left(\xi-4+\frac{1}{\xi}\right)\log(\xi) + \frac{5}{3}\left(\xi+\frac{1}{\xi}\right) - \frac{22}{3}\right] - i\pi\Theta(k^2-4m^2)(k^2+2m^2)\sqrt{1-\frac{4m^2}{k^2}}\right\}. \quad (114)$$

Additionally, since $\hat{\Pi}(k)$ has singular order $\omega_- = 2$, its general expression is

$$\tilde{\Pi}(k) = \hat{\Pi}(k) + C_0 + C_2 k^2, \quad (115)$$

because a term such as $c_a k^a$ is forbidden due to parity invariance of the QED. In order to fix the values of C_0 and C_2 we study Møller's scattering with vacuum polarization insertions. By a procedure identical to the one developed for the scattering of two fermions in Yukawa's model in Ref. [61], we find that the total radiation field propagator is the solution of the equation:

$$\hat{D}^{ab}_{\mathrm{tot}} = \hat{d}^{ac}(\delta^b_c + (2\pi)^4\tilde{\Pi}_{cd}\tilde{D}^{db}_{\mathrm{tot}}), \quad (116)$$

with $\hat{d}^{ab}$ the normalized distribution for Møller's scattering given in Eq. (62). Equation (116) can also be put in the following form:

$$(\delta^a_d - (2\pi)^4\hat{d}^{ac}\tilde{\Pi}_{cd})\hat{D}^{db}_{\mathrm{tot}} = \hat{d}^{ab}. \quad (117)$$

The usual technique [52] to solve this equation consists in inverting the distribution $\hat{d}^{ab}$. However, in our case this distribution has no inverse due to the noncovariant terms contained in it [74]. Nonetheless, with Eqs. (62) and (113) we can form the inter-parenthetical expression of Eq. (117), which we will write as

$$L^a{}_d = \pi_1\delta^a_d + \pi_2 k^a \eta_d; \quad (118)$$

$$\pi_1 = \frac{k^2 - (2\pi)^{-2}\tilde{\Pi} + i0^+}{k^2 + i0^+}, \qquad \pi_2 = \frac{(2\pi)^{-2}\tilde{\Pi}}{k_-(k^2+i0^+)}.$$

It turns out that this tensor does have an inverse, which we will call ${E^c}_a$:

$$ {E^c}_a = \sigma_1 \delta^c_a + \sigma_2 k^c k_a + \sigma_3 k^c \eta_a + \sigma_4 \eta^c k_a + \sigma_5 \eta^c \eta_a. \qquad (119) $$

Then the coefficients σ_i are found by the set of equations ${E^c}_a {L^a}_d = \delta^c_d$; the solution is

$$ \sigma_1 = \frac{1}{\pi_1}, \qquad \sigma_2 = 0, \qquad \sigma_3 = -\frac{\sigma_1 \pi_2}{\pi_1 + k_- \pi_2}, \qquad \sigma_4 = 0 = \sigma_5. \qquad (120) $$

Substituting Eq. (120) with the values of π_i given in Eq. (118) into Eq. (119) we find

$$ {E^c}_a = \frac{1}{k^2 - (2\pi)^{-2}\tilde{\Pi} + i0^+} \left\{ k^2 \delta^c_a - \frac{(2\pi)^{-2}\tilde{\Pi}}{k_-} k^c \eta_a \right\}. \qquad (121) $$

Now we can solve Eq. (117) by multiplying it by ${E^c}_a$. We obtain that the total photon propagator is

$$ \hat{D}^{cb}_{\mathrm{tot}}(k) = -\frac{(2\pi)^{-2}}{k^2 - (2\pi)^{-2}\tilde{\Pi}(k) + i0^+} \times \left(g^{cb} - \frac{k^c \eta^b + \eta^c k^b}{k_-} \right). \qquad (122) $$

As we can see, the total propagator preserves the same tensor structure of the distribution $\hat{d}^{ab}(k)$. This is different to what occurs when normalizing the total photon propagator in a covariant approach in instant dynamics, when the total propagator is split into two terms. The one which contains $\tilde{\Pi}$ is transversal to the momentum k, while the part parallel to the momentum remains independent of $\tilde{\Pi}$; see Ref. [52]. However, the two propagators reduce to the covariant one and are equal to each other once the conservation of the current is taken into account, eliminating all the terms proportional to k^a; this is an expression of gauge invariance.

The vacuum polarization scalar $\tilde{\Pi}$ appears in the denominator of $\hat{D}^{ab}_{\mathrm{tot}}(k)$, so that it is possible to impose the physical requirements: (i) The physical mass of the photon is zero, so that the propagator must have a pole in $k^2 = 0$. (ii) The physical value of the electric charge is the coupling constant e of the one-point distribution T_1. These two requirements are translated, respectively, into

$$ \lim_{k^2 \to 0} \tilde{\Pi}(k) = 0 \quad \text{and} \quad \lim_{k^2 \to 0} \frac{d\tilde{\Pi}(k)}{d(k^2)} = 0. \qquad (123) $$

These two conditions are already satisfied by $\hat{\Pi}(k)$ in Eq. (114), so that the coefficients in Eq. (115) must be $C_0 = 0 = C_2$, and the right normalized solution is the central one.

VII. CONCLUSIONS

We have formulated QED in light-front dynamics in the causal framework, for which we used the quantized field operators obtained by direct construction of Fock's space; it was proved that the equal-time (anti) commutation relations for them are the same as that obtained in Refs. [16,28] and by the usage of Dirac-Bergmann's method and the correspondence principle in Ref. [39].

We proved that Møller's and Compton's scattering processes are equivalent to those in instant dynamics if the right normalization terms are chosen, and, in the first case, if the conservation of current is taken into account—an extension off the mass shell of the S operator would lead to a difference with instant dynamics, but that is not manifest in the real world. We can interpret this result by saying that the instantaneous terms in Feynman's propagators are not physical ones, but a consequence of the splitting procedure according to a time variable whose isochronic surfaces intersect the light cone on the entire x^- axis. Such a splitting procedure, by construction, cannot tell anything about the value of the retarded distribution at the x^- axis [60], so the instantaneous terms that arise in it cannot be relied on, but must be fixed by other conditions besides causality. As we have seen, Lorentz covariance implies that they must not be there. We see here that the intrinsic richness of the possible normalization terms in light-front dynamics allows one to start with an invariant T_1 distribution, without instantaneous interaction terms that are unnecessary in order to obtain a covariant theory. They can be recovered, however, by defining the Lagrangian density as containing all the normalization terms of the higher-order transition distributions, which establishes a direct link to the usual approach, and showing in passing, and without the necessity of any combinatoric argument, that in a perturbation series based on $\mathscr{L}$, the instantaneous terms in it cancel exactly the ones in the field propagators.

In the study of vacuum polarization, the calculation is greatly simplified by the factorization of a second-order polynomial, leading to a result which is equal to the one obtained in instant dynamics. For its normalization we have considered Møller's scattering with vacuum polarization insertions. This requires one to define the total photon propagator, which has the same tensor structure as the commutation distribution of this field. Again, although different to the instant dynamics total propagator, it leads to the same physical results because the current conservation holds in the real world, as an expression of gauge invariance. The imposition of the zero mass of the photon and the value of the electric charge imply that the central solution is the right one.

Along this study we have encountered gauge invariance at two points: in the study of Møller's scattering and in the study of vacuum polarization. We have explicitly shown that the equivalence of these two results with instant dynamics relies on the gauge invariance property, expressed as the conservation of the electric current. Consequently, it is mandatory to study the complete implementation of quantum gauge invariance in null-plane QED. Our study of QED in the null-plane CPT will continue by addressing this problem and by considering other radiative corrections, Ward-Takahashi's identities, and so on.

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