

## Inverse-Reynolds-dominance approach to transient fluid dynamics

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We consider the evolution equations for the bulk viscous pressure, diffusion current, and shear tensor derived within second-order relativistic dissipative hydrodynamics from kinetic theory. By matching the higher-order moments directly to the dissipative quantities, all terms which are of second order in the Knudsen number  $\text{Kn}$  vanish, leaving only terms of order  $\mathcal{O}(\text{Re}^{-1}\text{Kn})$  and  $\mathcal{O}(\text{Re}^{-2})$  in the relaxation equations, where  $\text{Re}^{-1}$  is the inverse Reynolds number. We therefore refer to this scheme as the inverse-Reynolds-dominance (IREd) approach. The remaining (nonvanishing) transport coefficients can be obtained exclusively in terms of the inverse of the collision matrix. This procedure fixes unambiguously the relaxation times of the dissipative quantities, which are no longer related to the eigenvalues of the inverse of the collision matrix. In particular, we find that the relaxation times corresponding to higher-order moments grow as their order increases, thereby contradicting the *separation of scales* paradigm. The formal (up to second order) equivalence with the standard DNMR approach is proven and the connection between the IREd transport coefficients and the usual DNMR ones is established.

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### I. INTRODUCTION

Formulating a causal and stable framework for relativistic dissipative hydrodynamics has been a long-standing issue that has seen a series of improvements in the last decade [1–3]. This problem is not merely academic, as dissipative fluid dynamics has been proven to be a powerful effective theory in relativistic systems, such as heavy-ion collisions [4–7] and relativistic astrophysical processes [8–10].

While the relativistic Euler equations describing the dynamics of the perfect fluid are unambiguously formulated, their generalization to relativistic dissipative fluids proves to be a formidable problem. In the nonrelativistic case, the leading-order contribution to the Chapman-Enskog expansion, i.e., the Navier-Stokes equations, yield a suitable theory for viscous hydrodynamics which has seen tremendous success [11]. At this level, the dissipative quantities, otherwise known as thermodynamic fluxes [12], are fixed by constitutive equations to the thermodynamic forces (expressed as gradients of the fluid properties), thereby implying an instantaneous response and an infinite information propagation speed, thus violating causality [12–15].

An approach attracting much interest in recent years is to abandon the traditional (Landau or Eckart) matching conditions, by which the energy and particle number density of the system are equated to their fictitious local-equilibrium counterparts. In contrast, general matching conditions can be exploited in the frame of a first-order-like theory closely resembling the Navier-Stokes formulation in a way that guarantees causality and stability [2,3,16,17].

In this paper, we focus on the more traditional approach of formulating a causal and stable theory of dissipative hydrodynamics in the form of relaxation equations for the dissipative quantities appearing in the particle current and stress-energy tensor decompositions, namely the bulk-viscous pressure  $\Pi$ , the particle diffusion current  $n^\mu$ , and the shear-stress tensor  $\pi^{\mu\nu}$ . Such second-order theories introduce relaxation times governing the response of the dissipative quantities with respect to changes in the fluid properties (e.g., pressure  $P$ , ratio  $\alpha = \mu/T$  between the chemical potential  $\mu$  and temperature  $T$ , and four-velocity  $u^\mu$ ). This procedure sets finite relaxation timescales of the approach towards the corresponding asymptotic Navier-Stokes limits, thereby rendering the formulation causal [18].

Naturally, due to the microscopic nature of the coefficients involved in second-order theories, an underlying formulation has to be provided. Most works employ kinetic theory, since it provides a suitable limit of quantum field theories in the semiclassical limit [19]. From a thermodynamical perspective, the entropy current describing the

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entropy flow in second-order hydrodynamics exhibits second-order terms, which are in principle calculable from kinetic theory [19–21] or can be postulated within the frame of extended irreversible thermodynamics [22–27].

Even though the second-order formalism by Müller, Israel and Stewart [20,28] has long been the most widely used second-order theory, its equations of motion were obtained by employing a nonorthogonal momentum space basis which cannot be used to systematically account for all second-order terms, leading to inaccurate expressions for the transport coefficients. This issue can be resolved by considering the exact equations of motion for the irreducible moments of the Boltzmann equation, as has been done in the celebrated DNMR formulation of relativistic dissipative hydrodynamics [1,29–31]. In addition, this formulation provided a way to improve the truncation by increasing the number of moments considered for a given tensor rank, thereby moving from the (lowest-order) 14-moment approximation to 23 moments and beyond.

In the DNMR formulation [1,32], the main idea consists of obtaining a system of relaxation equations for the eigenmodes of the linearized collision kernel, which can then be related to the irreducible moments and thus to the dissipative quantities. The evolution equations for the dissipative quantities obtained in this procedure read

$$\tau_{\Pi}\dot{\Pi} + \Pi = -\zeta\theta + \mathcal{J} + \mathcal{K} + \mathcal{R}, \quad (1a)$$

$$\tau_n\dot{n}^{(\mu)} + n^\mu = \kappa I^\mu + \mathcal{J}^\mu + \mathcal{K}^\mu + \mathcal{R}^\mu, \quad (1b)$$

$$\tau_\pi\dot{\pi}^{(\mu\nu)} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{R}^{\mu\nu}, \quad (1c)$$

where  $\tau_{\Pi}$ ,  $\tau_n$  and  $\tau_\pi$  are the relaxation time corresponding to  $\Pi$ ,  $n^\mu$  and  $\pi^{\mu\nu}$ , respectively, while  $\zeta$ ,  $\kappa$  and  $\eta$  are the bulk viscosity, diffusivity and shear viscosity coefficients, constituting the so-called first-order (Navier-Stokes) transport coefficients. In this procedure, the relaxation equations for the dissipative quantities are derived on the basis of a hierarchical truncation with respect to the inverse Reynolds number  $\text{Re}^{-1}$ , related to the magnitude of the dissipative quantities ( $\text{Re}^{-1} \sim |\Pi|/P_0, |n^\mu|/n_0, |\pi^{\mu\nu}|/P_0$ ), as well as to the Knudsen number  $\text{Kn}$ , which can be related to the magnitude of gradients ( $\text{Kn} \sim \ell\nabla A/A_0$ , where  $\ell$  is a characteristic microscopic scale and  $A$  is a fluid property) or to the microscopic mean free path ( $\text{Kn} \sim \tau_{\Pi}/L, \tau_n/L, \tau_\pi/L$ , where  $L$  is a macroscopic length scale). By the above definitions, the second-order terms  $\mathcal{J}^{\mu_1\cdots\mu_\ell}$ ,  $\mathcal{K}^{\mu_1\cdots\mu_\ell}$ , and  $\mathcal{R}^{\mu_1\cdots\mu_\ell}$  collect all contributions that are of orders  $\mathcal{O}(\text{Re}^{-1}\text{Kn})$ ,  $\mathcal{O}(\text{Kn}^2)$  and  $\mathcal{O}(\text{Re}^{-2})$ , respectively. All third-order terms with respect to  $\text{Kn}$  and  $\text{Re}^{-1}$ , i.e., the terms of orders  $\mathcal{O}(\text{Re}^{-3})$ ,  $\mathcal{O}(\text{Re}^{-2}\text{Kn})$ ,  $\mathcal{O}(\text{Re}^{-1}\text{Kn}^2)$ , and  $\mathcal{O}(\text{Kn}^3)$ , are neglected. It is thus implied that, besides the requirement that both  $\text{Re}^{-1}$  and  $\text{Kn}$  are small, the above power-counting scheme also assumes that they are of the same magnitude, i.e.,  $\text{Kn} \sim \text{Re}^{-1}$ . It

should be noted that in general the magnitudes of  $\text{Kn}$  and  $\text{Re}^{-1}$  need not be the same, hence the regimes where only one of these quantities is small and the other one is large lie outside the domain of applicability of second-order hydrodynamics. Such regimes may be probed using third-order hydrodynamics [33,34] or directly kinetic theory [15,35–40], however in this paper we focus only on the second-order theory.

The  $\mathcal{R}^{\mu_1\cdots\mu_\ell}$  terms, derived in Ref. [29], arise due to quadratic terms appearing in the collision term. These terms will not be discussed further in this work. The  $\mathcal{K}^{\mu_1\cdots\mu_\ell}$  terms involve quadratic terms in the first-order gradients of the flow properties (e.g.,  $\sigma^{\lambda(\mu}\sigma^{\nu)\lambda}$ ) or second-order gradients (e.g.,  $\Delta_\lambda^\mu\nabla_\nu\sigma^{\lambda\nu}$ ). Their transport coefficients were derived in Ref. [29], however they are usually disregarded because they give rise to parabolic equations [39]. On the other hand, the terms in  $\mathcal{J}^{\mu_1\cdots\mu_\ell}$  are hyperbolic in nature and are fully compatible with special relativity.

In this paper we show that it is possible to formulate a theory of dissipative relativistic hydrodynamics setting the noncausal contribution  $\mathcal{K}^{\mu_1\cdots\mu_\ell}$  to zero by construction. The basis of our analysis is the asymptotic matching scheme proposed in Ref. [39] in the context of multiple dynamical moments, as well as in Ref. [41] for the case of multi-component fluids. The scheme finds its nonrelativistic analogue in the work of Struchtrup [42], and it is sometimes called the *order of magnitude approach*.

Except in the case of the lowest-order truncation, the transport coefficients and the relaxation times obtained in this scheme are different compared to those obtained in DNMR. The two theories thus seem to yield, in general, different equations. In this paper we establish the connection between the two schemes and show that they are equivalent up to second order in  $\text{Kn}$  and  $\text{Re}^{-1}$ . By consistently using the matching conditions to express thermodynamic forces in terms of dissipative quantities, we show that all terms contained in  $\mathcal{K}^{\mu_1\cdots\mu_\ell}$  in DNMR can be reabsorbed into the transport coefficients in  $\mathcal{J}^{\mu_1\cdots\mu_\ell}$  and the relaxation times, thus modifying the usual DNMR transport coefficients. We therefore call our approach the *inverse-Reynolds-dominance* (IReD) approach, as it consists, effectively, in replacing  $\mathcal{O}(\text{Kn}^2)$  terms in favor of  $\mathcal{O}(\text{Re}^{-1}\text{Kn})$ , making the inverse Reynolds number “dominant” over the Knudsen number. The IReD equations are formally equivalent to the DNMR ones. We will show this by analytically establishing the connection between the transport coefficients appearing in the two formulations.

The outline of this paper is as follows. In Sec. II, we review the DNMR formalism introduced in Ref. [1], while in Sec. III we discuss the IReD scheme, leading to vanishing  $\mathcal{K}^{\mu_1\cdots\mu_\ell}$  terms [41]. Section IV addresses the connection between the transport coefficients arising in the IReD approach compared to the DNMR ones (technical details are relegated to Appendix A). Section V discusses the connection between the approach introduced in Ref. [39] for

the case of 23 dynamical moments and our proposed IReD approach. In Sec. VI we list the explicit values for the transport coefficients in the limit of an ultrarelativistic ideal gas of hard spheres, demonstrating the convergence of the method when including higher-order moments. The general expressions for the transport coefficients in the IReD approach are summarized in Appendix B. Section VII concludes this paper. Throughout this paper, we use Planck units ( $c = \hbar = k_B = 1$ ) and the  $(+, -, -, -)$  metric convention. Our analysis is restricted to second order with respect to Kn and  $\text{Re}^{-1}$  and we work under the assumption that  $\text{Kn} \sim \text{Re}^{-1}$ .

## II. DNMR APPROACH

In this section, we review the DNMR formalism introduced in Ref. [1]. The starting point of the analysis is the Boltzmann equation,

$$k^\mu \partial_\mu f_{\mathbf{k}} = C[f], \quad (2)$$

where  $f_{\mathbf{k}} \equiv f_{\mathbf{k}}(x)$  is the one-particle distribution function,  $k^\mu = (k^0, \mathbf{k})$  is the on-shell four-momentum ( $k^2 = (k^0)^2 - \mathbf{k}^2 = m^2$ ), while  $C[f]$  is the collision term. By the  $H$ -theorem [19,43,44],  $C[f]$  acts by drawing the system towards local thermodynamic equilibrium, described by the equilibrium distribution  $f_{0\mathbf{k}}$ .

The deviation from equilibrium  $\delta f_{\mathbf{k}} = f_{\mathbf{k}} - f_{0\mathbf{k}}$  can be characterized in terms of its irreducible moments  $\rho_r^{\mu_1 \dots \mu_\ell}$ , defined as

$$\rho_r^{\mu_1 \dots \mu_\ell} = \int dK E_{\mathbf{k}} k^{\langle \mu_1} \dots k^{\mu_\ell \rangle} \delta f_{\mathbf{k}}, \quad (3)$$

where  $dK = g d^3 k / [(2\pi)^3 k^0]$  is the Lorentz-invariant integration measure ( $g$  is the number of internal degrees of freedom), while  $A^{\langle \mu_1 \dots \mu_\ell \rangle} = \Delta_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell} A^{\nu_1 \dots \nu_\ell}$  is the symmetrized and (for  $\ell > 1$ ) traceless projection of the tensor  $A^{\mu_1 \dots \mu_\ell}$  with respect to the fluid four-velocity  $u^\mu$ . In particular, the  $r = 0$  moments can be related to the bulk pressure  $\Pi$ , diffusion current  $n^\mu$  and shear stress  $\pi^{\mu\nu}$  as follows:

$$\rho_0 = -\frac{3}{m^2} \Pi, \quad \rho_0^\mu = n^\mu, \quad \rho_0^{\mu\nu} = \pi^{\mu\nu}. \quad (4)$$

In the Landau frame, the charge current  $N^\mu$  and stress-energy tensor  $T^{\mu\nu}$  admit the following decomposition:

$$\begin{aligned} N^\mu &= n u^\mu + n^\mu, \\ T^{\mu\nu} &= \varepsilon u^\mu u^\nu - (P + \Pi) \Delta^{\mu\nu} + \pi^{\mu\nu}, \end{aligned} \quad (5)$$

where  $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ . Since the particle-number density  $n$  and energy density  $\varepsilon$  are equal to their fictitious equilibrium values ( $n = n_0$ ,  $\varepsilon = \varepsilon_0$ ), the moments  $\rho_1 = \delta n$  and  $\rho_2 = \delta \varepsilon$  both vanish. In addition, the heat flow  $W^\mu = \Delta_\nu^\mu u_\lambda T^{\nu\lambda} = \rho_1^\mu$  also vanishes by the Landau matching condition,  $T_\nu^\mu u^\nu = \varepsilon u^\mu$ . Summarizing, in the Landau frame the following moments are automatically zero:

$$\rho_1 = \rho_2 = \rho_1^\mu = 0. \quad (6)$$

Starting from the Boltzmann equation (2) and defining  $\nabla^\mu = \Delta_\nu^\mu \partial^\nu$  and  $\dot{f} = Df = u \cdot \partial f$  for an arbitrary function  $f$ , the equations of motion for the irreducible moments  $\rho_r$ ,  $\rho_r^\mu$  and  $\rho_r^{\mu\nu}$  can be derived as shown in Ref. [1], leading to

$$\begin{aligned} \dot{\rho}_r - C_{r-1} &= \alpha_r^{(0)} \theta - \frac{G_{2r}}{D_{20}} \Pi \theta + \frac{G_{2r}}{D_{20}} \pi^{\mu\nu} \sigma_{\mu\nu} + \frac{G_{3r}}{D_{20}} \partial_\mu n^\mu + (r-1) \rho_{r-2}^{\mu\nu} \sigma_{\mu\nu} + r \rho_{r-1}^\mu \dot{u}_\mu - \nabla_\mu \rho_{r-1}^\mu \\ &\quad - \frac{1}{3} [(r+2) \rho_r - (r-1) m^2 \rho_{r-2}] \theta, \end{aligned} \quad (7a)$$

$$\begin{aligned} \dot{\rho}_r^{(\mu)} - C_{r-1}^{(\mu)} &= \alpha_r^{(1)} I^\mu + \rho_r^\nu \omega^\mu{}_\nu + \frac{1}{3} [(r-1) m^2 \rho_{r-2}^\mu - (r+3) \rho_r^\mu] \theta - \Delta_\lambda^\mu \nabla_\nu \rho_{r-1}^{\lambda\nu} + r \rho_{r-1}^{\mu\nu} \dot{u}_\nu \\ &\quad + \frac{1}{5} [(2r-2) m^2 \rho_{r-2}^\nu - (2r+3) \rho_r^\nu] \sigma_\nu^\mu + \frac{1}{3} [m^2 r \rho_{r-1} - (r+3) \rho_{r+1}] \dot{u}^\mu \\ &\quad + \frac{\beta J_{r+2,1}}{\varepsilon + P} (\Pi \dot{u}^\mu - \nabla^\mu \Pi + \Delta_\nu^\mu \partial_\lambda \pi^{\lambda\nu}) - \frac{1}{3} \nabla^\mu (m^2 \rho_{r-1} - \rho_{r+1}) + (r-1) \rho_{r-2}^{\mu\nu\lambda} \sigma_{\lambda\nu}, \end{aligned} \quad (7b)$$

$$\begin{aligned} \dot{\rho}_r^{(\mu\nu)} - C_{r-1}^{(\mu\nu)} &= 2\alpha_r^{(2)} \sigma^{\mu\nu} - \frac{2}{7} [(2r+5) \rho_r^{\lambda(\mu} - 2m^2 (r-1) \rho_{r-2}^{\lambda(\mu} \sigma_\lambda^{\nu)}] + 2\rho_r^{\lambda(\mu} \omega^{\nu)}{}_\lambda \\ &\quad + \frac{2}{15} [(r+4) \rho_{r+2} - (2r+3) m^2 \rho_r + (r-1) m^4 \rho_{r-2}] \sigma^{\mu\nu} + \frac{2}{5} \nabla^{(\mu} (\rho_{r+1}^{\nu)} - m^2 \rho_{r-1}^{\nu)} \\ &\quad - \frac{2}{5} [(r+5) \rho_{r+1}^{(\mu} - r m^2 \rho_{r-1}^{(\mu} \dot{u}^{\nu)}] - \frac{1}{3} [(r+4) \rho_r^{\mu\nu} - m^2 (r-1) \rho_{r-2}^{\mu\nu}] \theta + (r-1) \rho_{r-2}^{\mu\nu\lambda\rho} \sigma_{\lambda\rho} - \Delta_{\alpha\beta}^{\mu\nu} \nabla_\lambda \rho_{r-1}^{\alpha\beta\lambda} + r \rho_{r-1}^{\mu\nu\lambda} \dot{u}_\lambda, \end{aligned} \quad (7c)$$

where  $I^\mu = \nabla^\mu \alpha$ . Furthermore,  $\sigma_{\mu\nu} = \nabla_{(\mu} u_{\nu)}$ ,  $\omega_{\mu\nu} = \frac{1}{2}(\partial_\mu u_\nu - \partial_\nu u_\mu)$  and  $\theta = \partial_\mu u^\mu$  denote the shear tensor, vorticity tensor and expansion scalar, respectively, while  $C_{r-1}^{(\mu_1 \dots \mu_\ell)}$  represents an irreducible moment of tensor-rank  $\ell$  of the collision term, defined in analogy to Eq. (3):

$$C_r^{(\mu_1 \dots \mu_\ell)} = \int dK E_{\mathbf{k}}^r k_{(\mu_1} \dots k_{\mu_\ell)} C[f]. \quad (8)$$

Furthermore,  $G_{nm} = J_{n0} J_{m0} - J_{n-1,0} J_{m+1,0}$ ,  $D_{nq} = J_{n+1,q} J_{n-1,q} - J_{nq}^2$ , while  $J_{nq} = (\partial I_{nq} / \partial \alpha)_\beta$  represents the derivative of  $I_{nq}$  with respect to  $\alpha = \mu/T$  at constant inverse temperature  $\beta$ , with

$$I_{nq} = \frac{1}{(2q+1)!!} \int dK E_{\mathbf{k}}^{n-2q} (-\Delta^{\alpha\beta} k_\alpha k_\beta)^q f_{0\mathbf{k}}. \quad (9)$$

The first terms appearing on the right-hand sides of Eqs. (7) are given as

$$\begin{aligned} \alpha_r^{(0)} &= (1-r)I_{r1} - I_{r0} - \frac{1}{D_{20}} [G_{2r}(\varepsilon + P) - G_{3r}n], \\ \alpha_r^{(1)} &= J_{r+1,1} - \frac{n}{\varepsilon + P} J_{r+2,1}, \\ \alpha_r^{(2)} &= I_{r+2,1} + (r-1)I_{r+2,2}. \end{aligned} \quad (10)$$

The relations (7) constitute a system of infinitely many coupled equations, where the unknowns are the irreducible moments  $\rho_r^{\mu_1 \dots \mu_\ell}$ . In order to extract from here the equations of motion for the dissipative quantities  $\Pi$ ,  $n^\mu$  and  $\pi^{\mu\nu}$ , the collision term  $C_r^{(\mu_1 \dots \mu_\ell)}$  must be expressed in terms of  $\rho_r^{\mu_1 \dots \mu_\ell}$ . This can be achieved by introducing a decomposition of  $\delta f_{\mathbf{k}}$  with respect to the irreducible moments,

$$\delta f_{\mathbf{k}} = f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}_{\mathbf{k}n}^{(\ell)} \rho_n^{\mu_1 \dots \mu_\ell} k_{(\mu_1} \dots k_{\mu_\ell)}, \quad (11)$$

where  $\tilde{f}_{0\mathbf{k}} = 1 - a f_{0\mathbf{k}}$  ( $a = 1$  for fermions,  $-1$  for bosons and  $0$  for classical particles) and  $N_\ell$  represent truncation orders which in principle should be taken to infinity. The functions  $\mathcal{H}_{\mathbf{k}n}^{(\ell)}$  are polynomials of order  $N_\ell$  in  $E_{\mathbf{k}}$  and are defined such that Eq. (3) holds exactly for  $0 \leq n \leq N_\ell$  [1].

Ignoring quadratic or higher-order terms in deviations from equilibrium, the collision term can be represented (to linear order) as

$$C_{r-1} = - \sum_{n=0, \neq 1, 2}^{N_0} \mathcal{A}_{rn}^{(0)} \rho_n, \quad (12a)$$

$$C_{r-1}^{(\mu)} = - \sum_{n=0, \neq 1}^{N_1} \mathcal{A}_{rn}^{(1)} \rho_n^\mu, \quad (12b)$$

$$C_{r-1}^{(\mu\nu)} = - \sum_{n=0}^{N_2} \mathcal{A}_{rn}^{(2)} \rho_n^{\mu\nu}, \quad (12c)$$

where  $\mathcal{A}_{rn}^{(\ell)}$  can be interpreted as the collision matrix. The sums appearing above skip the moments which vanish due to the Landau matching, as shown in Eq. (6).

The final step is to relate the irreducible moments  $\rho_{r \neq 0}^{\mu_1 \dots \mu_\ell}$  to those of order  $r = 0$ . This is the branching point between the DNMR approach and the IReD approach presented in Sec. III. In the DNMR approach, the basis of this construction is to seek a diagonalization of  $\mathcal{A}_{rn}^{(\ell)}$  ensured by the matrix of eigenvectors  $\Omega_{rn}^{(\ell)}$ , such that

$$(\Omega^{(\ell)})^{-1} \mathcal{A}^{(\ell)} \Omega^{(\ell)} = \text{diag}(\chi_0^{(\ell)}, \chi_1^{(\ell)}, \dots), \quad (13)$$

where the columns of the diagonalization matrix  $\Omega^{(\ell)}$  are chosen such that the eigenvalues  $\chi_r^{(\ell)}$  appear in ascending order,

$$\chi_0^{(\ell)} \leq \chi_1^{(\ell)} \leq \dots \quad (14)$$

With the above convention, it is possible to enforce a *separation of scales* by which only the eigenvectors

$$X_0^{\mu_1 \dots \mu_\ell} = \sum_{j=0}^{N_\ell} (\Omega^{(\ell)})_{0j}^{-1} \rho_j^{\mu_1 \dots \mu_\ell} \quad (15)$$

corresponding to the slowest scale  $\chi_0^{(\ell)}$  remain in the transient regime (the normalization of  $\Omega_{ij}^{(\ell)}$  is such that  $\Omega_{00}^{(\ell)} = 1$ ). The eigenvectors  $X_{r>0}^{\mu_1 \dots \mu_\ell}$ , corresponding to larger eigenvalues  $\chi_{r>0}^{(\ell)}$ , are approximated by their asymptotic (Navier-Stokes) values

$$X_{r>0} \simeq \frac{\beta_r^{(0)}}{\chi_r^{(0)}} \theta, \quad X_{r>0}^\mu \simeq \frac{\beta_r^{(1)}}{\chi_r^{(1)}} I^\mu, \quad X_{r>0}^{\mu\nu} \simeq \frac{\beta_r^{(2)}}{\chi_r^{(2)}} \sigma^{\mu\nu}, \quad (16)$$

where

$$\beta_i^{(0)} = \sum_{j=0, \neq 1, 2}^{N_0} (\Omega^{(0)})_{ij}^{-1} \alpha_j^{(0)}, \quad (17a)$$

$$\beta_i^{(1)} = \sum_{j=0, \neq 1}^{N_1} (\Omega^{(1)})_{ij}^{-1} \alpha_j^{(1)}, \quad (17b)$$

$$\beta_i^{(2)} = 2 \sum_{j=0}^{N_2} (\Omega^{(2)})_{ij}^{-1} \alpha_j^{(2)}. \quad (17c)$$

By this approximation, the irreducible moments  $\rho_r^{\mu_1 \dots \mu_\ell} = \sum_{n=0}^{N_\ell} \Omega_{rn}^{(\ell)} X_n^{\mu_1 \dots \mu_\ell}$  take the following form:



$$\rho_i \simeq -\frac{3}{m^2} [\Omega_{i0}^{(0)} \Pi - (\zeta_i - \Omega_{i0}^{(0)} \zeta) \theta], \quad (18a)$$

$$\rho_i^\mu \simeq \Omega_{i0}^{(1)} n^\mu + (\kappa_i - \Omega_{i0}^{(1)} \kappa) I^\mu, \quad (18b)$$

$$\rho_i^{\mu\nu} \simeq \Omega_{i0}^{(2)} \pi^{\mu\nu} + 2(\eta_i - \Omega_{i0}^{(2)} \eta) \sigma^{\mu\nu}, \quad (18c)$$

$$\rho_i^{\mu\nu\lambda\dots} \simeq \mathcal{O}(\text{Kn}^2, \text{KnRe}^{-1}), \quad (18d)$$

where the property  $X_0^{\mu_1\dots\mu_\ell} = \rho_0^{\mu_1\dots\mu_\ell} - \sum_{n>0}^{N_\ell} \Omega_{rn}^{(\ell)} X_n^{\mu_1\dots\mu_\ell}$  was employed. In the above, the first-order transport coefficients  $\zeta_i$ ,  $\kappa_i$  and  $\eta_i$  are computed via

$$\zeta_n = \frac{m^2}{3} \sum_{r=0, \neq 1, 2}^{N_0} \tau_{nr}^{(0)} \alpha_r^{(0)}, \quad (19a)$$

$$\kappa_n = \sum_{r=0, \neq 1}^{N_1} \tau_{nr}^{(1)} \alpha_r^{(1)}, \quad (19b)$$

$$\eta_n = \sum_{r=0}^{N_2} \tau_{nr}^{(2)} \alpha_r^{(2)}, \quad (19c)$$

with  $\zeta = \zeta_0$ ,  $\kappa = \kappa_0$  and  $\eta = \eta_0$ . The inverse collision matrix  $\tau_{rn}^{(\ell)}$  appearing above satisfies

$$\tau_{rn}^{(\ell)} = (\mathcal{A}^{(\ell)})_{rn}^{-1} = \sum_{m=0}^{N_\ell} \Omega_{rm}^{(\ell)} \frac{1}{\chi_m^{(\ell)}} (\Omega^{(\ell)})_{mn}^{-1}. \quad (20)$$

In what concerns the moments of negative order  $\rho_{-r}^{\mu_1\dots\mu_\ell}$  (with  $r > 0$ ), they can also be related to the dissipative quantities via

$$\rho_{-r}^{\mu_1\dots\mu_\ell} = \sum_{n=0}^{N_\ell} \mathcal{F}_{rn}^{(\ell)} \rho_n^{\mu_1\dots\mu_\ell}, \quad (21)$$

where the functions  $\mathcal{F}_{rn}^{(\ell)}$  are defined as

$$\mathcal{F}_{rn}^{(\ell)} = \frac{\ell!}{(2\ell+1)!!} \int dK f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} E_{\mathbf{k}}^{-r} \mathcal{H}_{\mathbf{k}\mathbf{n}}^{(\ell)} (\Delta^{\alpha\beta} k_\alpha k_\beta)^\ell, \quad (22)$$

which follows after introducing Eq. (11) into Eq. (3). Using now the asymptotic matching in Eqs. (18), we arrive at

$$\rho_{-r} = -\frac{3}{m^2} (\gamma_r^{(0)} \Pi - \hat{\gamma}_r^{(0)} \theta), \quad (23a)$$

$$\rho_{-r}^\mu = \gamma_r^{(1)} n^\mu + \hat{\gamma}_r^{(1)} I^\mu, \quad (23b)$$

$$\rho_{-r}^{\mu\nu} = \gamma_r^{(2)} \pi^{\mu\nu} + 2\hat{\gamma}_r^{(2)} \sigma^{\mu\nu}. \quad (23c)$$

The coefficients  $\gamma_r^{(\ell)}$  and  $\hat{\gamma}_r^{(\ell)}$  can be computed using the functions  $\mathcal{F}_{rn}^{(\ell)}$ ,

$$\begin{aligned} \gamma_r^{(0)} &= \sum_{n=0, \neq 1, 2}^{N_0} \mathcal{F}_{rn}^{(0)} \Omega_{n0}^{(0)}, & \hat{\gamma}_r^{(0)} &= \sum_{n=0, \neq 1, 2}^{N_0} \mathcal{F}_{rn}^{(0)} (\zeta_n - \Omega_{n0}^{(0)} \zeta), \\ \gamma_r^{(1)} &= \sum_{n=0, \neq 1}^{N_1} \mathcal{F}_{rn}^{(1)} \Omega_{n0}^{(1)}, & \hat{\gamma}_r^{(1)} &= \sum_{n=0, \neq 1}^{N_1} \mathcal{F}_{rn}^{(1)} (\kappa_n - \Omega_{n0}^{(1)} \kappa), \\ \gamma_r^{(2)} &= \sum_{n=0}^{N_2} \mathcal{F}_{rn}^{(2)} \Omega_{n0}^{(2)}, & \hat{\gamma}_r^{(2)} &= \sum_{n=0}^{N_2} \mathcal{F}_{rn}^{(2)} (\eta_n - \Omega_{n0}^{(2)} \eta). \end{aligned} \quad (24)$$

At this point, we remark that in the DNMR approach [1] and in later papers [29], the terms  $\hat{\gamma}_r^{(\ell)}$  are neglected, such that the  $\mathcal{O}(\text{Kn})$  contributions to  $\rho_{-r}^{\mu_1\dots\mu_\ell}$  that should later appear in the  $\mathcal{K}^{\mu_1\dots\mu_\ell}$  terms are disregarded completely [29]. In order to conform with the DNMR notation and still stay accurate at first order with respect to both Kn and  $\text{Re}^{-1}$ , the coefficient  $\gamma_r^{(\ell)}$  should be replaced by

$$\bar{\gamma}_r^{(0)} = \gamma_r^{(0)} + \frac{1}{\zeta} \hat{\gamma}_r^{(0)} = \sum_{n=0, \neq 1, 2}^{N_0} \mathcal{F}_{rn}^{(0)} \mathcal{C}_n^{(0)}, \quad (25a)$$

$$\bar{\gamma}_r^{(1)} = \gamma_r^{(1)} + \frac{1}{\kappa} \hat{\gamma}_r^{(1)} = \sum_{n=0, \neq 1}^{N_1} \mathcal{F}_{rn}^{(1)} \mathcal{C}_n^{(1)}, \quad (25b)$$

$$\bar{\gamma}_r^{(2)} = \gamma_r^{(2)} + \frac{1}{\eta} \hat{\gamma}_r^{(2)} = \sum_{n=0}^{N_2} \mathcal{F}_{rn}^{(2)} \mathcal{C}_n^{(2)}, \quad (25c)$$

where we introduced the notation (also to be used in the following section)

$$\mathcal{C}_n^{(0)} = \frac{\zeta_n}{\zeta_0}, \quad \mathcal{C}_n^{(1)} = \frac{\kappa_n}{\kappa_0}, \quad \mathcal{C}_n^{(2)} = \frac{\eta_n}{\eta_0}. \quad (26)$$

The same quantities are denoted in Ref. [41] by  $\bar{\zeta}_n = \mathcal{C}_n^{(0)}$ ,  $\bar{\kappa}_n = \mathcal{C}_n^{(1)}$  and  $\bar{\eta}_n = \mathcal{C}_n^{(2)}$ . With the above convention, Eqs. (23) becomes

$$\rho_{-r} = -\frac{3}{m^2} \bar{\gamma}_r^{(0)} \Pi, \quad \rho_{-r}^\mu = \bar{\gamma}_r^{(1)} n^\mu, \quad \rho_{-r}^{\mu\nu} = \bar{\gamma}_r^{(2)} \pi^{\mu\nu}, \quad (27)$$

which is similar, but not identical, to Eq. (67) in Ref. [1].

Finally, the evolution equations (1) for  $\Pi$ ,  $n^\mu$  and  $\pi^{\mu\nu}$  can be obtained by multiplying Eqs. (7) by  $\tau_{0r}^{(\ell)}$  and summing over  $r$ . The relaxation times  $\tau_\Pi$ ,  $\tau_n$  and  $\tau_\pi$  are given by the inverse of the smallest eigenvalues  $\chi_0^{(\ell)}$  of the collision matrices  $\mathcal{A}_{rn}^{(\ell)}$  [see Eq. (13)]:

$$\tau_{\Pi} = \frac{1}{\chi_0^{(0)}} = \sum_{r=0, \neq 1, 2}^{N_0} \tau_{0r}^{(0)} \Omega_{r0}^{(0)}, \quad (28a)$$

$$\tau_n = \frac{1}{\chi_0^{(1)}} = \sum_{r=0, \neq 1}^{N_1} \tau_{0r}^{(1)} \Omega_{r0}^{(1)}, \quad (28b)$$

$$\tau_{\pi} = \frac{1}{\chi_0^{(2)}} = \sum_{r=0}^{N_2} \tau_{0r}^{(2)} \Omega_{r0}^{(2)}, \quad (28c)$$

where we remind that the normalization of  $\Omega_{rn}^{(\ell)}$  is such that  $\Omega_{00}^{(\ell)} = 1$ . For completeness and for future reference, we display below the complete expressions for the  $\mathcal{J}^{\mu_1 \dots \mu_\ell}$  terms [1],

$$\begin{aligned} \mathcal{J} = & -\ell_{\Pi n} \nabla \cdot n - \tau_{\Pi n} n \cdot F - \delta_{\Pi \Pi} \Pi \theta - \lambda_{\Pi n} n \cdot I \\ & + \lambda_{\Pi \pi} \pi^{\mu\nu} \sigma_{\mu\nu}, \end{aligned} \quad (29a)$$

$$\begin{aligned} \mathcal{J}^\mu = & -\tau_n n_\nu \omega^{\nu\mu} - \delta_{nn} n^\mu \theta - \ell_{n\Pi} \nabla^\mu \Pi \\ & + \ell_{n\pi} \Delta^{\mu\nu} \nabla_\lambda \pi^\lambda{}_\nu + \tau_{n\Pi} \Pi F^\mu - \tau_{n\pi} \pi^{\mu\nu} F_\nu \\ & - \lambda_{nn} n_\nu \sigma^{\mu\nu} + \lambda_{n\Pi} \Pi I^\mu - \lambda_{n\pi} \pi^{\mu\nu} I_\nu, \end{aligned} \quad (29b)$$

$$\begin{aligned} \mathcal{J}^{\mu\nu} = & 2\tau_\pi \pi_\lambda^{(\mu} \omega^{\nu)\lambda} - \delta_{\pi\pi} \pi^{\mu\nu} \theta - \tau_{\pi\pi} \pi^{\lambda(\mu} \sigma_\lambda^{\nu)} + \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu} \\ & - \tau_{\pi n} n^{(\mu} F^{\nu)} + \ell_{\pi n} \nabla^{(\mu} n^{\nu)} + \lambda_{\pi n} n^{(\mu} I^{\nu)}, \end{aligned} \quad (29c)$$

where  $F^\mu = \nabla^\mu P$  and  $I^\mu = \nabla^\mu \alpha$ . We also display the  $\mathcal{K}^{\mu_1 \dots \mu_\ell}$  terms, following the conventions of Ref. [29]:

$$\begin{aligned} \mathcal{K} = & \tilde{\zeta}_1 \omega_{\mu\nu} \omega^{\mu\nu} + \tilde{\zeta}_2 \sigma_{\mu\nu} \sigma^{\mu\nu} + \tilde{\zeta}_3 \theta^2 + \tilde{\zeta}_4 I \cdot I + \tilde{\zeta}_5 F \cdot F \\ & + \tilde{\zeta}_6 I \cdot F + \tilde{\zeta}_7 \nabla \cdot I + \tilde{\zeta}_8 \nabla \cdot F, \end{aligned} \quad (30a)$$

$$\begin{aligned} \mathcal{K}^\mu = & \tilde{\kappa}_1 \sigma^{\mu\nu} I_\nu + \tilde{\kappa}_2 \sigma^{\mu\nu} F_\nu + \tilde{\kappa}_3 I^\mu \theta + \tilde{\kappa}_4 F^\mu \theta \\ & + \tilde{\kappa}_5 \omega^{\mu\nu} I_\nu + \tilde{\kappa}_6 \Delta_\lambda^\mu \nabla_\nu \sigma^{\lambda\nu} + \tilde{\kappa}_7 \nabla^\mu \theta, \end{aligned} \quad (30b)$$

$$\begin{aligned} \mathcal{K}^{\mu\nu} = & \tilde{\eta}_1 \omega^{\lambda(\mu} \omega^{\nu)\lambda} + \tilde{\eta}_2 \theta \sigma^{\mu\nu} + \tilde{\eta}_3 \sigma^{\lambda(\mu} \sigma_\lambda^{\nu)} \\ & + \tilde{\eta}_4 \sigma_\lambda^{(\mu} \omega^{\nu)\lambda} + \tilde{\eta}_5 I^{(\mu} I^{\nu)} + \tilde{\eta}_6 F^{(\mu} F^{\nu)} \\ & + \tilde{\eta}_7 I^{(\mu} F^{\nu)} + \tilde{\eta}_8 \nabla^{(\mu} I^{\nu)} + \tilde{\eta}_9 \nabla^{(\mu} F^{\nu)}. \end{aligned} \quad (30c)$$

To understand the origin of the  $\mathcal{O}(\text{Re}^{-1}\text{Kn})$  and  $\mathcal{O}(\text{Kn}^2)$  terms, we note that the asymptotic matching in Eqs. (18) replaces the irreducible moments [originally of order  $\mathcal{O}(\text{Re}^{-1})$ ] with  $\mathcal{O}(\text{Re}^{-1})$  and  $\mathcal{O}(\text{Kn})$  terms proportional to  $(\Pi, n^\mu, \pi^{\mu\nu})$  and  $(\theta, I^\mu, \sigma^{\mu\nu})$ , respectively. At the level of Eqs. (1), the former terms make  $\mathcal{O}(\text{Re}^{-1}\text{Kn})$  contributions, while the latter ones give rise to  $\mathcal{O}(\text{Kn}^2)$  terms. This can be easily seen in what concerns the terms appearing on the right-hand side of Eqs. (7), since there the irreducible moments always come with  $\mathcal{O}(\text{Kn})$  coefficients. Additional contributions arise from the comoving derivative of the irreducible

moments appearing on the left-hand side of Eqs. (7). We illustrate this by considering the particular example of the tensor moments  $\dot{\rho}_r^{(\mu\nu)}$ . Taking the comoving derivative of Eq. (18c) leads to

$$\begin{aligned} \dot{\rho}_r^{(\mu\nu)} = & \Omega_{r0}^{(2)} \dot{\pi}^{(\mu\nu)} + \dot{\Omega}_{r0}^{(2)} \pi^{\mu\nu} + 2D[\eta(C_r^{(2)} - \Omega_{r0}^{(2)})] \sigma^{\mu\nu} \\ & + 2\eta(C_r^{(2)} - \Omega_{r0}^{(2)}) \dot{\sigma}^{(\mu\nu)} + \mathcal{O}(\text{Re}^{-1}\text{Kn}^2), \end{aligned} \quad (31)$$

where  $C_r^{(2)} = \eta_r/\eta$  was introduced in Eqs. (26). The first term in Eq. (31) gives rise to the relaxation time  $\tau_\pi$  via Eq. (28c). To leading order, the comoving derivative  $Df = \dot{f}$  of a thermodynamic function  $f \equiv f(\alpha, \beta)$  is of order  $\mathcal{O}(\text{Kn})$ , since

$$\dot{f} = \frac{\partial f}{\partial \alpha} \dot{\alpha} + \frac{\partial f}{\partial \beta} \dot{\beta} = \left( \mathcal{H} \frac{\partial f}{\partial \alpha} + \tilde{\mathcal{H}} \frac{\partial f}{\partial \beta} \right) \theta + \mathcal{O}(\text{Re}^{-1}\text{Kn}), \quad (32)$$

where  $\mathcal{H}$  and  $\tilde{\mathcal{H}}$  are defined in Eq. (A2b), while  $\dot{\alpha}$  and  $\dot{\beta}$  are given in Eqs. (A1a) and (A1b). Thus, the second term of Eq. (31) is of order  $\mathcal{O}(\text{Re}^{-1}\text{Kn})$ , contributing to  $\mathcal{J}^{\mu\nu}$ . In contrast, the third and fourth terms are of order  $\mathcal{O}(\text{Kn}^2)$ , thus contributing to  $\mathcal{K}^{\mu\nu}$ .

As mentioned in the introduction, the  $\mathcal{K}^{\mu_1 \dots \mu_\ell}$  terms are traditionally ignored in the literature, either because they vanish in the 14 moment limit, or because they lead to parabolic equations of motion [39]. In the following section, we rederive the evolution equations (1) such that  $\mathcal{K}^{\mu_1 \dots \mu_\ell}$  vanish identically by construction.

### III. INVERSE-REYNOLDS-DOMINANCE (IRED) APPROACH

In this section, we discuss the derivation of the evolution equations (1) for the case when the terms of second order with respect to Kn vanish,  $\mathcal{K}^{\mu_1 \dots \mu_\ell} = 0$ . The derivation is identical to that presented in the previous section, up to Eqs. (12). The main difference compared to the DNMR approach is at the level of the asymptotic matching. In this section, we bypass the diagonalization of the collision matrix via the matrix  $\Omega_{rn}^{(\ell)}$ . Multiplying Eqs. (7) by  $\tau_{nr}^{(\ell)}$  and summing over  $r$ , we arrive at [41]

$$\sum_{r=0, \neq 1, 2}^{N_0} \tau_{nr}^{(0)} \dot{\rho}_r + \rho_n = \frac{3}{m^2} \zeta_n \theta + \mathcal{O}(\text{KnRe}^{-1}), \quad (33a)$$

$$\sum_{r=0, \neq 1}^{N_1} \tau_{nr}^{(1)} \dot{\rho}_r^{(\mu)} + \rho_n^\mu = \kappa_n I^\mu + \mathcal{O}(\text{KnRe}^{-1}), \quad (33b)$$

$$\sum_{r=0}^{N_2} \tau_{nr}^{(2)} \dot{\rho}_r^{(\mu\nu)} + \rho_n^{\mu\nu} = 2\eta_n \sigma^{\mu\nu} + \mathcal{O}(\text{KnRe}^{-1}), \quad (33c)$$

where the first-order transport coefficients  $\zeta_n$ ,  $\kappa_n$  and  $\eta_n$  were introduced in Eqs. (19). Note that the comoving derivatives on the left-hand sides of Eqs. (33) are of order  $\mathcal{O}(\text{KnRe}^{-1})$  as well. Neglecting terms of this order, we obtain straightforwardly from Eqs. (33)

$$\rho_n \simeq \frac{3}{m^2} \zeta_n \theta, \quad \rho_n^\mu \simeq \kappa_n I^\mu, \quad \rho_n^{\mu\nu} \simeq 2\eta_n \sigma^{\mu\nu}, \quad (34)$$

while  $\rho_n^{\mu\lambda\dots} \simeq \mathcal{O}(\text{Kn}^2, \text{KnRe}^{-1})$ . The above relations establish the correspondence between quantities of orders  $\mathcal{O}(\text{Re}^{-1})$  and  $\mathcal{O}(\text{Kn})$  appearing on the left- and right-hand sides, respectively. We now exploit this correspondence in order to eliminate the  $\mathcal{O}(\text{Kn})$  terms appearing in the DNMR matching prescription shown in Eqs. (18). Specializing the above relations to the case  $n = 0$  and using Eqs. (4) allows the thermodynamic forces  $\theta$ ,  $n^\mu$  and  $\sigma^{\mu\nu}$  to be replaced by the dissipative quantities  $\Pi$ ,  $n^\mu$  and  $\pi^{\mu\nu}$ , leading to the asymptotic matching equations

$$\rho_n \simeq -\frac{3}{m^2} C_n^{(0)} \Pi, \quad \rho_n^\mu \simeq C_n^{(1)} n^\mu, \quad \rho_n^{\mu\nu} \simeq C_n^{(2)} \pi^{\mu\nu}, \quad (35)$$

where the coefficients  $C_n^{(\ell)}$  were introduced in Eqs. (26). Equations (35) naturally hold also when  $n = -r < 0$  by identifying

$$C_{-r}^{(\ell)} = \tilde{\gamma}_r^{(\ell)} = \sum_{n=0}^{N_\ell} \mathcal{F}_{rn}^{(\ell)} C_n^{(\ell)}, \quad (36)$$

where  $\tilde{\gamma}_r^{(\ell)}$  was introduced in Eqs. (25) and the function  $\mathcal{F}_{rn}^{(\ell)}$  is defined in Eq. (22). Equations (35) relate the higher-order moments  $\rho_{r>0}^{\mu_1\dots\mu_\ell}$  to the zeroth-order ones. As mentioned in the introduction, a similar approach was proposed under the name of the *order of magnitude* approach in Ref. [42] in the case of nonrelativistic fluids, as well as in Ref. [41] for multicomponent relativistic fluids. In the following, we will refer to this approach as the *inverse-Reynolds-dominance* (IReD) approach, for reasons that will become apparent.

We first remark that Eqs. (35) is equivalent to the original DNMR matching in Eqs. (18). This can be seen by replacing  $\theta = -\Pi/\zeta$ ,  $I^\mu = n^\mu/\kappa$  and  $\sigma^{\mu\nu} = \pi^{\mu\nu}/(2\eta)$  and noting that the error introduced by these replacements can be neglected since it is of higher order than the terms shown in Eqs. (18). By using the relations (35) in the equations of motion (33), we can replace all irreducible moments appearing on the right-hand side by the dissipative quantities  $\Pi$ ,  $n^\mu$ , and  $\pi^{\mu\nu}$ , with the neglected terms being of order  $\mathcal{O}(\text{Kn}^2 \text{Re}^{-1})$ . Furthermore, setting the index  $n = 0$  in Eqs. (33), we obtain the relaxation equations (1) with  $\mathcal{K}^{\mu_1\dots\mu_\ell} = 0$ . The  $\mathcal{J}^{\mu_1\dots\mu_\ell}$  terms retain the form in Eqs. (29) and the transport coefficients arising there are identical in form to those derived in the DNMR formalism and reported

in Ref. [1], with the exception that all instances of  $\Omega_{r0}^{(\ell)}$  should be replaced by  $C_r^{(\ell)}$  (also  $\gamma_r^{(\ell)}$  should be replaced by  $\tilde{\gamma}_r^{(\ell)} \equiv C_{-r}^{(\ell)}$ )<sup>1</sup>:

$$\text{(DNMR)} \quad \text{(IReD)} \\ \Omega_{r0}^{(\ell)} \rightarrow C_r^{(\ell)}, \quad (37a)$$

$$\gamma_r^{(\ell)} \rightarrow C_{-r}^{(\ell)}, \quad (37b)$$

$$\mathcal{K}^{\mu_1\dots\mu_\ell} \rightarrow 0. \quad (37c)$$

The expressions for the transport coefficients obtained using the IReD approach are summarized in Appendix B. The above prescription holds also for the computation of the relaxation times. Replacing  $\Omega_{r0}^{(\ell)}$  with  $C_r^{(\ell)}$  in Eqs. (28), we arrive at

$$\tau_\Pi = \sum_{r=0, \neq 1}^{N_0} \tau_{0r}^{(0)} C_r^{(0)}, \quad (38a)$$

$$\tau_n = \sum_{r=0, \neq 1}^{N_1} \tau_{0r}^{(1)} C_r^{(1)}, \quad (38b)$$

$$\tau_\pi = \sum_{r=0}^{N_2} \tau_{0r}^{(2)} C_r^{(2)}. \quad (38c)$$

Upon performing the replacements in Eqs. (37), the values of the transport coefficients arising in the IReD approach will be different from those computed using the DNMR approach. This is clearly the case for the coefficients of the  $\mathcal{O}(\text{Kn}^2)$  terms, which vanish identically in the IReD approach. We will come back to the relation between the IReD and DNMR transport coefficients in the next section.

The matching procedure in Eqs. (35) eliminates the  $\mathcal{K}^{\mu_1\dots\mu_\ell}$  terms which are of order  $\mathcal{O}(\text{Kn}^2)$ , retaining the  $\mathcal{J}^{\mu_1\dots\mu_\ell}$  terms of order  $\mathcal{O}(\text{KnRe}^{-1})$  and thereby trading one power of  $\text{Kn}$  for a power of  $\text{Re}^{-1}$ . This is clear when considering the terms appearing on the right-hand side of Eqs. (7) [see also the discussion before Eq. (31)]. The comoving derivatives of the irreducible moments appearing on the left-hand side of Eqs. (7) make only  $\mathcal{O}(\text{Re}^{-1} \text{Kn})$  contributions. To see this, we reconsider the comoving derivative of the tensor moments with the asymptotic matching in Eqs. (35),

$$\dot{\rho}^{(\mu\nu)} = C_r^{(2)} \dot{\pi}^{(\mu\nu)} + \dot{C}_r^{(2)} \pi^{\mu\nu} + \mathcal{O}(\text{Re}^{-1} \text{Kn}^2). \quad (39)$$

The first term contributes to the relaxation time  $\tau_\pi$  via Eq. (38c). As indicated in Eq. (32),  $\dot{C}_r^{(2)}$  is of order

<sup>1</sup>See Appendix C of Ref. [41] for explicit expressions in the case of a multicomponent fluid.

$\mathcal{O}(\text{Kn})$ , such that the second term is of order  $\mathcal{O}(\text{KnRe}^{-1})$ , contributing only to  $\mathcal{J}^{\mu\nu}$ . We have thus established that the  $\mathcal{O}(\text{Kn}^2)$  terms vanish identically under the asymptotic matching in Eqs. (35). For this reason, we refer to this approach as the inverse-Reynolds-dominance approach.

The connection between the IReD relaxation times in Eqs. (38) and the eigenvalues of  $\mathcal{A}_m^{(\ell)}$  is lost, therefore one may wonder about the fate of the *separation of scales*. In order to analyze the timescales associated with higher-order moments, it is convenient to introduce the coefficients  $\mathcal{C}_{n;r}^{(\ell)}$  via

$$\mathcal{C}_{n;r}^{(0)} = \frac{\zeta_n}{\zeta_r}, \quad \mathcal{C}_{n;r}^{(1)} = \frac{\kappa_n}{\kappa_r}, \quad \mathcal{C}_{n;r}^{(2)} = \frac{\eta_n}{\eta_r}, \quad (40)$$

such that  $\mathcal{C}_{n;0}^{(\ell)} = \mathcal{C}_n^{(\ell)}$  reduces to the coefficients introduced in Eqs. (26). To obtain the evolution equations for the irreducible moments  $\rho_r^{\mu_1 \dots \mu_\ell}$ , all the other irreducible moments should be written in terms of these ones via formulas analogous to Eqs. (35),

$$\rho_n \simeq \mathcal{C}_{n;r}^{(0)} \rho_r, \quad \rho_n^\mu \simeq \mathcal{C}_{n;r}^{(1)} \rho_r^\mu, \quad \rho_n^{\mu\nu} \simeq \mathcal{C}_{n;r}^{(2)} \rho_r^{\mu\nu}. \quad (41)$$

With these relations, we can apply the same procedure that was employed to yield Eqs. (1) and obtain

$$\tau_{\Pi;r} \dot{\rho}_r + \rho_r = \frac{3}{m^2} \zeta_r \theta + \mathcal{O}(\text{KnRe}^{-1}), \quad (42a)$$

$$\tau_{n;r} \dot{\rho}_r^{(\mu)} + \rho_r^\mu = \kappa_r I^\mu + \mathcal{O}(\text{KnRe}^{-1}), \quad (42b)$$

$$\tau_{\pi;r} \dot{\rho}_r^{(\mu\nu)} + \rho_r^{\mu\nu} = 2\eta_r \sigma^{\mu\nu} + \mathcal{O}(\text{KnRe}^{-1}), \quad (42c)$$

where the omitted terms on the right-hand side are of the same structure as Eqs. (29). The relaxation times appearing above are given by equations analogous to Eqs. (38), with  $\mathcal{C}_r^{(\ell)} \equiv \mathcal{C}_{r;0}^{(\ell)}$  replaced by  $\mathcal{C}_{r;n}^{(\ell)}$ :

$$\tau_{\Pi;n} = \sum_{r=0, \neq 1, 2}^{N_0} \tau_{nr}^{(0)} \mathcal{C}_{r;n}^{(0)}, \quad (43a)$$

$$\tau_{n;n} = \sum_{r=0, \neq 1}^{N_1} \tau_{nr}^{(1)} \mathcal{C}_{r;n}^{(1)}, \quad (43b)$$

$$\tau_{\pi;n} = \sum_{r=0}^{N_2} \tau_{nr}^{(2)} \mathcal{C}_{r;n}^{(2)}. \quad (43c)$$

Setting  $n = 0$  in the above equations reproduces Eqs. (38). The ordering of the relaxation times thus obtained clearly depends on the details of the (inverse of the) collision matrix. For definiteness, we report in Table I the first four relaxation times in comparison to the first four eigenvalues  $\chi_n^{(\ell)}$  obtained for the case of an ultrarelativistic ideal gas interacting via a constant cross section  $\sigma$  (to be discussed in Sec. VI). It can be seen that the separation of scales principle invoked in the DNMR approach no longer holds, being in fact reversed. The relaxation times obey the inequality  $\tau_{*;0} \leq \tau_{*;1} \leq \dots$ , for all  $* \in \{n, \pi\}$  (the bulk sector does not contribute to the dynamics for a gas of massless particles).

Based on the above analysis, it becomes evident that demanding that the  $\mathcal{O}(\text{Kn}^2)$  terms vanish gives relaxation times which are not compatible with the separation of

TABLE I. (left) Relaxation times  $\tau_{n;r}$  and  $\tau_{\pi;r}$  corresponding to the vector and tensor moments  $\rho_r^\mu$  and  $\rho_r^{\mu\nu}$ , respectively, obtained for various values of the truncation orders  $N_1$  and  $N_2 = N_1 - 1$ . (right) Inverse eigenvalues  $[\chi_r^{(1)}]^{-1}$  and  $[\chi_r^{(2)}]^{-1}$  shown in descending order. The relaxation times and inverse eigenvalues are expressed in units of the mean free path  $\lambda_{\text{mfp}} = 1/(n\sigma)$ , where  $n$  is the local particle-number density and  $\sigma$  is the (constant) collision cross section.

[IReD]		Diffusion: Relaxation times			[DNMR]		Diffusion: Inverse eigenvalues		
$N_1$	$\tau_{n,0}[\lambda_{\text{mfp}}]$	$\tau_{n,2}[\lambda_{\text{mfp}}]$	$\tau_{n,3}[\lambda_{\text{mfp}}]$	$\tau_{n,4}[\lambda_{\text{mfp}}]$	$N_1$	$[\chi_0^{(1)}]^{-1}[\lambda_{\text{mfp}}]$	$[\chi_2^{(1)}]^{-1}[\lambda_{\text{mfp}}]$	$[\chi_3^{(1)}]^{-1}[\lambda_{\text{mfp}}]$	$[\chi_4^{(1)}]^{-1}[\lambda_{\text{mfp}}]$
1	9/4	...	...	...	1	9/4	...	...	...
2	2.076	2.419	...	...	2	2.59	1.629	...	...
3	2.076	2.435	2.565	...	3	2.575	1.961	1.413	...
4	2.079	2.438	2.568	2.680	4	2.573	1.85	1.597	1.304
$\infty$	2.084	2.440	2.570	2.681	$\infty$	2.572	1.847	1.586	1.451

  

[IReD]		Shear: Relaxation times			[DNMR]		Shear: Inverse eigenvalues		
$N_2$	$\tau_{\pi,0}[\lambda_{\text{mfp}}]$	$\tau_{\pi,1}[\lambda_{\text{mfp}}]$	$\tau_{\pi,2}[\lambda_{\text{mfp}}]$	$\tau_{\pi,3}[\lambda_{\text{mfp}}]$	$N_2$	$[\chi_0^{(2)}]^{-1}[\lambda_{\text{mfp}}]$	$[\chi_1^{(2)}]^{-1}[\lambda_{\text{mfp}}]$	$[\chi_2^{(2)}]^{-1}[\lambda_{\text{mfp}}]$	$[\chi_3^{(2)}]^{-1}[\lambda_{\text{mfp}}]$
0	5/3	...	...	...	0	5/3	...	...	...
1	1.649	1.785	...	...	1	2	1.364	...	...
2	1.654	1.788	1.902	...	2	2	1.646	1.241	...
3	1.655	1.789	1.902	2.001	3	2	1.650	1.477	1.176
$\infty$	1.656	1.789	1.902	2.001	$\infty$	2	1.650	1.484	1.386



scales concept. Conversely, enforcing the separation of scales as done in DNMR (by setting  $\tau_{\Pi} = [\chi_0^{(0)}]^{-1}$ , etc.) introduces in principle terms of order  $\mathcal{O}(\text{Kn}^2)$  in the evolution equations for the dissipative quantities. Despite this difference, the DNMR and the IReD approaches are equivalent, as we will show in the next section.

#### IV. CONNECTION TO DNMR

As discussed in Secs. II and III, the IReD approach yields relaxation equations for  $\Pi$ ,  $n^\mu$  and  $\pi^{\mu\nu}$  for which  $\mathcal{K}^{\mu_1 \dots \mu_\ell} = 0$ . Since the DNMR and IReD approaches are both exact to second order in  $\text{Kn}$  and  $\text{Re}^{-1}$ , they must coincide up to (and including) terms of second order. In order to distinguish between the transport coefficients arising in the two approaches, we will use a tilde  $\tilde{\phantom{x}}$  to denote transport coefficients computed in the DNMR approach. Keeping in mind that the first-order transport coefficients  $\zeta_n$ ,  $\kappa_n$  and  $\eta_n$  are exactly the same in the two approaches, being given by Eqs. (19), the goal of this section is to prove the following equivalence:

$$\tau_{\Pi} \dot{\Pi} - \mathcal{J} = \tilde{\tau}_{\Pi} \dot{\Pi} - \tilde{\mathcal{J}} - \tilde{\mathcal{K}}, \quad (44a)$$

$$\tau_n \dot{n}^{(\mu)} - \mathcal{J}^\mu = \tilde{\tau}_n \dot{n}^{(\mu)} - \tilde{\mathcal{J}}^\mu - \tilde{\mathcal{K}}^\mu, \quad (44b)$$

$$\tau_\pi \dot{\pi}^{(\mu\nu)} - \mathcal{J}^{\mu\nu} = \tilde{\tau}_\pi \dot{\pi}^{(\mu\nu)} - \tilde{\mathcal{J}}^{\mu\nu} - \tilde{\mathcal{K}}^{\mu\nu}, \quad (44c)$$

where the  $\mathcal{O}(\text{Kn}^2)$  terms are absent on the left-hand side by virtue of the IReD asymptotic matching.

The detailed comparison will be carried out in Appendix A. Here we will put forth the key points and focus on the modification of the relaxation times. This modification arises due to terms in  $\tilde{\mathcal{K}}^{\mu_1 \dots \mu_\ell}$  that originate from  $\dot{\rho}_r^{(\mu_1 \dots \mu_\ell)}$ . Focusing on the DNMR asymptotic matching for the case of the tensor moments, we multiply Eq. (31) by  $\tau_{0r}^{(2)}$  and sum with respect to  $r$ :

$$\begin{aligned} \sum_{r=0}^{N_2} \tau_{0r}^{(2)} \dot{\rho}_r^{(\mu\nu)} &= \dot{\pi}^{(\mu\nu)} \sum_{r=0}^{N_2} \tau_{0r}^{(2)} \Omega_{r0}^{(2)} \\ &+ 2\eta \dot{\sigma}^{(\mu\nu)} \sum_{r=0}^{N_2} \tau_{0r}^{(2)} (\mathcal{C}_r^{(2)} - \Omega_{r0}^{(2)}) + \dots, \end{aligned} \quad (45)$$

where we omitted second-order terms proportional to  $\pi^{\mu\nu}$  and  $\sigma^{\mu\nu}$  that lead to contributions to  $\tilde{\mathcal{J}}^{\mu\nu}$  and  $\tilde{\mathcal{K}}^{\mu\nu}$ . The summation with respect to  $r$  can be performed in favor of the DNMR and IReD relaxation times  $\tilde{\tau}_\pi$  and  $\tau_\pi$ , introduced in Eqs. (28c) and (38c), respectively. Performing the same steps for the scalar and vector moments, we arrive at

$$\begin{aligned} \sum_{r=0, \neq 1, 2}^{N_0} \tau_{0r}^{(0)} \dot{\rho}_r &= -\frac{3}{m^2} [\tilde{\tau}_{\Pi} \dot{\Pi} - \zeta(\tau_{\Pi} - \tilde{\tau}_{\Pi}) \dot{\theta} + \dots], \\ \sum_{r=0, \neq 1}^{N_1} \tau_{0r}^{(1)} \dot{\rho}_r^{(\mu)} &= \tilde{\tau}_n \dot{n}^{(\mu)} + \kappa(\tau_n - \tilde{\tau}_n) \dot{I}^{(\mu)} + \dots, \\ \sum_{r=0}^{N_2} \tau_{0r}^{(2)} \dot{\rho}_r^{(\mu\nu)} &= \tilde{\tau}_\pi \dot{\pi}^{(\mu\nu)} + 2\eta(\tau_\pi - \tilde{\tau}_\pi) \dot{\sigma}^{(\mu\nu)} + \dots. \end{aligned} \quad (46)$$

Employing now the first-order (Navier-Stokes) relations

$$\zeta \theta = -\Pi + \mathcal{O}(\text{Kn}^2, \text{KnRe}^{-1}), \quad (47a)$$

$$\kappa I^\mu = n^\mu + \mathcal{O}(\text{Kn}^2, \text{KnRe}^{-1}), \quad (47b)$$

$$2\eta \sigma^{\mu\nu} = \pi^{\mu\nu} + \mathcal{O}(\text{Kn}^2, \text{KnRe}^{-1}), \quad (47c)$$

to eliminate the thermodynamic forces in favor of the corresponding fluxes, it can be seen that the second terms in Eqs. (46) lead to the replacement of the DNMR relaxation times ( $\tilde{\tau}_{\Pi}$ ,  $\tilde{\tau}_n$ ,  $\tilde{\tau}_\pi$ ) by the IReD ones ( $\tau_{\Pi}$ ,  $\tau_n$ ,  $\tau_\pi$ ), e.g.,

$$\begin{aligned} \tilde{\tau}_\pi \dot{\pi}^{(\mu\nu)} + 2\eta(\tau_\pi - \tilde{\tau}_\pi) \dot{\sigma}^{(\mu\nu)} \\ = \tau_\pi \dot{\pi}^{(\mu\nu)} - (\tau_\pi - \tilde{\tau}_\pi) \pi^{\mu\nu} \frac{\dot{\eta}}{\eta} + \dots, \end{aligned} \quad (48)$$

where the neglected terms are of third order.

The above discussion hints that the key to connecting the DNMR transport coefficients to the IReD ones is to look at the comoving derivatives of  $\theta$ ,  $I^\mu$  and  $\sigma^{\mu\nu}$ . The full expressions are derived in Appendix A. Here we just reproduce the terms that hold the key to establishing the connection between the DNMR and IReD relaxation times, namely

$$\dot{\theta} = \omega^{\mu\lambda} \omega_{\mu\lambda} + \dots, \quad (49a)$$

$$\dot{I}^{(\mu)} = -\omega^{\mu\nu} I_\nu + \dots, \quad (49b)$$

$$\dot{\sigma}^{(\mu\nu)} = -\omega^{\lambda(\mu} \omega^{\nu)\lambda} + \dots. \quad (49c)$$

The terms shown on the right-hand sides have no correspondent in the  $\tilde{\mathcal{J}}^{\mu_1 \dots \mu_\ell}$  terms (except for the case of  $\omega^{\mu\nu} I_\nu$ , which can be related to  $\omega^{\mu\nu} n_\nu / \kappa$ ), therefore the coefficients of these terms appearing in  $\tilde{\mathcal{K}}^{\mu_1 \dots \mu_\ell}$  will modify the relaxation times appearing on the left-hand side of Eqs. (1). Focusing on the tensor sector, one can use Eq. (49c) together with  $\sigma^{\mu\nu} \simeq \pi^{\mu\nu} / 2\eta$  to establish

$$\tilde{\eta}_1 \omega^{\lambda(\mu} \omega^{\nu)\lambda} \simeq -\frac{\tilde{\eta}_1}{2\eta} \dot{\pi}^{(\mu\nu)} + \frac{\tilde{\eta}_1 \dot{\eta}}{2\eta^2} \pi^{\mu\nu} + \dots, \quad (50)$$

where the dots indicate the  $\mathcal{O}(\text{Kn}^2)$  terms which were omitted in Eq. (49c). The coefficient  $\tilde{\eta}_1 / 2\eta$  of  $-\dot{\pi}^{(\mu\nu)}$  represents exactly the difference between the IReD and DNMR relaxation times. Performing the same steps for the scalar sector, we arrive at

$$\tau_{\Pi} = \tilde{\tau}_{\Pi} + \frac{\tilde{\zeta}_1}{\zeta}, \quad (51a)$$

$$\tau_{\pi} = \tilde{\tau}_{\pi} + \frac{\tilde{\eta}_1}{2\eta}. \quad (51b)$$

In the case of the vector relaxation time, the term  $\tilde{\kappa}_5 \omega^{\mu\nu} I_{\nu}$  must simultaneously account for the change of the relaxation time on the left-hand side (in the term  $\tilde{\tau}_n \dot{n}^{(\mu)}$ ), as well as in the first term appearing in  $\tilde{\mathcal{J}}^{\mu}$ , namely  $\tilde{\tau}_n \omega^{\mu\nu} n_{\nu}$ . Since both terms have equal weights, they get one half of  $\tilde{\kappa}_5 \omega^{\mu\nu} I_{\nu}$  each, such that

$$\tau_n = \tilde{\tau}_n + \frac{\tilde{\kappa}_5}{2\kappa}. \quad (51c)$$

Likewise, the term  $\tilde{\eta}_4 \sigma_{\lambda}^{(\mu} \omega^{\nu)\lambda}$  in  $\tilde{\mathcal{K}}^{\mu\nu}$  acts by changing  $\tilde{\tau}_{\pi}$  in the term  $2\tilde{\tau}_{\pi} \pi_{\lambda}^{(\mu} \omega^{\nu)\lambda}$  appearing in  $\tilde{\mathcal{J}}^{\mu\nu}$ . The resulting relaxation time is indeed equal to  $\tau_{\pi}$  given in Eqs. (51) by virtue of the equality  $\tilde{\eta}_4 = 2\tilde{\eta}_1$  established by Eq. (I22) of Ref. [29].

The relations in (51) can be explicitly checked by noting that [29]

$$\tilde{\zeta}_1 = \sum_{r=0, \neq 1, 2}^{N_0} \tau_{0r}^{(0)} (\zeta_r - \Omega_{r0}^{(0)} \zeta) = \zeta (\tau_{\Pi} - \tilde{\tau}_{\Pi}), \quad (52a)$$

$$\tilde{\kappa}_5 = 2 \sum_{r=0, \neq 1}^{N_1} \tau_{0r}^{(1)} (\kappa_r - \Omega_{r0}^{(1)} \kappa) = 2\kappa (\tau_n - \tilde{\tau}_n), \quad (52b)$$

$$\tilde{\eta}_1 = 2 \sum_{r=0}^{N_2} \tau_{0r}^{(2)} (\eta_r - \Omega_{r0}^{(2)} \eta) = 2\eta (\tau_{\pi} - \tilde{\tau}_{\pi}), \quad (52c)$$

where the DNMR (with tilde) and IReD (without tilde) relaxation times arise by virtue of Eqs. (28) and (38), respectively.

Table II summarizes the connection between the transport coefficients appearing in the IReD and DNMR formulations. While in this section we focused the discussion only on the relaxation times, the procedure to obtain the results reported in Table II is similar in spirit, involving straightforward but tedious algebra, which is sketched in Appendix A.

## V. CONNECTION TO DENICOL *et al.* [39]

In this section, we discuss the connection with Ref. [39], where the parabolic  $\mathcal{K}^{\mu_1 \dots \mu_{\ell}}$  are eliminated in the context of multiple dynamic moments. Without reviewing all the details of this work, we recall only the matching formulas given in Eq. (20) of Ref. [39],

$$\rho_r^{\mu} = \lambda_{r0}^{(1)} n^{\mu} + \lambda_{r2}^{(1)} \rho_2^{\mu}, \quad (53a)$$

$$\rho_r^{\mu\nu} = \lambda_{r0}^{(2)} \pi^{\mu\nu} + \lambda_{r1}^{(2)} \rho_1^{\mu\nu}, \quad (53b)$$

which address only the vector and tensor moments, since the work is focused on massless constituents for which the scalar moments are irrelevant. Equations (53a) and (53b)

TABLE II. Comparison between the transport coefficients arising in the IReD approach (left column) and those arising in the DNMR approach. The partial derivatives are taken by considering  $\alpha = \beta\mu$  and  $\beta$  as independent variables and  $h = (\varepsilon + P)/n$  is the specific enthalpy. The notation  $\mathcal{H}$  and  $\tilde{\mathcal{H}}$  is introduced in Eq. (A2b).

IReD	DNMR
$\tau_{\Pi}$	$\tilde{\tau}_{\Pi} + \frac{\tilde{\zeta}_1}{\zeta}$
$\tau_n$	$\tilde{\tau}_n + \frac{\tilde{\kappa}_5}{2\kappa}$
$\tau_{\pi}$	$\tilde{\tau}_{\pi} + \frac{\tilde{\eta}_1}{2\eta}$
$\ell_{\Pi n}$	$\tilde{\ell}_{\Pi n} - \frac{\tilde{\zeta}_7}{\kappa}$
$\tau_{\Pi n}$	$\tilde{\tau}_{\Pi n} - \frac{\tilde{\zeta}_1 D_{20} \mathcal{H}}{\kappa(\varepsilon + P)^3} - \frac{\tilde{\zeta}_6}{\kappa} - \frac{\tilde{\zeta}_7}{\kappa^2 (\varepsilon + P)} \frac{\partial \kappa}{\partial \ln \beta}$
$\delta_{\Pi \Pi}$	$\tilde{\delta}_{\Pi \Pi} - \frac{\tilde{\zeta}_1}{\zeta^2} \left( \mathcal{H} \frac{\partial \zeta}{\partial \alpha} + \tilde{\mathcal{H}} \frac{\partial \zeta}{\partial \beta} - \frac{\zeta}{3} \right) + \frac{\tilde{\zeta}_3}{\zeta}$
$\lambda_{\Pi n}$	$\tilde{\lambda}_{\Pi n} - \frac{\tilde{\zeta}_4}{\kappa} + \frac{\tilde{\zeta}_7}{\kappa^2} \left( \frac{\partial \kappa}{\partial \alpha} + \frac{1}{h} \frac{\partial \kappa}{\partial \beta} \right)$
$\lambda_{\Pi \pi}$	$\tilde{\lambda}_{\Pi \pi} + \frac{\tilde{\zeta}_1 + \tilde{\zeta}_2}{2\eta}$
$\delta_{nn} + \frac{\zeta}{\kappa} \lambda_{n\Pi}$	$\tilde{\delta}_{nn} + \frac{\zeta}{\kappa} \tilde{\lambda}_{n\Pi} - \frac{\tilde{\kappa}_3}{\kappa} + \frac{\mathcal{H} \tilde{\kappa}_5 + 2\tilde{\kappa}_7}{2\kappa \zeta} \left( \frac{\partial \zeta}{\partial \alpha} + \frac{1}{h} \frac{\partial \zeta}{\partial \beta} \right) - \frac{\tilde{\kappa}_5}{2\kappa^2} \left( \tilde{\mathcal{H}} \frac{\partial \kappa}{\partial \beta} + \frac{\kappa}{h} \frac{\partial \mathcal{H}}{\partial \beta} + \frac{\partial (\kappa \mathcal{H})}{\partial \alpha} - \frac{\kappa}{3} \right)$
$\ell_{n\Pi}$	$\tilde{\ell}_{n\Pi} + \frac{\mathcal{H} \tilde{\kappa}_5 + 2\tilde{\kappa}_7}{2\zeta}$
$\ell_{n\pi}$	$\tilde{\ell}_{n\pi} + \frac{\tilde{\kappa}_6}{2\eta}$
$\tau_{n\Pi}$	$\tilde{\tau}_{n\Pi} - \frac{\tilde{\kappa}_4}{\zeta} - \frac{\mathcal{H} \tilde{\kappa}_5 + 2\tilde{\kappa}_7}{2\zeta^2 (\varepsilon + P)} \frac{\partial \zeta}{\partial \ln \beta} + \frac{\tilde{\kappa}_5 / 2\zeta}{\varepsilon + P} \frac{\partial (\beta \mathcal{H})}{\partial \beta}$
$\tau_{n\pi}$	$\tilde{\tau}_{n\pi} - \frac{\tilde{\kappa}_2}{2\eta} - \frac{\tilde{\kappa}_6}{2\eta^2 (\varepsilon + P)} \frac{\partial \eta}{\partial \ln \beta}$
$\lambda_{nn} + \frac{2\eta}{\kappa} \lambda_{n\pi}$	$\tilde{\lambda}_{nn} + \frac{2\eta}{\kappa} \tilde{\lambda}_{n\pi} - \frac{\tilde{\kappa}_1}{\kappa} + \frac{\tilde{\kappa}_5}{2\kappa} + \frac{\tilde{\kappa}_6}{\eta \kappa} \left( \frac{\partial \eta}{\partial \alpha} + \frac{1}{h} \frac{\partial \eta}{\partial \beta} \right)$
$\delta_{\pi\pi} + \frac{\zeta}{2\eta} \lambda_{\pi\Pi}$	$\tilde{\delta}_{\pi\pi} + \frac{\zeta}{2\eta} \tilde{\lambda}_{\pi\Pi} + \frac{\tilde{\eta}_1}{3\eta} - \frac{\tilde{\eta}_2}{2\eta} - \frac{\tilde{\eta}_1}{2\eta^2} \left( \mathcal{H} \frac{\partial \eta}{\partial \alpha} + \tilde{\mathcal{H}} \frac{\partial \eta}{\partial \beta} \right)$
$\tau_{\pi\pi}$	$\tilde{\tau}_{\pi\pi} + \frac{\tilde{\eta}_1 - \tilde{\eta}_3}{2\eta}$
$\tau_{\pi n}$	$\tilde{\tau}_{\pi n} + \frac{\tilde{\eta}_1 D_{20} \mathcal{H}}{\kappa(\varepsilon + P)^3} - \frac{\tilde{\eta}_7}{\kappa} - \frac{\tilde{\eta}_8}{\kappa^2 (\varepsilon + P)} \frac{\partial \kappa}{\partial \ln \beta}$
$\ell_{\pi n}$	$\tilde{\ell}_{\pi n} + \frac{\tilde{\eta}_8}{\kappa}$
$\lambda_{\pi n}$	$\tilde{\lambda}_{\pi n} + \frac{\tilde{\eta}_5}{\kappa} - \frac{\tilde{\eta}_8}{\kappa^2} \left( \frac{\partial \kappa}{\partial \alpha} + \frac{1}{h} \frac{\partial \kappa}{\partial \beta} \right)$

can be supplemented naturally with an equivalent equation for the scalar moments,

$$\rho_r = -\frac{3}{m^2} \lambda_{r0}^{(0)} \Pi + \lambda_{r3}^{(0)} \rho_3. \quad (53c)$$

The coefficients  $\lambda_{rs}^{(\ell)}$  appearing above are given in Eq. (21) of Ref. [39] for  $\ell = 1, 2$  as

TABLE III. Transport coefficients for the diffusion current  $n^\mu$  arising in the IReD approach for an ultrarelativistic ideal gas interacting via a constant cross section  $\sigma$  for various truncation orders. We use the convention  $N_0 = N_1 + 1 = N_2 + 2$  and the total number of moments is  $5N_2 + 3N_1 + N_0 + 9$ .

Number of moments	$\kappa$	$\tau_n[\lambda_{\text{mfp}}]$	$\delta_{nn}[\tau_n]$	$\lambda_{nn}[\tau_n]$	$\lambda_{n\pi}[\tau_n]$	$\ell_{n\pi}[\tau_n]$	$\tau_{n\pi}[\tau_n]$
14	$3/(16\sigma)$	$9/4$	1	$3/5$	$\beta/20$	$\beta/20$	$\beta/80P$
23	$21/(128\sigma)$	2.0759	1	0.85806	$0.067742\beta$	$0.030645\beta$	$0.0076613\beta/P$
32	$0.16054/\sigma$	2.0761	1	0.88847	$0.069060\beta$	$0.029064\beta$	$0.0072661\beta/P$
41	$0.15959/\sigma$	2.0794	1	0.89501	$0.069240\beta$	$0.028677\beta$	$0.0071692\beta/P$
$\infty$	$0.158925/\sigma$	2.0838	1	0.89862	$0.069273\beta$	$0.028371\beta$	$0.0070927\beta/P$

TABLE IV. Same as Table III for the shear stress  $\pi^{\mu\nu}$ .

Number of moments	$\eta$	$\tau_\pi[\lambda_{\text{mfp}}]$	$\tau_{\pi\pi}[\tau_\pi]$	$\lambda_{\pi\pi}[\tau_\pi]$	$\delta_{\pi\pi}[\tau_\pi]$	$\ell_{\pi\pi}[\tau_\pi]$	$\tau_{\pi\pi}[\tau_\pi]$
14	$4/(3\sigma\beta)$	$5/3$	$10/7$	0	$4/3$	0	0
23	$14/(11\sigma\beta)$	1.6494	1.6850	$0.23622/\beta$	$4/3$	$-0.47244/\beta$	$-0.47244/(\beta P)$
32	$1.2685/(\sigma\beta)$	1.6540	1.6936	$0.21580/\beta$	$4/3$	$-0.54342/\beta$	$-0.54342/(\beta P)$
41	$1.2678/(\sigma\beta)$	1.6552	1.6944	$0.20890/\beta$	$4/3$	$-0.56014/\beta$	$-0.56014/(\beta P)$
$\infty$	$1.2676/(\sigma\beta)$	1.6557	1.6945	$0.20503/\beta$	$4/3$	$-0.56960/\beta$	$-0.56960/(\beta P)$

$$\lambda_{r0}^{(1)} = \frac{\Omega_{20}^{(1)}\kappa_r - \Omega_{r0}^{(1)}\kappa_2}{\Omega_{20}^{(1)}\kappa_0 - \kappa_2}, \quad \lambda_{r2}^{(1)} = \frac{\Omega_{r0}^{(1)}\kappa_0 - \kappa_r}{\Omega_{20}^{(1)}\kappa_0 - \kappa_2}, \quad (54a)$$

$$\lambda_{r0}^{(2)} = \frac{\Omega_{10}^{(2)}\eta_r - \Omega_{r0}^{(2)}\eta_1}{\Omega_{10}^{(2)}\eta_0 - \eta_1}, \quad \lambda_{r1}^{(2)} = \frac{\Omega_{r0}^{(2)}\eta_0 - \eta_r}{\Omega_{10}^{(2)}\eta_0 - \eta_1}. \quad (54b)$$

In the case of the scalar moments, the relevant coefficients read

$$\lambda_{r0}^{(0)} = \frac{\Omega_{30}^{(0)}\zeta_r - \Omega_{r0}^{(0)}\zeta_3}{\Omega_{30}^{(0)}\zeta_0 - \zeta_3}, \quad \lambda_{r3}^{(0)} = \frac{\Omega_{r0}^{(0)}\zeta_0 - \zeta_r}{\Omega_{30}^{(0)}\zeta_0 - \zeta_3}. \quad (54c)$$

As shown in Ref. [39], the above matching prescription succeeds in reproducing  $\mathcal{K} = \mathcal{K}^\mu = \mathcal{K}^{\mu\nu} = 0$ , which is identical to the desideratum of our IReD approach. The connection with the current approach can be established by downgrading the moments  $\rho_3$ ,  $\rho_2^\mu$  and  $\rho_1^{\mu\nu}$  from being dynamical (i.e., separate degrees of freedom) by using the matching formulas  $\rho_3 = -(3/m^2)\mathcal{C}_3^{(0)}\Pi$ ,  $\rho_2^\mu = \mathcal{C}_2^{(1)}n^\mu$  and  $\rho_1^{\mu\nu} = \mathcal{C}_1^{(2)}\pi^{\mu\nu}$  given in Eqs. (35). Noting that

$$\lambda_{r0}^{(0)} + \mathcal{C}_3^{(0)}\lambda_{r3}^{(0)} = \mathcal{C}_r^{(0)}, \quad (55a)$$

$$\lambda_{r0}^{(1)} + \mathcal{C}_2^{(1)}\lambda_{r2}^{(1)} = \mathcal{C}_r^{(1)}, \quad (55b)$$

$$\lambda_{r0}^{(2)} + \mathcal{C}_1^{(2)}\lambda_{r1}^{(2)} = \mathcal{C}_r^{(2)}, \quad (55c)$$

it is clear that Eqs. (53) reduce to Eqs. (35) for all values of  $r$ .

## VI. EXPLICIT VALUES IN THE ULTRARELATIVISTIC LIMIT

We now explicitly evaluate the IReD transport coefficients reported in Appendix B for an ultrarelativistic ideal fluid of hard spheres, interacting via a constant cross section  $\sigma$ . The procedure for performing the calculations is identical to the one introduced in Ref. [1] and will therefore not be repeated here. Following Ref. [1], we report the values of the coefficients obtained by employing 14, 23, 32 and 41 moments. In addition, we report convergence ( $\infty$ ) values for the transport coefficients, which are obtained by employing high-precision arithmetics using MATHEMATICA [45] with  $N_0 - 2 = N_1 - 1 = N_2 = 100$ . The values of the transport coefficients related to the diffusion current  $n^\mu$  and shear stress  $\pi^{\mu\nu}$  are reported in Tables III and IV, respectively. The tables showing these transport coefficients for  $0 \leq N_2 \leq 100$ , as well as the relaxation time and inverse eigenvalues listed in Table I, can be accessed as Supplemental Material [46]. Naturally, we do not report transport coefficients for the bulk viscous pressure  $\Pi$ , since, for massless particles, the bulk sector does not make any contribution.

## VII. CONCLUSION

In this paper, we considered the connection between the transport coefficients arising in the standard DNMR and the IReD approach. We show that the transport coefficients appearing in the  $\mathcal{J}^{\mu_1 \dots \mu_\ell}$  terms [accounting for all  $\mathcal{O}(\text{KnRe}^{-1})$  contributions] receive modifications coming from the original  $\mathcal{K}^{\mu_1 \dots \mu_\ell}$  terms. Moreover, the relaxation times in the IReD approach differ from the DNMR ones, being given as a combination of the DNMR

relaxation time and a second-order transport coefficient coming from the  $\mathcal{K}^{\mu_1 \dots \mu_\ell}$  terms.

In the process of absorbing the  $\mathcal{K}^{\mu_1 \dots \mu_\ell}$  terms, we obtained relaxation times which are no longer constrained to satisfy the separation of scales. In particular, for the case of the ultrarelativistic hard sphere ideal gas, we found that the relaxation times of the dissipative quantities  $\Pi$ ,  $n^\mu$  and  $\pi^{\mu\nu}$  are smaller than those corresponding to higher-order moments. For the same system, we also reported accurate values for all transport coefficients (corresponding to the limits  $N_0, N_1, N_2 \rightarrow \infty$ ) appearing in the vector and tensor sectors.

Due to their parabolic nature, the  $\mathcal{K}^{\mu_1 \dots \mu_\ell}$  terms which are quadratic in Kn may lead to violations of causality, as pointed out in Refs. [1,39], and are therefore customarily omitted. Our work provides the foundation for hydrodynamical theories which are free of such terms, while retaining second-order accuracy with respect to Kn and  $\text{Re}^{-1}$ . The absence of parabolic terms in the IReD approach may help in deriving the entropy current from kinetic theory. Such an analysis was performed in the 14-moment approximation [19–21,23], where the parabolic terms are absent also in the DNMR approach. Extending the analysis beyond 14 moments (e.g., when  $N_\ell \rightarrow \infty$ ) remains an open problem representing an interesting avenue for future research.

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## APPENDIX A: EQUIVALENCE BETWEEN IReD AND DNMR

In this appendix we report the calculations leading to Table II. We will manipulate the terms appearing in the  $\tilde{\mathcal{K}}$ ,  $\tilde{\mathcal{K}}^\mu$  and  $\tilde{\mathcal{K}}^{\mu\nu}$  terms (30) with the purpose of absorbing them into the corresponding  $\tilde{\mathcal{J}}^{\mu_1 \dots \mu_\ell}$  terms, thus inferring the connection to the coefficients obtained in the IReD

approach. We will employ the same notation as in Sec. IV, by which the DNMR quantities will be denoted with a tilde  $\tilde{\cdot}$ . The main idea is to trade one power of Kn for one power of  $\text{Re}^{-1}$ . This is done using the Navier-Stokes asymptotic matching (47) between the thermodynamic fluxes  $\Pi$ ,  $n^\mu$  and  $\pi^{\mu\nu}$  and the thermodynamic forces  $\theta$ ,  $I^\mu$  and  $\sigma^{\mu\nu}$ .

As already mentioned in Sec. IV, all terms appearing in  $\tilde{\mathcal{K}}^{\mu_1 \dots \mu_\ell}$  can be related to those appearing in  $\tilde{\mathcal{J}}^{\mu_1 \dots \mu_\ell}$ , with the exception of  $\tilde{\zeta}_1 \omega_{\mu\nu} \omega^{\mu\nu}$  and  $\tilde{\eta}_1 \omega_\lambda^{(\mu} \omega^{\nu)\lambda}$  appearing in  $\tilde{\mathcal{K}}$  and  $\tilde{\mathcal{K}}^{\mu\nu}$ , respectively. We also include here the  $\tilde{\kappa}_5 \omega^{\mu\nu} I_\nu$  term for reasons that will become apparent. These terms can be related with the comoving derivatives of the thermodynamic forces, as suggested in Eqs. (49). We start this section by deriving this latter equation.

We first recall Eqs. (39)–(41) from Ref. [1],

$$\dot{\alpha} = \mathcal{H}\theta + \frac{J_{20}\Pi\theta - J_{30}\partial_\mu n^\mu}{D_{20}} - \frac{J_{20}}{D_{20}} \pi^{\mu\nu} \sigma_{\mu\nu}, \quad (\text{A1a})$$

$$\dot{\beta} = \tilde{\mathcal{H}}\theta + \frac{J_{10}\Pi\theta - J_{20}\partial_\mu n^\mu}{D_{20}} - \frac{J_{10}}{D_{20}} \pi^{\mu\nu} \sigma_{\mu\nu}, \quad (\text{A1b})$$

$$\dot{i}^\mu = \frac{F^\mu + \nabla^\mu \Pi - \Delta_\alpha^\mu \nabla_\beta \pi^{\alpha\beta} - \Pi \dot{i}^\mu - \pi^{\mu\nu} \dot{i}_\nu}{\varepsilon + P}, \quad (\text{A1c})$$

where  $\mathcal{H}$  [introduced in Eq. (I18) of Ref. [29]] and  $\tilde{\mathcal{H}}$  are defined as

$$\mathcal{H} = \frac{J_{20}(\varepsilon + P) - J_{30}n}{D_{20}}, \quad (\text{A2a})$$

$$\tilde{\mathcal{H}} = \frac{J_{10}(\varepsilon + P) - J_{20}n}{D_{20}}, \quad (\text{A2b})$$

while  $J_{nq}$  and  $D_{nq}$  are introduced above Eq. (9).

The comoving derivative of  $\theta = \partial_\mu u^\mu$  can be computed as follows:

$$\dot{\theta} = \partial_\mu \dot{i}^\mu - (\partial_\mu u_\lambda)(\partial^\lambda u^\mu). \quad (\text{A3})$$

Noting that  $\partial_\mu \dot{i}^\mu = \nabla_\mu \dot{i}^\mu - \dot{i}_\mu \dot{i}^\mu$  and  $(\partial_\mu u_\lambda)(\partial^\lambda u^\mu) = (\nabla_\mu u_\lambda)(\nabla^\lambda u^\mu)$ , we find

$$\dot{\theta} = -\dot{i} \cdot \dot{i} + \nabla_\alpha \dot{i}^\alpha - (\nabla_\alpha u^\rho)(\nabla_\rho u^\alpha). \quad (\text{A4})$$

In the case of  $I^\mu = \nabla^\mu \alpha$ , the comoving derivative gives

$$\dot{I}^\mu = \dot{\Delta}_\nu^\mu \partial^\nu \alpha + \nabla^\mu \dot{\alpha} - (\nabla^\mu u^\nu)(\partial_\nu \alpha). \quad (\text{A5})$$

Projecting the above using  $\Delta_\nu^\mu$  and noting that  $\Delta_\nu^\mu \dot{\Delta}_\lambda^\nu = -\dot{i}^\mu u_\lambda$ , we arrive at

$$\dot{I}^{(\mu)} = -\dot{i}^\mu \dot{\alpha} + \nabla^\mu \dot{\alpha} - (\nabla^\mu u^\nu) I_\nu. \quad (\text{A6})$$



Finally, the comoving derivative of  $\sigma^{\mu\nu} = \Delta_{\alpha\beta}^{\mu\nu}\partial^\alpha u^\beta$  can be written as

$$\dot{\sigma}^{\mu\nu} = \dot{\Delta}_{\alpha\beta}^{\mu\nu}\partial^\alpha u^\beta + \nabla^{(\mu} \dot{u}^{\nu)} - \Delta_{\alpha\beta}^{\mu\nu}(\partial^\alpha u^\lambda)(\partial_\lambda u^\beta). \quad (\text{A7})$$

Using  $\nabla^\alpha u^\rho = \sigma^{\alpha\rho} + \omega^{\alpha\rho} + \frac{1}{3}\theta\Delta^{\alpha\rho}$ , the last term can be expressed as

$$\Delta_{\alpha\beta}^{\mu\nu}(\partial^\alpha u^\lambda)(\partial_\lambda u^\beta) = \sigma^{\lambda(\mu} \sigma_{\lambda}^{\nu)} + \omega^{\lambda(\mu} \omega_{\lambda}^{\nu)} + \frac{2}{3}\sigma^{\mu\nu}\theta. \quad (\text{A8})$$

Projecting Eq. (A7) using  $\Delta_{\alpha\beta}^{\mu\nu}$  and using  $\dot{\Delta}_{\alpha\beta}^{\langle\mu\nu\rangle} = \dot{\Delta}_{\alpha\lambda}^{\mu\nu}u^\lambda u_\beta + \dot{\Delta}_{\beta\lambda}^{\mu\nu}u^\lambda u_\alpha$ , we arrive at

$$\dot{\sigma}^{\langle\mu\nu\rangle} = -\dot{u}^{\langle\mu} \dot{u}^{\nu\rangle} + \nabla^{\langle\mu} \dot{u}^{\nu\rangle} - \sigma^{\lambda(\mu} \dot{\sigma}_{\lambda}^{\nu)} - \omega^{\lambda(\mu} \dot{\omega}_{\lambda}^{\nu)} - \frac{2}{3}\sigma^{\mu\nu}\theta, \quad (\text{A9})$$

where we also used the property  $\dot{\Delta}_{\alpha\beta}^{\mu\nu}\partial^\alpha u^\beta = \dot{\Delta}_{\alpha\beta}^{\mu\nu}\nabla^\alpha u^\beta - \dot{u}^{\langle\mu} \dot{u}^{\nu\rangle}$ .

Using Eqs. (A1) to leading order in Kn and  $\text{Re}^{-1}$  leads to

$$\begin{aligned} \dot{\theta} &= \omega^{\mu\nu} \omega_{\mu\nu} - \sigma^{\mu\nu} \sigma_{\mu\nu} - \frac{1}{3}\theta^2 - \frac{2(\varepsilon + P) + \beta J_{30}}{(\varepsilon + P)^3} F \cdot F \\ &\quad - \frac{D_{20}\mathcal{H}}{(\varepsilon + P)^3} I \cdot F + \frac{\nabla \cdot F}{\varepsilon + P}, \\ \dot{I}^{\langle\mu\rangle} &= -\sigma^{\mu\nu} I_\nu + I^\mu \theta \left( \frac{\partial \mathcal{H}}{\partial \alpha} + \frac{1}{h} \frac{\partial \mathcal{H}}{\partial \beta} - \frac{1}{3} \right) - \frac{F^\mu \theta}{\varepsilon + P} \frac{\partial(\beta \mathcal{H})}{\partial \beta} \\ &\quad - \omega^{\mu\nu} I_\nu + \mathcal{H} \nabla^\mu \theta, \\ \dot{\sigma}^{\langle\mu\nu\rangle} &= -\omega^{\lambda(\mu} \dot{\omega}_{\lambda}^{\nu)} - \frac{2}{3}\theta \sigma^{\mu\nu} - \sigma^{\lambda(\mu} \dot{\sigma}_{\lambda}^{\nu)} \\ &\quad - \frac{2(\varepsilon + P) + \beta J_{30}}{(\varepsilon + P)^3} F^{\langle\mu} F^{\nu\rangle} - \frac{D_{20}\mathcal{H}}{(\varepsilon + P)^3} I^{\langle\mu} F^{\nu\rangle} \\ &\quad + \frac{\nabla^{\langle\mu} F^{\nu\rangle}}{\varepsilon + P}. \end{aligned} \quad (\text{A10})$$

Using Eqs. (A10) to replace  $\omega^{\mu\nu} \omega_{\mu\nu}$ ,  $\omega^{\mu\alpha} I_\alpha$ , and  $\omega^{\lambda(\mu} \dot{\omega}_{\lambda}^{\nu)}$  in Eqs. (30) gives

$$\begin{aligned} \tilde{\mathcal{K}} &= \tilde{\zeta}_1 \dot{\theta} + (\tilde{\zeta}_2 + \tilde{\zeta}_1) \sigma_{\mu\nu} \sigma^{\mu\nu} + \left( \tilde{\zeta}_3 + \frac{1}{3} \tilde{\zeta}_1 \right) \theta^2 \\ &\quad + \tilde{\zeta}_4 I \cdot I + \left( \tilde{\zeta}_5 + \tilde{\zeta}_1 \frac{2(\varepsilon + P) + \beta J_{30}}{(\varepsilon + P)^3} \right) F \cdot F \\ &\quad + \left( \tilde{\zeta}_6 + \tilde{\zeta}_1 \frac{D_{20}\mathcal{H}}{(\varepsilon + P)^3} \right) I \cdot F + \tilde{\zeta}_7 \nabla \cdot I \\ &\quad + \left( \tilde{\zeta}_8 - \frac{\tilde{\zeta}_1}{\varepsilon + P} \right) \nabla \cdot F, \end{aligned} \quad (\text{A11a})$$

$$\begin{aligned} \tilde{\mathcal{K}}^\mu &= -\frac{\tilde{\kappa}_5}{2} \dot{I}^\mu + \left( \tilde{\kappa}_1 - \frac{\tilde{\kappa}_5}{2} \right) \sigma^{\mu\nu} I_\nu + \tilde{\kappa}_2 \sigma^{\mu\nu} F_\nu \\ &\quad + \left[ \tilde{\kappa}_3 + \frac{\tilde{\kappa}_5}{2} \left( \frac{\partial \mathcal{H}}{\partial \alpha} + \frac{1}{h} \frac{\partial \mathcal{H}}{\partial \beta} - \frac{1}{3} \right) \right] I^\mu \theta \\ &\quad + \left[ \tilde{\kappa}_4 - \frac{\tilde{\kappa}_5}{2(\varepsilon + P)} \frac{\partial(\beta \mathcal{H})}{\partial \beta} \right] F^\mu \theta + \frac{\tilde{\kappa}_5}{2} \omega^{\mu\nu} I_\nu \\ &\quad + \tilde{\kappa}_6 \Delta_\lambda^\mu \nabla_\nu \sigma^{\lambda\nu} + \left( \tilde{\kappa}_7 + \frac{\tilde{\kappa}_5}{2} \mathcal{H} \right) \nabla^\mu \theta, \end{aligned} \quad (\text{A11b})$$

$$\begin{aligned} \tilde{\mathcal{K}}^{\mu\nu} &= -\tilde{\eta}_1 \dot{\sigma}^{\langle\mu\nu\rangle} + \left( \tilde{\eta}_2 - \frac{2}{3} \tilde{\eta}_1 \right) \theta \sigma^{\mu\nu} \\ &\quad + (\tilde{\eta}_3 - \tilde{\eta}_1) \sigma^{\lambda(\mu} \dot{\sigma}_{\lambda}^{\nu)} + \tilde{\eta}_4 \sigma_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda} + \tilde{\eta}_5 I^{\langle\mu} I^{\nu\rangle} \\ &\quad + \left( \tilde{\eta}_6 - \tilde{\eta}_1 \frac{2(\varepsilon + P) + \beta J_{30}}{(\varepsilon + P)^3} \right) F^{\langle\mu} F^{\nu\rangle} \\ &\quad + \left( \tilde{\eta}_7 - \tilde{\eta}_1 \frac{D_{20}\mathcal{H}}{(\varepsilon + P)^3} \right) I^{\langle\mu} F^{\nu\rangle} \\ &\quad + \tilde{\eta}_8 \nabla^{\langle\mu} I^{\nu\rangle} + \left( \tilde{\eta}_9 + \frac{\tilde{\eta}_1}{\varepsilon + P} \right) \nabla^{\langle\mu} F^{\nu\rangle}. \end{aligned} \quad (\text{A11c})$$

Using the relations (15) and (18) in Ref. [29] relating  $\tilde{\zeta}_5$  and  $\tilde{\zeta}_8$  to  $\tilde{\zeta}_1$ , one can see that the coefficients in front of  $F \cdot F$  and  $\nabla \cdot F$  vanish identically. Similarly, the relations (I24) and (I27) in Ref. [29] between  $\tilde{\eta}_6$ ,  $\tilde{\eta}_9$  and  $\tilde{\eta}_1$  imply that the coefficients in front of  $F^{\langle\mu} F^{\nu\rangle}$  and  $\nabla^{\langle\mu} F^{\nu\rangle}$  also vanish. This is consistent with, and indeed required by, the equivalence between the IReD and DNMR approaches, since no such terms appear in either  $\mathcal{J}$  or  $\mathcal{J}^{\mu\nu}$ . For this reason, the coefficients  $\tilde{\zeta}_5$ ,  $\tilde{\zeta}_8$ ,  $\tilde{\eta}_6$  and  $\tilde{\eta}_9$  do not appear in Table II.

Comparing the above to Eqs. (30), it can be seen that aside from the new terms proportional to  $\theta$ ,  $\dot{I}^{\langle\mu\rangle}$  and  $\dot{\sigma}^{\langle\mu\nu\rangle}$ , the coefficients of these terms ( $\tilde{\zeta}_1$ ,  $\tilde{\kappa}_5$  and  $\tilde{\eta}_1$ ) appear in several other terms. To compare with the coefficients obtained in the IReD approach, the thermodynamic forces  $\theta$ ,  $I^\mu$  and  $\sigma^{\mu\nu}$  can be expressed in terms of the thermodynamic fluxes  $\Pi$ ,  $n^\mu$  and  $\pi^{\mu\nu}$  via the asymptotic Navier-Stokes constitutive relations in Eqs. (47). During this procedure, the comoving derivatives of the thermodynamic forces give rise to comoving derivatives of the thermodynamic fluxes, as well as to derivatives of the transport coefficients:

$$\dot{\theta} = -\frac{1}{\zeta} \dot{\Pi} + \frac{\Pi}{\zeta^2} \dot{\zeta}, \quad (\text{A12a})$$

$$\dot{I}^{\langle\mu\rangle} = \frac{1}{\kappa} \dot{n}^{\langle\mu\rangle} - \frac{n^\mu}{\kappa^2} \dot{\kappa}, \quad (\text{A12b})$$

$$\dot{\sigma}^{\langle\mu\nu\rangle} = \frac{1}{2\eta} \dot{\pi}^{\langle\mu\nu\rangle} - \frac{\pi^{\mu\nu}}{2\eta^2} \dot{\eta}, \quad (\text{A12c})$$

where the comoving derivative of a function depending on the fluid properties  $\beta$  and  $\alpha$  can be computed via Eq. (32).

The emergence of comoving derivatives of the thermodynamic forces in Eqs. (A12) leads to modifications of the relaxation times  $\tau_{\Pi}$ ,  $\tau_n$  and  $\tau_\pi$ , as indicated in Eqs. (51). Furthermore, since the quantities in  $\tilde{\mathcal{K}}^{\mu_1 \dots \mu_\ell}$  are of second order in Kn, the matching in Eqs. (47) reduces them to quantities of order  $\mathcal{O}(\text{KnRe}^{-1})$ , which are then absorbed into the  $\tilde{\mathcal{J}}^{\mu_1 \dots \mu_\ell}$  terms. By this procedure, the original transport coefficients appearing in  $\tilde{\mathcal{J}}^{\mu_1 \dots \mu_\ell}$  are modified. Since the procedure stays accurate at second order with respect to Kn and  $\text{Re}^{-1}$ , the modified transport coefficients must exactly agree with those obtained in the IReD approach. To illustrate the connection between the original and the modified transport coefficients, let us focus on some examples concerning the terms in  $\tilde{\mathcal{K}}^\mu$ . Starting from

$$\tilde{\kappa}_6 \Delta^\mu_\lambda \nabla_\nu \sigma^{\lambda\nu} \simeq \frac{\tilde{\kappa}_6}{2\eta} \Delta^\mu_\lambda \nabla_\nu \pi^{\lambda\nu} - \tilde{\kappa}_6 \frac{\pi^{\mu\nu}}{2\eta^2} \nabla_\nu \eta, \quad (\text{A13})$$

it can be seen that the first term is of the same form as  $\tilde{\ell}_{n\pi} \Delta^\mu_\lambda \nabla_\nu \pi^{\lambda\nu}$  and will thus lead to the following modification of this transport coefficient:

$$\ell_{n\pi} = \tilde{\ell}_{n\pi} + \frac{\tilde{\kappa}_6}{2\eta}. \quad (\text{A14})$$

The relation above can be validated using the explicit expressions for  $\tilde{\ell}_{n\pi}$  and  $\tilde{\kappa}_6$ , which we reproduce from Eq. (C9) in Ref. [1] and Eq. (I15) in Ref. [29], respectively,

$$\begin{aligned} \tilde{\ell}_{n\pi} &= -\tau_{00}^{(1)} \gamma_1^{(2)} + \sum_{r=0, \neq 1}^{N_1} \tau_{0r}^{(1)} \frac{\beta J_{r+2,1}}{\varepsilon + P} \\ &\quad - \sum_{r=0}^{N_1-2} \tau_{0,r+2}^{(1)} \Omega_{r+1,0}^{(2)}, \end{aligned} \quad (\text{A15a})$$

$$\tilde{\kappa}_6 = -2 \sum_{r=1}^{N_1-1} \tau_{0,r+1}^{(1)} (\eta_r - \Omega_{r0}^{(2)} \eta). \quad (\text{A15b})$$

Replacing  $\gamma_1^{(2)}$  in the expression for  $\tilde{\ell}_{n\pi}$  with  $\tilde{\gamma}_1^{(2)}$  defined in Eqs. (25) (see also the discussion around this equation), we arrive at

$$\begin{aligned} \tilde{\ell}_{n\pi} + \frac{\tilde{\kappa}_6}{2\eta} &= -\tau_{00}^{(1)} \tilde{\gamma}_1^{(2)} + \sum_{r=0, \neq 1}^{N_1} \tau_{0r}^{(1)} \frac{\beta J_{r+2,1}}{\varepsilon + P} \\ &\quad - \sum_{r=0}^{N_1-2} \tau_{0,r+2}^{(1)} \mathcal{C}_{r+1}^{(2)}, \end{aligned} \quad (\text{A16})$$

which is exactly the expression for  $\ell_{n\pi}$  following the identification given in Eqs. (37) applied to Eq. (A15a) [see also Eq. (B8)].

The second term in Eq. (A13) gives rise to

$$\pi^{\mu\nu} \nabla_\nu \eta = -\frac{\pi^{\mu\nu} F_\nu}{\varepsilon + P} \frac{\partial \eta}{\partial \ln \beta} + \pi^{\mu\nu} I_\nu \left( \frac{\partial \eta}{\partial \alpha} + \frac{1}{h} \frac{\partial \eta}{\partial \beta} \right). \quad (\text{A17})$$

The terms on the right-hand side have the same form as the terms  $-\tilde{\lambda}_{n\pi} \pi^{\mu\nu} I_\nu$  and  $-\tilde{\tau}_{n\pi} \pi^{\mu\nu} F_\nu$  appearing in  $\tilde{\mathcal{J}}^\mu$ , thus leading to a modification to these latter two transport coefficients ( $\tilde{\lambda}_{n\pi}$  and  $\tilde{\tau}_{n\pi}$ ).

It is worth noting that using the above procedure may lead to ambiguities. To illustrate such situations, let us focus on the term  $\tilde{\kappa}_1 \sigma^{\mu\nu} I_\nu$ , which can contribute to both  $-\tilde{\lambda}_{n\pi} \pi^{\mu\nu} I_\nu$  and to  $-\tilde{\lambda}_{nn} n_\nu \sigma^{\mu\nu}$ , since

$$\sigma^{\mu\nu} I_\nu \simeq \frac{\pi^{\mu\nu}}{2\eta} I_\nu \simeq \sigma^{\mu\nu} \frac{n_\nu}{\kappa}. \quad (\text{A18})$$

Taking the first equality would modify only  $\tilde{\lambda}_{n\pi}$ , whereas taking the second equality modifies  $\tilde{\lambda}_{nn}$ . The decision on how to distribute the contribution from  $\tilde{\kappa}_1$  to  $\tilde{\lambda}_{n\pi}$  and  $\tilde{\lambda}_{nn}$  can in principle be made by looking at the explicit expression for  $\tilde{\kappa}_1$ , reported in Eq. (I10) of Ref. [29]. Another possibility is to acknowledge that this apparent ambiguity can be identified also in the form of  $\tilde{\mathcal{J}}^\mu$ , allowing the two terms  $\tilde{\lambda}_{n\pi} \pi^{\mu\nu} I_\nu$  and  $\tilde{\lambda}_{nn} \sigma^{\mu\nu} n_\nu$  to be merged into a single one:

$$\begin{aligned} \tilde{\lambda}_{nn} \sigma^{\mu\nu} n_\nu + \tilde{\lambda}_{n\pi} \pi^{\mu\nu} I_\nu &\simeq \left( \frac{\kappa}{2\eta} \tilde{\lambda}_{nn} + \tilde{\lambda}_{n\pi} \right) \pi^{\mu\nu} I_\nu \\ &\simeq \left( \tilde{\lambda}_{nn} + \frac{2\eta}{\kappa} \tilde{\lambda}_{n\pi} \right) \sigma^{\mu\nu} n_\nu. \end{aligned} \quad (\text{A19})$$

Choosing to express all terms in the form  $\sigma^{\mu\nu} n_\nu$ , we obtain

$$\begin{aligned} \lambda_{nn} + \frac{2\eta}{\kappa} \lambda_{n\pi} &= \tilde{\lambda}_{nn} + \frac{2\eta}{\kappa} \tilde{\lambda}_{n\pi} + \frac{\tilde{\kappa}_6}{\eta \kappa} \left( \frac{\partial \eta}{\partial \alpha} + \frac{1}{h} \frac{\partial \eta}{\partial \beta} \right) \\ &\quad - \frac{1}{\kappa} \left( \tilde{\kappa}_1 - \frac{\tilde{\kappa}_5}{2} \right). \end{aligned} \quad (\text{A20})$$

The above discussion summarizes the key points required to obtain the relations presented in Table II. While Eqs. (A14) and (A20) refer only to the modifications of the  $\ell_{n\pi}$ ,  $\lambda_{nn}$  and  $\lambda_{n\pi}$  coefficients, the relations involving the other coefficients can be derived following the same steps using straightforward but lengthy algebra, which we do not present here explicitly.

## APPENDIX B: SECOND-ORDER TRANSPORT COEFFICIENTS IN THE IReD APPROACH

In this appendix we give the transport coefficients of the IReD formalism. In what follows, we identify  $\mathcal{C}_{-n}^{(\ell)} = \tilde{\gamma}_n$ , as in Eq. (36). For the bulk pressure we have

$$\ell_{\Pi n} = -\frac{m^2}{3} \sum_{r=0, \neq 1, 2}^{N_0} \tau_{0r}^{(0)} \left( \mathcal{C}_{r-1}^{(1)} - \frac{G_{3r}}{D_{20}} \right), \quad (\text{B1})$$

$$\tau_{\Pi n} = \sum_{r=0, \neq 1, 2}^{N_0} \frac{m^2 \tau_{0r}^{(0)}}{3(\varepsilon + P)} \left( r \mathcal{C}_{r-1}^{(1)} + \frac{\partial \mathcal{C}_{r-1}^{(1)}}{\partial \ln \beta} - \frac{G_{3r}}{D_{20}} \right), \quad (\text{B2})$$

$$\delta_{\Pi \Pi} = \sum_{r=0, \neq 1, 2}^{N_0} \tau_{0r}^{(0)} \left[ \frac{r+2}{3} \mathcal{C}_r^{(0)} + \mathcal{H} \frac{\partial \mathcal{C}_r^{(0)}}{\partial \alpha} + \bar{\mathcal{H}} \frac{\partial \mathcal{C}_r^{(0)}}{\partial \beta} - \frac{m^2}{3} (r-1) \mathcal{C}_{r-2}^{(0)} - \frac{m^2}{3} \frac{G_{2r}}{D_{20}} \right], \quad (\text{B3})$$

$$\lambda_{\Pi n} = -\frac{m^2}{3} \sum_{r=0, \neq 1, 2}^{N_0} \tau_{0r}^{(0)} \left( \frac{\partial \mathcal{C}_{r-1}^{(1)}}{\partial \alpha} + \frac{1}{h} \frac{\partial \mathcal{C}_{r-1}^{(1)}}{\partial \beta} \right), \quad (\text{B4})$$

$$\lambda_{\Pi \pi} = -\frac{m^2}{3} \sum_{r=0, \neq 1, 2}^{N_0} \tau_{0r}^{(0)} \left[ \frac{G_{2r}}{D_{20}} + (r-1) \mathcal{C}_{r-2}^{(2)} \right]. \quad (\text{B5})$$

For the particle-diffusion current

$$\delta_{nm} = \sum_{r=0, \neq 1}^{N_1} \tau_{0r}^{(1)} \left[ \frac{r+3}{3} \mathcal{C}_r^{(1)} + \mathcal{H} \frac{\partial \mathcal{C}_r^{(1)}}{\partial \alpha} + \bar{\mathcal{H}} \frac{\partial \mathcal{C}_r^{(1)}}{\partial \beta} - \frac{m^2}{3} (r-1) \mathcal{C}_{r-2}^{(1)} \right], \quad (\text{B6})$$

$$\ell_{n\Pi} = \sum_{r=0, \neq 1}^{N_1} \tau_{0r}^{(1)} \left( \frac{\beta J_{r+2,1}}{\varepsilon + P} - \mathcal{C}_{r-1}^{(0)} + \frac{1}{m^2} \mathcal{C}_{r+1}^{(0)} \right), \quad (\text{B7})$$

$$\ell_{n\pi} = \sum_{r=0, \neq 1}^{N_1} \tau_{0r}^{(1)} \left( \frac{\beta J_{r+2,1}}{\varepsilon + P} - \mathcal{C}_{r-1}^{(2)} \right), \quad (\text{B8})$$

$$\tau_{n\Pi} = \sum_{r=0, \neq 1}^{N_1} \frac{\tau_{0r}^{(1)}}{\varepsilon + P} \left[ \frac{\beta J_{r+2,1}}{\varepsilon + P} - r \mathcal{C}_{r-1}^{(0)} + \frac{1}{m^2} (r+3) \mathcal{C}_{r+1}^{(0)} - \frac{\partial \mathcal{C}_{r-1}^{(0)}}{\partial \ln \beta} + \frac{1}{m^2} \frac{\partial \mathcal{C}_{r+1}^{(0)}}{\partial \ln \beta} \right], \quad (\text{B9})$$

$$\tau_{n\pi} = \sum_{r=0, \neq 1}^{N_1} \frac{\tau_{0r}^{(1)}}{\varepsilon + P} \left( \frac{\beta J_{r+2,1}}{\varepsilon + P} - r \mathcal{C}_{r-1}^{(2)} - \frac{\partial \mathcal{C}_{r-1}^{(2)}}{\partial \ln \beta} \right), \quad (\text{B10})$$

$$\lambda_{nn} = \sum_{r=0, \neq 1}^{N_1} \frac{\tau_{0r}^{(1)}}{5} [(2r+3) \mathcal{C}_r^{(1)} - 2m^2 (r-1) \mathcal{C}_{r-2}^{(1)}], \quad (\text{B11})$$

$$\lambda_{n\Pi} = \sum_{r=0, \neq 1}^{N_1} \tau_{0r}^{(1)} \left[ \left( \frac{\partial \mathcal{C}_{r-1}^{(0)}}{\partial \alpha} + \frac{1}{h} \frac{\partial \mathcal{C}_{r-1}^{(0)}}{\partial \beta} \right) \right. \quad (\text{B12})$$

$$\left. - \frac{1}{m^2} \left( \frac{\partial \mathcal{C}_{r+1}^{(0)}}{\partial \alpha} + \frac{1}{h} \frac{\partial \mathcal{C}_{r+1}^{(0)}}{\partial \beta} \right) \right], \quad (\text{B13})$$

$$\lambda_{n\pi} = \sum_{r=0, \neq 1}^{N_1} \tau_{0r}^{(1)} \left( \frac{\partial \mathcal{C}_{r-1}^{(2)}}{\partial \alpha} + \frac{1}{h} \frac{\partial \mathcal{C}_{r-1}^{(2)}}{\partial \beta} \right). \quad (\text{B14})$$

Finally, for the shear-stress tensor we have

$$\delta_{\pi\pi} = \sum_{r=0}^{N_2} \tau_{0r}^{(2)} \left[ \frac{r+4}{3} \mathcal{C}_r^{(2)} + \mathcal{H} \frac{\partial \mathcal{C}_r^{(2)}}{\partial \alpha} + \bar{\mathcal{H}} \frac{\partial \mathcal{C}_r^{(2)}}{\partial \beta} - \frac{m^2}{3} (r-1) \mathcal{C}_{r-2}^{(2)} \right], \quad (\text{B15})$$

$$\tau_{\pi\pi} = \frac{2}{7} \sum_{r=0}^{N_2} \tau_{0r}^{(2)} [(2r+5) \mathcal{C}_r^{(2)} - 2m^2 (r-1) \mathcal{C}_{r-2}^{(2)}], \quad (\text{B16})$$

$$\lambda_{\pi\Pi} = -\frac{2}{5m^2} \sum_{r=0}^{N_2} \tau_{0r}^{(2)} [(r+4) \mathcal{C}_{r+2}^{(0)} - m^2 (2r+3) \mathcal{C}_r^{(0)} + m^4 (r-1) \mathcal{C}_{r-2}^{(0)}], \quad (\text{B17})$$

$$\tau_{\pi n} = \frac{2}{5(\varepsilon + P)} \sum_{r=0}^{N_2} \tau_{0r}^{(2)} \left[ (r+5) \mathcal{C}_{r+1}^{(1)} - m^2 r \mathcal{C}_{r-1}^{(1)} + \frac{\partial \mathcal{C}_{r+1}^{(1)}}{\partial \ln \beta} - m^2 \frac{\partial \mathcal{C}_{r-1}^{(1)}}{\partial \ln \beta} \right], \quad (\text{B18})$$

$$\ell_{\pi n} = \frac{2}{5} \sum_{r=0}^{N_2} \tau_{0r}^{(2)} (\mathcal{C}_{r+1}^{(1)} - m^2 \mathcal{C}_{r-1}^{(1)}), \quad (\text{B19})$$

$$\lambda_{\pi n} = \frac{2}{5} \sum_{r=0}^{N_2} \tau_{0r}^{(2)} \left[ \left( \frac{\partial \mathcal{C}_{r+1}^{(1)}}{\partial \alpha} + \frac{1}{h} \frac{\partial \mathcal{C}_{r+1}^{(1)}}{\partial \beta} \right) - m^2 \left( \frac{\partial \mathcal{C}_{r-1}^{(1)}}{\partial \alpha} + \frac{1}{h} \frac{\partial \mathcal{C}_{r-1}^{(1)}}{\partial \beta} \right) \right]. \quad (\text{B20})$$

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