# $\mathcal{PT}$ symmetric fermionic field theories with axions: Renormalization and dynamical mass generation

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We consider the renormalization properties of non-Hermitian Yukawa theories involving a pseudoscalar (axion) field at or near four dimensions. The non-Hermiticity is $\mathcal{PT}$ symmetric where $\mathcal{P}$ is a linear operator (such as parity) and $\mathcal{T}$ is an antilinear idempotent operator (such as time reversal). The coupling constants of the Yukawa and quartic scalar coupling terms reflect this non-Hermiticity. The path integral representing the field theory is used to discuss the Feynman rules associated with the field theory. The fixed point structure associated with the renormalization group has $\mathcal{PT}$ symmetric and Hermitian fixed points. At two loops in the massless theory, we demonstrate the flow from Hermitian to non-Hermitian fixed points. From the one-loop renormalization of a massive Yukawa theory, a self-consistent Nambu–Jona-Lasinio gap equation is established and its real solutions are discussed.

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## I. INTRODUCTION

Quantum field theories have provided successful theories of fundamental (high-energy particle) physics, of condensed matter, and (of aspects) of gravitational physics. In order to go beyond the field-theoretic description provided by the Standard Model (SM) of particle physics, it is necessary to extend the framework. Two such extensions are

(i) First, within the familiar quantum mechanical assumption of Hermiticity, the SM framework is embedded in more general approaches, exemplified by grand unified theories, supersymmetry, supergravity, and string/brane theory (in higher spatial dimensions).

(ii) Second, starting in 1998 [1], in the context of quantum mechanics (one-dimensional quantum field theory), non-Hermitian $\mathcal{PT}$ symmetric theories [2] were shown to allow unitary time evolution.[1] $\mathcal{P}$ is a linear idempotent operator (such as parity) and $\mathcal{T}$ is an antilinear idempotent operator (such as time reversal). In $\mathcal{PT}$ symmetric quantum mechanics the energy eigenvalues are real and bounded below. This development has led to the study of $\mathcal{PT}$ symmetric quantum field theories [3–12].[2]

A major driver for the explosion of interest in quantum mechanical $\mathcal{PT}$ symmetry has been the massive activity, both theoretical and experimental, in material science and optics [15].

It is the purpose of this work to discuss the foundations of a $(3+1)$-dimensional $\mathcal{PT}$ symmetric quantum field theory of fermion and axion fields, from the point of view of its renormalization and dynamical mass generation for both axions and fermions. The structure of the article is as follows: in Sec. II we set up in detail the path-integral formalism describing the quantum field theory of our model, paying special attention to its $\mathcal{PT}$ symmetric nature. In Sec. III, we present the renormalization, in $(3+1)$ dimensions, of our field-theoretic model, which involves a chiral Yukawa interaction of a fermion field $\psi$ with a pseudoscalar (axionlike) field $\phi$ in the presence of a quartic self-interaction for $\phi$. The fields have bare masses. In Sec. IV, we discuss the renormalization group for this massive Yukawa theory, allowing for appropriately defined non-Hermitian fixed points in the space of couplings of the

[1]A new inner product on the Hilbert space is used, which replaces the conventional Dirac inner product.

[2]A more general form of non-Hermiticity, known as pseudo-Hermiticity, is discussed in the Appendix and also can lead to unitary time evolution [13,14] on a Hilbert space with an unconventional inner product.

model. We study the behavior of the various (perturbative) coupling parameters at one loop, derive the beta functions of the renormalization group (RG), and determine the RG fixed points and study their stability. We also discuss the RG flows of the couplings and masses. In Sec. V, following the approach of Nambu and Jona–Lasinio [16,17] (who considered a nonrenormalizable model with quartic fermion interactions, a prototype for dynamical mass generation for fermions), we study the dynamical mass generation in our model, by letting the bare mass terms go to zero. We compare the resultant masses with those from the nonperturbative ones obtained in [9,10] following Schwinger-Dyson (SD) methods (in the absence of the quartic scalar coupling). In Sec. VI we discuss briefly the results of the renormalization of the model at two loops and demonstrate a renormalization-group flow also from Hermitian to non-Hermitian couplings. Finally, conclusions and outlook are given in Sec. VII. A technical discussion on pseudo-Hermiticty, giving background essential for understanding $\mathcal{PT}$ symmetry and non-Hermiticity, is given in the Appendix.

## II. OUR MODEL, MOTIVATION, AND FORMULATION

Our axion field theory is a generalization of the simplest scalar quantum field theory with a non-Hermitian $\mathcal{PT}$ symmetric potential defined by the $D$-dimensional Hamiltonian density:

$$H_{PT} = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + g\phi^2(i\phi)^\delta. \tag{1}$$

$\phi$ is a pseudoscalar field and $\delta > 0$ is real. $H$ is non-Hermitian but $\mathcal{PT}$ symmetric because $\phi$ changes sign under $\mathcal{P}$; $\phi$ remains unchanged under $\mathcal{T}$ and $i$ changes sign under $\mathcal{T}$. This Hamiltonian density is the field-theoretic analogue of the $\mathcal{PT}$ symmetric quantum mechanical Hamiltonian [1]

$$H = p^2 + x^2(ix)^\delta \tag{2}$$

which launched the field of $\mathcal{PT}$ symmetry. Dorey *et al.* [18,19] demonstrated the surprising feature that for $\delta > 0$ the eigenvalues of $H$ are all discrete, real, and positive even though it is not Dirac Hermitian. It is necessary to broaden the class covering the Hamiltonian equation (1) in order to be able to apply the ideas of $\mathcal{PT}$ symmetry to phenomenologically interesting models.[3] Furthermore it is now realized that $\mathcal{PT}$ symmetry is part of a much *broader* class of models which is denoted as pseudo-Hermitian [13] (see the Appendix).

[3]There is no proof that, in all cases for which a $\mathcal{PT}$ symmetry can be defined, the spectrum is purely real. The spectrum will depend on the precise boundary conditions in the problem. If the entire energy spectrum is not real, the symmetry is said to be broken. However, for a large class of models, it has been found that the spectrum is real (and bounded below).

Recently, we have discussed dynamical mass generation [9–11] for fermions and pseudoscalar gravitational axion fields in effective field theories containing Yukawa type interactions between the axions and the fermions. These models arise in scenarios for radiative Majorana sterile neutrino masses [10]. These Yukawa interactions can be both Hermitian and non-Hermitian but $\mathcal{PT}$ symmetric. Although motivated by the issue of dynamical mass generation, a nonperturbative phenomena, our emphasis in this work is on understanding the model within the context of a more fundamental non-Hermitian quantum field theory where the effects of renormalization need to be considered. A more fundamental model goes beyond the Yukawa interactions in the effective theory to include quartic and cubic couplings in the scalar field. This leads us to consider the generalized Lagrangian $\mathcal{L}$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{M^2}{2}\phi^2 + \overline{\psi}(i\not{\partial} - m)\psi - ig\overline{\psi}\gamma^5\psi\phi + \frac{u}{4!}\phi^2(i\phi)^\delta = L_B + L_F, \tag{3}$$

where

$$L_B = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{M^2}{2}\phi^2 + \frac{u}{4!}\phi^2(i\phi)^\delta, \tag{4}$$

and

$$L_F = \overline{\psi}(i\not{\partial} - m)\psi - ig\overline{\psi}\gamma^5\psi\phi\psi. \tag{5}$$

The scalar model of Eq. (1) is contained within this Lagrangian as $L_B$. In [10] the analysis was done in the spirit of effective Lagrangians and so the quartic coupling, which emerges from the requirement of renormalizability, was not included. When we discuss renormalizability, we require that $\delta = 2$.[4] Equation (3) represents the most general renormalizable Lagrangian involving our $\phi$ and $\psi$ fields in four dimensions. For $u < 0$ the quartic interaction is Hermitian; for $u > 0$ the quartic coupling is non-Hermitian.[5] At the nonperturbative level, $\mathcal{PT}$ symmetric quartic scalar Hamiltonians lead to one-point and, more generally, odd-number-point Greens functions. When $\delta = 1$, $\mathcal{L}$ will have a term $ih\phi^3$ where $h$ is a coupling constant; unlike $\phi^3$ theory, with a real coupling such a theory is $\mathcal{PT}$ symmetric and is a sensible theory.

In order to define a quantum theory, whether we do this through a Schrödinger equation (for quantum mechanical theories) or more generally through path integrals for $D$-dimensional quantum field theories, we need to *specify boundary conditions*. So if we analytically continue a

[4]Renormalizability for general $\delta$ is discussed in [20,21].

[5]This remark has to be understood within the context of the deformation implied by $\delta$ and boundary conditions in path integrals.

coupling in some way (the particular sense is specified for the relevant coupling) it is absolutely necessary to show how the boundary conditions are affected. Once these boundary conditions are determined then it will be clear what Hermiticity or non-Hermiticity means.

The introduction of fermions in a $\mathcal{PT}$ context needs a discussion and is a comparatively unexplored area within the study of $\mathcal{PT}$ quantum field theories. For applications of $\mathcal{PT}$ symmetry to fundamental physics it is important to incorporate fermions [9–12,22–25]. In this new area it is not possible currently to match the rigor applied to conventional Hermitian field theories. However, we plan to lay some foundations. In the discussion of Feynman rules, $\mathcal{P}$ and $\mathcal{T}$ symmetries are connected to the issue of path integrals [26] and their boundary conditions. In the context of Lorentz invariant pseudoHermitian field theories, we shall touch on the role $\mathcal{CPT}$ symmetry [27] (where $\mathcal{C}$ is the charge conjugation operator), and is a fundamental symmetry of Hermitian Lorentz invariant field theories. The understanding of the role of $\mathcal{PT}$ type symmetry in relativistic quantum mechanics and field theory for fermions is less developed than for the bosonic case. All the above mentioned $\mathcal{PT}$ symmetric field-theoretic systems are relativistic, and for which the generation of real masses can, in principle, be understood as a consequence of the existence of an underlying antilinear symmetry [28–30].

### A. Bosonic path integrals and boundary conditions

We shall start off in the simplest context: bosonic path integrals with discrete $\mathcal{P}$ and $\mathcal{T}$ symmetries. The action that will be considered is of the following type:

$$S(\varphi) = \int d^D x \left( \frac{1}{2} (\partial_\mu \varphi)^2 + V(\varphi) \right). \tag{6}$$

The canonical form of $V(\varphi)$ used in the study of $\mathcal{PT}$ symmetry is

$$V(\varphi) = \frac{u}{4!} \varphi^2 (i\varphi)^\delta \tag{7}$$

with $u$ and $\delta$ real. The action of $\mathcal{PT}$ on $V(\varphi)$ is determined through the following:

$$\begin{aligned} &\mathcal{P}\colon\ \varphi \to -\varphi \\ &\mathcal{T}\colon\ \varphi \to \varphi \\ &\mathcal{T}\colon\ i \to -i. \end{aligned} \tag{8}$$

The potential $V(\varphi)$ is $\mathcal{PT}$ symmetric for all values of $\delta$. $\mathcal{PT}$ is an example of an antilinear symmetry (since $\mathcal{T}$ is antilinear). Pseudo-Hermiticity relies on the presence of an antilinear symmetry. For $\delta = 2$ we have the negative quartic potential which is conventionally an unstable potential and energies of states have an imaginary part. The above $\mathcal{PT}$ symmetric formulation, involving a complex deformation of the potential, leads to a theory in $D = 0$ and $D = 1$ with real energies. There are strong grounds to expect this to hold for $D > 1$. The purpose of this section is to formulate the analysis in $D = 0$ in such a way that the generalization to $D > 0$ is clear (but may have complications such as renormalization). The path in $\varphi$ space, because of the deformation parametrized by $\delta$, is taken (and required) to explore the complex $\varphi$ plane. The presence of $\mathcal{PT}$ symmetry results in a left-right symmetry of the deformed path, i.e., a reflection symmetry in the imaginary $\varphi$ axis. This left-right symmetry is responsible for real energy eigenvalues. If, for example, we have $\mathcal{T}\colon\ \varphi \to -\varphi$ then we do not have $\mathcal{PT}$ symmetry for general $\delta$, the boundary conditions are different, and the left-right symmetry of the deformed paths no longer holds. If the Lagrangian (e.g., for $\delta = 2$) formally shows $\mathcal{PT}$ symmetry for $\mathcal{T}\colon\ \varphi \to -\varphi$ the physical consequences of the different assignments of $\mathcal{P}$ and $\mathcal{T}$ are entirely different; one case may give an acceptable physical theory with left-right symmetry and real eigenvalues, while the other case with up-down symmetry would not have real eigenvalues which are bounded below. We will consider below the Euclidean version of the path integral to improve the convergence of the path integral.

#### *1. The quartic potential*

The partition function for $D = 0$ has the form

$$Z = \int_C d\varphi \, \exp\left( -\left( \frac{1}{2} m^2 \varphi^2 - \frac{1}{4!} u \varphi^4 \right) \right). \tag{9}$$

$Z$ represents a zero-dimensional field theory [2] and the path-integral measure is the measure for contour integration. The study of this toy model (which can formally be investigated as a field theory with Feynman rules) will help in understanding the role of Stokes wedges [31] in path integrals. For $u > 0$ the integral with the contour $-\infty < \varphi < \infty$ does not exist. For $u < 0$ the integral with the contour exists in the Stokes wedges $-\frac{\pi}{8} < \arg\varphi < \frac{\pi}{8}$ and $\frac{7\pi}{8} < \arg\varphi < \frac{9\pi}{8}$. Hence, the *conventional* Hermitian theory can use the contour $-\infty < \varphi < \infty$ which goes through the center of both Stokes wedges. It is straightforward to see that there are four possible Stokes wedges, each with an opening of $\pi/4$. In a $\mathcal{PT}$ symmetric context, the partition function can exist for a contour $C$ in the complex $\varphi$ plane, chosen to lie in the Stokes wedges: $-\frac{3\pi}{8} < \arg\varphi < -\frac{\pi}{8}$ and $-\frac{7\pi}{8} < \arg\varphi < -\frac{5\pi}{8}$ (see Fig. 1). These Stokes wedges are left-right symmetric and so the $\mathcal{PT}$ symmetric theory has real eigenvalues which are bounded below.

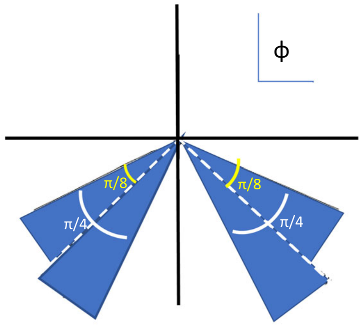


FIG. 1. $\mathcal{PT}$ symmetric Stokes wedges for quartic potential.

### 2. The cubic potential

An analysis similar to that for the quartic potential can be carried out for the cubic potential [32] partition function $Z_3^{PT}$

$$Z_3^{PT} = \int_{\mathcal{C}} dz \, \exp - \left( \frac{1}{2} m^2 \varphi^2 + \frac{i}{3!} \tilde{g} \varphi^3 \right), \tag{10}$$

where $\tilde{g}$ is real. The associated Stokes wedges are $\{-\frac{\pi}{3} < \arg(\varphi) < 0\}$ and $\{-\pi < \arg(\varphi) < -\frac{2\pi}{3}\}$ (see Fig. 2) and the integral converges along the real $\varphi$ axis. If $\varphi$ is $\mathcal{T}$ odd, the $\mathcal{PT}$ conjugate of $\{-\frac{\pi}{3} < \arg(\varphi) < 0\}$ is $\{0 < \arg(\varphi) < \frac{\pi}{3}\}$. The two Stokes wedges are contiguous and so the contour $\mathcal{C}$ can be deformed off to $\infty$ and the theory would be trivial.

From the above discussions it should be clear that in the presence of both cubic and quartic potentials, the Stokes wedges are determined by the quartic potential.

### 3. Fermionic path integrals and their role in $\mathcal{PT}$ symmetry

An essential feature of our model is the presence of fermions [33]. Since our method of analysis is based on path integrals we need to estimate whether the findings on bosonic path integrals are modified by the presence of fermions. The fermionic part of the path integral is in terms of Grassmann numbers which are anticommuting numbers and so Gaussians of Grassmann numbers truncate; at this level there should not be any additional convergence issues in the fermionic theory. To investigate further, since fermions

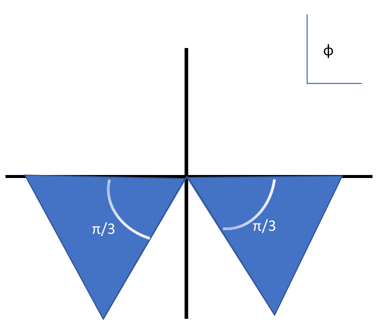


FIG. 2. $\mathcal{PT}$ symmetric Stokes wedges for cubic potential.

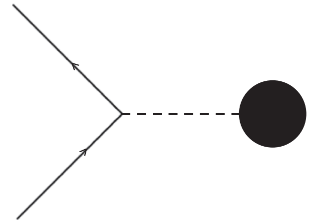

FIG. 3. The master vertex for the functional determinant. Continuous lines with arrows denote fermions. The dashed line ending in the dark blob denotes an external scalar field source.

appear quadratically in $L_F$, they can be formally integrated out in the partition function $Z_{\mathrm{eff}}$ associated with Eq. (3):

$$Z_{\mathrm{eff}} = \int D\phi \, \exp\left[-S_B(\varphi)\right] \det\left(\gamma^\mu \partial_\mu + im + ig\gamma_5 \varphi\right) \tag{11}$$

where

$$\begin{aligned} &\det\left(\gamma^\mu \partial_\mu + im + ig\gamma_5 \varphi\right) \\ &\quad = \int D\psi^\dagger D\psi \, \exp\left(-\psi^\dagger [\gamma^\mu \partial_\mu + im + ig\gamma_5 \varphi] \psi\right). \end{aligned} \tag{12}$$

These fermionic determinants have been widely studied using Feynman-diagram representations (see Figs. 3 and 4), and are complicated.

The formal expressions for these determinants are generally nonlocal; these determinants are approximated using semiclassical methods but even this is nontrivial to do rigorously. On making approximations there is an indication that corrections to the bosonic part of the Lagrangian is of the form $-u^2\varphi^4$ and $g^4\varphi^4$ [33]. Consequently quantum fluctuations may lead to non-Hermitian behavior even if the starting Lagrangian is Hermitian [34]. This issue has relevance within the context of the renormalization group.

## B. The measure of the path integral

The path integral formulation is regularly used to compute correlation (Schwinger) functions in conventional

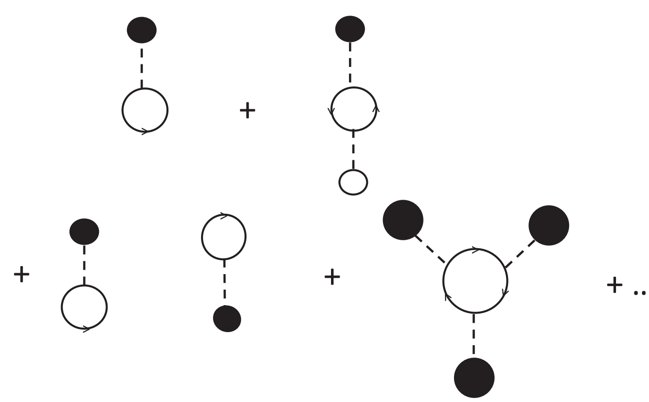


FIG. 4. Lowest functional vertices for the determinant, including disconnected graphs. The symbols are as in Fig. 3.

quantum field theories involving scalar, vector, and spinor fields. The advantage of this approach is that there is no need to construct a Hamiltonian, the Hilbert space, and equation of motion. For these very same reasons this approach is being advocated by us for $\mathcal{PT}$ symmetric field theories and is, in our opinion, the way forward. However, in the earlier discussion we have been somewhat nonspecific about details of the path-integral measure except to assume that the properties that we are accustomed to in contour integration and complex analysis continue to serve us well. Since many works on $\mathcal{PT}$ quantum mechanics rely on Hilbert-space methods and modified inner products in the non-Hermitian $\mathcal{PT}$-symmetry context, we will give additional supporting arguments for the path-integral approach which makes a connection with the Hilbert-space methods used in discussing pseudo-Hermiticity (see the Appendix).

#### 1. The calculation of Green's functions

Since the Dirac inner product of quantum mechanics (the $L^2$ norm), when applied to $\mathcal{PT}$-symmetric Hamiltonians, leads to a nonunitary quantum-mechanical theory, in the canonical operator approach unitarity is restored through the introduction of a modified inner product [13].[6] Unlike the inner product in Hermitian quantum mechanics, this modified inner product is *not* uniquely determined and is dependent on the Hamiltonian. Through examples, we shall compare the calculation of a two-point function in the path-integral and canonical approaches. Such examples provide evidence for the conjecture that for the calculation of Green's functions (within both a Minkowski and an Euclidean framework), the determination of the $\mathcal{C}$ operator (or equivalently the Hilbert-space metric) is not necessary (at least for a class of models where the non-Hermiticity lies in the interaction part of the Hamiltonian). This evidence stimulated a formal justification of these findings [35].

The evidence is based on both an exactly soluble $\mathcal{PT}$-symmetric quantum-mechanical model (the Swanson model) and also on an a perturbative treatment of the imaginary cubic potential [32]. We shall then outline an argument which justifies, in a general context, conclusions deduced from these two models.

The $\mathcal{C}$ operator [36] can be written as

$$\mathcal{C} = \exp(\mathcal{Q})\mathcal{P}, \tag{13}$$

where $\mathcal{Q}$ is Hermitian and $\mathcal{P}$ is the linear parity operator [2]. Moreover $\eta = \exp(-\mathcal{Q})$, where $\eta$ is the metric in the pseudo-Hermitian formulation of $\mathcal{PT}$ symmetry. The conventional adjoint $H^\dagger$ of a Hamiltonian $H$ satisfies

$$H^\dagger = \exp(-\mathcal{Q})H\exp(\mathcal{Q}) \tag{14}$$

and this leads to an associated Hermitian Hamiltonian $h$ which is defined as

$$h = \exp\left(-\frac{\mathcal{Q}}{2}\right)H\exp\left(\frac{\mathcal{Q}}{2}\right). \tag{15}$$

### C. The Swanson model

The classical Swanson Hamiltonian (with $a > 0$, $b > 0$, and $c$ pure imaginary) [35,37] $H_S$ is

$$H_S = ax^2 + bp^2 + 2cxp \tag{16}$$

from which we deduce (up to a total derivative) a classical Lagrangian $L_S = \frac{\dot{x}^2}{4b} - \tilde{a}x^2$ where $\tilde{a} = a - \frac{c^2}{b}$. $L_S$ is a scaled Lagrangian for a Hermitian harmonic oscillator and leads to a Hermitian Hamiltonian

$$h_S(x, P) = bP^2 + \tilde{a}x^2 \tag{17}$$

with $P = p + (c/b)x$. Using conventional techniques such as path integrals for Hermitian Hamiltonians, the time-ordered $n$-point Greens functions $G_n(t_1, t_2, \ldots, t_n)$ can be calculated.

From the theory of pseudo-Hermitian Hamiltonians there is a similarity transformation (determined by a $Q$ function) which relates the two. The ground states of the two Hamiltonians are also related by this similarity transformation:

$$|\Omega_{h_S}\rangle = \exp\left(-\frac{\mathcal{Q}}{2}\right)|\Omega_{H_S}\rangle. \tag{18}$$

Since $G_2(t,t) = \langle\Omega_{h_S}|x^2|\Omega_{h_S}\rangle$, we can rewrite it as

$$G_2(t,t) = \langle\Omega_{H_S}|\exp\left(-\frac{\mathcal{Q}}{2}\right)x^2\exp\left(-\frac{\mathcal{Q}}{2}\right)|\Omega_{H_S}\rangle. \tag{19}$$

Moreover, in this case, it is possible to find a $\mathcal{Q}$ which depends just on $x$, and so

$$G_2(t,t) = \langle\Omega_{H_S}|\exp(-\mathcal{Q})x^2|\Omega_{H_S}\rangle, \tag{20}$$

the form expected in a non-Hermitian framework. Hence, in this simple case a path-integral computation and one using a $\mathcal{PT}$-symmetric framework with a $\mathcal{PT}$ inner product give the same result.

### D. The imaginary cubic potential

An example of a model which is not exactly soluble but can be solved by using perturbation theory is provided by the Hamiltonian

[6]The formulations in terms of either pseudo-Hermiticity (see the Appendix) or an invertible metric operator or a $\mathcal{C}$ operator [32] are entirely equivalent.

$$H_C = \frac{1}{2}p^2 + \frac{1}{2}x^2 + i\tilde{g}x^3 \tag{21}$$

with $\tilde{g}$ real.

The associated classical Lagrangian is

$$L_C = \frac{1}{2}(\dot{x}^2 - x^2) - i\tilde{g}x^3. \tag{22}$$

From our earlier discussion of Stoke's wedges, we know that the edge of the Stoke's wedges coincides with the real $x$ axis; so conventional Feynman rules are valid and lead to [38]

$$G_1 = -\frac{3}{2}i\tilde{g} + \frac{33}{2}i\tilde{g}^3 + O(\tilde{g}^5). \tag{23}$$

This result coincides with the quantum mechanical calculation using $H_C$ and the $\mathcal{PT}$-symmetric inner product, i.e., $\langle 0|\exp(-\mathcal{Q})x|0\rangle$ for a suitable operator $\mathcal{Q}$ [39]. Hence, we have further evidence supporting the conjecture which prompted the investigation of Jones and Rivers [37] to be discussed next.

### E. General argument

A quantum field theory is characterized by its Green's functions. The Schwinger-Dyson equations (SDEs) [26], which are $c$-number equations, determine the Green's functions of a field theory. The SDEs can be derived from the partition function $Z[j]$ where $j(x)$ denotes a source field. For definiteness we will consider a pseudoscalar field $\phi(x)$ (in a spacetime dimension $D$) with an action $S[\phi] = \int L(\phi(x), \partial_\mu\phi(x))d^Dx$ where $L(\phi(x), \partial_\mu\phi(x))$ is the Lagrangian density. We can take for definiteness

$$L(\phi(x), \partial_\mu\phi(x)) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - U(\phi) \tag{24}$$

where $U(\phi)$ is, in general, a non-Hermitian Hamiltonian. $Z[j]$ can be represented in two ways: one in terms of a path integral denoted by $Z_1[j]$ and the other in terms of a time-ordered product of operators denoted by $Z_2[j]$. We shall use the relation between the two expressions to argue that the path integral approach does not require, as far as the computation of Green's functions is concerned, explicit knowledge of the non-Hermitian metric. The expressions for $Z_1[j]$ and $Z_2[j]$ are

$$Z[j] = Z_1[j] = \int D\phi \exp\left(-S[\phi] + \int j(x)\phi(x)\right) \tag{25}$$

and

$$Z[j] = Z_2[j] = \langle\Omega|\eta T\left(\exp\left[i\int dx j(x)\phi(x)\right]\right)|\Omega\rangle, \tag{26}$$

where $|\Omega\rangle$ denotes the vacuum state. The metric operator $\eta$ is time independent.

From (25) we obtain the SDE on requiring that

$$\int D\phi \frac{\delta}{\delta\phi(x)}\exp\left[-S[\phi] + \int dy\phi(y)j(y)\right] = 0. \tag{27}$$

The Green's functions are obtained from

$$G_n(x_1, x_2, \ldots, x_n) = \frac{1}{Z[j]}\left(-i\frac{\delta}{\delta j(x_1)}\right)\left(-i\frac{\delta}{\delta j(x_2)}\right) \ldots \left(-i\frac{\delta}{\delta j(x_n)}\right)Z(j)|_{j=0}. \tag{28}$$

The path- integral measure is formally encoded in $D\phi$, but, in the derivation of the SDEs, it is not necessary to specify this measure precisely. The main assumption is that the path integral exists. From (27) we deduce that

$$\left(-\frac{\delta S}{\delta\phi(x)}\Big|_{\phi(x')=\frac{\delta}{\delta j(x')}} + j(x)\right)Z(j) = 0. \tag{29}$$

Alternatively we can derive the SDEs using $Z_2[j]$. The derivation starts from the Heisenberg equations of motion. $\mathcal{H}$ is given by

$$\mathcal{H} = \int dx\left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi) + \frac{1}{2}m^2\phi^2 + U(\phi)\right] \tag{30}$$

with $\pi = \partial_0\phi$ and we assume $[\mathcal{H}, \eta] = 0$. The Heisenberg equations of motion (in natural units and using notation which does not distinguish between classical and operator fields) can be shown to be:

$$(\partial^2 + m^2)\phi(x) + U'(\phi(x)) = 0 \tag{31}$$

where

$$\partial_0\phi = i[\mathcal{H}, \phi] \tag{32}$$

and

$$\partial_0\pi = i[\mathcal{H}, \pi]. \tag{33}$$

The assumption that these Heisenberg equations [(32), (33)] are valid for a pseudo-Hermitian Hamiltonian $\mathcal{H}$ will now be justified. For $\mathcal{H}$ to be pseudo-Hermitian with respect to an inner product $\eta$ (discussed earlier) a necessary condition is that

$$\eta\exp(i\mathcal{H}) = \exp(i\mathcal{H}^\dagger)\eta. \tag{34}$$

In Hilbert space a ket $|\psi_S\rangle$ in the Schrödinger picture is related to a ket $|\phi_H\rangle$ in the Heisenberg picture by

$$|\phi_H\rangle = \exp(i\mathcal{H}t)|\psi_S\rangle. \tag{35}$$

Similarly, for the corresponding bras

$$\langle\phi_H| = \langle\psi_S|\exp(-i\mathcal{H}^\dagger t). \tag{36}$$

For an operator $\mathcal{O}$ in the Schrödinger picture

$$\begin{aligned}\langle\psi_S|\eta\mathcal{O}|\psi_S\rangle &= \langle\phi_H|\exp(i\mathcal{H}^\dagger t)\eta\mathcal{O}\exp(-i\mathcal{H}t)|\phi_H\rangle\\ &= \langle\phi_H|\eta\exp(i\mathcal{H}t)\mathcal{O}\exp(-i\mathcal{H}t)|\phi_H\rangle\\ &= \langle\phi_H|\eta\mathcal{O}_{\mathcal{H}}|\phi_H\rangle\end{aligned} \tag{37}$$

where

$$\mathcal{O}_H = \exp(i\mathcal{H}t)\mathcal{O}\exp(-i\mathcal{H}t). \tag{38}$$

Hence, even in the case of pseudo-Hermitian Hamiltonians, the Heisenberg picture operator $\mathcal{O}_H$ obeys the standard form of the Heisenberg equations of motion.

In the canonical formulation $Z_2[j]$, we shall adapt the Symanzik construction for SDE in the presence of a source. The metric appears explicitly in this formulation. In this construction the expectation values of fields are determined by the known classical equations of motion and the equal time commutation relations. If we find that the SDE are the same in the path integral and canonical approaches, then we can deduce that the path integral formulation (without an explicit implementation of $\eta$) leads to a correct calculation of Green's functions in the case of pseudo-Hermitian Hamiltonians. For a given $x^\mu = (x_0, \vec{x})$ (and with all fields below operator-valued) [26]

$$Z_2[j] = \langle\Omega|\eta(\mathcal{E}(\infty, x_0)\mathcal{E}(x_0, -\infty))|\Omega\rangle \tag{39}$$

and formally

$$\left(-i\frac{\delta}{\delta j(x)}\right)^p Z_2[j] = \langle\Omega|\eta\mathcal{E}(\infty, x_0)\phi(x)^p\mathcal{E}(x_0, -\infty)|\Omega\rangle. \tag{40}$$

where

$$\mathcal{E}(x_0', x_0) = T\left[\exp\left(i\int_{x_0}^{x_0'} dy_0 \int d\vec{y}\, j(y_0, \vec{y})\phi(y_0, \vec{y})\right)\right]. \tag{41}$$

Hence,

$$0 = \langle\Omega|\eta\mathcal{E}(\infty, x_0)\left(-\frac{\delta S}{\delta\phi(x)}\right)\mathcal{E}(x_0, -\infty)|\Omega\rangle, \tag{42}$$

and

$$\begin{aligned}0 = &\left[(\partial^2 + m^2)\left(-i\frac{\delta}{\delta j(x)}\right) + U'\left(-i\frac{\delta}{\delta j(x)}\right)\right]Z_2[j]\\ &+ \langle\Omega|\eta\mathcal{E}(\infty, x_0)\partial_0^2\phi(x)\mathcal{E}(x_0, -\infty)|\Omega\rangle\\ &- \partial_0^2\langle\Omega|\eta\mathcal{E}(\infty, x_0)\phi(x)E(x_0, -\infty)|\Omega\rangle.\end{aligned} \tag{43}$$

The last two terms in (43) can be simplified further. We first note that

$$\begin{aligned}&\partial_0\langle\Omega|\eta(\mathcal{E}(\infty, x_0)\phi(x_0, \vec{x})\mathcal{E}(x_0, -\infty))|\Omega\rangle\\ &\quad= \langle\Omega|\eta(\mathcal{E}(\infty, x_0)\pi(x_0, \vec{x})\mathcal{E}(x_0, -\infty))|\Omega\rangle\end{aligned}$$

since $\phi(x_0, \vec{x})$ commutes with itself at equal times. Differentiating again with respect to $x_0$ we obtain

$$\begin{aligned}&\partial_0^2\langle\Omega|\eta(\mathcal{E}(\infty, x_0)\phi(x_0, \vec{x})\mathcal{E}(x_0, -\infty))|\Omega\rangle\\ &\quad= \langle\Omega|\eta(\mathcal{E}(\infty, x_0)\partial_0^2\phi(x_0, \vec{x})\mathcal{E}(x_0, -\infty))|\Omega\rangle + j(x).\end{aligned} \tag{44}$$

Using (44) in (43), we finally find (29) again but using the inner product $\eta$ this time. This demonstration has been confined to scalar theories. In our earlier discussion the fermions did not bring any qualitatively different issues into defining the boundary conditions of the path integral; consequently this derivation should formally still be valid.

### F. Renormalization and the path integral measure

Recent works [23–25] on local $\mathcal{PT}$-fermionic quantum field theory have *not* addressed the essential issue of renormalization [34] which arise due to quantum fluctuations. Now that, in the context of $\mathcal{PT}$ field theories, we have presented a full discussion of the quantization procedure through path integrals, we will give a detailed analysis of renormalization and the RG within our simple model theory studied as an effective theory in the context of axion physics [10,11] (involving a Yukawa coupling of a Dirac fermion field $\psi$ to $\phi$, a real pseudoscalar field). The Yukawa coupling can be real or imaginary.

Given the possibility that some non-Hermitian field theories may be a basis for fundamental theories, we aim to study the model as a quantum field theory. The renormalization group leads to coupling constants running with the energy scale. The connection between Hermitian and non-Hermitian couplings through these flows will be examined. In addition, on noting that the existence of an underlying antilinear ($\mathcal{PT}$) [1,10,11] symmetry [28–30] in the models allows for real energy eigenvalues, we shall examine, using the renormalizability of the model, dynamical mass generation for the fermion and pseudoscalar fields. In the case of small couplings, our analysis yields nonperturbative results for the generated (real) masses, which agree with a (one-loop) SD analysis [10,11] for the model without axion self-interaction.

### G. Issues on $\mathbb{C}\mathcal{PT}$ and $\mathcal{CPT}$ invariance

$\mathbb{C}\mathcal{PT}(\equiv\Theta)$ invariance [27], where $\mathbb{C}$ is the conventional charge conjugation operator of Dirac [41,42], should not be confused with the $\mathcal{C}$ operator (13) discussed in Sec. II B. This invariance has been proved for Hermitian Lorentz invariant theories and is sometimes referred to as the (Hermitian) $\mathbb{C}\mathcal{PT}$ theorem. $\Theta$ is an important symmetry that relates matter and antimatter, for example, the equality of masses and lifetimes, and opposite charges, for particles and antiparticles. It is important to discuss the fate of this symmetry in local relativistic non-Hermitian quantum field theories. This is still an open issue. For the conventional $\mathbb{C}\mathcal{PT}$ theorem, one expects the violation of this symmetry in the presence of general non-Hermitian couplings. Purely phenomenological considerations were adopted in an early study [40], to discuss potential experimental searches for general non-Hermitian $\Theta$-violating quantum field theories. In what follows we shall concentrate only on our type of Lagrangian (3). For the case of a purely imaginary $g$ coupling and positive $u$ the theory is pseudo-Hermitian (see the discussion of pseudo-Hermiticity in the Appendix and boundary conditions on a Feynman path integral in Sec. II A).

The Yukawa interaction, if non-Hermitian with the Dirac inner product, turns out to be $\mathbb{C}\mathcal{PT}$ odd [41,42]. This can lead, in principle, to observable consequences [40]. In this work and in [10,11] the antiparticle state can be defined perturbatively in the Yukawa coupling. However, from a foundational view point it would be interesting to see whether, on restricting to pseudo-Hermitian theories, one can define, in principle, a *new* set of $\mathbb{C}$, $\mathcal{P}$, and $\mathcal{T}$ operators such that the resultant *new* $\Theta$ operator is an antilinear symmetry of the non-Hermitian theory. In a general $\mathcal{PT}$ symmetric (or pseudo-Hermitian) case, the Hamiltonian $H$ is not Hermitian and so the conventional antiunitary $\mathbb{C}\mathcal{PT}$ operator does not map a particle state into an antiparticle state. As noted in Sec. II B (see also the Appendix), corresponding to a pseudo-Hermitian $H$ there is a Hermitian $h$ related by a Hermitian Hilbert-space automorphism [$\tilde{\eta} = \eta^{1/2}$, where $\eta$ is defined in Eq. (34)]:

$$h = \tilde{\eta} H \tilde{\eta}^{-1}. \tag{45}$$

The new inner product $\langle\langle .|.\rangle\rangle$ for the non-Hermitian operators is given by $\langle .|\tilde{\eta}^2 .\rangle$.

This statement is strictly valid for finite dimensional quantum mechanical pseudo-Hermitian systems. Were we to assume the validity of the mapping (45), though, in quantum field-theoretic systems, it would be possible in principle [using a similarity transformation $\tilde{\eta}$ cf. Eq. (55)] to construct a $\mathbb{C}\mathcal{PT}$ operator that can define the antiparticle state nonperturbatively in pseudo-Hermitian field theories.[7] However the existence of well-defined similarity transformations which lead to a useful $\mathbb{C}\mathcal{PT}$ operator needs further investigation.

There is another perspective on $\mathbb{C}\mathcal{PT}$ invariance of pseudo-Hermitian field theories, which uses complex Lorentz transformations, see [30], to claim that the conventional $\mathbb{C}\mathcal{PT}$ operator ($\widehat{\theta_{\mathbb{C}\mathcal{PT}}}$) is the correct $\mathbb{C}\mathcal{PT}$ operator for pseudo-Hermitian Hamiltonians. We believe this assertion to be unproven. In that work, it is asserted that there are two conditions under which a $\mathbb{C}\mathcal{PT}$ invariance theorem for non-Hermitian systems would be valid. The first is the existence of an antilinear symmetry, which replaces Hermiticity, and ensures the time independence of the appropriate inner products that enter the non-Hermitian theory. The second condition is the extension of the requirement of Lorentz invariance, to encompass invariance under complex Lorentz transformations. This approach is not applicable to our case. In the work of [30], the field operator $i\overline{\psi}\gamma^5\psi$ is $\mathbb{C}\mathcal{PT}$ and picks up a $+1$ phase (under appropriate normalization of the phases in the definition of $\mathcal{P}$, $\mathcal{T}$, and $\mathbb{C}$) under the application of the pertinent transformation. For conventional [30] $\mathbb{C}\mathcal{PT}$ invariance to hold, the Yukawa interaction with a pseudoscalar $\phi$ would require a $\mathcal{T}$-odd transformation of $\phi$ (see II A) and a *real* coupling. However, our term with purely imaginary Yukawa coupling is $\mathbb{C}\mathcal{PT}$ odd under our assumed transformations (8); so the considerations of [30] do *not* apply. On the other hand, the non-Hermitian self-interactions of axions in our model satisfy the criteria of [30] for $\mathbb{C}\mathcal{PT}$ invariance.

The first criterion of [30], the existence of an antilinear symmetry, such as $\mathcal{PT}$, is guaranteed in our case as well, thus leading to either reality of the energy spectrum, or at least the appearance of the energy eigenvalues in complex conjugate pairs. In what follows we shall examine dynamical mass generation with real eigenvalues in our Yukawa-system with axion self-interactions (3) [(4), (5)] and demonstrate (in Sec. V) that this is possible in the model with *Hermitian* Yukawa interactions and *non-Hermitian* $\mathbb{C}\mathcal{PT}$ even axion self-interaction couplings, under some circumstances, which we shall specify (see also [10,11]). In the non-Hermitian perturbative Yukawa-interaction case, however, as we shall see, dynamical mass generation, when applied naively, i.e., via the replacement of the real Yukawa couplings by the purely imaginary ones, leads to unacceptably large masses (above the UV cutoff), which are thus not self-consistent. It should be noted that in [10,11], for a model with an attractive $\mathbb{C}\mathcal{PT}$-even four-fermion interaction and an anti-Hermitian Yukawa interaction, one can obtain dynamical masses for fermions and pseudoscalars, of approximately equal magnitude proportional to $|g|\Lambda$, where $\Lambda$ is the ultraviolet cutoff, and $g$ is the Yukawa coupling [as in (3)]. We shall now give a simple argument to show how this might be understood using the conventional $\widehat{\theta_{\mathbb{C}\mathcal{PT}}}$.

[7] Lattice field theories on a finite lattice are finite dimensional and are a viable regularization of continuum field theories.

In quantum mechanics we have

$$(\widehat{\theta_{\mathbb{C}\mathcal{PT}}}\widehat{H}^* - \widehat{H}\,\widehat{\theta_{\mathbb{C}\mathcal{PT}}})|E\rangle = \delta\overline{|E\rangle} \neq 0, \tag{46}$$

where $\widehat{H}$ is a non-Hermitian Hamiltonian operator with an energy eigenstate $|E\rangle$, and $*$ denotes standard complex conjugation and $\overline{|E\rangle}$ denotes the antiparticle energy eigenstate, defined by $\widehat{\theta_{\mathbb{C}\mathcal{PT}}}|E\rangle = \overline{|E\rangle}$.

Hence,

$$\widehat{H}\,\overline{|E\rangle} = (E^\star - \delta)\overline{|E\rangle}. \tag{47}$$

In our case, $E = E_1 + i\mu$, with $E_1$, $\mu \in \mathbb{R}$. Here, $E_1$ corresponds to the real dynamical masses from the earlier SD analysis [10,11], while $i\mu \propto \langle\overline{\psi}\gamma^5\psi\rangle$, $\mu \in \mathbb{R}$ would represent the purely imaginary chiral condensate, corresponding to an anti-Hermitian chiral mass for the fermions.

In [9] we have argued that it is *not* possible, for *energetic* reasons, to generate dynamically, using SD analysis, a nonzero non-Hermitian condensate $\langle\overline{\psi}\gamma^5\psi\rangle$; i.e., dynamically one should have $\mu = 0$. We interpret this result as implying that, in the massive phase of the system, the mass eigenvalue of the non-Hermitian operator $\overline{\psi}\gamma^5\psi$ on an energy eigenstate $|E\rangle >$ would vanish

$$\overline{\psi}\gamma^5\psi|E\rangle = 0. \tag{48}$$

For our anti-Hermitian Yukawa model (3), we obtain

$$\begin{aligned}
&\widehat{\theta_{\mathbb{C}\mathcal{PT}}}\,\overline{\psi}(x)\gamma^5\psi(x)\widehat{\theta_{\mathbb{C}\mathcal{PT}}}^{-1} = -\overline{\psi}(-x)\gamma^5\psi(-x)\\
&\Rightarrow \int d^4x[\widehat{\theta_{\mathbb{C}\mathcal{PT}}}, \overline{\psi}(x)\gamma^5\psi(x)]|E\rangle\\
&= 2\int d^4x\widehat{\theta_{\mathbb{C}\mathcal{PT}}}\,\overline{\psi}(-x)\gamma^5\psi(-x)|E\rangle = 0
\end{aligned} \tag{49}$$

where in the last equality we took into account (48), and the interpretation that the mass eigenvalues of the Hamiltonian operator are associated with the dynamically generated masses for the various fields (fermions and axions) in the system.

The result (49) implies that the non-Hermitian Yukawa interactions do *not affect* the equality of the dynamically generated masses between particle and antiparticle (fermion or boson) states, in this system. Thus, the SD treatment of [10,11] and also the analysis by Nambu and Jona-Lasinio (NJL) [16] in this work provides a correct framework within our anti-Hermitian Yukawa framework for a description of dynamical mass generation for both fermions and (pseudo)scalars. The calculation of the correct form of the $\Theta$ operator using such ideas as the similarity transformation remains to be done for non-Hermitian theories, having antilinear symmetries such as $\mathcal{PT}$.

## III. THE YUKAWA MODEL

The massive Yukawa model is given by the bare Lagrangian in three-space and one-time dimensions in terms of bare parameters (emphasised through the use of the subscript 0)[8]:

$$\begin{aligned}
\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{M_0^2}{2}\phi_0^2 + \overline{\psi}_0(i\not\partial - m_0)\psi_0\\
- ig_0\overline{\psi}_0\gamma^5\psi_0\phi_0 + \frac{u_0}{4!}\phi_0^4.
\end{aligned} \tag{50}$$

$\mathcal{L}$ is renormalized through mass, coupling constant, and wave function renormalizations; we will take the spacetime dimensionality $D$ to be $4 - \epsilon$ where $\epsilon$ is a small parameter. Furthermore, $\epsilon$ is a useful small parameter in the analysis of fixed points. It is the simplest nontrivial renormalizable model of a Dirac fermion field $\psi_0$ interacting with a pseudoscalar field $\phi_0$. If $g_0$ is real then the Yukawa term is Hermitian and $g_0^2 > 0$. If $g_0$ is purely imaginary, then the Yukawa term is non-Hermitian but it is $\mathcal{PT}$ symmetric, with our definitions of the discrete symmetries $\mathcal{P}$, $\mathcal{T}$ to be discussed below [cf. (51) and (53)], and $g_0^2 < 0$. $u_0$ is real but it can be positive or negative. If $u_0 > 0$ the quartic term is non-Hermitian (in a Minkowski formulation). If $u_0 < 0$ the quartic term is Hermitian. Thus, both couplings allow the possibility of showing non-Hermitian but $\mathcal{PT}$ symmetric behavior. For the non-Hermitian case for $u$ (in the Euclidean picture) the path integral contour in the $\phi$ plane has a pair of $\mathcal{PT}$ symmetric Stokes wedges (see Fig. 1) which means that the contour asymptotically needs to end up in these wedges. Moreover, in this non-Hermitian case there are three $\phi$-saddle points (or configurations such as bounces depending on $D$); the fluctuations around the trivial saddle point give standard perturbation theory and Feynman rules. The nontrivial saddle points give rise to (non-perturbative) instanton-like contributions. So renormalization flows near Gaussian fixed points and quartic Hermitian fixed points using Feynman rules should be a reasonable indicator of the scale dependence of couplings of the theory.

In the Dirac representation of gamma matrices, the conventional discrete transformations on $\psi_0$ that we use [41] are[9]

[8]Our Minkowski-metric signature convention is $(+,-,-,-)$. For our discussion of dynamical mass generation the $\phi^3$ coupling is not going to be considered any further since our original Yukawa model does not require it for a consistent perturbative renormalization, i.e., $\tilde{g}$ is not generated through renormalization.

[9]We remark that in Ref. [23] a rather different $\mathcal{T}$ transformation was used (related to the discrete symmetries of the Dirac equation [41]); the action of $\mathcal{T}$ on a spinor wave function $\psi$ produces the complex conjugate field $i\gamma^1\gamma^3\psi^\star(-t,\vec{x})$. Such a transformation is a symmetry of the Lagrangian. The form of these transformations in the Weyl representation of the $\gamma$ matrices are discussed in [42]. The Lagrangian is invariant under these transformations.

$$\mathcal{P}\psi_0(t,\vec{x})\mathcal{P}^{-1} = \gamma^0\psi_0(t,-\vec{x}),$$
$$\mathcal{T}\psi_0(t,\vec{x})\mathcal{T}^{-1} = i\gamma^1\gamma^3\psi_0(-t,\vec{x}),$$
$$\mathbb{C}\psi(t,\vec{x})\mathbb{C}^{-1} = i\gamma^2\psi^\dagger(t,\vec{x}), \tag{51}$$

where $\mathbb{C}$ denotes the charge conjugation operator [41] (*not* to be confused with the $\mathcal{C}$ operator) and $\mathcal{T}$ is the antilinear operator time-reversal operator. Also, under the action of $\mathcal{P}$ and $\mathcal{T}$, the pseudoscalar field $\phi(t,\vec{x})$ transforms as[10]

$$\mathcal{P}\phi_0(t,\vec{x})\mathcal{P}^{-1} = -\phi_0(t,-\vec{x}), \qquad \mathcal{T}\phi_0(t,\vec{x})\mathcal{T}^{-1} = \phi_0(-t,\vec{x}). \tag{53}$$

In this article we are interested in determining the conditions under which there is dynamical mass generation. We stress that the results of our study here (and also those in [9–11]) demonstrate the possibility of generating dynamically *real masses* in non-Hermitian theories. We use Feynman rules (discussed earlier in this article) at a perturbative level, which are valid for weak pseudo-Hermitian interactions.

We now come to the pseudoscalar self-interaction term in (50). For $u$, $\delta > 0$, in any space dimension, the non-Hermitian scalar potential,

$$u\phi^2(i\phi)^\delta, \tag{54}$$

is $\mathcal{PT}$ symmetric.

For $\delta \neq 0$ the choice of $\mathcal{T}$ odd for $\phi$ would spoil $\mathcal{PT}$ symmetry, and so is not of interest. We have seen in our earlier discussion of Stokes wedges that the limit $\delta \to 2$, where $u = u_0$, gives a *non-Hermitian* but $\mathcal{PT}$ symmetric theory; of course the term (54) is non-Hermitian for every $4 > \delta > 0$ and $u > 0$. $\mathcal{PT}$ symmetric Hamiltonians are pseudo-Hermitian.

Explicitly, the Lagrangian interaction term (54) is pseudo-Hermitian and can be derived by a similarity transformation from the Hermitian interaction $\phi^{2+\delta}$ [30][11]:

$$-\phi(t=0,\vec{x})^2(i\phi(t=0,\vec{x}))^\delta = S\phi(t=0,\vec{x})^{2+\delta}S^{-1},$$
$$S = \exp\left(-\frac{\pi}{2}\int d^3x\,\Pi(t=0,\vec{x})\phi(t=0,\vec{x})\right), \tag{55}$$

where $\Pi(t,\vec{x})$ is the canonical momentum of $\phi(t,\vec{x})$, in the free theory, satisfying the (equal-time) canonical commutation relations $[\phi(t,\vec{x}),\Pi(t,\vec{x}')] = i\delta^{(3)}(\vec{x}-\vec{x}')$.[12]

In this work we will study the renormalization of the Yukawa theory (50) and determine its fixed point structure and the corresponding stability of the fixed points. We shall discuss RG flows that interpolate between Hermitian and non-Hermitian fixed points, and discuss mass generation for the fermion and axion fields using the method of Nambu and Jona-Lasinio [16,17]. Hence, we shall study mass generation for both axions and fermion fields using mass renormalizations calculated perturbatively, and examine the effects of the self-interaction on mass generation by comparing our results with some of the nonperturbative masses discussed in [10,11]. This discussion will be a prelude to the full Schwinger-Dyson treatment, with the inclusion of axion self interactions reserved for a future publication.

The renormalized Lagrangian (where the renormalized parameters are without the subscript 0) is given by

$$\begin{aligned}\mathcal{L} = {}&\frac{1}{2}(1+\delta Z_\phi)\partial_\mu\phi\partial^\mu\phi - \frac{M_0^2}{2}(1+\delta Z_\phi)\phi^2 \\ &+ (1+\delta Z_\psi)\overline{\psi}(i\partial\!\!\!/ - m_0)\psi \\ &- ig_0(1+\delta Z_\psi)\sqrt{1+\delta Z_\phi}\,\overline{\psi}\gamma^5\psi\phi + \frac{u_0}{4!}(1+\delta Z_\phi)^2\phi^4\end{aligned} \tag{56}$$

where we have introduced the multiplicative renormalizations $Z_\phi$, $Z_\psi$, $Z_g$, $Z_u$, $Z_m$, and $Z_M$ defined through

$$\phi_0 = \sqrt{Z_\phi}\phi = \sqrt{1+\delta Z_\phi}\phi, \tag{57}$$

$$\psi_0 = \sqrt{Z_\psi}\psi = \sqrt{1+\delta Z_\psi}\psi, \tag{58}$$

[10]We note that, in *quantum mechanics* [1], under $\mathcal{T}$ one has $i \to -i$ as a consequence of the action of $\mathcal{T}$ on the Heisenberg commutator between position ($\hat{x}$) and momentum ($\hat{p}$) operators: $\mathcal{T}[\hat{x},\hat{p}]\mathcal{T}^{-1} = -[\hat{x},\hat{p}] = -i$ ($\hbar = 1$ in natural units), from which it follows immediately that

$$\mathcal{T}i\mathcal{T}^{-1} = -i. \tag{52}$$

In canonical quantization of field theory [6,41], the above Heisenberg-commutator argument is extended to equal-time canonical commutators between fields and their canonical conjugate momenta, and, thus, the property (52) is understood to be valid for quantum field-theoretic systems as well, and should be imposed when considering time-reversal transformations in the field theory lagrangian.

[11]In fact, exploiting the time independence of the respective Hamiltonian, it suffices to evaluate the similarity transformation only for $t = 0$.

[12]In arriving at (55), we use the Baker-Hausdorff formula

$$e^A B e^{-A} = B + [A,B] + \sum_{n=2}^{\infty}\frac{1}{n!}[A,[A,\ldots[A,B],\ldots],$$

and took into account the following result of the canonical field-theoretic (equal-time) commutation relation:

$$-\frac{\pi}{2}\left[\int d^3x\,\Pi(t=0,\vec{x})\phi(t=0,\vec{x}),\phi(t=0,\vec{y})^{2+\delta}\right] = +i\frac{\pi}{2}(2+\delta)\phi(t=0,\vec{y})^{2+\delta}, \qquad \delta > 0,$$

which stems from the fact that [, ] is a linear operator that behaves like a derivative with respect to the field $\phi$.

$$M_0^2 Z_\phi = M^2 + \delta M^2 = M^2 Z_M, \tag{59}$$

$$m_0 Z_\psi = m + \delta m = m Z_m, \tag{60}$$

$$g_0 Z_\psi \sqrt{Z_\phi} = g + \delta g = g Z_g, \tag{61}$$

$$u_0 (Z_\phi)^2 = u + \delta u = u Z_u. \tag{62}$$

This Yukawa model is the natural field-theoretic version of the quantum mechanical model considered in [23], which can be considered as a free theory but with both conventional and axial mass terms. The axial mass becomes a dynamical field in our model within a Yukawa term; the Yukawa coupling $g$ can be purely imaginary $g^2 < 0$ [9–11] due to the aforementioned $\mathcal{PT}$ symmetry of the relativistic theory [9,30].[13] It was noted in [23] that the *absence* of a conventional mass term led to *broken* $\mathcal{PT}$ symmetry[14] and so it is natural, at the perturbative level, to consider a massive theory. Moreover, one approach to dynamical mass generation [16], the one that we will follow, is to consider a theory with a mass which is then determined self-consistently through a gap equation [17].

The massless variant of the Yukawa theory ($M_0 = m_0 = 0$), with no quartic term, has recently been studied using unrenormalized Schwinger-Dyson equations [10,11] with a momentum cutoff $\Lambda$. The emphasis was on effective theory for energy and momentum scales below $\Lambda$. Earlier work has suggested a link between renormalization and emergence of non-Hermiticity [34]. Our fermionic model (3), a natural generalization of the canonical scalar model (1), shows the interplay of non-Hermiticity and $\mathcal{PT}$ symmetry. The associated renormalization group allows us to discuss the energy dependence of the couplings and the Hermiticity of the theory. We should note that in [10,11] non-Hermiticity was examined only for the coupling $g$. Here, we examine $\mathcal{PT}$ symmetric non-Hermiticity in the self-interaction of the pseudoscalar field (54) as well. Nonetheless, as we will find that the dynamically generated fermion masses will acquire a nonperturbative form, similar in structure to the one derived in [10,11] for real coupling $g$.

## IV. RENORMALZATION GROUP ANALYSIS OF THE MASSIVE YUKAWA THEORY

We shall use dimensional regularization of the massive Yukawa theory with spacetime dimension $D = 4 - \epsilon$ and $\epsilon > 0$. At one loop the renormalization group equations are

$$\frac{dg}{dt} = \frac{5g^3}{16\pi^2} - \frac{\epsilon g}{2} \tag{63}$$

$$\frac{du}{dt} = \frac{48g^4 - 3u^2 + 8g^2 u}{16\pi^2} - u\epsilon \tag{64}$$

$$\frac{dm}{dt} = -\frac{g^2 m^2}{16\pi^2} \tag{65}$$

$$\frac{dM}{dt} = \frac{1}{32\pi^2 M}[4g^2(M^2 - 2m^2) - um^2] \tag{66}$$

where $\frac{d}{dt} \equiv \mu \frac{d}{d\mu}$, with $\mu$ being the mass scale introduced in the method of dimensional regularization. The study of $\epsilon$-dependent fixed points was initiated by Wilson and Fisher [43].

It is also interesting to consider a change of variables from $(m, M, g)$ to $(\sigma, M, y)$ where $m = \sigma M$ and $y = g^2$ in order to see any correlation in the behavior of $m$ and $M$ under renormalization. The $\beta$ functions are then polynomials in these variables. From (63), (64), (65), and (66) we deduce that

$$\frac{dM}{dt} = \frac{M}{32\pi^2}[4y(1 - 2\sigma^2) - \sigma^2 u] \tag{67}$$

$$\frac{d\sigma}{dt} = -\frac{\sigma}{32\pi}\left[4y\left(1 - 2\sigma^2\left(1 + \frac{M}{128\pi}\right)\right) - u\sigma^2\right] \tag{68}$$

$$\frac{dy}{dt} = \frac{5}{8\pi^2} y^2 - \epsilon y \tag{69}$$

$$\frac{du}{dt} = \frac{48y^2 - 3u^2 + 8yu}{16\pi^2} - u\epsilon. \tag{70}$$

Equations (63) and (64) form a closed set; their solutions feed into Eqs. (65) and (66). Similarly (69) and (70) form a closed set.

Our strategy will be to use a combination of dominant-balance ideas for equations [31], geometric methods from the theory of dynamical systems and direct solution of differential equations to determine flows to the Hermitian and non-Hermitian regions of parameter space. Our conclusions will be valid within the context of the above (approximate) one-loop renormalization group equations.

In the next section we shall analyze the two-loop renormalization group flows for a massless Yukawa theory, a model that was previously used in a Schwinger-Dyson analysis of mass generation [10,11].

[13]We note that purely imaginary couplings, upon renormalization, are consistent in our models, in the sense that the respective counterterms $Z_g = 1 + \mathcal{O}(g^2)$, $Z_\psi$, and $Z_\phi$ in (61) are real.

[14]A system with broken $\mathcal{PT}$ symmetry is one in which there are some energy eigenvalues which occur in complex conjugate pairs. A system with unbroken $\mathcal{PT}$ symmetry is one in which all the energy eigenvalues are real.

### A. The behavior of the $g$ and $u$ coupling constants

The fixed points of $g$ are $g^*$ where $g^* = g^*_\pm = \pm\sqrt{\frac{8\pi^2\epsilon}{5}}$ and the trivial fixed point $g^* = 0$.[15] The related fixed points $u^*$ for $u$ are determined by

$$48g^{*4} - 3u^{*2} + 8g^{*2}u^* = 16\pi^2\epsilon u^*. \tag{71}$$

The solutions for $u^*$ are $u^* = 0$ and $u^*_\pm = u_\pm\epsilon$ where

$$u_\pm = \frac{8}{3}g_0^2 \pm \sqrt{\frac{64}{9}g_0^4 + 64} \tag{72}$$

and $g_0 = \sqrt{\frac{8\pi^2}{5}} \sim 3.97$, which gives $u_+ \sim 84.97$ and $u_- \sim -0.75$. Thus, we observe that $u_-$ is negative and, therefore, according to our discussion below (54) in Sec. III, is a Hermitian fixed point. On the other hand, $u_+$ is positive and, thus, is a non-Hermitian fixed point.[16] At the level of fixed points, non-Hermiticity is therefore introduced through the $u$ coupling. On the other hand, $g$ remains real at the fixed points. In summary, the various fixed points [in the $(g, u)$ plane], denoted by $(g^*, u^*)$, are given below:

(1) (0,0)
(2) $(0, -\frac{16\pi^2}{3}\epsilon)$
(3) $(g^*_+, u^*_+)$
(4) $(g^*_+, u^*_-)$
(5) $(g^*_-, u^*_+)$
(6) $(g^*_-, u^*_-)$

which will be denoted by $f_i$, $i = 1, \ldots, 6$. The $\epsilon$-dependent fixed points are examples of Wilson-Fisher fixed points [43]. Let us first discuss the linear stability of these fixed points.

#### 1. Stability of fixed points in the (u,g) plane

We denote deviations from the fixed points by $\delta g = g - g^*$ and $\delta u = u - u^*$. Linear stability analysis around the fixed point gives

$$\frac{d}{dt}\begin{pmatrix}\delta g\\ \delta u\end{pmatrix} = M(g^*, u^*, \epsilon)\begin{pmatrix}\delta g\\ \delta u\end{pmatrix}. \tag{73}$$

In order to avoid algebraic complexity we will consider the stability using a numerical value for $\epsilon = .0101321$ which leads to $g^*_\pm = \pm.4$, $u^*_+ = .887953$ and $u^*_- = -.461286$. Eigenvalues for $M$ at the fixed point $f_i$ are $\lambda_{i1}$ and $\lambda_{i2}$. The corresponding (un-normalized) two-dimensional eigenvectors are $e_{i1}$ and $e_{i2}$. Thus, we have

(1) $\lambda_{11} = -.0101321$ and $\lambda_{12} = -.00506605$ with $e_{11} = \binom{0}{1}$ and $e_{12} = \binom{1}{0}$,
(2) $\lambda_{21} = .0101321$ and $\lambda_{12} = -.00506605$ with $e_{21} = \binom{0}{1}$ and $e_{22} = \binom{1}{0}$,
(3) $\lambda_{31} = -.0357646$ and $\lambda_{32} = .0101321$ with $e_{31} = \binom{0}{1}$ and $e_{32} = \binom{.37403}{.927417}$,
(4) $\lambda_{41} = .0155004$ and $\lambda_{42} = .0101321$ with $e_{41} = \binom{0}{1}$ and $e_{42} = \binom{0904311}{-.995903}$,
(5) $\lambda_{51} = -.0357646$ and $\lambda_{52} = .0101321$ with $e_{51} = \binom{0}{1}$ and $e_{52} = \binom{.37403}{-.927417}$,
(6) $\lambda_{61} = .0155004$ and $\lambda_{62} = .0101321$ with $e_{61} = \binom{0}{1}$ and $e_{62} = \binom{0.0904311}{.995903}$.

A fixed point $f_i$ is
(i) a sink if $\lambda_{i1} < 0$ and $\lambda_{i2} < 0$
(ii) a source if $\lambda_{i1} > 0$ and $\lambda_{i2} > 0$
(iii) a saddle point if $\lambda_{i1} > 0$ and $\lambda_{i2} < 0$ or vice versa.

These fixed points help to organize the renormalization group flow through their basins of attraction. If an eigenvalue is 0, then a nonlinear analysis is required around the fixed point to determine its stability. We note that $t \to \infty$ is a flow to high energy; the flow $t \to -\infty$ is a flow to low energy. An energy of $O(1)$ corresponds to $t = 0$. We shall consider the fixed points for $m$ and $M$ later.

#### 2. The renormalization group flow for u and g

We shall consider the solutions of the coupled flow equations (63) and (64) [and the closely related equations (71) and (72)]. We can rewrite (63) as

$$\frac{dg}{dt} = \frac{5g}{16\pi^2}(g - g_+)(g - g_-). \tag{74}$$

For $g \gg g_+$ (74) simplifies to

$$\frac{dg}{dt} = \frac{5}{16\pi^2}g^3 \tag{75}$$

and leads to

$$g^2 = y = -\frac{1}{2(c + \frac{5t}{16\pi^2})} \tag{76}$$

where $c$ is a constant of integration. At $t = 0$, if the theory is Hermitian, then $c$ is negative. As $t$ increases, $g$ increases but remains Hermitian until at finite time $t = \frac{16\pi^2|c|}{5}$ the approximation of small $g$, and thus perturbative renormalization, breaks down.

For $g \ll g_-$ we again have (75) and $c$ is negative for a theory which is Hermitian at a scale $\mu \sim 1$. In the IR, $g$ remains small. In the UV, $g$ moves towards $g = 0$ but then veers away to large positive values of $g$ where perturbation theory is not trustworthy.

For $0 < g < g_+$ it is clear that $g \to 0$ as $t \to \infty$. As $t \to -\infty$ we have $g \to g_+$. As $\epsilon \to 0$ there is a bifurcation where the fixed points $g_+$, $g_-$, 0 coalesce. The trivial fixed point is unstable both in the IR and the UV.

[15] The sign of $g$ distinguishes separate parts of "theory" space.
[16] In another Hermitian model, to be discussed later, the possibility of flow to a non-Hermitian quartic self-coupling fixed point has also been noticed [44].

We will now consider the flow of $u$ using (70). The solution of (69) is

$$y(t) = -\frac{8d(\epsilon)\pi^2\epsilon}{5(e^{\epsilon t} - d(\epsilon)} \tag{77}$$

where $d(\epsilon) = e^{8\pi^2\epsilon c_1}(> 0)$ and $c_1$ is a constant of integration. The resultant solution of (70) for $u(t)$ is

$$u(t) = 8d(\epsilon)\pi^2\epsilon\left(1 + \sqrt{145} - \frac{c_2(\sqrt{145}-1)e^{\sqrt{29/5}\epsilon t}}{(e^{\epsilon t} - d(\epsilon))^{\sqrt{29/5}})}\right)\Big/ \left(15(e^{\epsilon t} - d(\epsilon))\left(1 + \frac{c_2 e^{\sqrt{\frac{29}{5}}\epsilon t}}{(e^{\epsilon t} - d(\epsilon))^{\sqrt{29/5}}}\right)\right) \tag{78}$$

where $c_2$ is an integration constant. This is complicated to analyze. If we keep away from the region of the fixed points near the origin, by considering $\epsilon \to 0$, the solutions in (77) and (78) can be simplified to

$$y(t) = -\frac{1}{\frac{5}{8\pi^2}t + c_1} \tag{79}$$

and

$$u(t) = \frac{8\pi^2[1 - \sqrt{145} + (1 + \sqrt{145})(8c_1\pi^2 + 5t)^{\sqrt{\frac{29}{5}}}c_2]}{3(8c_1\pi^2 + 5t)(1 + c_2(8c_1\pi^2 + 5t)^{\sqrt{29/5}})}. \tag{80}$$

For $g$ to be non-Hermitian at $t = 0$, we have $y < 0$, and so $c_1 > 0$; so as $t \to \infty$, $y$ remains non-Hermitian but slowly vanishes. In the infrared (IR), as $t$ decreases from $t = 0$, $y$ increases but remains non-Hermitian; perturbation theory becomes unreliable.

In the Hermitian case, $y > 0$ at $t = 0$ and so $c_1$ is negative. As $t$ increases from $t = 0$ $y$ remains Hermitian but increases until perturbation theory is invalid.

For $u$ to be real we need $(8c_1\pi^2 + 5t)$ to be non-negative. This requires $c_2 = 0$ and so

$$u(t) = -\frac{8\pi^2(\sqrt{145} - 1)}{3(8c_1\pi^2 + 5t)}. \tag{81}$$

The implication of a non-Hermitian $g$ ($c_1 > 0$) for $u$ is that it is Hermitian (i.e., $u < 0$) at $t = 0$. As $t \to \infty$, $u$ falls off to 0 but remains Hermitian. In the IR, $u$ increases until perturbation theory is unreliable.

The implication of a Hermitian $g$ ($c_1 < 0$) is that $u$ is non-Hermitian (i.e., $u > 0$) and remains so in the IR. The self-interaction coupling $u$ increases in the UV until the perturbative analysis becomes unreliable. In the IR, $u$ falls off but remains non-Hermitian.

### *3. The renormalization group flow for m and M*

The fixed points for $M$ and $\sigma$ can be deduced[17] from (67) and (68). Let $(u^*, y^*)$ denote any of the fixed points that we have already found. The possible fixed points are

(i) $(M^* = 0,\ \sigma^* = 0)$
(ii) $(M^* = 0,\ l(y^*, \sigma^*) = 0)$
(iii) $(l(y^*, \sigma^*) = 0,\ \sigma^* = 0)$

where $l(y, \sigma) = 4y(1 - 2\sigma^2) - u\sigma^2$. On analyzing these possibilities, we find the fixed points are $\sigma^* = M^* = 0$ in addition to the fixed points for $u$ and $g$.

The beta functions associated with (67)–(70) are

$$\beta_y(y, \epsilon) = \frac{5y^2}{8\pi^2} - \epsilon y \tag{82}$$

$$\beta_u(y, u, \epsilon) = \frac{48y^2 - 3u^2 + 8yu}{16\pi^2} - u\epsilon \tag{83}$$

$$\beta_\sigma(y, u, \sigma, M) = -\frac{\sigma}{32\pi}\left[4y\left(1 - 2\sigma^2\left(1 + \frac{M}{128\pi}\right)\right) - u\sigma^2\right] \tag{84}$$

$$\beta_M(y, u, \sigma, M) = \frac{M}{32\pi^2}[4y(1 - 2\sigma^2) - \sigma^2 u]. \tag{85}$$

In $(y, u, \sigma, M)$ space consider $\delta y \equiv y - y^*$, $\delta u \equiv u - u^*$, $\delta\sigma \equiv \sigma - \sigma^*$, and $\delta M \equiv M - M^*$. These linear deviations around a generic fixed point $(y^*, u^*, \sigma^*, M^*)$ satisfy

$$\frac{d}{dt}\begin{pmatrix} \delta y \\ \delta u \\ \delta\sigma \\ \delta M \end{pmatrix} = \underline{\underline{N}}\begin{pmatrix} \delta y \\ \delta u \\ \delta\sigma \\ \delta M \end{pmatrix} \tag{86}$$

where $\underline{\underline{N}}$ is a $4 \times 4$ matrix whose nonzero elements are given by

$$N_{11} = \partial_y\beta_y(y^*, \epsilon) = \frac{5y^*}{4\pi^2} - \epsilon \tag{87}$$

$$N_{21} = \partial_y\beta_u(y^*, u^*, \epsilon) = \frac{8u^* + 96y^*}{16\pi^2} \tag{88}$$

$$N_{22} = \partial_u\beta_u(y^*, u^*, \epsilon) = \frac{8y^* - 6u^*}{16\pi^2} - \epsilon \tag{89}$$

$$N_{31} = \partial_y\beta_\sigma(y^*, u^*, \sigma^*, M^*) = -\frac{M^*\sigma^{*2}}{16\pi^2} + \frac{\sigma^{*3}}{4\pi^2} - \frac{\sigma^*}{8\pi^2} \tag{90}$$

$$N_{32} = \partial_u\beta_\sigma(y^*, u^*, \sigma^*, M^*) = \frac{\sigma^{*3}}{32\pi^2} \tag{91}$$

[17] In principle, the poles of the Green's function and $m$ and $M$ are distinct.

$$N_{33} = \partial_\sigma \beta_\sigma(y^*, u^*, \sigma^*, M^*) = -\frac{M^* \sigma^* y^*}{8\pi^2} + \frac{3\sigma^{*2}(\frac{u^*}{8} + y^*)}{4\pi^2} - \frac{y^*}{8\pi^2} \quad (92)$$

$$N_{34} = \partial_M \beta_\sigma(y^*, u^*, \sigma^*, M^*) = -\frac{\sigma^{*2} y^*}{16\pi^2} \quad (93)$$

$$N_{41} = \partial_y \beta_M(y^*, u^*, \sigma^*, M^*) = \frac{M^*(1 - 2\sigma^{*2})}{8\pi^2} \quad (94)$$

$$N_{42} = \partial_u \beta_M(y^*, u^*, \sigma^*, M^*) = -\frac{M^* \sigma^{*2}}{32\pi^2} \quad (95)$$

$$N_{43} = \partial_\sigma \beta_M(y^*, u^*, \sigma^*, M^*) = -\frac{M^*(\sigma^* u^* + 8\sigma^* y^*)}{16\pi^2} \quad (96)$$

$$N_{44} = \partial_M \beta_M(y^*, u^*, \sigma^*, M^*) = \frac{4(1 - 2\sigma^{*2}) y^* - \sigma^{*2} u^*}{32\pi^2}. \quad (97)$$

The fixed points in the space $(y, u, \sigma, M)$ are

(i) $f_7 = (0, 0, 0, 0)$
(ii) $f_8 = (0, -\frac{\epsilon}{2}, 0, 0)$
(iii) $f_9 = (y_+^*, u_+^*, 0, 0)$
(iv) $f_{10} = (y_+^*, u_-^*, 0, 0)$

Recall the values of the fixed points used earlier in the $(g, u)$ plane: ${y_+}^* = {g_+}^{*2} = 0.16$, ${u_+}^* = 0.887953$, and ${u_-}^* = -0.461286$. In terms of $m$ and $M$, the fixed points are $m = m^* = 0$ and $M = M^* = 0$. From (65) and (66) we note that near these fixed points, for nontrivial $g$,

$$\frac{dm}{dt} = -\frac{g^{*2}}{8\pi^2} m, \quad (98)$$

$$\frac{dM}{dt} = \frac{g^{*2}}{8\pi^2} M. \quad (99)$$

Since (98) and (99) do not depend on $u^*$, they are not affected by non-Hermiticity. We note that

(i) A small deviation of $m$ increases (decreases) in the IR (UV).
(ii) A small deviation of $M$ increases (decreases) in the UV (IR).

## V. DYNAMICAL MASS GENERATION

A nonperturbative approach to dynamical mass generation was pioneered by NJL [16], extending the mechanism for the generation of a mass gap in superconductivity to relativistic particle physics. In their model, NJL discussed dynamical chiral symmetry breaking via the generation of fermion masses through appropriate bilinear fermion condensates that were formed as a result of attractive (*nonrenormalizable*) four-fermion contact interactions. They restricted their discussion to one loop, and found that fermion mass generation was possible when the coupling of the four-fermion interactions exceeded a critical value.

In our approach, we shall use the model (50), which, in contrast to the NJL model [16], is renormalizable and does not contain four-fermion interactions. It contains, however, Yukawa interactions and pseudoscalar self-interactions. As shown below, these interactions suffice to generate dynamical masses for both fermion and pseudoscalar fields, for small values of the Hermitian (real) coupling of the Yukawa interaction. We shall make use of the arguments of [9], according to which the dynamical generation of a non-Hermitian chiral-fermion-mass term $m_5 \overline{\psi} \gamma_5 \psi$ is not energetically favorable, to discuss only dynamical generation of a Dirac-type mass for the fermion $\psi$ and a mass for the pseudoscalar (axionlike) field $\phi$.

We shall use the simplified (one-loop) [17] NJL approach to check the possibility of dynamical mass generalization in our model (50). Within the context of our renormalizable theory, the NJL approach starts with a Lagrangian with nonzero renormalized masses $m$ and $M$ [17]. We will denote the massless free Lagrangian by $L_0$ (which contains the kinetic terms for the $\phi$ and $\psi$ fields) and the interaction Lagrangian by $L_{\rm int}$ (which contains the Yukawa and pseudoscalar self-interaction terms); thus the full Lagrangian $L$ is

$$\left(L_0 - m\overline{\psi}\psi - \frac{1}{2} M^2 \phi^2\right) + \left(L_{\rm int} + \Delta m \overline{\psi}\psi + \frac{1}{2} \Delta M^2 \phi^2\right). \quad (100)$$

At the end of the calculation of the two-point one-particle-irreducible (1PI) functions for the scalar and the fermion, we set $M^2 = \Delta M^2$ and $\Delta m = m$ [17]. The renormalized two-point 1PI functions for the fermion and scalar are *assumed* to behave like

$$\Gamma_f^{(2)}(p) = \widetilde{Z_f}(\gamma^\nu p_\nu - m) \quad (101)$$

$$\Gamma_s^{(2)}(p) = \tilde{Z}_s(p^2 - M^2) \quad (102)$$

for $p^2 \ll m^2, M^2$, where $\tilde{Z}_f$ and $\tilde{Z}_s$ are finite renormalizations. From renormalized *one-loop* perturbation theory for the fermion two-point function we can readily show that[18]

$$m + \frac{g^2 m}{16\pi^2} \int_0^1 dx \left(\gamma + \log\left(\frac{\Delta(x, m, M)}{4\pi\mu^2}\right)\right) = 0 \quad (103)$$

where $\gamma$ is the Euler constant and

$$\Delta(x, m, M) = x m^2 + (1 - x) M^2. \quad (104)$$

[18]One loop involving two Yukawa vertices contributes to the fermion self-energy [see Fig. 5(a)].

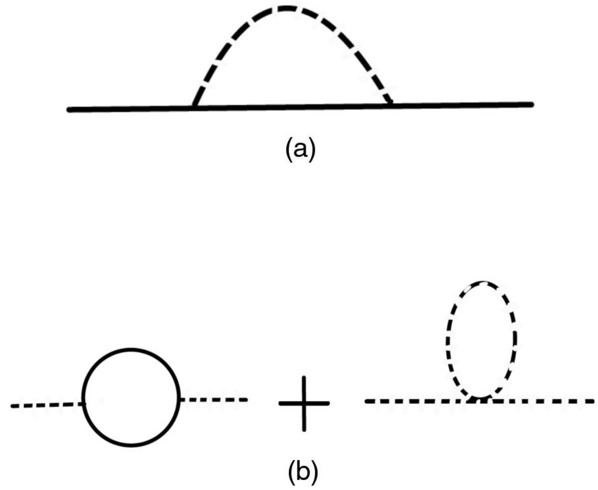


FIG. 5. (a) One-loop diagram contributing to the quantum corrections of the fermion self-energy in the model (50). Dashed lines correspond to pseudoscalar fields, while continuous lines indicate fermions. (b) One-loop diagrams contributing to the quantum corrections of the pseudoscalar self-energy.

From renormalized one-loop perturbation theory for the scalar two-point function we can also show that[19]

$$M^2 + \frac{uM^2}{2}\left(\frac{\gamma - 1 + \log\frac{M^2}{4\pi\mu^2}}{16\pi^2}\right) + \frac{g^2}{4\pi^2}\left[m^2\left(\gamma - 1 + \log\left(\frac{m^2}{4\pi\mu^2}\right)\right)\right] = 0. \tag{105}$$

This approach [17] to dynamical mass generation is approximate and relies on perturbative renormalizability. In order to analyze (104) and (105) it is convenient first to introduce dimensionless variables

$$a = \frac{m}{2\sqrt{\pi}\mu} \quad \text{and} \quad b = \frac{M^2}{4\pi\mu^2}. \tag{106}$$

In terms of $a$ and $b$, (104) and (105) read

$$1 = -\frac{g^2}{16\pi^2}\int_0^1 dx(\gamma + \log[xa^2 + (1-x)b]) \tag{107}$$

and

$$b + \frac{ub}{32\pi^2}(\gamma - 1 + \log b) + \frac{g^2a^2}{4\pi^2}(\gamma - 1 + \log a^2) = 0, \tag{108}$$

respectively. It is straightforward to show that

$$\int_0^1 dx \log[xa^2 + (1-x)b] = \frac{1}{a^2 - b}[a^2\log a^2 - a^2 - b\log b + b]$$

which has a vanishing limit as $a^2 \to b$ (a consistency requirement). For notational simplification we let

$$\overline{g}^2 \equiv \frac{g^2}{4\pi^2} \quad \text{and} \quad \overline{u} \equiv \frac{u}{32\pi^2}. \tag{109}$$

Then, Eq. (107) becomes

$$\left(1 + \frac{\overline{g}^2\gamma}{4} - \frac{\overline{g}^2}{4}(1 - \log a^2)\right)a^2 = b\left(\frac{\overline{g}^2}{4}(\log b - 1) + 1 + \frac{\overline{g}^2\gamma}{4}\right) \tag{110}$$

and (108) becomes

$$\overline{u}b\log b = -(-\overline{u}(1-\gamma) + 1)b - \overline{g}^2a^2(\gamma - 1 + \log a^2). \tag{111}$$

We shall study the possible solutions of (110) and (111) *in various limits*. Since our fixed points for $u$ and $g$ have been found in perturbation theory, we will not strictly adhere to their values at fixed points in considering the landscape of regimes where dynamical mass generation may be possible. This landscape will guide future nonperturbative studies using the Schwinger-Dyson equations, which will appear in a forthcoming publication. We will consider the following limiting cases:

(i) If $a^2$ and $b$ are both small, then from (110) we have approximately the leading behavior

$$\frac{\overline{g}^2}{4}a^2\log a^2 = \frac{\overline{g}^2}{4}b\log b \tag{112}$$

which is certainly compatible with $a^2 = b$.

Assuming then that $a^2 \simeq b$, we observe from (111) that the leading behavior gives

$$\overline{u}b\log b = -\overline{g}^2a^2\log a^2 \tag{113}$$

which would imply that a solution with small $a^2$ and small $b$ is possible if

$$\overline{u} \simeq -\overline{g}^2. \tag{114}$$

From (110) we can also deduce that

$$b \approx \exp\left(-\frac{4}{\overline{g}^2} - \gamma\right). \tag{115}$$

[19]Two Feynman diagrams contribute to the (pseudo)scalar self-energy, one involving a fermion loop and the other a (pseudo) scalar loop [see Fig. 5(b)].

So $g$ and $u$ would both have to be Hermitian and small for $b(\simeq a^2)$ to remain small. It is assumed that $\overline{g}^2$ is positive to avoid getting a $b$ which is too large, and so we stay here within the Hermitian Yukawa interactions. Hence, in this approach a *solution with small* $a^2$ *and* $b$ is only compatible with small $\overline{g}$ in the Hermitian case [We remind the reader that, according to our discussion below (54) in Sec. III, negative $u$ corresponds to the Hermitian theory].

Hence, mass generation can take place with $\overline{u}$ small and negative, for small $\overline{g}^2$. A Wilson-Fisher point that is qualitatively similar is $\overline{g}^2 = .4\epsilon$ and $\overline{u} = -.0783\epsilon$.

By substituting $a^2 = b$ in (108) and using (115), we obtain

$$\overline{u} = -\overline{g}^2 \frac{\overline{g}^2 + 3}{\overline{g}^2 + 4}, \tag{116}$$

which corrects (114) with higher orders in the Yukawa coupling.

Using definitions (106) and (109), we then arrive at the following expression for the dynamically generated fermion and axion masses, assumed to be approximately equal in magnitude:

$$m \simeq M = \tilde{\mu} \exp\left(-\frac{8\pi^2}{g^2}\right), \quad \tilde{\mu} \equiv \sqrt{4\pi} e^{-\gamma/2} \mu. \tag{117}$$

$\tilde{\mu}$ is the transmutation mass parameter redefined, in the standard way, to absorb the Euler's constant $\gamma$. Given that $g$ is perturbatively small in our analysis, the dynamically generated masses (117) are *non-perturbative* in the real Yukawa coupling $g$.

It must be noted that the form of the solution (117) is *identical* to the one generated through a Schwinger-Dyson approach in [10], in the Hermitian-Yukawa-interaction case, upon the replacement of the UV cutoff $\Lambda$ in the effective theory of that work with the transmutation mass $\tilde{\mu}$ in our approach. However, there is an *essential difference* in our case from that of [10], in that there is a nontrivial self interaction, which is necessarily nonvanishing, and its coupling is proportional to the negative of the square of the Yukawa coupling (116). The couplings are both in the Hermitian regime. Any possible non-Hermitian regime in this analysis would lead to the generation of very large masses, which is physically unacceptable in perturbation theory.

(ii) Let us look for solutions with $b \ll a^2$. We deduce from (110) that

$$\log a^2 = \left(-\frac{4}{\overline{g}^2} + 1 - \gamma\right) - \frac{b}{a^2}\left(1 - \gamma - \frac{4}{\overline{g}^2}\right) + \frac{b \log b}{a^2} \tag{118}$$

and from (108) that

$$\overline{u} b \log b = -b(1 + \overline{u}(\gamma - 1)) - \overline{g}^2 a^2 (\gamma - 1 + \log a^2). \tag{119}$$

For $b \ll a^2$ we have approximately from (118) that

$$a^2 \simeq \exp\left(1 - \gamma - \frac{4}{\overline{g}^2}\right). \tag{120}$$

Hence

$$\overline{u} b \log b = -(\overline{u}(\gamma - 1) + 1) b + 4a^2. \tag{121}$$

In order to be compatible with $b \ll a^2$, dominant balance requires

$$\overline{u} b \log b \simeq 4a^2 \tag{122}$$

and also

$$|\overline{u} \log b| \gg |\overline{u}(1 - \gamma) - 1|. \tag{123}$$

Since $a^2$ is positive and $b$ is small, from (122) we have $\overline{u} < 0$ and so

$$-\log b \gg 1 - \gamma + \frac{1}{|\overline{u}|}. \tag{124}$$

From (120) and (122) we can show that for Hermitian $\overline{u}$ and $\overline{g}$ there is a possibility of generating masses in this regime when $|\overline{u}|$ is an order of magnitude smaller than $\overline{g}$.

(iii) Let us look for solutions with $b \gg a^2$. This case includes the possibility of zero fermion mass as well. From (111) we have on invoking dominant balance

$$b \approx \exp\left[1 - \gamma - \frac{1}{\overline{u}}\right]. \tag{125}$$

For $b$ to be small we require small positive $\overline{u}$, i.e., $u$ is non-Hermitian. From (110) we have

$$a^2 = b\left[\frac{4 - \frac{\overline{g}^2}{u}}{4 + \gamma \overline{g}^2}\right]. \tag{126}$$

Consequently we require

$$\overline{u} \approx \frac{\overline{g}^2}{4}. \tag{127}$$

This indicates that solutions with $b \gg a^2$ may be viable when $\overline{u}$ and $\overline{g}^2$ have the same positive sign. Hence, in this mass regime we have a

Hermitian Yukawa coupling case, $g^2 > 0$, and an anti-Hermitian scalar self-interaction. In this case, we deduce that both axion and fermion dynamically generated masses are extremely suppressed for perturbatively small real Yukawa couplings $g$. As already mentioned, zero fermion masses are also compatible with this scenario. For anti-Hermitian Yukawa couplings, as considered in [9–11], $g^2 < 0$ (i.e., purely imaginary $g = i\breve{g}$, $\breve{g} \in \mathbb{R}$), this case leads to very large axion masses for $|\breve{g}| \ll 1$. We stress once again that our results above point to an essential difference from the studies in [9–11], where self-interactions of the axions were not considered; here the pseudoscalar self-interaction coupling $u$ is necessarily nontrivial for consistency of the quantum theory.

## VI. TWO-LOOP RENORMALIZATION GROUP ANALYSIS: RENORMALIZATION GROUP FLOWS BETWEEN HERMITIAN AND NON-HERMITIAN FIXED POINTS

The presence of both Hermitian and non-Hermitian fixed points within our models might be the result of the one-loop nature of our approximation. It is of course, in general, difficult to rule out this possibility without some parameter in the theory which can control the contributions of higher loops. However we have analyzed a two-loop renormalization flow [45,46] for a similar, but massless, Yukawa model given by the Lagrangian $\mathcal{L}_{MY}$. In what follows, we shall demonstrate that in such a model, there is a renormalization-group flow between Hermitian and non-Hermitian fixed points.

The Lagrangian $\mathcal{L}_{MY}$ is

$$\mathcal{L}_{MY} = \frac{1}{2}(\partial\phi)^2 + i\overline{\psi}\gamma^\mu\partial_\mu\psi + ig\phi\overline{\psi}\gamma_5\psi - \frac{u}{4!}\phi^4 \tag{128}$$

where $\psi$ is a massless Dirac-fermion field and $\phi$ is a massless pseudoscalar field, $g$ denotes the Yukawa coupling, and $u$ denotes the self-interaction of $\phi$. We shall consider $u > 0$ (the Hermitian case) but allow $g$ to be real or imaginary. From the consideration of the convergence of path integrals given earlier, we know that the usual Feynman rules are valid. If $u$ were to go towards a negative $u$ fixed point, according to the renormalization group flow, a resulting $|u|$, which is not small, might be indicative of an interesting behavior. If $|u|$ is small then the Feynman rules would be still valid since the Feynman rules give an approximation to the behavior near the trivial saddle point of the path integral.

We define for notational convenience

$$\tilde{g} \equiv \frac{g^2}{16\pi^2} \quad \text{and} \quad h \equiv \frac{u}{16\pi^2}. \tag{129}$$

The loop calculation involves 14 topologically distinct graphs. In [45,46] the calculation of the beta function $\beta_{\tilde{g}}$ for $\tilde{g}$ gives[20]

$$\beta_{\tilde{g}} = 10\tilde{g}^2 + \frac{1}{6}h^2\tilde{g} - 4h\tilde{g}^2 - \frac{57}{2}\tilde{g}^3 \tag{130}$$

and the calculation of the beta function $\beta_h$ for $h$ gives

$$\beta_h = 3h^2 + 8h\tilde{g} - 48\tilde{g}^2 - \frac{17}{3}h^3 - 12\tilde{g}h^2 + 28h\tilde{g}^2 + 384\tilde{g}^3. \tag{131}$$

We can show that there are four fixed points $(\tilde{g}_i, h_i)$, $i = 1, \ldots, 4$ where

$$\tilde{g}_1 = 0 \qquad h_1 = 0 \tag{132}$$

$$\tilde{g}_2 = 0 \qquad h_2 = 0.529412 \tag{133}$$

$$\tilde{g}_3 = -0.00570795 \qquad h_3 = 0.525424 \tag{134}$$

$$\tilde{g}_4 = 0.234024 \qquad h_4 = 1.01657. \tag{135}$$

In this two-loop calculation we note the appearance of a non-Hermitian [purely imaginary, cf. (129)] Yukawa coupling $g$ at the $i = 3$ fixed point.

### A. Stability analysis

A linear stability analysis at the fixed point $(\tilde{g}_i{}^*, h_i{}^*)$ in the $(\tilde{g}, h)$ coupling space gives
(1) for $i = 2$ the eigenvalues $-1.58824$, $0.0467128$
(2) for $i = 3$ the eigenvalues $-1.51242$, $-0.0479488$
(3) for $i = 4$ the eigenvalues $-13.1652$, $-2.34048$.
None of these fixed points are IR stable (where all the eigenvalues are positive).

The trivial fixed point $i = 1$ requires a separate nonlinear analysis. The flow near the trivial fixed point is approximated by

$$\frac{d\tilde{g}}{dt} = 10\tilde{g}^2 \tag{136}$$

$$\frac{dh}{dt} = 3h^2 + 8h\tilde{g} - 48\tilde{g}^2. \tag{137}$$

The solution of (136) [for $\tilde{g}(t)$] is

$$\tilde{g}(t) = \frac{\tilde{g}_0}{1 - 10\tilde{g}_0 t} \tag{138}$$

[20]In $D = 4 - \epsilon$ the beta functions for the couplings in the model would have $\epsilon$-dependent terms determined by the engineering dimensions of the couplings in the noninteger $D$ dimensions.

with $\tilde{g}(0) = \tilde{g}_0$. For $\tilde{g}_0 > 0$ (the Hermitian case) the flow is away from $\tilde{g} = 0$ in the UV ($t \to \infty$); in fact, we observe from (138), that in the UV limit, the coupling $\tilde{g}$ approaches 0, but from negative values, that is, there is a *flow* from Hermitian to non-Hermitian Yukawa couplings. The point $\tilde{g} = 0$ is IR ($t \to 0$) stable though. For the *non-Hermitian* case, $\tilde{g}_0 < 0$, the renormalization group flow in the UV is towards $\tilde{g} = 0$. The Yukawa coupling stays non-Hermitian during the flow. In the IR, the renormalization flow is away from $\tilde{g} = 0$.

Let us consider the behavior of $h(t)$ with $h(0) \equiv h_0$; the solution of (137) is

$$h(t) = \tilde{g}_0 \frac{11.0416 + 13.0416(1 - 10\tilde{g}_0 t)^{2.40832} c}{3(1 - 10\tilde{g}_0 t)[1 + (1 - 10\tilde{g}_0 t)^{2.40832} c]} \tag{139}$$

where $c = \frac{3.68053\tilde{g}_0 - h_0}{-4.3472\tilde{g}_0 + h_0}$. (The sign of $c$ is not important for the stability analysis.) For the *Hermitian* case, $\tilde{g}_0 > 0$, the renormalization group leads to a flow away from $h = 0$ towards a non-Hermitian value of $h$ in the UV and a flow towards $h = 0$ in the IR. For the *non-Hermitian* case, $\tilde{g}_0 < 0$, $h(t)$ flows to 0 in the UV and in the IR $h(t)$ flows away from $h = 0$ through Hermitian values of $h$. Hence, we see an interplay of Hermitian and non-Hermtian behavior in the renormalization group behavior.[21]

The possible connection between Hermitian and non-Hermitian fixed points that we have noticed is unlikely to be an artifact. There is some independent evidence that this happens in other theories although the possible connection with $\mathcal{PT}$ symmetry was not realized. This independent evidence has been found in a more complicated model, a chiral Yukawa model [44], with the Standard Model symmetry implemented only at the global level. The flow of the quartic scalar coupling from positive to negative values was observed. However whether a $\mathcal{PT}$ interpretation is valid in detail remains to be explored.

## VII. CONCLUSIONS AND OUTLOOK

In this work we have laid the foundation for the analysis of field theories involving $\mathcal{PT}$ symmetric interactions between a pseudoscalar and a Dirac fermion using a path-integral formulation.. We have studied a perturbative renormalization-group analysis of the model, given in (3), involving a self-interacting pseudoscalar (axionlike) field coupled to fermions. The model, without axion self-interactions (and axion potentials) has previously been considered from the point of view of dynamical mass generation within a Schwinger-Dyson framework [10,11]. We have noted here that quantum consistency at the perturbative level requires the presence of a nontrivial quartic $\phi^4$ coupling, which is proportional to the square of the Yukawa interaction $g^2$, to leading order in perturbation theory in $g$.

Motivated by the models of [10,11], we have considered both Hermitian ($g^2 > 0$) and anti-Hermitian ($g^2 < 0$) Yukawa couplings. As a preparation for a full Schwinger-Dyson treatment, which we postpone to a future publication, we have studied here the possibilities for a one-loop dynamical mass generation for both axions and fermions using a method due to Nambu and Joan-Lasinio.

We have focused here on a one-loop dynamical mass generation for both axion and fermion fields in the model (3), which we have studied in various limits. Depending on the sign of the squared coefficient, one can obtain Hermitian or anti-Hermitian axion self-interactions. In the Hermitian Yukawa interaction case, for the case of equal masses of axions and fermions, we have recovered the nonperturbative expression for the dynamically generated masses of axions and fermions, discussed in [10,11] (but in the presence of the axion self-interaction quartic coupling $u$).

We have also managed to demonstrate that there is a renormalization-group flow between Hermitian and non-Hermitian fixed points in the theory, which also manifests itself at two loops in a model with no bare axion and fermion masses. This last result might be interpreted as a spontaneous appearance of non-Hermiticity in this class of models, in analogy to the situation characterizing Nambu–Jona-Lasinio theories with four-fermion interactions. The reader should also recall that, as far as the quartic (pseudoscalar) coupling $u$ is concerned, there is a flow from the Hermitian case ($u < 0$), corresponding to a positive $\phi^4$ potential [cf. (50)], to the non-Hermitian ($\mathcal{PT}$ symmetric) case ($u > 0$) that corresponds to an upside down $\phi^4$ self-interaction potential.

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## APPENDIX: ASPECTS OF PSEUDOHERMITICITY

It was pointed out by Mostafazadeh [13] that $\mathcal{PT}$ symmetry was part of a more general framework known as pseudo-Hermiticity. We know that $\mathcal{PT}$ symmetry implies

[21]It would interesting to compare the non-Hermiticity in our model with spontaneous non-Hermiticity discussed in [25] within the traditional NJL model.

$$H = \mathcal{P}^{-1}H^{\dagger}\mathcal{P} = \mathcal{P}H^{\dagger}\mathcal{P} \tag{A1}$$

since $\mathcal{P} = \mathcal{P}^{\dagger}$ and $\mathcal{P}^2 = 1$.

In quantum mechanics any operator $H$ (acting on a Hilbert space $\mathcal{H}$) is pseudo-Hermitian if it can be related to its adjoint by

$$H^{\dagger} = \eta H \eta^{-1} \tag{A2}$$

where $\eta$ is a bounded automorphism of the Hilbert space ($\eta$ can be chosen to be Hermitian). Pseudo-Hermiticity is a generalization of both Hermiticity and $\mathcal{PT}$ symmetry. If the usual model-independent inner product on $\mathcal{H}$ is written as $\langle | \rangle$ then

$$\langle \phi | H\psi \rangle = \langle H^{\dagger}\phi | \psi \rangle. \tag{A3}$$

The pseudo-Hermitian $H$ can have both real and complex conjugate eigenvalues. Let us consider a form on $\mathcal{H}$ defined by

$$\langle \phi | \psi \rangle_{\eta} \equiv \langle \phi | \eta | \psi \rangle = \langle \phi | \eta\psi \rangle = \langle \eta\phi | \psi \rangle. \tag{A4}$$

The adjoint with respect to this inner product, $\widehat{H}$ say, is defined by

$$\langle \widehat{H}\phi | \psi \rangle_{\eta} = \langle \phi | H\psi \rangle_{\eta} = \langle \phi | \eta H \psi \rangle \tag{A5}$$

$$= \langle H^{\dagger}\eta\phi | \psi \rangle = \langle \eta\eta^{-1}H^{\dagger}\eta\phi | \psi \rangle \tag{A6}$$

$$= \langle \eta^{-1}H^{\dagger}\eta\phi | \psi \rangle_{\eta}. \tag{A7}$$

So $\widehat{H} = \eta^{-1}H^{\dagger}\eta$ and

$$\widehat{H} = H. \tag{A8}$$

In order to have a probabilistic interpretation for the quantum mechanics in terms of this inner product, it is necessary to choose $\eta$ to be a positive operator $\eta = \tilde{\eta}^{\dagger}\tilde{\eta}$. We can write

$$H = \tilde{\eta}^{-1}(\tilde{\eta}^{-1})^{\dagger}H^{\dagger}\tilde{\eta}^{\dagger}\tilde{\eta} \tag{A9}$$

and so

$$\tilde{\eta}H\tilde{\eta}^{-1} = (\tilde{\eta}^{-1})^{\dagger}H^{\dagger}\tilde{\eta}^{\dagger} = (\tilde{\eta}H\tilde{\eta}^{-1})^{\dagger}. \tag{A10}$$

If we identify

$$h = \tilde{\eta}H\tilde{\eta}^{-1} \tag{A11}$$

then from (A10) we find

$$h = h^{\dagger}. \tag{A12}$$

Hence, for any pseudo-Hermitian $H$ we can, in principle, find a corresponding Hermitian $h$ using a similarity transformation $q$ which may not be unique. This argument would also be valid for other pseudo-Hermitian operators. This result has been found in a quantum system with a finite number of degrees of freedom. For a field theory with an infinite number of degrees of freedom it may be possible to formally construct $\tilde{\eta}$ and requires further investigation [cf. Eq. (55)].


[1] C. M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians Having $\mathcal{PT}$ Symmetry, Phys. Rev. Lett. **80**, 5243 (1998).

[2] C. M. Bender *et al.*, *PT Symmetry in Quantum and Classical Physics* (World Scientific, Singapore, 2019).

[3] C. M. Bender, D. C. Brody, and H. F. Jones, Extension of $\mathcal{PT}$ symmetric quantum mechanics to quantum field theory with cubic interactions, Phys. Rev. D **70**, 025001 (2004); Erratum, Phys. Rev. D **71**, 049901 (2005); Scalar Quantum Field Theory with Cubic Interaction, Phys. Rev. Lett. **93**, 251601 (2004).

[4] C. M. Bender, S. F. Brandt, J. H. Chen, and Q. Wang, Ghost busting: $\mathcal{PT}$ symmetric interpretation of the Lee model, Phys. Rev. D **71**, 025014 (2005).

[5] J. Alexandre, J. Ellis, and P. Millington, Discrete spacetime symmetries and particle mixing in non-Hermitian scalar quantum field theories, Phys. Rev. D **102**, 125030 (2020); J. Alexandre, J. Ellis, P. Millington, and D. Seynaeve, Spontaneously breaking non-Abelian gauge symmetry in non-Hermitian field theories, Phys. Rev. D **101**, 035008 (2020); Gauge invariance and the Englert-Brout-Higgs mechanism in non-Hermitian field theories, Phys. Rev. D **99**, 075024 (2019); Spontaneous symmetry breaking and the Goldstone theorem in non-Hermitian field theories, Phys. Rev. D **98**, 045001 (2018); P. D. Mannheim, Extension of the Goldstone and the Englert-Brout-Higgs mechanisms to non-Hermitian theories, Goldstone bosons and the Englert-Brout-Higgs mechanism in non-Hermitian theories, Phys. Rev. D **99**, 045006 (2019); A. Fring and T. Taira, Non-Hermitian gauge field theories and BPS limits, 't Hooft-Polyakov monopoles in non-Hermitian quantum field theory, Phys. Lett. B **807**, 135583 (2020); Massive gauge particles versus Goldstone bosons in non-Hermitian non-Abelian gauge theory, Pseudo-Hermitian approach to

Goldstone's theorem in non-Abelian non-Hermitian quantum field theories, Phys. Rev. D **101**, 045014 (2020); Goldstone bosons in different PT-regimes of non-Hermitian scalar quantum field theories, Nucl. Phys. **B950**, 114834 (2020); J. Alexandre, P. Millington, and D. Seynaeve, Symmetries and conservation laws in non-Hermitian field theories, Phys. Rev. D **96**, 065027 (2017).

[6] C. M. Bender, H. F. Jones, and R. J. Rivers, Dual PT-symmetric quantum field theories, Phys. Lett. B **625**, 333 (2005).

[7] J. Alexandre, C. M. Bender, and P. Millington, Non-Hermitian extension of gauge theories and implications for neutrino physics, J. High Energy Phys. 11 (2015) 111.

[8] P. N. Meisinger and M. C. Ogilvie, PT symmetry in classical and quantum statistical mechanics, Phil. Trans. R. Soc. A **371**, 20120058 (2013).

[9] J. Alexandre and N. E. Mavromatos, On the consistency of a non-Hermitian Yukawa interaction, Phys. Lett. B **807**, 135562 (2020).

[10] J. Alexandre, N. E. Mavromatos, and A. Soto, Dynamical Majorana neutrino masses and axions I, Nucl. Phys. **B961**, 115212 (2020).

[11] N. E. Mavromatos and A. Soto, Dynamical Majorana neutrino masses and axions II: Inclusion of anomaly terms and axial background, Nucl. Phys. **B962**, 115275 (2021).

[12] J. Alexandre, J. Ellis, and P. Millington, $\mathcal{PT}$ -symmetric non-Hermitian quantum field theories with supersymmetry, Phys. Rev. D **101**, 085015 (2020).

[13] A. Mostafazadeh, Pseudo-Hermiticity versus PT symmetry. The necessary condition for the reality of the spectrum, J. Math. Phys. (N.Y.) **43**, 205 (2002); Exact PT symmetry is equivalent to Hermiticity, J. Phys. A **36**, 7081 (2003).

[14] F. G. Scholtz, H. B. Geyer, and F. J. W. Hahne, Quasi-Hermitian operators in quantum mechanics and the variational principle, Ann. Phys. (N.Y.) **213**, 74 (1992).

[15] *Parity-Time Symmetry and Its Applications*, edited by D. Christodoulides and J. Yang (Springer, New York, 2018).

[16] Y. Nambu and G. Jona-Lasinio, Dynamical model of elementary particles based on an analogy with superconductivity. 1, Phys. Rev. **122**, 345 (1961).

[17] L. N. Chang and N. P. Chang, The renormalization group analysis of dynamical symmetry breaking in QCD, Phys. Rev. D **29**, 312 (1984).

[18] P. Dorey, C. Dunning, and R. Tateo, Spectral equivalences, Bethe Ansatz equations, and reality properties in PT-symmetric quantum mechanics, J. Phys. A **34**, 5679 (2001).

[19] P. Dorey, C. Dunning, and R. Tateo, The ODE/IM correspondence, J. Phys. A **40**, R205 (2007).

[20] C. M. Bender, N. Hassanpour, S. P. Klevansky, and S. Sarkar, $PT$-symmetric quantum field theory in $D$ dimensions, Phys. Rev. D **98**, 125003 (2018).

[21] A. Felski, C. M. Bender, S. P. Klevansky, and S. Sarkar, Towards perturbative renormalization of $\phi^2(i\phi)^\epsilon$ quantum field theory, Phys. Rev. D **104**, 085011 (2021).

[22] A. Beygi, S. P. Klevansky, and C. M. Bender, Two- and four-dimensional representations of the $PT$- and $CPT$-symmetric fermionic algebras, Phys. Rev. A **97**, 032128 (2018).

[23] A. Beygi, S. P. Klevansky, and C. M. Bender, Relativistic $PT$-symmetric fermionic theories in $1+1$ and $3+1$ dimensions, Phys. Rev. A **99**, 062117 (2019).

[24] A. Felski, A. Beygi, and S. P. Klevansky, Non-Hermitian extension of the Nambu–Jona-Lasinio model in $3+1$ and $1+1$ dimensions, Phys. Rev. D **101**, 116001 (2020).

[25] M. N. Chernodub, A. Cortijo, and M. Ruggieri, Spontaneous non-Hermiticity in the Nambu–Jona-Lasinio model, Phys. Rev. D **104**, 056023 (2021).

[26] R. J. Rivers, *Path Integral Methods in Quantum Field Theory* (Cambridge University Press, Cambridge, England, 1987).

[27] R. Streater and A. S. Wightman, *PCT, Spin, Statistics and All That* (Princeton University Press, Princeton, NJ, 2000).

[28] A. Mostafazadeh, Pseudo-Hermitian representation of quantum mechanics, Int. J. Geom. Methods Mod. Phys. **07**, 1191 (2010).

[29] P. D. Mannheim, Appropriate inner product for PT-symmetric Hamiltonians, Phys. Rev. D **97**, 045001 (2018).

[30] P. D. Mannheim, Antilinearity rather than hermiticity as a guiding principle for quantum theory, J. Phys. A **51**, 315302 (2018); Antilinear symmetry and the ghost problem in quantum field theory, arXiv:2104.03708.

[31] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (Springer, New York, 1999).

[32] C. M. Bender, D. C. Brody, and H. F. Jones, Extension of PT-symmetric quantum mechanics to quantum field theory with cubic interaction, Phys. Rev. D **70**, 025001 (2004).

[33] P. Ramond, *Field Theory: A Modern Primer* (Westview Press, 1997).

[34] C. M. Bender, A. Felski, S. P. Klevansky, and S. Sarkar, *PT* symmetry and renormalisation in quantum field theory, J. Phys. Conf. Ser. **2038**, 012004 (2021).

[35] H. F. Jones and R. J. Rivers, Which green functions does the path integral for quasi-Hermitian Hamiltonians represent?, Phys. Lett. A **373**, 3304 (2009).

[36] C. M. Bender and H. F. Jones, Semiclassical calculation of the $\mathcal{C}$ operator in $\mathcal{PT}$ symmetric quantum mechanics, Phys. Lett. A **328**, 102 (2004).

[37] H. F. Jones and R. J. Rivers, Disappearing $Q$ operator, Phys. Rev. D **75**, 025023 (2007).

[38] C. M. Bender, J.-H. Chen, and K. A. Milton, PT-symmetric versus Hermitian formulations of quantum mechanics, J. Phys. A **39**, 1657 (2006).

[39] H. F. Jones, On pseudo-Hermitian Hamiltonians and their Hermitian counterparts, J. Phys. A **38**, 1741 (2005).

[40] L. B. Okun, C, P, T are broken. Why not CPT?, arXiv:hep-ph/0210052.

[41] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields*, 1st ed. (McGraw-Hill College, New York, 1965), ISBN-10:0070054940.

[42] M. D. Schwartz, *Quantum Field Theory and the Standard Model* (Cambridge University Press, UK, 2014), ISBN-13: 978-1107034730.

[43] K. G. Wilson and M. E. Fisher, Critical Exponents in 3.99 Dimensions, Phys. Rev. Lett. **28**, 240 (1972).

[44] E. Mølgaard and R. Shrock, Renormalization-group flows and fixed points in Yukawa theories, Phys. Rev. D **89**, 105007 (2014).

[45] C. Manuel, Differential renormalization of a Yukawa model with $\gamma_5$, Int. J. Mod. Phys. A **08**, 3223 (1993).

[46] C. Schubert, The Yukawa model as an example for dimensional renormalization with $\gamma_5$, Nucl. Phys. **B323**, 478 (1989).