Induced topological gravity and anomaly inflow from Kähler-Dirac fermions in odd dimensions

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We show that the effective action that results from integrating out massive Kähler-Dirac fermions propagating on a curved three-dimensional space is a topological gravity theory of Chern-Simons type. In the presence of a domain wall, massless, two-dimensional Kähler-Dirac fermions appear that are localized to the wall. Potential gravitational anomalies arising for these domain wall fermions are canceled via anomaly inflow from the bulk gravitational theory. We also study the invariance of the theory under large gauge transformations. The analysis and conclusions generalize straightforwardly to higher dimensions.

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I. INTRODUCTION

An alternative to the Dirac equation for describing fermions was proposed many years ago by Kähler [1]. It is based on the simple observation that a natural square root of the Laplacian is the operator $d - d^\dagger$ where d is the exterior derivative and $d^\dagger$ its adjoint. A key difference that distinguishes the Kähler-Dirac operator from its Dirac cousin is the fact that the former can be defined without reference to a local frame and spin connection. Thus Kähler-Dirac fermions are not globally equivalent to Dirac fermions and are well-defined on any smooth manifold. Nevertheless there is a close connection between the two in flat space. The Kähler-Dirac equation in D-dimensional flat space takes the form

$$(d - d^\dagger - m)\Phi = 0, \tag{1}$$

where $\Phi = (\omega, \omega_\mu, \omega_{\mu\nu}, \ldots, \omega_{\mu_1\cdots\mu_D})$ is a collection of p-form fields with p running from 0 to D. In this paper we will work in Euclidean space. It is straightforward to show that in even dimensions this can be mapped into a Dirac equation describing $2^{D/2}$ degenerate Dirac spinors corresponding to the columns of a $2^{D/2} \times 2^{D/2}$ matrix Ψ [2,3],

$$(\gamma^\mu \partial_\mu - m)\Psi = 0, \tag{2}$$

where

$$\Psi = \sum_{\mu=0}^{D} \omega_{\mu_1\cdots\mu_p}\gamma^{\mu_1}\cdots\gamma^{\mu_p}. \tag{3}$$

Kähler-Dirac fields arise naturally in twisted supersymmetric theories [4] and are closely related to staggered fermions [2,5]. Recently, there has been renewed interest in them in connection with Dai-Freed anomalies [6,7], topological insulators [8], and symmetric mass generation in staggered fermion lattice models [9–13]. They have also been proposed as an ingredient in the construction of chiral lattice theories [14].

One consequence of this work has been the realization that massless Kähler-Dirac theories in even dimensions suffer from a gravitational anomaly, which breaks a global $U(1)$ symmetry, unique to Kähler-Dirac fermions, down to Z_4 [15,16]. In four dimensions this anomaly is given by the Euler density $\int \epsilon_{abcd} R^{ab} \wedge R^{cd}$ in contrast to the usual gravitational anomaly of Dirac fermions given by $\int R^{ab} \wedge R^{ab}$ with R the Riemann tensor [17,18]. It should be noted that since Kähler-Dirac fermions can be decomposed into Dirac fermions in flat space they do not possess conventional γ_5 anomalies.

Remarkably this new anomaly survives discretization since it depends only on the topology of the background which can be captured exactly in a simplicial approximation to the space. This Z_4 symmetry prohibits bare mass terms but allows for four fermion interactions which can gap fermions without breaking symmetries for multiples of two Kähler-Dirac fields [16]. In flat space each such Kähler-Dirac field can be decomposed into $2^{D/2+1}$ Majorana spinors, and we deduce that such theories contain eight and sixteen Majorana spinors in two and four dimensions, respectively. These magic fermion numbers that allow for symmetric mass generation are in agreement with the cancellation of certain discrete anomalies for Weyl

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fermions—chiral fermion parity in two dimensions and spin-Z_4 symmetry in four [6,19].

In this paper we will show that *massive* Kähler-Dirac fermions in odd dimensions exhibit further interesting properties; they yield gravitational Chern-Simons (CS) theories at low energies. Furthermore, in the presence of domain walls, these theories contain massless Kähler-Dirac fields localized to the domain wall. We show that potential anomalies for these domain wall fermions, of the type discussed above, are canceled via anomaly inflow from the bulk gravitational theory.

II. KÄHLER-DIRAC FERMIONS IN THREE DIMENSIONS

Following our earlier discussion the massless Kähler-Dirac (KD) action in three dimensions can be written as

$$\int d^3x\sqrt{g}\bar{\Phi}(d-d^\dagger)\Phi, \tag{4}$$

where $\Phi = (\phi, \phi_\mu, \phi_{\mu\nu}, \phi_{\mu\nu\lambda})$ is a collection of p-forms (antisymmetric tensors). Notice that such a field possesses eight (complex) components in three dimensions.

This action is invariant under a $U(1)$ symmetry of the form

$$\Phi \to e^{i\alpha\Gamma}\Phi, \qquad \bar{\Phi} \to \bar{\Phi}e^{i\alpha\Gamma}, \tag{5}$$

where the linear operator Γ acts on the component p-forms ϕ_p according to whether it carries an even or odd number of indices $\phi_p \to (-1)^p\phi_p$. This property implies that Γ anticommutes with the Kähler-Dirac operator which then ensures the $U(1)$ symmetry of the action. Furthermore, the operator Γ can be used to construct projectors $P_\pm = \frac{1}{2}(I \pm \Gamma)$ which act naturally on a Kähler-Dirac field to yield a pair of so-called reduced Kähler-Dirac fields $\Phi_\pm = P_\pm\Phi$. The Kähler-Dirac operator maps between Φ_+ and Φ_- and vice versa.

If we want to map three-dimensional Kähler-Dirac fermions into a set of spinors, we will need the analog of the matrix expansion given in Eq. (3). Clearly one cannot map the eight component fields of a Kähler-Dirac fermion in three dimensions using just the minimal Dirac matrices corresponding to the Pauli matrices. Instead one must double the number of components of the spinor with the resulting matrix representation of the three-dimensional Kähler-Dirac field employing 4×4 gamma matrices.[1] Naively such a field carries 16 degrees of freedom but this can be reduced to 8 using the projection operators $P_\pm$ described earlier. These can be implemented in the matrix representation as

$$\Psi_\pm = P_\pm\Psi = \frac{1}{2}(\Psi \pm \gamma_5\Psi\gamma_5). \tag{6}$$

The use of this four-dimensional representation allows one to write down a *massive* three-dimensional Kähler-Dirac action which preserves the $U(1)$ symmetry provided the mass term is taken proportional to γ^4:

$$S = \int d^3x \mathrm{Tr}[\bar{\Psi}(\gamma^\mu\partial_\mu - i\gamma^4 M)P_+\Psi]. \tag{7}$$

This action is invariant under a global spin(3) Lorentz symmetry L and a global spin(4) flavor symmetry F which act on the fields as

$$\Psi_+ \to L\Psi_+F^\dagger, \qquad \bar{\Psi}_- \to F\bar{\Psi}_-L^\dagger. \tag{8}$$

Notice that F should contain L as a subgroup to reflect the Kähler-Dirac nature of the fermions since under a Lorentz transformation a Kähler-Dirac field must transform as a sum of tensor representations. However, to facilitate the computation of the effective action in the next section we will treat both symmetries as independent when gauging the action and impose the Kähler-Dirac condition relating the corresponding gauge fields only after the fermion integration. On a curved space as well as having gauged the flavor symmetry the action is modified to [22]

$$S = \int d^3x\, \hat{E}\, \mathrm{Tr}[\bar{\Psi}(\hat{E}^\mu_{\mathbb{A}}\gamma^{\mathbb{A}}D_\mu - i\gamma^4 M)P_+\Psi], \tag{9}$$

where $\hat{E}_\mu$ is a 3-frame corresponding to the background metric and D_μ the associated covariant derivative which acts on the field Ψ as

$$D_\mu\Psi = \partial_\mu\Psi + \Omega_\mu\Psi - \Psi\hat{\Omega}_\mu, \tag{10}$$

where Ω_μ is the three-dimensional spin connection and $\hat{\Omega}_\mu$ is a spin(4) flavor gauge field. Notice that while Kähler-Dirac fermions do not require the use of a spin connection, it is necessary to introduce such an object to do calculations in the matrix basis where the Kähler-Dirac field is represented in terms of Ψ.

III. INTEGRATING OUT THE FERMIONS

We will focus on deriving an effective action for $\hat{\Omega}_\mu$ perturbatively in the limit $M \to \infty$. If we integrate out the fermions, we obtain an effective action which can be written

[1]Three-dimensional fermions of this type are called reducible fermions and correspond to a sum of the two irreducible spinor representations for spin(3)—see [20,21].

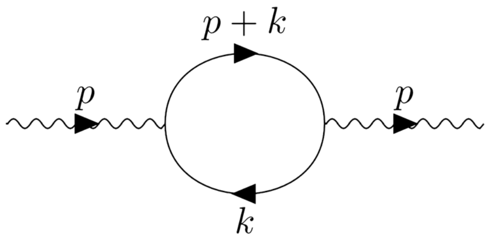

FIG. 1. One loop contribution to vacuum polarization.

$$\begin{aligned} S_{\rm eff} &= \mathrm{Tr}\log\left[(\partial\!\!\!/ - i\gamma^4 M + V\!\!\!/)P_+\right] \\ &= \mathrm{Tr}\log\left[(\partial\!\!\!/ - i\gamma^4 M)\left(I + \frac{V\!\!\!/}{\partial\!\!\!/ - i\gamma^4 M}\right)P_+\right] \\ &= \mathrm{Tr}\log\left[\left(I + \frac{V\!\!\!/}{\partial\!\!\!/ - i\gamma^4 M}\right)P_+\right] \\ &\quad + \text{terms independent of } V. \end{aligned} \tag{11}$$

Expanding the logarithm the leading term is

$$-\frac{1}{2}\mathrm{Tr}\left[\left(\frac{V\!\!\!/}{\partial\!\!\!/ - i\gamma^4 M}\right)^2 P_+\right] \tag{12}$$

corresponding to the diagram in Fig. 1. Here $V\!\!\!/ = V_L + V_R$ where the subscripts indicate whether the gauge field acts on the left or right of the matrix fermion Ψ:

$$V_L = \hat{E}^\mu_{\mathbb{A}}\gamma^{\mathbb{A}}\Omega_\mu, \tag{13}$$

$$V_R = -\hat{E}^\mu_{\mathbb{A}}\gamma^{\mathbb{A}}\hat{\Omega}_\mu. \tag{14}$$

In momentum space this gives[2]

$$\begin{aligned} S^{\rm quad}_{\rm eff} &= \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\mathrm{Tr}\left(\frac{k\!\!\!/ - \gamma^4 M}{k^2+M^2}\gamma^\mu V_\mu \frac{k\!\!\!/ + p\!\!\!/ - \gamma^4 M}{(k+p)^2+M^2} \right. \\ &\quad \left. \times\gamma^\nu V_\nu P_+\right). \end{aligned} \tag{15}$$

If we focus on the contribution that is linear in the external momentum p, we find

$$\begin{aligned} S^{\rm quad}_{\rm eff} &= -\frac{1}{2}\times\frac{1}{2}\times\left(-\frac{1}{2}\right)^2 \mathrm{tr}(\hat{E}\gamma^5\gamma^4\gamma^{\mathbb{E}}\gamma^{\mathbb{F}}\gamma^{\mathbb{G}}) \\ &\quad\times\hat{E}^\mu_{\mathbb{E}}\hat{E}^\delta_{\mathbb{F}}\hat{E}^\nu_{\mathbb{G}}\times\mathcal{I}\times p_\delta\hat{\Omega}^{\mathbb{AB}}_\mu(-p)\hat{\Omega}^{\mathbb{CD}}_\nu(p) \\ &\quad\times\mathrm{tr}(\sigma_{\mathbb{AB}}\sigma_{\mathbb{CD}}\gamma_5). \end{aligned} \tag{16}$$

Notice that a nonvanishing contribution comes only from employing V_R at both vertices. The integral $\mathcal{I}$ is given by

$$\mathcal{I} = \int\frac{d^3k}{(2\pi)^3}\frac{M}{(k^2+M^2)((k+p)^2+M^2)}. \tag{17}$$

For $M \gg p$ and rescaling $k/M \to k$,

$$\mathcal{I} = \frac{M}{|M|}\times\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+1)^2} = \frac{M}{|M|}\times\frac{1}{8\pi}. \tag{18}$$

Employing the identity $\hat{E}\epsilon^{\mathbb{ABC}}\hat{E}^\mu_{\mathbb{A}}\hat{E}^\delta_{\mathbb{B}}\hat{E}^\nu_{\mathbb{C}} = \epsilon^{\mu\delta\nu}$ we find a contribution to the effective action of the form

$$\begin{aligned} S^{\rm quad}_{\rm eff} &= -i\frac{M}{|M|}\times 4\times\left(\frac{1}{2}\right)^4\times\frac{1}{8\pi} \\ &\quad\times\int d^3x\,\epsilon^{\mu\delta\nu}\hat{\Omega}^{\mathbb{AB}}_\mu(\partial_\delta\hat{\Omega}^{\mathbb{CD}}_\nu)\epsilon_{\mathbb{ABCD}}. \end{aligned} \tag{19}$$

This is not gauge invariant. There is, however, a contribution coming from next order in the expansion of the logarithm:

$$\frac{1}{3}\mathrm{Tr}\left[\left(\frac{V\!\!\!/}{\partial\!\!\!/ - i\gamma^4 M}\right)^3 P_+\right], \tag{20}$$

which corresponds to the Feynman diagram in Fig. 2. In momentum space this gives

$$\begin{aligned} S^{\rm cubic}_{\rm eff} &= \frac{i}{3}\int\frac{d^3k}{(2\pi)^3}\mathrm{Tr}\left(\frac{k\!\!\!/ - \gamma^4 M}{k^2+M^2}\gamma^\mu V_\mu\frac{k\!\!\!/ + q\!\!\!/ - \gamma^4 M}{(k+q)^2+M^2}\right. \\ &\quad\left.\times\gamma^\delta V_\delta\frac{k\!\!\!/ - p\!\!\!/ - \gamma^4 M}{(k-p)^2+M^2}\gamma^\nu V_\nu P_+\right). \end{aligned} \tag{21}$$

Extracting the leading term which again uses only V_R yields

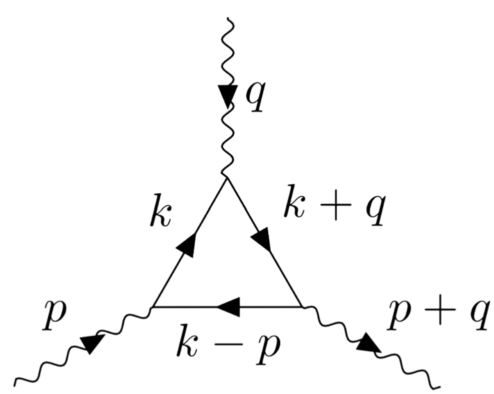

FIG. 2. One loop contribution to three gauge boson vertex.

[2] For more details see the Appendix.

$$S_{\rm eff}^{\rm cubic} = -i\times\frac{1}{3}\times\frac{1}{2}\times\left(-\frac{1}{2}\right)^3 {\rm tr}(\hat{E}\gamma^5\gamma^4\gamma^{\mathbb{G}}\gamma^4\gamma^{\mathbb{H}}\gamma^4\gamma^{\mathbb{I}})\hat{E}^\mu_{\mathbb{G}}\hat{E}^\delta_{\mathbb{H}}\hat{E}^\nu_{\mathbb{I}}\times\mathfrak{I}\times\hat{\Omega}^{\mathbb{AB}}_\mu(-p)\hat{\Omega}^{\mathbb{CD}}_\delta(-q)\hat{\Omega}^{\mathbb{EF}}_\nu(p+q){\rm tr}(\sigma_{\mathbb{AB}}\sigma_{\mathbb{CD}}\sigma_{\mathbb{EF}}\gamma_5)$$
$$= i\times 4\times\frac{1}{3}\times\left(\frac{1}{2}\right)^4\times\epsilon^{\mu\delta\nu}\times\mathfrak{I}\times\hat{\Omega}^{\mathbb{AB}}_\mu(-p)\hat{\Omega}^{\mathbb{CD}}_\delta(-q)\hat{\Omega}^{\mathbb{EF}}_\nu(p+q)\times\frac{1}{8}(2\delta_{\mathbb{AB}}\epsilon_{\mathbb{CDEF}}-3\delta_{\mathbb{AC}}\epsilon_{\mathbb{BDEF}}$$
$$+\delta_{\mathbb{BC}}\epsilon_{\mathbb{ADEF}}+3\delta_{\mathbb{AD}}\epsilon_{\mathbb{BCEF}}-\delta_{\mathbb{BD}}\epsilon_{\mathbb{ACEF}}-2\delta_{\mathbb{BE}}\epsilon_{\mathbb{ACDF}}-\delta_{\mathbb{CE}}\epsilon_{\mathbb{ABDF}}+\delta_{\mathbb{DE}}\epsilon_{\mathbb{ABCF}}+2\delta_{\mathbb{BF}}\epsilon_{\mathbb{ACDE}}+\delta_{\mathbb{CF}}\epsilon_{\mathbb{ABDE}}-\delta_{\mathbb{DF}}\epsilon_{\mathbb{ABCE}})$$
$$= i\times 4\times\frac{1}{3}\times\left(\frac{1}{2}\right)^4\times\epsilon^{\mu\delta\nu}\times\mathfrak{I}\times\hat{\Omega}^{\mathbb{AM}}_\mu(-p)\hat{\Omega}^{\mathbb{MB}}_\delta(-q)\hat{\Omega}^{\mathbb{CD}}_\nu(p+q)(2\epsilon_{\mathbb{ABCD}}), \tag{22}$$

where

$$\mathfrak{I} = \int\frac{d^3k}{(2\pi)^3}\frac{M(k^2+M^2)}{(k^2+M^2)((k+q)^2+M^2)((k-p)^2+M^2)}. \tag{23}$$

For $M \gg p$, q and from rescaling $k/M \to k$

$$\mathfrak{I} = \frac{M}{|M|}\times\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+1)^2} = \mathcal{I}, \tag{24}$$

where $\mathcal{I}$ is given by Eq. (18). In real space this yields

$$S_{\rm eff}^{\rm cubic} = i\times\frac{M}{|M|}\times 4\times\frac{1}{3}\times\left(\frac{1}{2}\right)^4\times\frac{1}{8\pi}\times\int d^3x\epsilon^{\mu\delta\nu}\hat{\Omega}^{\mathbb{AM}}_\mu\hat{\Omega}^{\mathbb{MB}}_\delta\hat{\Omega}^{\mathbb{CD}}_\nu(2\epsilon_{\mathbb{ABCD}}). \tag{25}$$

Combining Eq. (25) and Eq. (19) gives the effective action

$$S_{\rm eff}^{\rm CS} = -\frac{M}{|M|}\times\frac{i}{4\times 8\pi}\int d^3x\epsilon^{\mu\delta\nu}\epsilon_{\mathbb{ABCD}}\times\left(\hat{\Omega}^{\mathbb{AB}}_\mu(\partial_\delta\hat{\Omega}^{\mathbb{CD}}_\nu)-\frac{2}{3}\hat{\Omega}^{\mathbb{AM}}_\mu\hat{\Omega}^{\mathbb{MB}}_\delta\hat{\Omega}^{\mathbb{CD}}_\nu\right)$$
$$= -\frac{M}{|M|}\times\frac{i}{32\pi}\int d^3x\epsilon^{\mu\delta\nu}\epsilon_{\mathbb{ABCD}}\times\hat{\Omega}^{\mathbb{AB}}_\mu\left(\frac{F^{\mathbb{CD}}_{\delta\nu}}{2}+\frac{1}{3}\hat{\Omega}^{\mathbb{CM}}_\delta\hat{\Omega}^{\mathbb{MD}}_\nu\right), \tag{26}$$

where F is the spin(4) curvature. It is the unique term in the effective action that survives the large M limit. Notice that while this piece of the effective action comes from a ultraviolet (U.V) convergent integral, this is not true of other terms arising in $S_{\rm eff}^{\rm CS}$ at finite M. Employing a Pauli-Villars regulator with mass Λ leads to the replacement $\frac{M}{|M|}\to(\frac{M}{|M|}+\frac{\Lambda}{|\Lambda|})$ in Eq. (26). This modification plays an important role in our later discussion of domain wall physics and invariance of the effective action under large gauge transformations in Sec. VII.

We now impose the condition that the original Lorentz symmetry be a subgroup of the spin(4) flavor symmetry by setting

$$\hat{\Omega}_\mu = \Omega^{AB}_\mu T_{AB} + 2E^A_\mu T_{4A}, \qquad A, B = 1\ldots 3, \tag{27}$$

where Ω_μ is the original spin connection with $T_{AB} = \frac{1}{4}[\gamma_A,\gamma_B]$ the generators while E_μ are the additional gauge fields needed for spin(4). In the next section we will see that E_μ can be interpreted as a dynamical frame for an emergent geometry. Equation (26) is hence a Chern-Simons term that ensures the effective action on a manifold without boundary is invariant under gauge transformations of the spin connection that can be smoothly deformed to the identity.

IV. GRAVITY INTERPRETATION

We can decompose the spin(4) curvature also under the original Lorentz group by computing the commutator of the corresponding spin(4) covariant derivative $[D_\mu, D_\nu]$ and expanding the resultant expression on the generators in a manner similar to that given in Eq. (27). This leads to the following expression:

$$F_{\mu\nu} = \left(\mathcal{R}^{AB}_{\mu\nu} - \frac{2}{\ell^2}E^A_{[\mu}E^B_{\nu]}\right)T_{AB} + \frac{4}{\ell}D_{[\mu}E^A_{\nu]}T_{4A}, \tag{28}$$

where $\mathcal{R}_{\mu\nu} = \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu + [\Omega_\mu,\Omega_\nu]$ is the spin(3) curvature and the remaining components $D_{[\mu}E_{\nu]}$ are recognized as the torsion $T_{\mu\nu}$. Notice that we have rescaled the gauge fields E_μ by an arbitrary length scale ℓ to make it possible to interpret E_μ as the dimensionless emergent frame. Substituting these expressions into Eq. (26) yields

$$S_{\rm eff}^{\rm CS} = -i\frac{M}{|M|}\times\frac{1}{32\pi}\int d^3x\epsilon^{\mu\nu\lambda}\epsilon_{ABC}\times\frac{1}{\ell}E^A_\mu\left(\mathcal{R}^{BC}_{\nu\lambda}-\frac{8}{3\ell^2}E^B_\nu E^C_\lambda\right), \tag{29}$$

where we have discarded boundary terms. Clearly the action rewritten in these variables contains both an Einstein-Hilbert and cosmological constant term as expected of a gravity theory [23,24]. However, the relative coefficients of these terms have been fixed by the requirement that the theory actually enjoys a local spin(4) symmetry now interpreted as a local de Sitter gauge symmetry. Notice that the equation of motion for the Chern-Simons theory $F_{\mu\nu} = 0$ now implies the pair of equations

$$\mathcal{R}_{\mu\nu} - \frac{2}{\ell^2} E_{[\mu} E_{\nu]} = 0, \qquad T_{\mu\nu} = 0, \tag{30}$$

corresponding to classical Euclidean de Sitter space and a torsion-free connection. Of course this identification between Einstein Hilbert and Chern-Simons theory is still problematic at the nonperturbative level since in the path integral the latter necessarily includes degenerate metrics with a vanishing frame. This is the origin of the topological character of the gravity theory as discussed in [25].

It is not a surprise that integrating out fermions in odd dimensions leads to a Chern-Simons theory—this is well-known in the case of Dirac fermions transforming under some internal symmetry. What is new here is that if those fermions are taken to be of Kähler-Dirac type propagating on a curved background geometry, then the induced Chern-Simons theory is actually a (topological) theory of gravity.

V. DOMAIN WALL CONSTRUCTION

In the previous section we assumed that the three-dimensional manifold was compact. It is interesting to ask what happens in the presence of a boundary or equivalently if a domain wall is introduced in the system. Our argument parallels the original discussion by Callan and Harvey for Dirac fermions and later employed by Kaplan in his construction of domain wall lattice fermions [26,27].

Let us imagine a manifold of the form $\mathcal{M} \times R$ with coordinates (x_μ, z) where x_μ parametrize the position on the domain wall. Let us also allow the fermion mass M to change sign as a function of the flat coordinate z as shown in Fig. 3:

$$M(z) = M_0 \frac{z}{|z|}. \tag{31}$$

One expects that massless states appear at $z = 0$. To see this let us rewrite the bulk Kähler-Dirac equation in the form

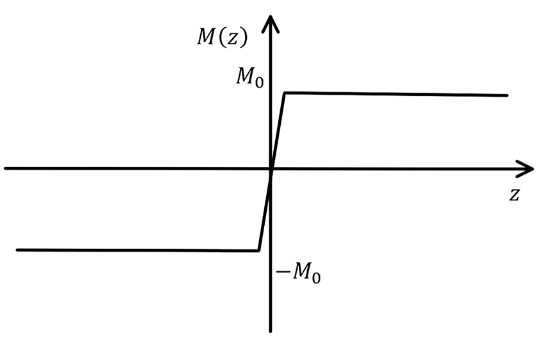

FIG. 3. Domain wall.

$$[\gamma^3\gamma^\mu D_\mu + \partial_z - i\gamma^3\gamma^4 M(z)]\Psi(x,z) = 0. \tag{32}$$

One can find zero mode solutions of this equation of the form

$$\Psi_{\rm DW} = \chi(z)\psi(x) \tag{33}$$

with $\gamma^3\gamma^\mu D_\mu \psi = 0$ and $\psi(x)$ an eigenvector of the Hermitian operator $H = -i\gamma^3\gamma^4$ with eigenvalue $+1$. The function $\chi(z)$ must then satisfy

$$\partial_z \chi(z) = -M(z)\chi(z). \tag{34}$$

Thus one finds $\chi(z) = e^{-M_0|z|}$ corresponding to zero modes exponentially localized to the domain wall at $z = 0$. Notice that $\Psi_{\rm DW}$ contains just 4 degrees of freedom—the original reduced field Ψ_+ contained 8 degrees of freedom while the restriction to fields with $H = +1$ further halves the number of degrees of freedom. Four degrees of freedom corresponds to the field content of a two-dimensional Kähler-Dirac field propagating on the wall. We can verify this explicitly by going to a (Euclidean) chiral basis for the gamma matrices corresponding to

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}, \tag{35}$$

where $\sigma_\mu = (i\sigma_i, I)$ and $\bar{\sigma}_\mu = (-i\sigma_i, I)$. This implies that Ψ_+ takes the 2×2 block form

$$\Psi_+ = \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}, \tag{36}$$

and the matrix H takes the form

$$H = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}. \tag{37}$$

The additional requirement that $\Psi_{\rm DW}$ be an eigenstate of H with eigenvalue $+1$ shows that ψ_1 contains two right-handed two-dimensional Weyl spinors while ψ_2 contains two left-handed spinors.

The constraint $H = 1$ for the domain wall fermions also breaks the original gauge symmetry to $\mathrm{spin}(2) \times \mathrm{spin}(2)$.[3] The first factor corresponds to the generator $\frac{1}{4}[\gamma^1, \gamma^2]$ and is associated with the two-dimensional spin connection $\hat{\Omega}_\mu^{12}$ needed to enforce local Lorentz invariance for the domain wall modes. The second factor corresponds to H itself. The covariant derivative associated with rotations generated by H takes the form

[3] Thus all gauge fields associated with the broken generators must vanish on the domain wall.

$$D_\mu \Psi_{\rm DW} = \partial_\mu \Psi_{\rm DW} + \frac{i}{2}\hat{\Omega}^{34}_\mu (H\Psi_{\rm DW} - \Psi_{\rm DW} H). \quad (38)$$

Using $H\Psi_{\rm DW} = \Psi_{\rm DW}$ this can be rewritten as

$$D_\mu \Psi_{\rm DW} = \partial_\mu \Psi_{\rm DW} + \frac{i}{2}\hat{\Omega}^{34}_\mu (\Psi_{\rm DW} - H\Psi_{\rm DW} H). \quad (39)$$

If we define the domain wall chiral operator $\hat{\gamma}_5 = \gamma_5 H$, the covariant derivative associated with H becomes

$$D_\mu \Psi_{\rm DW} = \partial_\mu \Psi^+_{\rm DW} + \partial_\mu \Psi^-_{\rm DW} + i\hat{\Omega}^{34}_\mu \Psi^-_{\rm DW}, \quad (40)$$

where the domain wall reduced field $\Psi^\pm_{\rm DW}$ is given by

$$\Psi^\pm_{\rm DW} = \hat{P}_\pm \Psi_{\rm DW} = \frac{1}{2}(\Psi_{\rm DW} \pm \hat{\gamma}^5 \Psi_{\rm DW} \hat{\gamma}^5). \quad (41)$$

Thus we find that the gauge field $\hat{\Omega}_{34}$ couples only to a two-dimensional reduced Kähler-Dirac field on the wall.

A similar feature is seen in the interaction of the domain wall fermion with the two-dimensional spin connection $\hat{\Omega}_{12}$. The corresponding term in the covariant derivative takes the form

$$\frac{i}{2}\hat{\Omega}^{12}_\mu (i\gamma^1\gamma^2 \Psi_{\rm DW} - \Psi_{\rm DW} i\gamma^1\gamma^2) = i\hat{\Omega}^{12}_\mu \hat{\gamma}^5 \Psi^-_{\rm DW}. \quad (42)$$

Thus all gauge interactions on the domain wall couple only to the reduced Kähler-Dirac field $\Psi^-_{\rm DW}$.

To summarize we find that the low energy excitations of the three-dimensional Kähler-Dirac theory in the presence of a domain wall are massless two-dimensional Kähler-Dirac fermions $\Psi_{\rm DW}$ localized to the wall and described by a Lorentz invariant action possessing an additional $U(1)$ symmetry generated by an operator $\hat{\Gamma} = \hat{\gamma}^5 \otimes \hat{\gamma}^5$. The operator $\hat{\Gamma}$ anticommutes with the two-dimensional Kähler-Dirac operator describing the domain wall fermions and allows the Kähler-Dirac field to be projected into two independent reduced Kähler-Dirac fields $\Psi^-_{\rm DW}$ and $\Psi^+_{\rm DW}$. Only one of these components $\Psi^-_{\rm DW}$ participates in the remaining $\mathrm{spin}(2) \times \mathrm{spin}(2)$ gauge symmetry.

VI. ANOMALY INFLOW FOR KÄHLER-DIRAC FERMIONS

At first glance the structure of the domain wall fermion action appears problematic since it is known that massless Kähler-Dirac fields in even dimensions suffer from a gravitational anomaly [15,16] that breaks this $U(1)_{\hat{\Gamma}}$ symmetry down to Z_4. The two-dimensional domain wall action we derived in the previous section includes a gauged version of this symmetry (since the gauge field couples to a $\hat{\Gamma}$ reduced fermion), and hence one naively expects an anomaly induced breaking of gauge invariance in the low energy theory. In this section we will show that there is an additional contribution which arises from the bulk action which restores gauge invariance via an anomaly inflow mechanism.

To show in detail how this occurs we first include a brief review of the derivation of the anomaly specialized to the case of two-dimensional domain wall fermions. Under a $U(1)_{\hat{\Gamma}}$ rotation with parameter $\alpha(x)$ the measure for the reduced Kähler-Dirac field $\Psi^-_{\rm DW}$ transforms by a factor $e^{i\int d^2x \alpha(x) A(x)}$ with

$$A(x) = \lim_{M\to\infty} \mathrm{Tr} \sum_n e\left(\bar{\phi}_n e^{\frac{1}{M^2}(\not{D})^2} \hat{P}_- P_+ \phi_n\right), \quad (43)$$

where we have regulated the UV divergence by inserting the factor $e^{\frac{1}{M^2}(\not{D})^2}$ where ϕ_n are eigenstates of the domain wall Kähler-Dirac operator $\not{D} = \gamma^3\gamma^\mu D_\mu$ and e represents the determinant of the frame restricted to the wall which we denote as e^a_μ. Cyclically permuting the trace we find

$$A = \lim_{M\to\infty} \mathrm{Tr}\left(e^{\frac{1}{M^2}(\not{D})^2} \hat{P}_- P_+ \sum_n e\phi_n \bar{\phi}_n\right)$$
$$= \lim_{x\to x'} \lim_{M\to\infty} \mathrm{Tr}\left(e^{\frac{1}{M^2}(\not{D})^2} \hat{P}_- P_+ \delta(x-x')\right). \quad (44)$$

Expanding $\not{D}^2$ we obtain

$$A = \lim_{x\to x'} \lim_{M\to\infty} -\frac{1}{4} \times \mathrm{Tr}\Big((-i\gamma^3\gamma^4) e\gamma^5 e^{\frac{1}{M^2}(\Box + \frac{1}{2} e^\mu_a e^\nu_b \sigma^{ab} F^{cd}_{\mu\nu}[\sigma_{cd},.])} \times \delta(x-x')\gamma^5(-i\gamma^3\gamma^4)\Big)$$
$$= \lim_{x\to x'} \lim_{M\to\infty} \mathrm{Tr}\Big((\sigma^{34}) e\gamma^5 e^{\frac{1}{M^2}(\Box + \frac{1}{2} e^\mu_a e^\nu_b \sigma^{ab} F^{cd}_{\mu\nu}[\sigma_{cd},.])} \times \delta(x-x')\gamma^5(\sigma_{34})\Big), \quad (45)$$

where F contains the surviving nonzero components of the $\mathrm{spin}(4)$ curvature corresponding to the symmetry $\mathrm{spin}(2) \times \mathrm{spin}(2)$. Expanding the exponential to $\mathcal{O}(1/M^2)$ to get a nonzero result for the trace over spinor and flavor indices and acting with $e^{\frac{1}{M^2}\Box^2}$ on the delta function yields

$$A = -\frac{1}{4\pi} \times \left(\frac{1}{2}\right) \mathrm{tr}(e\gamma^5 \sigma^{ab}\sigma^{34}) e^\mu_a e^\nu_b F^{cd}_{\mu\nu} \mathrm{tr}(\gamma_5 \sigma_{cd} \sigma_{34})$$
$$= -\frac{1}{8\pi} \epsilon^{\mu\nu} \epsilon_{cd} R^{cd}_{\mu\nu}, \quad (46)$$

where $R_{\mu\nu}$ corresponds to the curvature of the spin connection $\hat{\Omega}_{12}$. We have employed the result $e\epsilon^{ab} e^\mu_a e^\nu_b = \epsilon^{\mu\nu}$

in the last line. Hence, under a $U(1)$ transformation the measure for a reduced Kähler-Dirac field transforms as[4]

$$\int D\bar{\Psi}D\Psi \to e^{-\frac{i}{8\pi}\int d^2x\alpha(x)\epsilon^{\mu\nu}\epsilon_{cd}R^{cd}_{\mu\nu}}\int D\bar{\Psi}D\Psi. \quad (47)$$

This naively breaks gauge invariance. However, the anomaly we have computed for the domain wall fermions is not the whole story. We showed earlier that the bulk contains also an induced Chern-Simons term. In general this also undergoes a nonzero change under a gauge transformation. In general the variation of the bulk Chern-Simons action takes the form

$$\begin{aligned}\delta S^{\rm CS}_{\rm eff} &= -\frac{M}{|M|}\times\frac{i}{32\pi}\times\epsilon^{\mu\delta\nu}\epsilon_{\mathbb{ABCD}} \\ &\quad\times(\partial_\mu\hat{\Omega}^{\mathbb{AB}}_\delta-\partial_\delta\hat{\Omega}^{\mathbb{AB}}_\mu-2\hat{\Omega}^{\mathbb{AM}}_\mu\hat{\Omega}^{\mathbb{MB}}_\delta)\delta\hat{\Omega}^{\mathbb{CD}}_\nu \\ &= -\frac{M}{|M|}\times\frac{i}{32\pi}\times\epsilon^{\mu\delta\nu}\epsilon_{\mathbb{ABCD}}F^{\mathbb{AB}}_{\mu\delta}\delta\hat{\Omega}^{\mathbb{CD}}_\nu, \end{aligned} \quad (48)$$

where F is the spin(4) curvature. Under a gauge transformation $\hat{\Omega}^{\mathbb{AB}}_\mu\to\hat{\Omega}^{\mathbb{AB}}_\mu+D_\mu\zeta^{\mathbb{AB}}$ the effective action changes:

$$\begin{aligned}\delta S^{\rm CS}_{\rm eff} &= -\int d^3x\frac{M}{|M|}\times\frac{i}{32\pi}\times\epsilon^{\mu\delta\nu}\epsilon_{\mathbb{ABCD}}F^{\mathbb{AB}}_{\mu\delta}(D_\nu\zeta^{\mathbb{CD}}) \\ &= i\int d^3x\frac{1}{16\pi}\times\epsilon^{\mu\delta}\epsilon_{\mathbb{ABCD}}F^{\mathbb{AB}}_{\mu\delta}\zeta^{\mathbb{CD}}\partial_z\left(\frac{M}{2|M|}\right) \\ &= i\int d^3x\delta(z)\times\frac{1}{16\pi}\times\epsilon^{\mu\delta}\epsilon_{ab}\epsilon_{cd}F^{ab}_{\mu\delta}\zeta^{cd}, \end{aligned} \quad (49)$$

where $a,b=\{1,2\}$ while $c,d=\{3,4\}$ and these indices are to be contracted using two independent two-dimensional ϵ symbols corresponding to the product of the two invariant tensors for spin(2) × spin(2)—the surviving symmetry on the domain wall. Taking $\zeta^{34}=-\zeta^{43}=\alpha(x)$ we find

$$\delta S^{\rm CS}_{\rm eff}=\frac{i}{8\pi}\int d^2x\alpha(x)\epsilon^{\mu\delta}\epsilon_{ab}R^{ab}_{\mu\delta}. \quad (50)$$

Thus the gauge transformation of the Chern-Simons term in the presence of the domain wall generates a contribution that is equal in magnitude but opposite in sign to that coming from the anomalous variation of the measure for the domain wall fermions—Eq. (47). Thus the bulk and boundary variations cancel and the full theory is gauge invariant. This is anomaly inflow in action for Kähler-Dirac fields. That this should occur is guaranteed by the fact that the Euler characteristic of the bulk theory is zero if it is taken to be a product of a two-dimensional space and a circle since $\chi(S^1)=0$.[5]

One might worry that the previous argument ignores the fact that the Chern-Simons term was computed for constant mass which is certainly not the situation close to the domain wall. However, it is possible to avoid this problem if one simply computes the change in the Chern-Simons current $J^{34}_\mu=\frac{\delta S_{\rm eff}}{\delta A^{34}_\mu}$ between $z=\infty$ and $z=-\infty$. One then finds

$$\Delta J^{34}_3=2\times2\times\frac{1}{32}\epsilon^{\mu\nu3}\epsilon_{AB}R^{AB}_{\mu\nu}, \quad (51)$$

where the second factor of 2 arises from the double counting associated with the fact that $A^{AB}_\mu=-A^{BA}_\mu$. Comparing this to the divergence of the $U(1)$ current arising from the domain wall fermions $\partial^\mu J^{34}_\mu=-\frac{1}{8\pi}\epsilon^{\mu\nu}\epsilon_{AB}R^{AB}_{\mu\nu}$ we see that the net flow of charge off the domain wall is accounted for by the Chern-Simons current.

VII. INVARIANCE UNDER LARGE GAUGE TRANSFORMATIONS

It is of course interesting to ask about the invariance of the theory under large gauge transformations. To facilitate this analysis it is convenient to again adopt a Euclidean chiral basis for the gamma matrices. The spin(4) connection becomes

$$\hat{\Omega}=\Omega^i\begin{pmatrix} i\sigma_i & 0 \\ 0 & i\sigma_i\end{pmatrix}+E^i\begin{pmatrix} i\sigma_i & 0 \\ 0 & -i\sigma_i\end{pmatrix}, \quad (52)$$

while the fermion field takes the form

$$\Psi_+=\begin{pmatrix}\psi_1 & 0 \\ 0 & \psi_2\end{pmatrix}, \qquad \bar{\Psi}_-=\begin{pmatrix} 0 & \bar{\psi}_1 \\ \bar{\psi}_2 & 0\end{pmatrix}. \quad (53)$$

The Kähler-Dirac action then separates into two independent contributions

$$\begin{aligned} S=\int d^3x\hat{E}[&{\rm tr}(\bar{\psi}_1(D\!\!\!/(\Omega+E)+iM)\psi_1) \\ &+{\rm tr}(\bar{\psi}_2(D\!\!\!/(\Omega-E)-iM)\psi_2)], \end{aligned} \quad (54)$$

where tr denotes a trace over a two-dimensional block. Each such block will then generate its own $SU(2)$ Chern-Simons term on integration over the fermions:

[4]Taking $\alpha(x)$ to be a constant one finds the measure transforms by the phase $e^{-i\chi\alpha}$ where χ is the Euler characteristic of the two-dimensional space. If one further replaces the reduced field by a full Kähler-Dirac field and compactifies the space to S^2 where $\chi=2$, one obtains the original $U(1)$ global anomaly referred to in the Introduction. In such a background the phase is just $e^{-4i\alpha}$ which leaves an unbroken Z_4 subgroup.

[5]Our previous discussion assumed z extends from $-\infty$ to ∞ but we can replace this by a circle at a price of adding an antidomain wall at infinity.

$$S = I_{\mathrm{CS}}(\Omega + E) - I_{\mathrm{CS}}(\Omega - E), \tag{55}$$

where

$$I_{\mathrm{CS}}(A) = \frac{1}{32\pi}\mathrm{sign}(M)\int d^3x\epsilon^{\mu\nu\rho} \times \mathrm{tr}\left(A_\mu\partial_\nu A\rho - \frac{2}{3}A_\mu A_\nu A_\rho\right). \tag{56}$$

The relative minus sign in Eq. (55) arises because of differing signs of the mass in the two blocks [25].

Under a gauge transformation $A_\mu \to g(x)A_\mu g^{-1}(x) + g(x)\partial_\mu g^{-1}(x)$, each Chern-Simons term transforms, up to a boundary term, according to

$$\delta I_{\mathrm{CS}} = \int d^3x\epsilon^{\mu\nu\delta}\left(\frac{1}{96\pi}\frac{M}{|M|}\right)\mathrm{tr}(g\partial_\mu g^{-1}g\partial_\nu g^{-1}g\partial_\delta g^{-1}) = \frac{M}{|M|}\pi n, \tag{57}$$

where the winding number $n = \pi_3(SU(2)) = \mathbb{Z}$.[6]

Thus naively the level number of the CS term is $k = \pm 1/2$ with the partition function changing sign for odd n. However, once one regulates the theory with a Pauli-Villars field corresponding to a z-independent cutoff mass Λ the coefficients of the Chern-Simons terms (the level numbers) are shifted to $k = 0$ and $k = 1$ in the two regions $z < 0$ and $z > 0$, respectively. Thus we find that the theory is in fact also invariant under large gauge transformations.

VIII. SUMMARY

We have shown that integrating out massive Kähler-Dirac fermions in a curved three-dimensional background yields a Chern-Simons term. This Chern-Simons term corresponds to a topological theory of gravity in which both spin connection and frame emerge from an extended gauge symmetry—in this case Euclidean de Sitter symmetry. Gravity theories of this type were proposed many years ago [23,24,28] and generalize Witten's old observation that three-dimensional gravity can be formally written as a Chern-Simons gauge theory [29].

In the presence of a domain wall we have shown that massless two-dimensional Kähler-Dirac fermions appear on the wall. These are described by a single Kähler-Dirac field which can be decomposed into two independent components called reduced Kähler-Dirac fields which carry half the number of degrees of freedom. We find that just one of these reduced fields participates in the gauge interactions on the domain wall. Furthermore although the reduced Kähler-Dirac fermions on the wall suffer from a gravitational anomaly, there is no violation of local gauge invariance because of anomaly inflow from the bulk gravitational Chern-Simons term.

It is not hard to generalize this construction to higher dimensions. For example, the effective long distance action for massive Kähler-Dirac fermions in five dimensions is also a topological gravity theory of Chern-Simons type [23,24,28] with gauge group spin(6) in Euclidean space. Using the same arguments as for three dimensions it is clear that massless four-dimensional Kähler-Dirac fermions invariant under local spin(4) Lorentz transformations and an additional local $U(1)$ symmetry would then arise in the presence of a domain wall in such a theory. Again, the domain wall action will contain a coupling of the $U(1)$ gauge field to a single reduced Kähler-Dirac field. As in three dimensions gauge invariance of the theory remains intact since the gauge variation of the five-dimensional Chern-Simons term cancels the potentially anomalous variation arising from the four-dimensional reduced fermions.

One of the conclusions one can draw from our work is that coupling reduced Kähler-Dirac fermions to gravity in some even dimensional space is inconsistent due to a (mixed) gravitational anomaly *unless* the theory lives on a domain wall or boundary of a space of one higher dimension. If this additional dimension is finite, there will necessarily be an antidomain wall which localizes another reduced Kähler-Dirac fermion with the opposite eigenvalue of Γ. In this scenario the Chern-Simons current naturally flows between the two walls and the low energy theory is manifestly well-defined.

Much of our discussion for Kähler-Dirac fermions has paralleled existing arguments for Dirac fermions. In this paper we have focused on perturbative anomalies and anomaly inflow. But in [16] it was shown that Kähler-Dirac fermions also exhibit discrete 't Hooft anomalies. Canceling these anomalies is a necessary condition for symmetric mass generation and requires multiples of two Kähler-Dirac fields or four reduced Kähler-Dirac fields. If we take the flat space limit, this constraint translates into the requirement that the theory contains 16 Majorana spinors in four dimensions in perfect agreement with results for gapping edge modes in four-dimensional topological insulators which require cancellation of a seemingly unrelated 't Hooft anomaly for a spin-Z_4 symmetry acting on Weyl fermions. This makes it plausible that theories of Weyl fermions, which are free of all 't Hooft anomalies and hence capable of symmetric mass generation, can be written in terms of Kähler-Dirac fermions. Furthermore since the anomalies of Kähler-Dirac fermions survive intact under discretization, this suggests that they may be important for constructing lattice mirror models that target chiral theories in the continuum limit. Indeed numerical simulations of two flavors of interacting staggered fermions (which are obtained by discretization of Kähler-Dirac fermions) show evidence for the existence of a massive symmetric phase [30]. Further work is needed to understand these issues in more detail.

[6]The normalization of our CS term reflects the nonstandard trace $\mathrm{tr}(\sigma^a\sigma^b) = 2\delta^{ab}$.

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APPENDIX: DETAILS ON THE COMPUTATION OF S_{eff}

For a complete set of basis states ϕ_n,

$$\begin{aligned} S_{\mathrm{eff}} &= \int d^3x \mathrm{Tr} \sum_n \hat{E}\,(\bar{\phi}_n \log[(\not{D} - i\gamma^4 M)P_+]\phi_n) \\ &= \int d^3x \mathrm{Tr}\left(\log[(\not{D} - i\gamma^4 M)P_+] \sum_n \hat{E}\,\phi_n\bar{\phi}_n\right) \\ &= \lim_{x\to x'} \int d^3x \mathrm{Tr}(\log[(\not{D} - i\gamma^4 M)P_+]\delta(x - x')) \\ &= \lim_{x\to x'} \int d^3x \mathrm{Tr}\left(\log[(\not{D} - i\gamma^4 M)P_+] \right. \\ &\quad \left. \times \int \frac{d^3k}{(2\pi)^3} e^{ik_\mu D^\mu \sigma(x,x')}\right), \end{aligned} \tag{A1}$$

where $\sigma(x, x')$ is the geodesic biscalar [a generalization of $\frac{1}{2}(x - x')^2$ in flat space] defined by

$$\sigma(x, x') = \frac{1}{2} g^{\mu\nu} D_\mu \sigma(x, x') D_\nu \sigma(x, x'). \tag{A2}$$

Expanding the logarithm and exponential as power series and using the properties of the geodesic biscalar [31]

$$\begin{aligned} \sigma(x, x) &= 0, \\ \lim_{x\to x'} D_\mu D_\nu \sigma(x, x') &= g_{\mu\nu}, \\ \text{and} \quad \lim_{x\to x'} D_\mu D_\nu D_\alpha \sigma(x, x') &= 0, \end{aligned}$$

we get

$$S_{\mathrm{eff}} = \int \hat{E}\,d^3x \int \frac{1}{\hat{E}} \frac{d^3k}{(2\pi)^3} \mathrm{Tr}(\log[(\not{k} - i\gamma^4 M)P_+]), \tag{A3}$$

where $\not{k} = g_{\mu\nu}\gamma^\mu k^\nu$. The determinant $\hat{E}$ has been restored to make the invariance of real-space and k-space measures manifest. We can now choose locally flat coordinates to evaluate the k-space integral. This allows us to reduce the calculation of S_{eff} to an equivalent flat space problem:

$$\begin{aligned} S_{\mathrm{eff}} \equiv \lim_{x\to x'} \int d^3x \mathrm{Tr}\left(\log[(\not{\partial} - i\gamma^4 M)P_+] \right. \\ \left. \times \int \frac{d^3k}{(2\pi)^3} e^{ik_\mu(x^\mu - x'^\mu)}\right). \end{aligned} \tag{A4}$$

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