Remarks about weighted energy integrals over Minkowski spectral functions from Euclidean lattice data

Thomas DeGrand [*]
Department of Physics, University of Colorado, Boulder, Colorado 80309, USA

(Received 17 March 2022; accepted 22 June 2022; published 7 July 2022)

I make some simple observations about the calculation of weighted averages over energy of Minkowski space spectral densities from weighted averages over time of Euclidean space correlation functions, measured in lattice simulations. The correlator of two vector currents is used as an example, where it appears that a determination of a weighted average of the spectral function near the rho pole at the five percent level is possible from lattice simulations.

DOI: 10.1103/PhysRevD.106.014504

Finding connections between theoretical calculations done in Euclidean space and results of experiments done in Minkowski space is a longstanding problem in many areas of physics and involves many approaches. Lattice studies of QCD and other related systems are no exception. This short paper describes a simple technique for extracting weighted averages over energy of Minkowski space spectral densities from Euclidean space lattice correlation functions. Examples are motivated by calculations of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment a_μ^{HVP} (with Ref. [1] as my primary reference), though there are obvious applications to many similar processes [2].

Here, I introduce the quantities which are the subject of this paper. In Euclidean space we have a correlation function (x is a four-dimensional variable)

$$\mathcal{G}_E(x) = \langle O_E(x) O_E(0) \rangle \tag{1}$$

and its Fourier transform

$$\Pi_E(k) = \int d^4x e^{ikx} \mathcal{G}_E(x). \tag{2}$$

I have obviously suppressed indices on O_E and Π_E [$J_\mu(x)$, $\Pi_{\mu\nu}$, etc.]. I specialize to one dimension,

$$G_E(t) = \sum_{\vec{x}} \mathcal{G}_E(t = x_0, \vec{x}), \tag{3}$$

and write

$$\Pi(Q) = \Pi_E(Q = k_0, \vec{k} = 0), \tag{4}$$

with, of course,

$$G_E(t) = \int_{-\infty}^{\infty} \frac{dQ}{2\pi} e^{-iQt} \Pi(Q); \quad \Pi(Q) = \int_{-\infty}^{\infty} dt e^{iQt} G_E(t). \tag{5}$$

Here, $\Pi(Q)$ obeys a dispersion relation (subtracted once, in the case of hadronic vacuum polarization) which connects it to an integral of the spectral function $\rho(s) = 2\,\mathrm{Im}\,\Pi(s)$, over positive Minkowski energy-squared s,

$$\Pi(Q) - \Pi(0) = \frac{Q^2}{2\pi} \int_0^\infty ds \frac{\rho(s)}{s(s + Q^2)}. \tag{6}$$

In turn, $\rho(s)$ is related to a total cross section via the optical theorem. Combining Eqs. (5) and (6) gives the connection between a Euclidean space correlation function defined at Euclidean time t, $G_E(t)$, and the spectral function,

$$G_E(t) = \frac{1}{2\pi} \int_0^\infty d\omega [\omega^2 \rho(\omega)] \exp(-\omega t) \tag{7}$$

(slightly abusing notation by expressing the spectral function as a function of ω rather than of $s = \omega^2$).

Inverting Eq. (7) to predict $\rho(\omega)$ from $G_E(t)$ is a difficult problem. However, it seems easy to compare a weighted average of $G_E(t)$ to a weighted average of $\rho(\omega)$,

$$\hat{\rho}(Q_0) \equiv \int_0^\infty R_E(Q_0, t) G(t) dt = \int_0^\infty d\omega \rho(\omega) T(Q_0, \omega) \tag{8}$$

[*] thomas.degrand@colorado.edu

with the connection

$$T(\omega) = \frac{\omega^2}{2\pi} \int_0^\infty e^{-\omega t} R_E(Q_0, t) dt. \tag{9}$$

Note that Q_0 is shorthand for possible tunable parameter(s) in the weighting function. The most prominent present-day example of such a connection is the calculation of the hadronic vacuum polarization contribution for the anomalous magnetic moment of the muon. The Euclidean weighting function $R_E(Q_0, t)$ for $a_\mu^{\rm HVP}$ is specified by a QED calculation.

The point of this paper is to remark that one could imagine doing the weighting with any function $R_E(Q_0, t)$. Each choice of $R_E(Q_0, t)$ amounts to its own (indirect) comparison of theory $[G_E(t)]$ with experiment $[\rho(\omega)$, processed into $\hat{\rho}(Q_0)]$. Families of related $R_E(Q_0, t)$'s can be combined into more extensive views of the spectral function. For weighted integrals over the spectral function of the vector current, all the technology for computing $a_\mu^{\rm HVP}$ is waiting to be used.

Some choices of R_E are more interesting than others, and a desirable goal would be to find an $R_E(Q_0, t)$ whose $T(\omega)$ is peaked around some energy range. To jump to the conclusion, the dominant feature of an $R_E(Q_0, t)$ which does that is a restriction to a range of t values $t_{\rm min} < t < t_{\rm max}$; the overall shape of $R_E(Q_0, t)$ does not seem to be important for the examples I display. And given what is published about the precision of $a_\mu^{\rm HVP}$ lattice results, it seems possible to make a lattice determination of a weighted average of $\rho(\omega)$ with enough accuracy to be phenomenologically interesting. (I have in mind the few percent tension in the $\pi\pi$ channel in the 0.6–0.9 GeV range described in Ref. [1], between the KLOE experiment [3] and other groups.)

I have not yet found a discussion of this approach in the literature.

The idea described here is just a trivial variation on the "coordinate space representation" for $a_\mu^{\rm HVP}$: There is an implicit assumption that $R_E(t)$ is a smooth function of t, and replacing an integral over continuous t by a sum over a set of discrete lattice points is no different than replacing any continuous integral by a grid sum.

There is also a large amount of literature proposing weighting functions $T(\omega)$ through solutions to the "inverse problem": Given a $G_E(t_i)$ defined at a set of discrete t_i values, various approaches have different criteria for defining and constructing a weighted $\hat{\rho}$. Often, no smoothness assumptions go into the choice, and in fact the $R_E(t_i)$'s found in the literature are far from smooth. Recent references (a very incomplete set for this vast field) include Ref. [2], which uses the Backus-Gilbert method [4,5]; related work by Ref. [6]; and Chebyshev techniques by Refs. [7–9]. What I am proposing might be more stable than "inverse problem" techniques, but it probably cannot produce $T(\omega)$'s with a strongly peaked structure.

Here, I will continue focusing on $a_\mu^{\rm HVP}$. There is a small amount of literature associated with modifications to its R_E. The most prominent one is probably the "intermediate window method" of Ref. [10]. It is a time-sliced version of the $a_\mu^{\rm HVP}$ weighting:

$$R_E(Q_0, t) = R_E^{a_\mu}(Q_0, t)[\Theta(t, t_{\rm min}, \Delta) - \Theta(t, t_{\rm max}, \Delta)] \tag{10}$$

where $\Theta(t, t_0, \Delta)$ is a smoothed step function. Another approach to weighting, called "finite energy sum rules," starts by writing a dispersion relation for a reweighted $\Pi_E(Q)$. For a discussion, see Refs. [11,12].

To set conventions, I am interested in the correlator of two vector currents

$$\Pi(q)_{\mu\nu} = \int d^4 x e^{iqx} \langle 0 | J_\mu(x) J_\nu(0) | 0 \rangle. \tag{11}$$

I remove the indices with a transverse projection,

$$\Pi_{\mu\nu} = [q_\mu q_\nu - g_{\mu\nu} q^2] \Pi(q^2), \tag{12}$$

and then the spectral function $\rho(\omega)$ is proportional to the discontinuity of Π across the real energy axis [setting $q_\mu = (\omega, \vec{0})$]. It is also proportional to the R-ratio, $R(\omega) = \sigma(e^+e^- \to {\rm hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$. The standard lattice vector-vector correlator contracts $\rho_{\mu\nu}$ against polarization vectors $\epsilon_\mu^i \epsilon_\nu^j$ where typically $\epsilon_\mu^i = (0, \vec{\epsilon}^i)$ is a unit vector. This means that in Eq. (7), $\rho(\omega) = R(\omega)/(6\pi)$ and

$$G_E(t) = \sum_i \int d^3 x \langle J_i(\vec{x}, t) J_i(0, 0) \rangle \tag{13}$$

where $J_i(x, t) = e_q \bar{\psi}(x, t) \gamma_i \psi(x, t)$ for a quark of charge e_q (in units of the electric charge).

The two relevant pictures are shown in Fig. 1: the familiar plot of the R-ratio in panel (a) and the expected $G_E(t)$ in panel (b), computed using Eq. (7). "Experiment" in these figures means the phenomenological model for $\rho(\omega)$ from Ref. [13] (in black) and a compilation of $R(\omega)$ from a table in the Review of Particle Properties [14] (points with error bars, in red). Of course, the question to try to answer is the following: Given a calculation $G_E(t)$, what can one say about $\rho(\omega)$?

This question is partially answered by the one theoretical line in panel (b) of Fig. 1: The dashed line is the contribution of a stable rho meson at 770 MeV with a decay constant $f_V = 0.25$:

$$G_E^V(t) = \frac{(\langle q \rangle m_V^2 f_V)^2}{2 m_V} \exp(-m_V t). \tag{14}$$

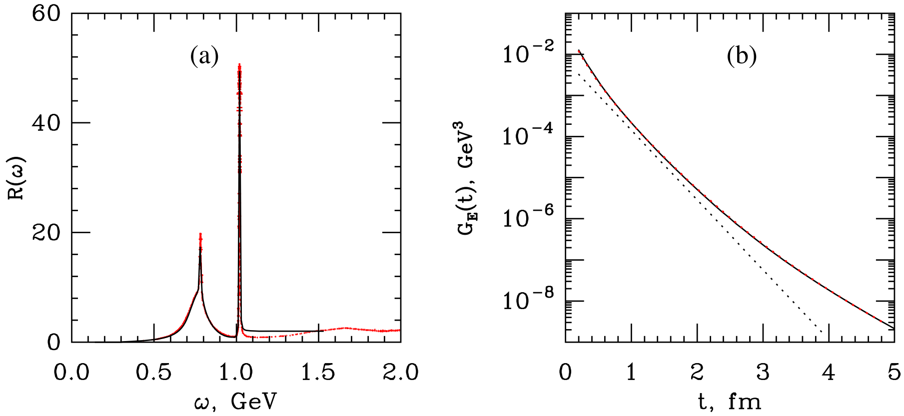

FIG. 1. (a) $R(\omega)$ the R-ratio from the phenomenological model for $\rho(\omega)$ from Ref. [13] (in black) and from the table in the Review of Particle Properties [14] (points with error bars, in red). (b) $G_E(t)$ from the phenomenological model for $\rho(\omega)$ from Ref. [13] (in black) and from the table in the Review of Particle Properties [14] (red bands, overlapping the black line). The dashed line is the contribution from a stable rho meson with $f_V = 0.25$.

The quantity $\langle q \rangle$ is the expectation value of the quarks' charges in the meson: $[2/3 - (-1/3)]/\sqrt{2} = 1/\sqrt{2}$ for the rho, $1/\sqrt{18}$ for the omega, $1/3$ for the phi, and so on. Here, $G_E(t)$ is flatter than $G_E^V(t)$ at very large t due to the contribution of two-pion states with an invariant mass smaller than the rho mass, and it is steeper than $G_E^V(t)$ at small t due to the phi meson and to the flat high energy part of $R(\omega)$. Nowhere does $G_E^V(t)$ saturate $G_E(t)$.

I can rephrase the question as follows: Given a lattice calculation of $G_E(t)$, what can one say about $\rho(\omega)$? Then, there are more constraints. The large ω region, where $\omega > 1/a$ and a is the lattice spacing, is contaminated by lattice artifacts and is inaccessible to a lattice calculation. Unfortunately, so is the small ω or large t region. The reason for this is that the lattice signal becomes noisy. This is a usual issue in lattice simulations [15–17]. The data in Ref. [18] provide an example—see their Fig. 2. The collaboration has data at lattice spacings between 0.15 and 0.06 fm. Their data are only usable out to distances $t \sim 2.5$ fm. This precludes, at least for the present time, studies of $\rho(\omega)$ near threshold. This situation is well known and documented in the a_μ^{HVP} literature [1].

Parenthetically, the vector correlator presents a somewhat special case compared to most lattice studies, where the lightest state in (continuum) $\rho(\omega)$ is an isolated pole. Then, simply going to large t gives a $G_E(t)$ which is dominated by properties of the pole. Standard lattice techniques (fits to exponentials) are more efficient at producing high quality results than the proposal of weighting $G_E(t)$ given here.

Lattice calculations are always done in finite volume, but the desired prediction is of an infinite volume observable. There are a variety of techniques for approaching the infinite volume limit used in the a_μ^{HVP} literature [1]. Since the proposal here is just to replace the weighing factor for a_μ^{HVP} by one which exposes a different average over the spectral function, dealing with the finite-to-infinite volume extrapolation will involve the same kind of analysis as for a_μ^{HVP}. I will continue the exposition assuming that this has been done.

Thus, we are pushed back to the region of ω near the rho mass. The physical rho meson is broad. Is it possible to say anything about $\rho(\omega)$ for ω near m_ρ? This seems to be a serious issue for a_μ^{HVP} determinations.

I divide up the contributions to $G_E(t)$ from different energy intervals. See Fig. 2, which shows the fractional contributions to $G_E(t)$ from different ω regions. Here $\rho(\omega)$ is taken from the phenomenological model of Ref. [13]. What is noticeable is that there is a fairly wide region at

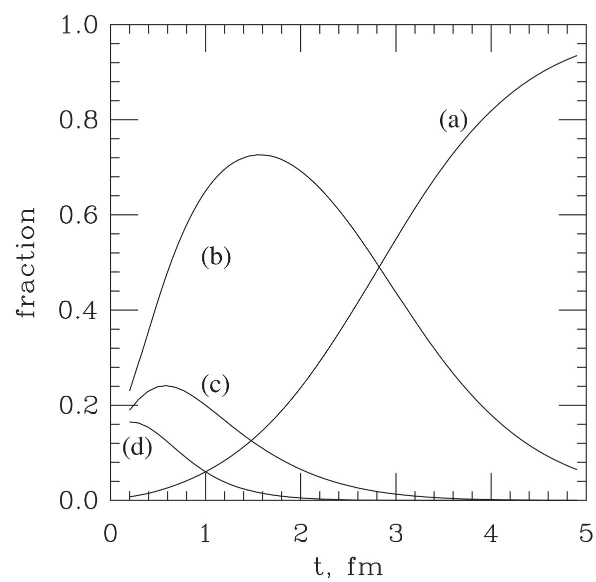

FIG. 2. Fractional contributions to $G_E(t)$ where $\rho(\omega)$ is taken from the phenomenological model of Ref. [13]. The curves label (a) $2m_\pi < \omega < 4m_\pi$, (b) $4m_\pi < \omega < 6m_\pi$, (c) $6m_\pi < \omega < 8m_\pi$, (d) $8m_\pi < \omega < 10m_\pi$.

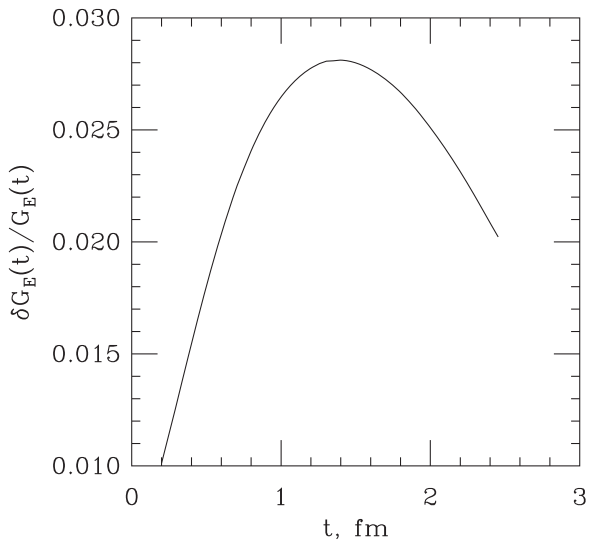

FIG. 3. Fractional change in $G_E(t)$ from a five percent variation in the phenomenological model for $\rho(\omega)$ of Ref. [13] over the range 0.6–0.9 GeV, as described in Eq. (15).

intermediate t where the region around the rho mass contributes heavily. Of course, this can be seen in Fig. 1. (This is basically just the phenomenon of vector meson dominance.)

As an application of this remark, suppose that the experimental $\rho(\omega)$ is not precisely known over some energy range, that two experiments differ by a fraction $\delta\rho(\omega)/\rho(\omega)$. Assuming that the difference is confined to some small region of ω, there will be a change in $G_E(t)$ [constructed from Eq. (7) with each experimental $\rho(\omega)$] of $\delta G_E(t)/G_E(t) \sim f\delta\rho/\rho$ where f is the fractional contribution of the ω region of $\rho(\omega)$ to $G_E(t)$.

A simple example comes from modifying the model for $\rho(\omega)$ from Ref. [13] over a range $\omega_{\min} < \omega < \omega_{\max}$, by multiplication by a weighting factor

$$w(\omega) = 1 + a \sin \pi \left(\frac{\omega - \omega_{\min}}{\omega_{\max} - \omega_{\min}} \right). \tag{15}$$

The fractional change in $G_E(t)$ is shown in Fig. 3 for the choice $\omega_{\min} = 0.6$ GeV, $\omega_{\max} = 0.9$ GeV, $a = 0.05$.

Notice the qualifier "assuming that the difference is confined to some small region of ω." The $G_E(t)$ at any t value is built of contributions from all ω, and a measurement of $G_E(t)$ at any t or for any range of t values does not make an absolute prediction about $\rho(\omega)$ at any particular ω value. However, lattice results could still be useful to distinguish between the different experimental $\rho(\omega)$'s.

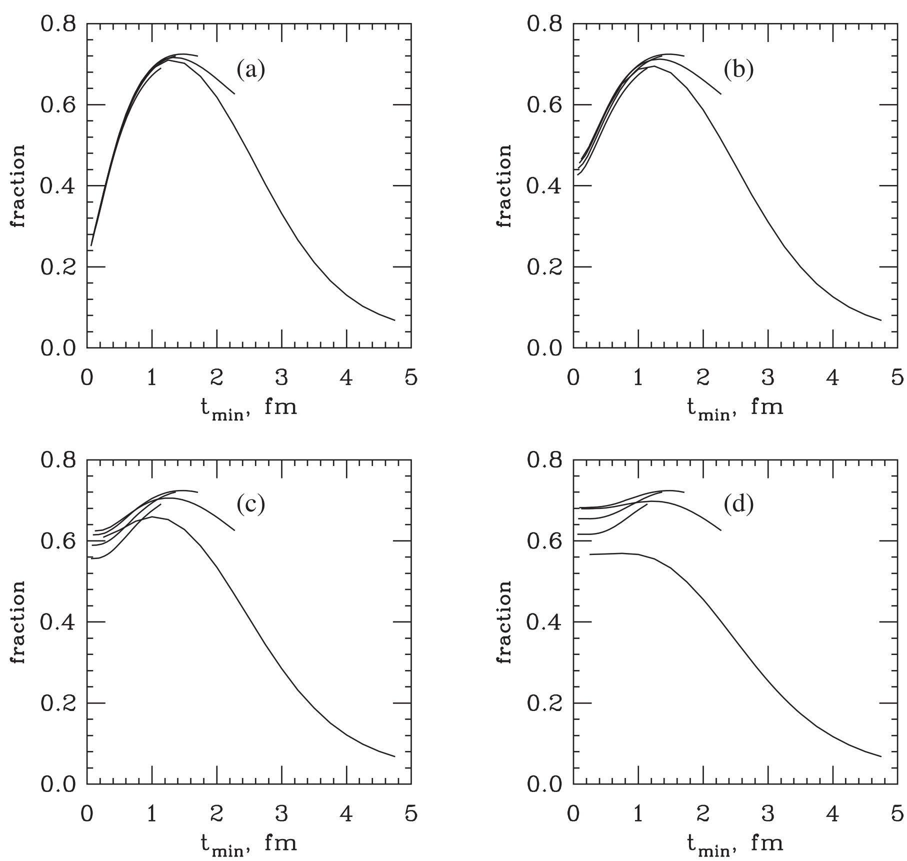

FIG. 4. Contribution of $4m_\pi < \omega < 6m_\pi$ to the integral of Eq. (8) for a power law $R_E(t) = (t/t_0)^n/n!$ with $t_0 = 0.15$ fm, for a range $t_{\min} < t < t_{\max}$ plotted versus $t_{\min}$ for $t_{\max} = 1.2$, 1.44, 1.8, 2.4, and 5 fm. (a) $n = 0$, (b) $n = 2$, (c) $n = 4$, and (d) $n = 6$.

Figure 3 shows that a five percent variation in $\rho(\omega)$ translates into a 2.7 percent variation in $G_E(t)$ over a fairly wide range of t. I cannot write about uncertainties in either $G_E(t)$ or $\rho(\omega)$. Lattice data for $G_E(t)$ are typically highly correlated, and it is almost impossible to estimate correlation uncertainties in a lattice data set without access to it. Similarly, the experimental data sets which give $\rho(\omega)$ are highly correlated. But, 2.7 percent seems to be an easy target, given that contemporary lattice measurements of a_μ^{HVP} are well under a percent. It seems likely that lattice calculations could determine $\rho(\omega)$ over the range 0.6–0.9 GeV at the five percent level.

To do this in practice, we need a weighting function. There seem to be many possible choices. But Figs. 2 and 3 indicate that all that is significant for $R_E(t)$ is that it be nonvanishing for the region of t where the desired ω value makes a large contribution to $G_E(t)$, and that it be small elsewhere. Figure 2 shows possible ranges of t's for different ω ranges. Two choices of $R_E(t)$ illustrate that claim.

First consider a family of power laws,

$$R_E(t) = \frac{1}{n!}\left(\frac{t}{t_0}\right)^n \tag{16}$$

for a range $t_{\mathrm{min}} < t < t_{\mathrm{max}}$. The t_0 and the $n!$ factor are just rescalings, which are useful for plots across n or for comparing weighted lattice data at different lattice spacings. Figure 4 shows the contribution of $4m_\pi < \omega < 6m_\pi$ to the integral of Eq. (8). Each panel is for a particular n value and shows a set of curves: Each curve is the fraction of the integral from t_{min} to t_{max} from this ω range, varying t_{min} at fixed t_{max}. A range of t in the range 1–2 fm gives an integral where the contribution of the $4m_\pi < \omega < 6m_\pi$ region of $\rho(\omega)$ approaches 70 percent, essentially independent of n. The curves extending out to $t_{\mathrm{max}} = 5$ fm show the obvious result that the contribution of the rho region to the integral becomes very small when taking $t_{\mathrm{min}} > 2$ fm.

This weighting, of course, has a hard cutoff in t. I have also repeated the analysis for soft cutoffs as in Eq. (10). There is little change in the result unless the smoothed step function $\Theta(t, t_{\mathrm{min}}, \Delta)$ becomes very broad.

Figure 5 shows the fractional change in the integral $\hat{\rho}$ from the model weighting factor of Eq. (15), using the power law weighting of Eq. (16). These curves all resemble the sensitivity of $G_E(t)$ itself to a variation in $\rho(\omega)$, as shown in Fig. 3.

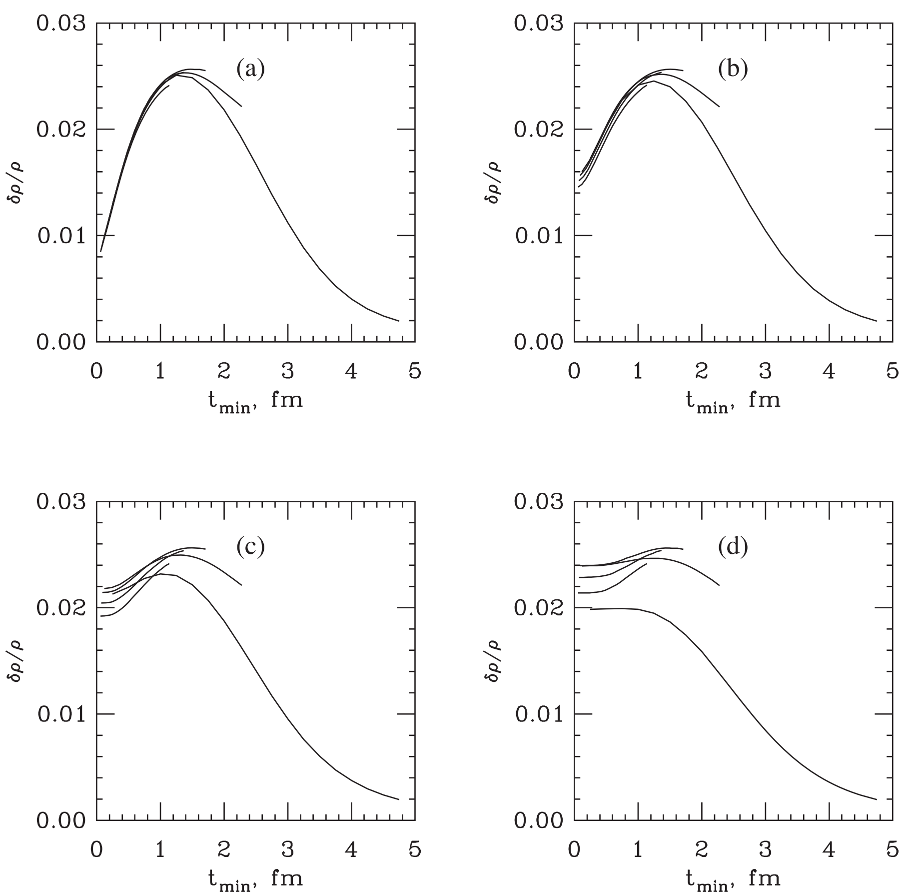

FIG. 5. Fractional change in the integral $\hat{\rho}$ of Eq. (8) under a five percent variation in $\rho(\omega)$ for $4m_\pi < \omega < 6m_\pi$ parametrized as in Eq. (15), for a power law $R_E(t) = (t/t_0)^n/n!$ with $t_0 = 0.15$ fm, for a range $t_{\mathrm{min}} < t < t_{\mathrm{max}}$ plotted versus t_{min} for $t_{\mathrm{max}} = 1.2$, 1.44, 1.8, 2.4, and 5 fm. (a) $n = 0$, (b) $n = 2$, (c) $n = 4$, and (d) $n = 6$.

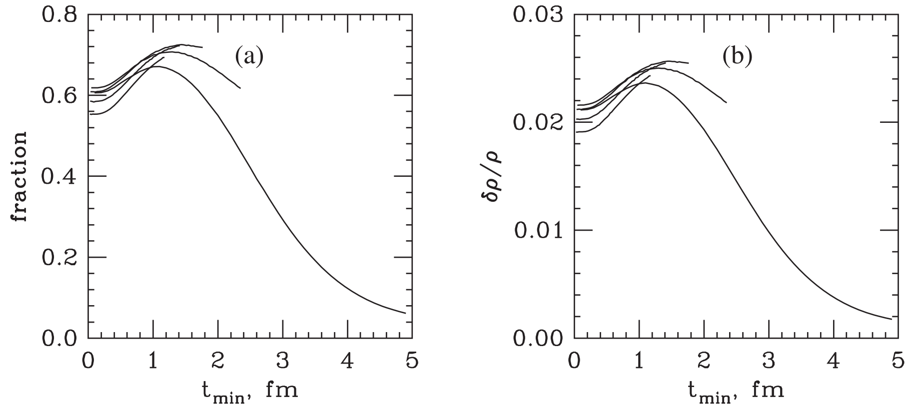

FIG. 6. Analog of Figs. 4 and 5 for the smearing kernel for a_μ^{HVP} with a sharp cutoff $t_{\min} \le t \le t_{\max}$, Eq. (10). Panel (a) is the contribution of $4m_\pi < \omega < 6m_\pi$ to the integral of Eq. (8). Panel (b) is the fractional change in the integral $\hat{\rho}$ of Eq. (8) under a five percent variation in $\rho(\omega)$ for $4m_\pi < \omega < 6m_\pi$ parametrized as in Eq. (15). Again, the curves are $t_{\min} < t < t_{\max}$ plotted versus $t_{\min}$ for $t_{\max} = 1.2$, 1.44, 1.8, 2.4, and 5 fm.

The conclusion to be drawn from these tests is that the feature of the smearing function $R_E(t)$ which is most sensitive to a variation in $\rho(\omega)$ over a limited ω range is the range of t values which the smearing function probes, rather than its precise shape.

Figure 6 shows similar results for the smearing kernel used for a_μ^{HVP} in its intermediate window guise, Eq. (10) [but with a sharp cutoff $t_{\min} \le t \le t_{\max}$)]. The figures are nearly identical to the ones for power law weighting. The conclusion seems to be that a lattice calculation of $G_E(t)$ with an accuracy of 2–3 percent (in the continuum limit, of course) over the range of 1–2 fm can distinguish a five percent variation in $\rho(\omega)$ in the rho region.

At this point I hope for an analysis by one of the lattice groups using their own data sets. The idea I have presented is trivial, but it also seems simple to implement. I think that $\rho(\omega)$ (and related quantities) are interesting in and of themselves, and that trying to extract features of $\rho(\omega)$ which have nothing to do with a_μ^{HVP} from $G_E(t)$ could be a useful project [19]. And, of course, identical weighting techniques can connect other inclusive processes with Euclidean correlators.

Questions at the end of a seminar by Finn Stokes at the MIT Virtual Lattice Colloquium Series motivated this paper. I would also like to thank Maarten Golterman for correspondence and Chris Polly for encouragement. This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of High Energy Physics under Award No. DE-SC-0010005.

[1] T. Aoyama, N. Asmussen, M. Benayoun, J. Bijnens, T. Blum, M. Bruno, I. Caprini, C. M. Carloni Calame, M. Cè, G. Colangelo *et al.*, Phys. Rep. **887**, 1 (2020).

[2] See the set of potential targets in M. T. Hansen, H. B. Meyer, and D. Robaina, Phys. Rev. D **96**, 094513 (2017).

[3] A. Anastasi *et al.* (KLOE-2 Collaboration), J. High Energy Phys. 03 (2018) 173.

[4] G. Backus and F. Gilbert, Phil. Trans. R. Soc. A **266**, 123 (1970).

[5] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes: The Art of Scientific Computing* (Cambridge University Press, Cambridge, England, 2007).

[6] M. Hansen, A. Lupo, and N. Tantalo, Phys. Rev. D **99**, 094508 (2019).

[7] P. Gambino and S. Hashimoto, Phys. Rev. Lett. **125**, 032001 (2020).

[8] H. Fukaya, S. Hashimoto, T. Kaneko, and H. Ohki, Phys. Rev. D **102**, 114516 (2020).

[9] G. Bailas, S. Hashimoto, and T. Ishikawa, Prog. Theor. Exp. Phys. **2020**, 043B07 (2020).

[10] T. Blum, P. A. Boyle, V. Gülpers, T. Izubuchi, L. Jin, C. Jung, A. Jüttner, C. Lehner, A. Portelli, and J. T. Tsang (RBC and UKQCD Collaborations), Phys. Rev. Lett. **121**, 022003 (2018).

[11] M. Davier, L. Girlanda, A. Hocker, and J. Stern, Phys. Rev. D **58**, 096014 (1998).

[12] K. Maltman, Phys. Lett. B **440**, 367 (1998).

[13] D. Bernecker and H. B. Meyer, Eur. Phys. J. A **47**, 148 (2011).

[14] P. A. Zyla *et al.* (Particle Data Group), Prog. Theor. Exp. Phys. **2020**, 083C01 (2020); see http://pdg.lbl.gov/current/xsect/ for the tabulation; an early description of the data set can be found in V. V. Ezhela, S. B. Lugovsky, and O. V. Zenin, arXiv:hep-ph/0312114.

[15] G. P. Lepage, The analysis of algorithms for lattice field theory, *Proceedings of the 1989 TASI Summer School, Boulder, CO, 1989* [Report No. CLNS-89-971] (World Scientific, Singapore, 1990).
[16] G. Parisi, Phys. Rep. **103**, 203 (1984).
[17] M. G. Endres, D. B. Kaplan, J. W. Lee, and A. N. Nicholson, Phys. Rev. Lett. **107**, 201601 (2011).
[18] C. T. H. Davies *et al.* (Fermilab Lattice, LATTICE-HPQCD, and MILC Collaborations), Phys. Rev. D **101**, 034512 (2020).
[19] See the discussion in G. Colangelo, A. X. El-Khadra, M. Hoferichter, A. Keshavarzi, C. Lehner, P. Stoffer, and T. Teubner, arXiv:2205.12963.