Three-loop QCD corrections to the electroweak boson masses

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I find the three-loop corrections at leading order in QCD to the physical masses of the Higgs, W, and Z bosons in the Standard Model. The results are obtained as functions of the $\overline{\mathrm{MS}}$ Lagrangian parameters only, using the tadpole-free scheme for the vacuum expectation value. The dependences of the computed masses on the renormalization scale are found to be smaller than present experimental uncertainties in each case. In the case of the Higgs boson mass, the new result is the state of the art, while the results for W and Z are in good numerical agreement with corresponding results in the on-shell and hybrid schemes. These results are now included in the Standard Model in Dimensional Regularization computer code.

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I. INTRODUCTION

Since the discovery of the Higgs boson in 2012, the Standard Model is a mathematically complete theory, for which precision calculations can be performed. In addition to providing a test of the agreement of the theory with experiment, this allows us to obtain accurate results for the short-distance Lagrangian parameters, suitable for matching to candidate ultraviolet completions. The goal of this paper is to report the three-loop QCD contributions to the pole masses of the W, Z, and Higgs bosons in the Standard Model. In the case of the Higgs boson mass, the result obtained is the new state-of-the-art result, including the complete set of two-loop effects as well as the three-loop terms proportional to $\alpha_S^2 y_t^2$, including all momentum-dependent effects, as well as the three-loop terms proportional to $\alpha_S y_t^4$ and y_t^6 in the approximation that $M_h^2 \ll M_t^2$.

The results below are given in the pure $\overline{\mathrm{MS}}$ renormalization scheme [1,2] based on dimensional regularization [3–7], so that all independent inputs are running Lagrangian parameters. The calculation is also based on the tadpole-free scheme for the Higgs vacuum expectation value (VEV), which is defined to be the minimum of the exact Landau gauge effective potential, currently known in an approximation at full three-loop order [8–11] with the leading four-loop order QCD part [12] and resummation of the Goldstone boson contributions [13,14]. The tadpole-free VEV scheme has a formally faster convergence in perturbation theory than schemes based on a tree-level VEV definition, since in the latter the tadpole diagrams necessarily introduce inverse powers of the Higgs self-coupling λ. The price to be paid for this improvement is that the validity of the calculations is restricted to the Landau gauge fixing prescription in the electroweak sector.

Previous two-loop calculations of the W and Z masses have been given in Refs. [15–18], using the tree-level definition for the VEV. In addition, there is a long history of calculations of the ρ parameter including up to four-loop order QCD contributions [19–41], which can be used to relate the W boson on-shell mass to the Z boson mass. The present paper relies on a quite different organization of perturbation theory, by taking all physical masses as outputs including the W and Z boson pole masses separately, rather than using the Z boson on-shell mass as an input. The complete two-loop W and Z boson pole squared masses in the scheme adopted in this paper were given in refs. [42] and [43] respectively. The present paper will add the three-loop QCD contributions to those results in a consistent way.

In the case of the Higgs boson pole squared mass, Ref. [44] provided the mixed QCD/electroweak parts, Ref. [45] gave results in the gaugeless limit in which g, g' are neglected in the two-loop part, and Ref. [46] gave an interpolating formula for the full two-loop approximation in a hybrid $\overline{\mathrm{MS}}$/on-shell scheme. In Ref. [47], the full two-loop corrections were extended to include the three-loop contributions in the gaugeless effective potential limit (formally, $g_3^2, y_t^2 \gg \lambda, g^2, g'^2$, where g_3, g, g' are the gauge couplings, y_t is the top-quark Yukawa coupling, and λ is the Higgs self-coupling) using the pure $\overline{\mathrm{MS}}$ tadpole-free scheme. The present paper will extend this further to include the momentum-dependent parts of the leading QCD contribution to the Higgs boson self-energy in the calculation of the pole squared mass.

To specify notation, the complex pole squared masses for the electroweak bosons are each given in the loop-expansion form

$$
\begin{aligned}
s^X_{\rm pole} &\equiv (M_X - i\Gamma_X/2)^2 \\
&= m_X^2 + \frac{1}{16\pi^2}\Delta_X^{(1)} + \frac{1}{(16\pi^2)^2}\Delta_X^{(2)} + \frac{1}{(16\pi^2)^3}\Delta_X^{(3)} + \ldots,
\end{aligned} \tag{1.1}
$$

with $X = W$, Z, and h. Note that all of the quantities appearing on the right-hand side of Eq. (1.1) depend only on the $\overline{\rm MS}$ input parameters of the theory. In particular, the tree-level $\overline{\rm MS}$ squared masses m_X^2 are given by $m_W^2 = g^2v^2/4$, $m_Z^2 = (g^2+g'^2)v^2/4$, and $m_h^2 = 2\lambda v^2$. The complete one- and two-loop contributions given in Refs. [42,43,47] were written in terms of master integrals defined in Refs. [48,49], the latter of which provided a computer program TSIL for their efficient numerical evaluation. The computer program SMDR [50] incorporates these calculations of the W, Z, and Higgs physical masses and many other results within the pure $\overline{\rm MS}$ tadpole-free scheme, matching observables to Lagrangian parameters. Another public code MR [51] provides similar functionality, but using the tree-level VEV scheme.

For the vector bosons, it is important to note that the standard practice in experimental papers and by the review of particle properties (RPPs) [52] from the Particle Data Group (PDG) is to report the on-shell masses found from a variable-width Breit-Wigner linewidth fit, which should be related to the complex pole mass and width M_X and Γ_X defined in Eq. (1.1) by

$$M^{\rm PDG} = M\frac{1+\delta}{\sqrt{1-\delta}}, \tag{1.2}$$

$$\Gamma^{\rm PDG} = \Gamma\frac{1+\delta}{(1-\delta)^{3/2}}, \tag{1.3}$$

where

$$\delta = \Gamma^2/4M^2. \tag{1.4}$$

(In this paper, the superscript "PDG" refers to the convention used by the PDG and not to the averaged experimental results produced by the PDG in the RPPs.) To add to the potential for confusion, in Refs. [42,43,47] by the present author, and many publications by other authors, a different parametrization for complex pole masses has been used, denoted here by

$$s_{\rm pole} = M'^2 - i\Gamma' M', \tag{1.5}$$

which is related to the M and Γ in Eq. (1.1) by

$$M' = M\sqrt{1-\delta}, \tag{1.6}$$

$$\Gamma' = \Gamma/\sqrt{1-\delta}. \tag{1.7}$$

The $(M^{\rm PDG}, \Gamma^{\rm PDG})$ and (M', Γ') parametrizations can be considered to contain the same information as (M, Γ), through the defining relations in Eqs. (1.2)–(1.4), (1.6), and (1.7). However, as emphasized in a recent paper [53], the (M, Γ) parametrization defined by Eq. (1.1) has the clear advantage that $\Gamma = 1/\tau$ is precisely the inverse mean lifetime of the particle, unlike $\Gamma^{\rm PDG}$ and Γ'. In the following $s_{\rm pole}$ will be computed, but the information that it contains must be converted to $M^{\rm PDG}$ to compare directly with the results quoted by the PDG and experimental collaborations. The W and Z PDG-convention masses that are almost always quoted are, respectively, about 0.020 and 0.026 GeV larger than the pole masses M_W and M_Z, and about 0.027 and 0.034 GeV larger than M'_W and M'_Z. The experimental values from the 2021 update of the 2020 RPPs are[1] $M_Z^{\rm PDG} = 91.1876 \pm 0.0021$ and $M_W^{\rm PDG} = 80.379 \pm 0.012$ and $M_h = 125.25 \pm 0.17$ GeV. The Higgs boson width (about 4.1 MeV, according to theory) is so small that the numerical distinction between the PDG-convention and complex pole mass versions of the real part M_h is negligible.

The three-loop integrals to be used below have been defined and discussed in Secs. IV, VI, and VII of Ref. [55]. The master integrals are given there as a renormalized ϵ-finite basis, defined so that expansions of integrals to positive powers in ϵ will never be needed, even when the results of the present paper are (eventually) extended to four-loop order or beyond. Denoting the lists of one-, two-, and three-loop renormalized ϵ-finite master integrals by $\mathcal{I}_j^{(1)}$, $\mathcal{I}_j^{(2)}$, and $\mathcal{I}_j^{(3)}$, respectively, then the general form of a three-loop contribution to the pole mass of $X = W$, Z, or h is

$$
\begin{aligned}
\Delta_X^{(3)} &= \sum_j c^{(3)}\mathcal{I}_j^{(3)} + \sum_{j,k} c_{j,k}^{(2,1)}\mathcal{I}_j^{(2)}\mathcal{I}_k^{(1)} \\
&\quad + \sum_{j,k,l} c_{j,k,l}^{(1,1,1)}\mathcal{I}_j^{(1)}\mathcal{I}_k^{(1)}\mathcal{I}_l^{(1)} + \sum_j c_j^{(2)}\mathcal{I}_j^{(2)} \\
&\quad + \sum_{j,k} c_{j,k}^{(1,1)}\mathcal{I}_j^{(1)}\mathcal{I}_k^{(1)} + \sum_j c_j^{(1)}\mathcal{I}_j^{(1)} + c^{(0)},
\end{aligned} \tag{1.8}
$$

where all of the coefficients $c^{(3)}, c_{j,k}^{(2,1)}, \ldots, c^{(0)}$ are dimensionless $\overline{\rm MS}$ couplings multiplied by rational functions of the $\overline{\rm MS}$ top-quark squared mass

$$t = y_t^2 v^2/2 \tag{1.9}$$

and either $s = W$, Z, or h as appropriate, where

[1]After the first version of the present paper, the CDF Collaboration released [54] a new measurement of the W mass that is substantially higher, $M_W^{\rm PDG} = 80.4335 \pm 0.0064_{\rm stat} \pm 0.0069_{\rm syst}$ GeV. See Figs. 5 and 6.

$$W = m_W^2 = g^2 v^2/4, \tag{1.10}$$

$$Z = m_Z^2 = (g^2 + g'^2) v^2/4, \tag{1.11}$$

$$h = m_h^2 = 2\lambda v^2. \tag{1.12}$$

The VEV v is defined to be the minimum of the $\overline{\rm MS}$ effective potential in Landau gauge at all orders in perturbation theory, so that the sum of all Higgs tadpole diagrams vanishes. Note that the name of each particle is being used as a synonym for the tree-level $\overline{\rm MS}$ squared mass in the tadpole-free scheme. (All other fermions are taken to be massless, except in the one-loop parts $\Delta_X^{(1)}$.) Note also that the tree-level $\overline{\rm MS}$ squared masses t, W, Z, and h are not gauge invariant, but are specific to Landau gauge, due to their dependence on the VEV. However, as is well known, the complex pole masses [and thus the PDG-convention masses for W and Z, defined by Eqs. (1.2)–(1.4)] are gauge invariant.

The loop integrals include logarithmic dependences on the $\overline{\rm MS}$ renormalization scale Q, written in this paper in terms of

$$L_t \equiv \ln(t/Q^2), \tag{1.13}$$

$$L_{-s} \equiv \ln(s/Q^2) - i\pi, \tag{1.14}$$

for the external momentum invariant s, which has a positive infinitesimal imaginary part. In the three-loop parts $\Delta_X^{(3)}$, the integrals will always be evaluated at external momentum invariant equal to the tree-level squared mass, $s = W$, Z, or h. This is just as consistent as choosing to evaluate them at the (real part of) the corresponding pole squared mass instead, as the difference is of four-loop order and numerically small.

In order to provide more opportunities for checks, the results below will be given in terms of $SU(3)_c$ group theory quantities,

$$C_G = N_c = 3, \qquad C_F = (N_c^2 - 1)/2N_c = 4/3,$$
$$T_F = 1/2, \qquad n_g = 3. \tag{1.15}$$

Here N_c is the number of colors, C_G and C_F are the quadratic Casimir invariants of the adjoint and fundamental representations, respectively, T_F is the Dynkin index of the fundamental representation, and n_g is the number of fermion generations.

For numerical results shown below, I will use a benchmark Standard Model designed to give output parameters in agreement with the current central values of the 2021 update of the 2020 RPPs [52],

$$\begin{aligned}
&M_t = 172.5\ \text{GeV}, \qquad M_h = 125.25\ \text{GeV}, \qquad M_Z^{\rm PDG} = 91.1876\ \text{GeV},\\
&G_F = 1.1663787 \times 10^{-5}\ \text{GeV}^2, \qquad \alpha_0 = 1/137.035999084, \qquad \alpha_S^{(5)}(M_Z) = 0.1179,\\
&m_b(m_b) = 4.18\ \text{GeV}, \qquad m_c(m_c) = 1.27\ \text{GeV}, \qquad m_s(2\ \text{GeV}) = 0.093\ \text{GeV},\\
&m_d(2\ \text{GeV}) = 0.00467\ \text{GeV}, \qquad m_u(2\ \text{GeV}) = 0.00216\ \text{GeV}, \qquad M_\tau = 1.77686\ \text{GeV},\\
&M_\mu = 0.1056583745\ \text{GeV}, \qquad M_e = 0.000510998946\ \text{GeV},\\
&\Delta\alpha_{\rm had}^{(5)}(M_Z) = 0.02766.
\end{aligned} \tag{1.16}$$

Using the latest version 1.2 of the computer program SMDR [50], which incorporates the new results of the present paper, these are best fit by the $\overline{\rm MS}$ input parameters (using the tadpole-free scheme for the Landau gauge VEV and writing g_3 for the QCD coupling in the full six-quark Standard Model theory),

$$\begin{aligned}
&Q_0 = 172.5\ \text{GeV},\\
&v(Q_0) = 246.603216913\ \text{GeV}, \qquad \lambda(Q_0) = 0.12639276585,\\
&g_3(Q_0) = 1.16300624875, \qquad g(Q_0) = 0.647606757306, \qquad g'(Q_0) = 0.358550211695,\\
&y_t(Q_0) = 0.93157701535, \qquad y_b(Q_0) = 0.015503239387, \qquad y_\tau(Q_0) = 0.0099944376213,\\
&y_c(Q_0) = 0.003394710569, \qquad y_s(Q_0) = 0.0002916507520, \qquad y_\mu(Q_0) = 0.0005883797990,\\
&y_d(Q_0) = 1.464523924362 \times 10^{-5}, \qquad y_u(Q_0) = 6.739112138367 \times 10^{-6},\\
&y_e(Q_0) = 2.792980305214 \times 10^{-6}.
\end{aligned} \tag{1.17}$$

Specifically, the values in Eq. (1.17) were obtained by applying the command-line utility calc_fit of SMDR to the values in Eq. (1.16). The code proceeds iteratively, converging to a stable relative precision of better than 10^{-12} in all outputs after a few iterations. Here I have included many more significant digits than justified by the theoretical errors, merely for the sake of reproducibility. These $\overline{\rm MS}$ quantities can be run to a different renormalization scale choice Q, where the pole squared masses can be recomputed. In the idealized case, the pole squared masses, being observables, would be independent of the scale Q at which they are computed.

Below, I will show figures illustrating the numerical results for the Z boson pole mass and width, the Higgs boson mass and width, the Higgs boson self-interaction, and the W boson mass and width. In the cases of the Z and Higgs boson masses, the numerical results shown are of course not predictions, in the sense that the numerical inputs were determined by the data in Eq. (1.16). Instead, they serve to show the dependence of the calculation on the choice of renormalization scale Q. In the case of the Higgs self-coupling, the results reflect the present state-of-the-art calculation, given the Higgs mass and other on-shell inputs. In the case of the W boson mass, the result is a genuine prediction, since it is not included in the data of Eq. (1.16). More generally, the results of this paper, as incorporated in SMDR, can be used to calculate the on-shell quantities for any chosen values of the $\overline{\rm MS}$ input quantities in Eq. (1.17). Or, conversely, the $\overline{\rm MS}$ parameters can be obtained iteratively by the SMDR code for any values of the on-shell quantities in Eq. (1.16).

The renormalization group running is carried out using the state-of-the-art beta functions for the Standard Model. The two- and three-loop beta functions were found in [56–60] and [61–69], respectively. The four-loop beta function for the QCD coupling g_3 was found in [70–74] in the approximation that only g_3, y_t, and λ are included. The pure QCD five-loop beta functions were obtained in [75,76], and the four- and five-loop QCD contributions to the quark Yukawa beta functions were obtained in Refs. [77,78] and Ref. [79] respectively, and the four-loop QCD contributions to the beta function of the Higgs self-coupling λ were obtained from [12,80]. Finally, the complete four-loop beta functions for the three gauge couplings have been provided by [81]. All of these results have been included in the latest version of the code SMDR, which was used to carry out the numerical computations described below. The code also implements results for multiloop threshold matching of electroweak couplings [17,18,82–85], the QCD coupling [86–91], and quark and lepton masses [92–103].

II. THE Z BOSON POLE MASS

Consider the Z boson complex pole squared mass $s^Z_{\rm pole}$ in the form of Eq. (1.1). The complete one- and two-loop contributions $\Delta_Z^{(1)}$ and $\Delta_Z^{(2)}$ were given in the tadpole-free pure $\overline{\rm MS}$ scheme in Ref. [43]. The three-loop QCD part can be split into contributions from 13 distinct classes of self-energy diagrams with different group theory structures, using the quantities defined in Eq. (1.15),

$$\begin{aligned}\Delta_Z^{(3),g_3^4} = g_3^4 N_c C_F\{&(a_{u_L}^2 + a_{u_R}^2)[C_G\Delta_Z^{(3,a)} + C_F\Delta_Z^{(3,b)} + T_F\Delta_Z^{(3,c)} + (2n_g-1)T_F\Delta_Z^{(3,d)}]\\ &+ 2a_{u_L}a_{u_R}[C_G\Delta_Z^{(3,e)} + C_F\Delta_Z^{(3,f)} + T_F\Delta_Z^{(3,g)} + (2n_g-1)T_F\Delta_Z^{(3,h)}] + (g^2+g'^2)T_F\Delta_Z^{(3,i)}\\ &+ [(n_g-1)(a_{u_L}^2+a_{u_R}^2) + n_g(a_{d_L}^2 + a_{d_R}^2)][C_G\Delta_Z^{(3,j)} + C_F\Delta_Z^{(3,k)} + T_F\Delta_Z^{(3,l)} + (2n_g-1)T_F\Delta_Z^{(3,m)}]\},\end{aligned} \tag{2.1}$$

where the tree-level couplings of the Z boson to up- and down-type quarks are

$$a_{u_R} = -\frac{2}{3}\frac{g'^2}{\sqrt{g^2+g'^2}}, \qquad a_{u_L} = \frac{1}{2}\sqrt{g^2+g'^2} + a_{u_R}, \tag{2.2}$$

$$a_{d_R} = \frac{1}{3}\frac{g'^2}{\sqrt{g^2+g'^2}}, \qquad a_{d_L} = -\frac{1}{2}\sqrt{g^2+g'^2} + a_{d_R}. \tag{2.3}$$

Most of the three-loop diagrams are straightforward to set up and can be carried out with a naive treatment of γ_5, taken to anticommute with all of the other gamma matrices. The known exception to this is the double triangle diagrams shown in Fig. 1, which feature two distinct triangle quark

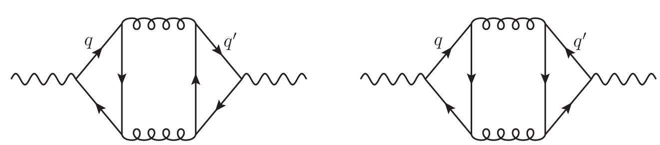

FIG. 1. Three-loop contribution to the Z boson mass from diagrams involving two triangle quark loops, which give a nonvanishing contribution with a consistent treatment of the axial vector coupling. These contributions are individually divergent for each of $(q,q') = (t,t), (t,b), (b,t), (b,b)$, but are finite and gauge invariant after the combination. Contributions involving sums over other (q,q') quark doublet combinations vanish in the massless quark limit.

loops each containing a γ_5 from the axial vector coupling to the Z boson. (The vector couplings to the Z boson give vanishing contributions for the sum of these diagrams.) The contributions from $(q, q') = (t, t), (t, b), (b, t), (b, b)$ are separately divergent, but their sum is finite and gauge invariant. Therefore, for these diagrams only, one can use the prescription [104,105]

$$\gamma^\mu \gamma_5 \to \frac{i}{6} \epsilon^{\mu\nu\rho\sigma} \gamma_\nu \gamma_\rho \gamma_\sigma, \tag{2.4}$$

based on the ‘t Hooft–Veltman treatment [6] of γ_5, and then carry out the Lorentz algebra in four dimensions before reducing to master integrals in d dimensions. The result is the contribution $\Delta_Z^{(3,i)}$ in Eq. (2.1). The contributions from diagrams with one or both of q and q' summed over the other quark doublets (u, d) and (c, s) vanish, because the axial couplings $a_{q_L} - a_{q_R}$ for down- and up-type quarks have the same magnitude and opposite sign, and they are being treated as mass degenerate (specifically, massless). The result for general nonzero $s = Z$ found here reduces to $\Delta_Z^{(3,i)} \to 21\zeta_3$ for $s = 0$, which agrees with the original calculation in that limit [106] and with the corresponding contribution to the ρ parameter obtained in [27,28,39].

The contributions from the diagrams in which the Z boson couples directly to a single massless (in the present approximation, nontop) quark loop are relatively simple, and can be written as

$$\Delta_Z^{(3,j)} = Z\left(-\frac{44215}{324} + \frac{908}{9}\zeta_3 + \frac{40}{3}\zeta_5 + \left[41 - \frac{88}{3}\zeta_3\right] L_{-Z} - \frac{11}{3} L_{-Z}^2\right), \tag{2.5}$$

$$\Delta_Z^{(3,k)} = Z\left(\frac{143}{9} + \frac{148}{3}\zeta_3 - 80\zeta_5 - L_{-Z}\right), \tag{2.6}$$

$$\begin{aligned}\Delta_Z^{(3,l)} = {} & \frac{16}{27} Z(7t + 3Z) I_{7c}(0,0,0,0,0,t,t) + \frac{16}{243}(128t + 43Z) I_{6c}(0,0,0,0,t,t) - \frac{8}{27}(18t + 7Z) I_{6f}(0,0,0,0,t,t) \\ & + \frac{32}{243t}(5Z - 17t) I_{5c}(0,0,0,t,t) + \frac{160}{81t} I_4(0,0,t,t) + \left(\frac{896}{243}\zeta_3 - \frac{8276}{729}\right) t + \frac{2599}{243} Z + \frac{80}{2187}\frac{Z^2}{t} \\ & + \left(\frac{11144}{243} t - \frac{224}{9}\zeta_3 t - \frac{3320}{243} Z\right) L_t - \frac{352}{81} t L_t^2 - \frac{112}{243} t L_t^3 + \frac{4}{243}(884t - 217Z) L_{-Z} + \left(\frac{160}{27} Z - \frac{32}{3} t\right) L_t L_{-Z} \\ & + \frac{272}{81} t L_t^2 L_{-Z} + \frac{20}{81} Z L_{-Z}^2 - \frac{16}{243}(17t + 10Z) L_t L_{-Z}^2,\end{aligned} \tag{2.7}$$

$$\Delta_Z^{(3,m)} = Z\left(\frac{3701}{81} - \frac{304}{9}\zeta_3 + \left[\frac{32}{3}\zeta_3 - \frac{44}{3}\right] L_{-Z} + \frac{4}{3} L_{-Z}^2\right). \tag{2.8}$$

Here, $\Delta_Z^{(3,l)}$ contains a top-quark loop that corrects a gluon propagator, rather than connecting to the external Z boson. The remaining contributions in Eq. (2.1) are much more complicated, and are given in the Supplemental Material [107]. Each of the contributions has the form of Eq. (1.8), with master integrals chosen in Ref. [55],

$$\mathcal{I}^{(1)} = \{A(t), B(0,0), B(t,t)\}, \tag{2.9}$$

$$\mathcal{I}^{(2)} = \{\zeta_3, V(t,t,0,t), M(t,t,t,t,0), M(0,t,0,t,t)\}, \tag{2.10}$$

$$\begin{aligned}\mathcal{I}^{(3)} = \{ & \zeta_5, H(0,0,t,0,t,t), H(0,t,t,t,0,t), I_4(t,t,t,t), I_{5a}(t,0,t,0,t), I_{5b}(0,t,t,t,t), \\ & I_{5c}(t,t,t,t,t), I_{6c}(t,t,t,0,t,t), I_{6c2}(t,t,t,0,0,0), I_{6d}(0,t,t,t,t,0), I_{6d}(t,0,t,0,t,0), \\ & I_{6d}(t,0,t,t,0,t), I_{6e}(0,0,0,0,t,t), I_{6e}(0,t,t,t,0,t), I_{6e}(t,t,t,0,t,t), I_{6f}(0,0,0,0,t,t), \\ & I_{6f5}(0,0,0,0,t,t), I_{7a}(0,0,t,t,t,t,t), I_{7a}(t,t,t,t,t,t,0), I_{7a3}(t,t,t,t,t,t,0), \\ & I_{7b}(0,t,t,t,t,0,0), I_{7b}(t,0,t,t,t,t,0), I_{7b4}(t,0,t,t,t,t,0), I_{7b4}(t,t,0,t,t,0,t), \\ & I_{7c}(t,t,t,t,0,0,0), I_{7d}(t,t,0,t,0,t,0), I_{7d}(t,t,0,t,t,0,t), I_{7e}(0,0,0,0,0,t,t), \\ & I_{7e}(0,0,t,t,t,0,0), I_{8a}(t,0,t,t,t,t,t,0), I_{8a}(t,t,t,t,t,0,0,0), I_{8b}(t,t,t,t,t,0,0,t), \\ & I_{8c}(t,0,t,t,t,t,t,0), I_{8c}^{pk}(t,t,t,t,t,0,0,t)\},\end{aligned} \tag{2.11}$$

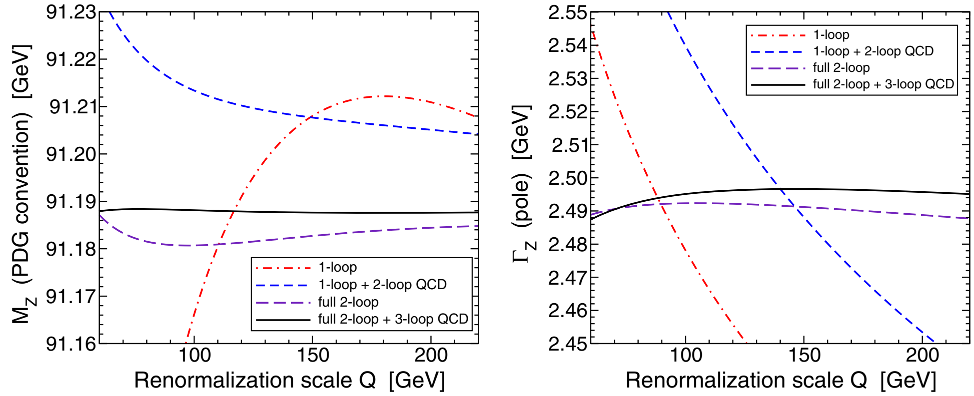

FIG. 2. The Z boson mass in the PDG-convention $M_Z^{\rm PDG}$ (left) and the width Γ_Z (right), obtained from the calculated complex pole mass $s_{\rm pole}^Z$, as a function of the renormalization scale Q. The different lines show various approximations as labeled. The $\overline{\rm MS}$ input parameters are as given in Eq. (1.17), which provide for $M_Z^{\rm PDG} = 91.1876$ GeV when calculated at the renormalization scale $Q = 160$ GeV using the full two-loop plus three-loop QCD approximation.

with $A(t) = t(L_t - 1)$ and $B(0,0) = 2 - L_{-Z}$. However, in Eq. (2.7) above, I have chosen to write the expression for $\Delta_Z^{(3,l)}$ in terms of candidate master integrals that were solved for in Ref. [55], rather than the master integrals listed above [which are a subset of the ones listed in Eq. (7.4) in Ref. [55], joined by $B(0,0)$ and ζ_3 and ζ_5 from the integrals with all propagators massless]. This simplifies the expression somewhat, because the integrals used in Eq. (2.7) have the same propagator structures as descendants of the underlying Feynman diagrams for the $\Delta_Z^{(3,l)}$ contribution.

As a check of Eq. (2.1), I have verified that the full expression for the observable $s_{\rm pole}^Z$ is renormalization group invariant through three-loop terms proportional to g_3^4, using the derivatives of the master integrals with respect to Q found in the ancillary file QddQ of Ref. [55].

For practical numerical evaluation, after using the Standard Model group theory values in Eq. (1.15) and applying the expansions for the master integrals in the ancillary file Ievenseries of Ref. [55], I find

$$\Delta_Z^{(3),g_3^4} = g_3^4 t\{(g^2+g'^2)(\delta_1^Z+\delta_2^Z) + a_{u_L}a_{u_R}\delta_3^Z + [2(a_{u_L}^2+a_{u_R}^2)+3(a_{d_L}^2+a_{d_R}^2)]\delta_4^Z\}, \tag{2.12}$$

where the series expansions of δ_1^Z, δ_2^Z, δ_3^Z, and δ_4^Z are given in the Supplemental Material [107] to order r_Z^{18}, where

$$r_Z \equiv \frac{Z}{4t} = \frac{g^2+g'^2}{8y_t^2}. \tag{2.13}$$

The contribution δ_1^Z isolates the results form the double triangle diagrams in Fig. 1. The series expansion coefficients are given both numerically and analytically in terms of L_t, L_{-Z}, and the constants ζ_3, ζ_5, and

$$c'_H = 32{\rm Li}_4(1/2) - 22\zeta_4 + \frac{4}{3}\ln^2(2)[\ln^2(2)-\pi^2)] \approx -13.2665092775\ldots. \tag{2.14}$$

The series converge for all $r_Z < 1$, which is clearly satisfied in actuality. The first few terms in the expansions are

$$\delta_1^Z = 50.486 + r_Z[79.645 + 49.333(L_t - L_{-Z}) + 8(L_t - L_{-Z})^2] + r_Z^2[-15.758 + 5.531(L_t - L_{-Z})] + r_Z^3[-3.066 - 1.493(L_t - L_{-Z})] + \cdots, \tag{2.15}$$

$$\delta_2^Z = 9.978 + 49.258L_t + 18L_t^2 - 30L_t^3 + r_Z(-113.200 - 90.222L_t + 28L_t^2) + r_Z^2(-42.485 - 63.002L_t - 4.8L_t^2) + r_Z^3(-45.813 - 74.011L_t - 32.914L_t^2) + \cdots, \tag{2.16}$$

$$\delta_3^Z = r_Z(-687.728 - 298.667L_t + 224L_t^2) + r_Z^2(-733.683 - 685.827L_t - 51.200L_t^2) + r_Z^3(-707.875 - 962.072L_t - 394.971L_t^2) + \cdots, \tag{2.17}$$

$$\delta_4^Z = r_Z(-56.799 - 14.758L_t - 10.667L_t^2 + 180.381L_{-Z} + 21.333L_tL_{-Z} - 122.667L_{-Z}^2) + r_Z^2[-88.570 - 33.375(L_t - L_{-Z}) - 3.793(L_t - L_{-Z})^2] + r_Z^3[4.074 + 2.521(L_t - L_{-Z}) + 0.406(L_t - L_{-Z})^2] + \cdots. \tag{2.18}$$

It is interesting to note that, in the expansion in small r_Z, the subleading contribution is numerically comparable to (or even larger than, for smaller Q) the leading contribution obtained by $r_Z = 0$. This is due mostly to the term proportional to $r_Z L_{-Z}^2$ in the contribution Eq. (2.18) from massless quark loops, because of the large magnitude of the coefficient -122.667 and because $L_{-Z}^2 = [-i\pi + \ln(Z/Q^2)]^2$ provides up to an order of magnitude enhancement.

The resulting contribution of Eq. (2.12) has now been included in the latest version 1.2 of the code SMDR [50]. Figure 2 shows the results for the PDG-convention mass $M_Z^{\rm PDG}$ and the width Γ_Z obtained from the pole mass, for the $\overline{\rm MS}$ input parameters given in Eq. (1.17). These benchmark parameters were chosen so that the calculated $M_Z^{\rm PDG}$, with all known contributions included and using the renormalization scale $Q = 160$ GeV, is equal to the experimental central value 91.1876 GeV. To obtain the results in the figure, the $\overline{\rm MS}$ input parameters are run to other $\overline{\rm MS}$ scales Q using the most complete available renormalization group equations (as listed at the end of the Introduction), and $s_{\rm pole}^Z$ is then recalculated. In the idealized case, the results should not depend on Q. I find that, with inclusion of the three-loop QCD corrections, the scale dependence of M_Z is remarkably small, less than 0.8 MeV as Q is varied between 50 and 220 GeV. However, given the larger scale dependence found in Sec. IV for the similar case of the W boson mass, I surmise that this very mild scale dependence is partly accidental, and the actual theoretical error due to neglecting higher order contributions is likely to be larger.

The scale dependence of Γ_Z shown in the right panel of Fig. 2 is less mild and not so much improved over the complete two-loop result, as it varies by a total of about 4 MeV (between minimum and maximum) as Q is varied between 80 and 220 GeV. Note that this determination of Γ_Z from the complex pole mass (in which the leading contribution arises only as a one-loop effect) is essentially one-loop order less accurate than a direction calculation of the Z boson decay width (in which the leading contribution is a tree-level effect).

III. THE HIGGS BOSON POLE MASS

Next, consider the complex pole mass $s_{\rm pole}^h$ for the Standard Model Higgs boson, written in the form of Eq. (1.1). In this section, I extend the results of Ref. [47] to include the momentum-dependent three-loop self-energy corrections to $\Delta_h^{(3)}$ that are proportional to $g_3^4 y_t^2 t$. Also included below are the three-loop contributions proportional to $g_3^2 y_t^4 t$ and $y_t^6 t$, in an effective potential approximation, which amounts to $g_3^2, y_t^2 \gg \lambda, g^2, g'^2$. For the $y_t^6 t$ part, I provide below an improvement over the result in [47]. Together with the full two-loop results, these constitute the most complete calculation of the Standard Model Higgs boson mass that is presently available.

The functions $\Delta_h^{(1)}$ and the QCD part of $\Delta_h^{(2)}$ were given in Eqs. (2.46) and (2.47) in Ref. [47] and are evaluated at $s = {\rm Re}[s_{\rm pole}^h]$, determined by iteration. The remaining, non-QCD part of $\Delta_h^{(2)}$ was given in an ancillary file of Ref. [47], where the master integrals were also evaluated at $s = {\rm Re}[s_{\rm pole}^h]$. However, in the present paper, I adopt a slightly different organization by evaluating the non-QCD part of $\Delta_h^{(2)}$ as exactly the same function but evaluated instead at $s = h$, which is consistent up to three-loop terms of order $y_t^6 t$. This allows an easier extension to three-loop order, as indicated below.

For the leading QCD part of $\Delta_h^{(3)}$ proportional to $g_3^4 y_t^2 t$, the new result can be written in terms of the contributions of four distinct classes of self-energy diagrams characterized by their group theory structures,

$$\Delta_h^{(3),g_3^4 y_t^2 t} = g_3^4 y_t^2 N_c C_F (C_G \Delta_h^{(3,a)} + C_F \Delta_h^{(3,b)} + T_F \Delta_h^{(3,c)} + (2n_g - 1) T_F \Delta_h^{(3,d)}). \tag{3.1}$$

The results for $\Delta_h^{(3,a)}$, $\Delta_h^{(3,b)}$, $\Delta_h^{(3,c)}$, and $\Delta_h^{(3,d)}$ are somewhat lengthy, and so are given in the Supplemental Material (file DeltaH3) provided with this paper [107]. They are written in terms of the same list of three-loop self-energy master integrals as for the Z boson, listed in Eqs. (2.9)–(2.11), with the exceptions that $I_{8b}(t,0,t,t,t,t,t,0)$ is also needed in $\mathcal{I}^{(3)}$, and ζ_5, $I_{6e}(0,0,0,0,t,t)$, $I_{6f}(0,0,0,0,t,t)$, $I_{6f5}(0,0,0,0,t,t)$, $I_{7e}(0,0,0,0,0,t,t)$, and $I_{7e}(0,0,t,t,t,0,0)$ are not needed, and of course one should use $s = h$ rather than $s = Z$.

Using the expansions of the master integrals given in Ref. [55], setting $s = h$ in $\Delta_h^{(3),g_3^4 y_t^2 t}$ (which is consistent up to terms of four-loop order), and plugging in the group theory constants from Eq. (1.15), the result becomes a power series in

$$r_h \equiv \frac{h}{4t} = \frac{\lambda}{y_t^2}, \tag{3.2}$$

with coefficients that depend on $L_t \equiv \ln(t/Q^2)$ and $L_{-h} \equiv \ln(h/Q^2) - i\pi$ and the constants ζ_3 and c'_H from Eq. (2.14). The expansion converges for $r_h < 1$ and does so rapidly for the value realized in the Standard Model. It is given to order r_h^{24} in Supplemental Material (file DeltaH3series) [107], both in analytic and numerical forms. The first few terms of the numerical form are

$$\Delta_h^{(3),g_3^4 y_t^2 t} = g_3^4 y_t^2 t(248.122 + 839.197L_t + 160L_t^2 - 736L_t^3 + r_h[-716.898 - 1546.064L_t + 336L_t^2 + 240L_t^3] + r_h^2[479.663 + 72.770L_t + 28.444L_{-h}] + r_h^3[-27.675 - 83.837L_t - 5.486L_t^2 + 13.274L_{-h}] + \cdots). \qquad (3.3)$$

As a nontrivial check, the result obtained with $r_h = 0$ agrees with that provided in the first line of Eq. (3.3) of Ref. [47]. The terms with positive powers of r_h are new in the present paper.

For the part of $\Delta_h^{(3)}$ proportional to $g_3^2 y_t^4 t$, the effective potential approximation gives the second line of Eq. (3.3) of Ref. [47], which is not improved on in the present paper, but is reproduced here for reference and comparison,

$$\Delta_h^{(3),g_3^2 y_t^4 t} = g_3^2 y_t^4 t(2764.365 + 1283.716L_t - 360L_t^2 + 240L_t^3). \qquad (3.4)$$

It is interesting that $\Delta_h^{(3),g_3^4 y_t^2 t}$ is numerically smaller than $\Delta_h^{(3),g_3^2 y_t^4 t}$, despite the parametric relative enhancement $N_c g_3^2/y_t^2$ of the former. In the approximation $r_h = 0$, this effect was noted in Refs. [10,47] [see the discussion involving Eqs. (6.21)–(6.28) of the former reference] as the result of an unexplained but dramatic near cancellation and is found here to be not changed by the inclusion of terms higher order in r_h.

Finally, for the part of $\Delta_h^{(3)}$ proportional to $y_t^6 t$, the effective potential approximation of Ref. [47] can be improved on slightly as follows. In the present paper, the non-QCD part of $\Delta_h^{(2)}$ is evaluated using master integrals with external momentum invariant h rather than $\mathrm{Re}[s^h_{\mathrm{pole}}]$. Then, due to the fortunate circumstance that the leading one-loop behavior of $s^h_{\mathrm{pole}} - h$ in the limit $y_t^2 \gg \lambda, g^2, g'^2$ is proportional to L_t,

$$s^h_{\mathrm{pole}} - h = \frac{1}{16\pi^2} 4N_c y_t^2 t L_t, \qquad (3.5)$$

we can fully repair the error in the three-loop part (caused by using h rather than $\mathrm{Re}[s^h_{\mathrm{pole}}]$ in the two-loop part), simply by requiring renormalization group invariance of the pole mass. This allows inference of the complete dependence proportional to $y_t^6 t L_t$, due to the explicit dependence on Q. By demanding (and checking) renormalization group invariance of s^h_{pole} through terms of three-loop order in the approximation $g_3^2, y_t^2 \gg \lambda, g^2, g'^2$, I find that the end result for the leading non-QCD three-loop contribution is that Eq. (3.4) of Ref. [47] should be replaced by

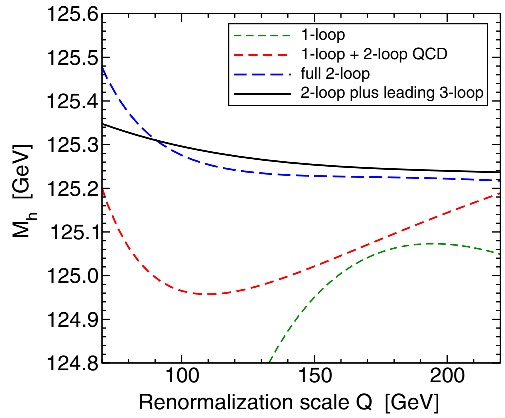

FIG. 3. The real part of the calculated Higgs boson pole mass, as a function of the renormalization scale Q. The different lines show various approximations as labeled. The $\overline{\mathrm{MS}}$ input parameters are as given in Eq. (1.17), which provide for $M_h = 125.25$ GeV when calculated at the renormalization scale $Q = 160$ GeV with the best available approximation as described in the text.

$$\Delta_{M_h^2}^{(3),y_t^6 t} = y_t^6 t[-3433.724 - 2426.808L_t - 101.016L_t^2 - 360L_t^3 + L_h(36 + 648L_t + 324L_t^2)], \qquad (3.6)$$

where the analytic forms of the decimal coefficients are

$$-3433.724 \approx -673 - \frac{17\pi^2}{2} - 1962\zeta_3 + 24c'_H, \qquad (3.7)$$

$$-2426.808 \approx -\frac{10491}{4} + 144\sqrt{3}\pi - 42\pi^2 - 144\zeta_3, \qquad (3.8)$$

$$-101.016 \approx -\frac{855}{2} + 60\sqrt{3}\pi. \qquad (3.9)$$

This result differs from Eq. (3.4) of Ref. [47] by terms that vanish when $L_t = 0$, consistent with the approximation made in that reference.

To recapitulate, in order to consistently include the three-loop results given above, the non-QCD part of $\Delta_h^{(2)}$ found in the ancillary file of Ref. [47] should use $s = h$ in the evaluation of the integrals, while $\Delta_h^{(1)}$ and the QCD part of $\Delta_h^{(2)}$ provided in that reference should use $s = \mathrm{Re}[s^h_{\mathrm{pole}}]$ determined by iteration. All of these results for the Higgs boson pole mass have now been implemented in version 1.2 of the computer code SMDR [50]. Figure 3 shows the results for M_h, for the benchmark $\overline{\mathrm{MS}}$ input parameters given in Eq. (1.17). Recall that these parameters were chosen so as to give the present experimental central value from the RPPs, $M_h = 125.25$ GeV, as the result of the calculation at renormalization scale $Q = 160$ GeV. The other results in the figure were obtained by running the $\overline{\mathrm{MS}}$ parameters in Eq. (1.17) from the input scale $Q_0 = 172.5$ GeV to each scale Q and redoing the calculation. The new contributions found in this paper give the best approximation available at this writing, but still imply a scale dependence of several tens of MeV. For example, the calculated M_h decreases by about 56 MeV when Q is varied from 100 to 200 GeV, for fixed values of the $\overline{\text{MS}}$ input parameters. This provides a lower bound on the theoretical error and suggests that a still more refined calculation of the Higgs pole mass, to include three-loop electroweak parts and even leading four-loop contributions, would be worthwhile, since the experimental uncertainty on M_h from future collider experiments may well be smaller [108]. It is also possible [109] to refine further the gaugeless limit by including momentum-dependent parts of the Higgs boson self-energy function.

A famous feature of the observed Higgs boson mass is that the Standard Model with no extensions can then have the Higgs self-coupling λ run negative at a scale that is far above the electroweak scale, but below the Planck scale, implying a possibly metastable electroweak vacuum. This is illustrated in Fig. 4, using the latest experimental values and the results of this paper to relate M_h to λ in the most accurate available way. As is well known (see, for example, Refs. [44–46,110]), the scale of possible instability is lowered if the top-quark mass is higher, or the QCD coupling is lower, or the Higgs mass is lower, than their benchmark values, while it is possible for the instability to be avoided up to the Planck scale if the deviations are in the opposite directions. While improved formulas and experimental values for M_h are welcome, the dominant uncertainty in these instability discussions comes from M_t (or y_t), and the second most important uncertainty is that of $\alpha_S^{(5)}(M_Z)$, through their renormalization group running influence on λ.

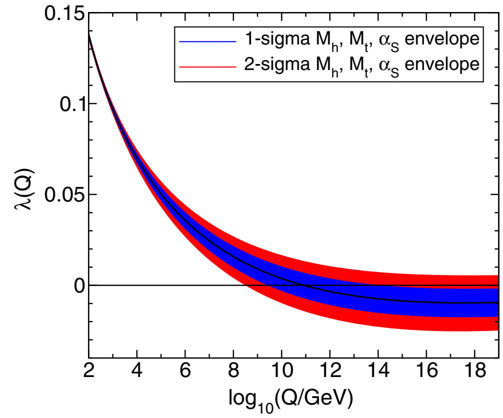

FIG. 4. The running Higgs self-coupling parameter λ as a function of the $\overline{\text{MS}}$ renormalization scale Q, using the results of this paper to relate it to M_h in the most accurate available way. The central value obtained from the present experimental data as in Eqs. (1.16) and (1.17) is the black line. The shaded envelopes are the envelopes obtained by varying $M_h = 125.25 \pm 0.17$ GeV, $M_t = 172.5 \pm 0.7$ GeV, and $\alpha_S^{(5)}(M_Z) = 0.1179 \pm 0.0010$ in their 1-sigma and 2-sigma ranges.

IV. THE W BOSON POLE MASS

Consider the W boson complex pole squared mass s^W_{pole} as in Eq. (1.1). The complete one- and two-loop parts $\Delta_W^{(1)}$ and $\Delta_W^{(2)}$ were given in Ref. [42]. The three-loop QCD part splits into eight distinct contributions with different group theory structures,

$$\Delta_W^{(3),g_3^4} = g_3^4 g^2 N_c C_F (C_G[\Delta_W^{(3,a)} + (n_g-1)\Delta_W^{(3,b)}] + C_F[\Delta_W^{(3,c)} + (n_g-1)\Delta_W^{(3,d)}] \\ + T_F[\Delta_W^{(3,e)} + (n_g-1)\Delta_W^{(3,f)} + (2n_g-1)\Delta_W^{(3,g)} + (2n_g-1)(n_g-1)\Delta_W^{(3,h)}]). \tag{4.1}$$

The four contributions from diagrams in which the W boson couples directly to massless quarks are relatively simple,

$$\Delta_W^{(3,b)} = W\left(-\frac{44215}{648} + \frac{454}{9}\zeta_3 + \frac{20}{3}\zeta_5 + \left[\frac{41}{2} - \frac{44}{3}\zeta_3\right]L_{-W} - \frac{11}{6}L_{-W}^2\right), \tag{4.2}$$

$$\Delta_W^{(3,d)} = W\left(\frac{143}{18} + \frac{74}{3}\zeta_3 - 40\zeta_5 - \frac{1}{2}L_{-W}\right), \tag{4.3}$$

$$\Delta_W^{(3,f)} = \frac{8}{27}W(7t+3W)I_{7c}(0,0,0,0,0,t,t) + \frac{8}{243}(128t+43W)I_{6c}(0,0,0,0,t,t) - \frac{4}{27}(18t+7W)I_{6f}(0,0,0,0,t,t) \\ + \frac{16}{243t}(5W-17t)I_{5c}(0,0,0,t,t) + \frac{80}{81t}I_4(0,0,t,t) + \left(\frac{448}{243}\zeta_3 - \frac{4138}{729}\right)t + \frac{2599}{486}W + \frac{40}{2187}\frac{W^2}{t} \\ + \left(\frac{5572}{243}t - \frac{112}{9}\zeta_3 t - \frac{1660}{243}W\right)L_t - \frac{176}{81}tL_t^2 - \frac{56}{243}tL_t^3 + \frac{2}{243}(884t - 217W)L_{-W} + \left(\frac{80}{27}W - \frac{16}{3}t\right)L_t L_{-W} \\ + \frac{136}{81}tL_t^2 L_{-W} + \frac{10}{81}WL_{-W}^2 - \frac{8}{243}(17t+10W)L_t L_{-W}^2, \tag{4.4}$$

$$\Delta_W^{(3,h)} = W\left(\frac{3701}{162} - \frac{152}{9}\zeta_3 + \left[\frac{16}{3}\zeta_3 - \frac{22}{3}\right]L_{-W} + \frac{2}{3}L_{-W}^2\right). \tag{4.5}$$

In fact, $\Delta_W^{(3,b)}$, $\Delta_W^{(3,d)}$, $\Delta_W^{(3,f)}$, and $\Delta_W^{(3,h)}$ can be obtained from, respectively, $\Delta_Z^{(3,j)}$, $\Delta_Z^{(3,k)}$, $\Delta_Z^{(3,l)}$, and $\Delta_Z^{(3,m)}$ in Eqs. (2.5)–(2.8) by replacing $Z \to W$ and dividing by 2. The reason for this is that they come from exactly the same Feynman diagram topologies.

The remaining four contributions $\Delta_W^{(3,a)}$, $\Delta_W^{(3,c)}$, $\Delta_W^{(3,e)}$, and $\Delta_W^{(3,g)}$ in Eq. (4.1) are more complicated and are relegated to the Supplemental Material (file DeltaW3) [107]. They each have the form of Eq. (1.8), with renormalized ϵ-finite master integrals that are a subset of Eqs. (6.2)–(6.4) of Ref. [55],

$$\mathcal{I}^{(1)} = \{A(t), B(0,t)\}, \tag{4.6}$$

$$\mathcal{I}^{(2)} = \{S(0,0,t), S(t,t,t), U(t,0,t,t), M(0,0,t,t,0)\}, \tag{4.7}$$

$$\begin{aligned}\mathcal{I}^{(3)} = \{&H(0,t,t,t,0,t), I_4(0,t,t,t), I_{6d}(0,0,t,0,t,0), I_{6d}(0,0,t,t,0,t), I_{6d}(t,0,0,0,0,0),\\ &I_{6e}(0,t,0,0,0,t), I_{6e}(t,0,t,0,0,t), I_{6f}(0,t,t,0,0,t), I_{6f1}(t,0,0,t,0,t),\\ &I_{7a}(0,0,0,0,t,t,t), I_{7a}(0,0,t,t,0,0,0), I_{7a}(0,t,t,0,0,t,0), I_{7a}(t,t,0,0,t,t,0),\\ &I_{7a5}(t,t,0,0,t,t,0), I_{7b}(0,0,t,0,t,0,0), I_{7c}(0,0,t,t,0,0,0), I_{7d}(0,t,0,t,t,0,t),\\ &I_{7e}(0,t,t,0,0,0,0), I_{8b}(0,0,0,t,t,0,0,t), I_{8c}(0,0,0,t,t,0,0,t), I_{8c}^{pk}(t,t,t,0,0,0,0,0)\}.\end{aligned} \tag{4.8}$$

I have checked that Eq. (4.1) gives a pole mass $s_{\rm pole}^W$ that is renormalization group invariant through three-loop terms of order g_3^4, using the derivatives of the master integrals with respect to Q found in the ancillary file QddQ of Ref. [55].

For practical numerical evaluation, after plugging in the Standard Model group theory values in Eq. (1.15) and applying the expansions for the master integrals in the Ref. [55] ancillary files Ioddseries and Ievenseries (the latter being needed only for the contribution $\Delta_W^{(3,f)}$ in which the W boson couplings are to a massless quark loop, with a top-quark loop correcting a gluon propagator), I obtain a series expansion

$$\Delta_W^{(3),g_3^4} = g_3^4 g^2 t(\delta_1^W + \delta_2^W), \tag{4.9}$$

where δ_1^W comes from $\Delta_W^{(3,a)}$, $\Delta_W^{(3,c)}$, $\Delta_W^{(3,e)}$, and $\Delta_W^{(3,g)}$, which follow from diagrams where the W boson couples directly to a top-bottom pair, and δ_2^W comes from $\Delta_W^{(3,b)}$, $\Delta_W^{(3,d)}$, $\Delta_W^{(3,f)}$, and $\Delta_W^{(3,h)}$ from diagrams in which the W boson couples directly to light-quark pairs. The Supplemental Material [107] (file DeltaW3series) provided with this paper gives the results, both analytically and numerically, to orders ρ_W^{30} and r_W^{16}, where

$$\rho_W \equiv \frac{W}{t} = \frac{g^2}{2y_t^2} \quad \text{and} \quad r_W = \frac{\rho_W}{4}, \tag{4.10}$$

and the coefficients involve $L_t = \ln(t/Q^2)$ and $L_{-W} = \ln(W/Q^2) - i\pi = 2 - B(0,0)|_{s=W+i\epsilon}$, as well as ζ_2, ζ_3, ζ_4, ζ_5, c'_H from Eq. (2.14), and

$$c_I = \sqrt{3}\,\mathrm{Im}[\mathrm{Li}_2(e^{2\pi i/3})] \approx 1.1719536193\ldots. \tag{4.11}$$

Note that δ_2^W is the same as δ_4^Z appearing in Eqs. (2.12) and (2.18) with the replacement $r_Z \to r_W$. The series for δ_1^W and δ_2^W converge for $\rho_W < 1$ and $r_W < 1$, respectively, which is clearly satisfied by the relevant value of W/t in the Standard Model.

The numerical form of the first few terms in the series are

$$\begin{aligned}\delta_1^W = {}& 12.8299 + 24.9541L_t + 63L_t^2 - 30L_t^3 + \rho_W(-23.800 - 54.693L_t + 14L_t^2)\\ &+ \rho_W^2(-2.327 - 17.873L_t - 1.5L_t^2) + \rho_W^3(-0.700 - 7.496L_t - 7.2L_t^2) + \cdots,\end{aligned} \tag{4.12}$$

$$\begin{aligned}\delta_2^W = {}& r_W(-56.799 - 14.758L_t - 10.667L_t^2 + 180.381L_{-W} + 21.333L_tL_{-W} - 122.667L_{-W}^2)\\ &+ r_W^2[-88.570 - 33.375(L_t - L_{-W}) - 3.793(L_t - L_{-W})^2]\\ &+ r_W^3[4.074 + 2.521(L_t - L_{-W}) + 0.406(L_t - L_{-W})^2] + \cdots.\end{aligned} \tag{4.13}$$

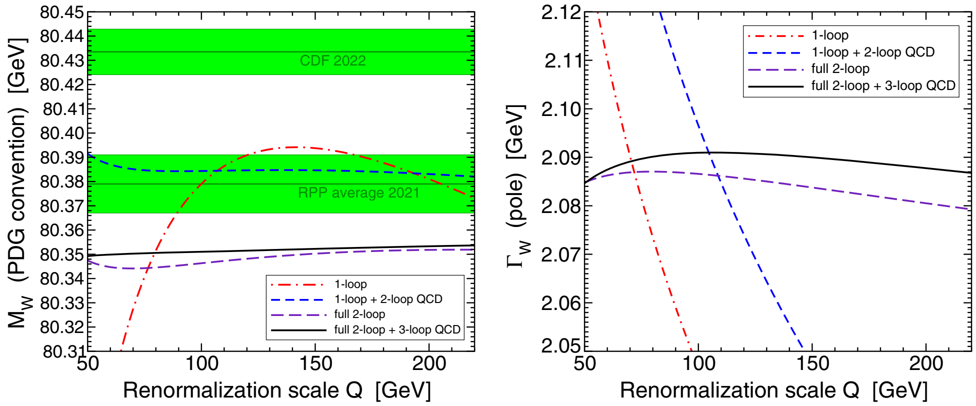

FIG. 5. The PDG-convention W boson mass, and the width Γ_W, obtained from the calculated complex pole mass $s_{\rm pole}^W$, as a function of the renormalization scale Q. The different lines show various approximations as labeled. The $\overline{\rm MS}$ input parameters are as given in Eq. (1.17). Also shown are the experimental central values and 1σ ranges for $M_W^{\rm PDF}$ as given by the 2021 update to the 2020 RPPs and from the 2022 result from CDF [54] with statistical and systematic errors combined in quadrature.

As in the case of the Z boson, it is interesting to note that, in this expansion in small W/t, the subleading contribution is numerically comparable to or larger than the leading contribution (obtained by $\rho_W = r_W = 0$), depending on the choice of Q. This is due mostly to the term proportional to $r_W L_{-W}^2$ in the contribution from massless quark loops, because of the large magnitude of the coefficient -122.667 and because $L_{-W}^2 = [-i\pi + \ln(W/Q^2)]^2$ provides up to an order of magnitude enhancement.

The contribution $\Delta_W^{(3),g_3^4}$ is now implemented in the new version 1.2 of the computer code SMDR [50]. Figure 5 shows the results for $M_W^{\rm PDG}$ and for Γ_W obtained from the complex pole squared mass $s_{\rm pole}^W$, for the $\overline{\rm MS}$ input parameters in Eq. (1.17) at the reference scale $Q_0 = 172.5$ GeV. The default scale used by SMDR v1.2 for the W mass calculation is $Q = 160$ GeV, which gives $M_W^{\rm PDG} = 80.3525$ and $\Gamma_W = 2.0896$ GeV. The results for other renormalization scales Q are obtained by first running the $\overline{\rm MS}$ parameters to Q and then recalculating $s_{\rm pole}^W$. The three-loop QCD contribution to $M_W^{\rm PDG}$ is seen to be as large as about 6 MeV. In the idealized case, the total $s_{\rm pole}^W$ would not depend on Q. The computed value of $M_W^{\rm PDG}$ varies by less than 2.4 MeV as Q is varied from 80 to 180 GeV. This is significantly larger than the scale dependence of the computed $M_Z^{\rm PDG}$ as found in Fig. 2, but compares quite favorably to the present experimental uncertainty of 12 MeV. The range for $M_W^{\rm PDG}$ from the average of experimental data released through 2021 is 80.379 ± 0.012 GeV. The CDF Collaboration has recently produced a result that is substantially higher, $80.4335 \pm 0.0064_{\rm stat} \pm 0.0069_{\rm syst}$ GeV, which is in stark disagreement with the Standard Model prediction. These results are also shown in Fig. 5. As seen in the right panel of Fig. 5, the total variation in Γ_W as Q varies from 60 to 220 GeV is about 3.5 MeV, but the spread is only about 2.3 MeV as Q varies from 80 to 180 GeV. These scale variations are improved over the full two-loop order calculation found in Ref. [42]. For comparison, the largest parametric uncertainty contributing to the M_W prediction is that of the top-quark pole mass M_t. If one fixes Eqs. (1.16) and (1.17) as a reference model and then adjusts the Standard Model inputs to fit varying $M_t, M_Z, \Delta\alpha_{\rm had}^{(5)}$, and $\alpha_S^{(5)}(M_Z)$, then one finds approximately

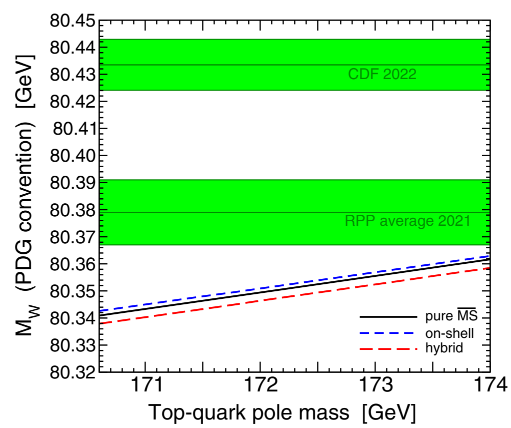

FIG. 6. Comparison of Standard Model predictions for the W boson mass in the PDG convention, as a function of the top-quark pole mass M_t, using data for $M_Z^{\rm PDG}$, G_μ, $\alpha_S^{(5)}(M_Z)$, $\Delta\alpha_{\rm had}^{(5)}$, and M_h from Eq. (1.16). The solid black line is the pure $\overline{\rm MS}$ scheme result, obtained using SMDR v1.2 incorporating the results of this paper. The short dashed (blue) line is the on-shell scheme result, obtained from the interpolating formula in Ref. [38]. The long dashed (red) line is the result from the hybrid $\overline{\rm MS}$-on-shell scheme of Ref. [17]. Also shown are the experimental central values and 1σ ranges for $M_W^{\rm PDF}$ as given by the 2021 update to the 2020 RPPs and from the 2022 result from CDF [54].

$$M_W^{\rm PDG} = M_W^{\rm PDG,ref} + 6.1\ {\rm MeV}\left(\frac{M_t - M_t^{\rm ref}}{\rm GeV}\right) + 1.3\ {\rm MeV}\left(\frac{M_Z^{\rm PDG} - M_Z^{\rm PDG,ref}}{\rm MeV}\right) - 1.8\ {\rm MeV}\left(\frac{\Delta\alpha_{\rm had}^{(5)} - \Delta\alpha_{\rm had}^{(5),{\rm ref}}}{0.0001}\right) - 0.7\ {\rm MeV}\left(\frac{\alpha_S^{(5)}(M_Z) - \alpha_S^{(5)}(M_Z)^{\rm ref}}{0.001}\right) \quad (4.14)$$

as the prediction for the W boson mass in the PDG convention, with $M_W^{\rm PDG,ref} = 80.3525$ GeV.

In Fig. 6, I compare the prediction for $M_W^{\rm PDF}$ from SMDR v1.2 (incorporating the results of this paper) in the pure $\overline{\rm MS}$ scheme to the corresponding results in the on-shell scheme using the interpolation formula in Ref. [38] and to those in the hybrid $\overline{\rm MS}$-on-shell scheme of Ref. [17], as a function of the top-quark pole mass. The other on-shell parameters $M_Z^{\rm PDG}$, G_μ, $\alpha_S^{(5)}(M_Z)$, $\Delta\alpha_{\rm had}^{(5)}$, and M_h are chosen to be the same and equal to the data given from Eq. (1.16) from the 2021 update to the 2020 RPPs, so that the results are directly comparable. (In the $\overline{\rm MS}$ scheme, this entails doing a fit to determine the Lagrangian parameters, which is readily accomplished using the C function SMDR_Fit_Inputs or the interactive command-line tool calc_fit-int.) The pure $\overline{\rm MS}$ scheme gives results between those of the on-shell and hybrid schemes, with a total spread between the three schemes of about 4.5 MeV.

V. OUTLOOK

In this paper, I have reported the three-loop QCD contributions to the W, Z, and Higgs boson physical masses in the Standard Model, in the pure $\overline{\rm MS}$ renormalization scheme with a tadpole-free treatment of the Higgs VEV. The results show improved renormalization group scale independence, especially for the W and Z boson cases, and in all three cases the scale variation is less than the present experimental uncertainty. Alternative methods based on on-shell type schemes have already included four-loop QCD contributions through the rho parameter, but it is not clear that these should be numerically more important than three-loop mixed and pure electroweak contributions. The results of this paper have all been incorporated in the latest version 1.2 of the code SMDR [50]. Further improvements in the approach of the present paper could come from computing all of the remaining three-loop self-energy contributions to the pole masses, which in the case of the most general diagrams will be a challenging, but perhaps not insurmountable, goal.

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