Letter

Decoherence limit of quantum systems obeying generalized uncertainty principle: New paradigm for Tsallis thermostatistics

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The generalized uncertainty principle (GUP) is a phenomenological model whose purpose is to account for a minimal length scale (e.g., Planck scale or characteristic inverse-mass scale in effective quantum description) in quantum systems. In this paper, we study possible observational effects of GUP systems in their decoherence domain. We first derive coherent states associated to GUP and unveil that in the momentum representation they coincide with Tsallis probability amplitudes, whose nonextensivity parameter q monotonically increases with the GUP deformation parameter β. Second, for $\beta < 0$ (i.e., $q < 1$), we show that, due to Bekner-Babenko inequality, the GUP is fully equivalent to information-theoretic uncertainty relations based on Tsallis-entropy-power. Finally, we invoke the maximal entropy principle known from estimation theory to reveal connection between the quasiclassical (decoherence) limit of GUP-related quantum theory and nonextensive thermostatistics of Tsallis. This might provide an exciting paradigm in a range of fields from quantum theory to analog gravity. For instance, in some quantum gravity theories, such as conformal gravity, aforementioned quasiclassical regime has relevant observational consequences. We discuss some of the implications.

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I. INTRODUCTION

There are indications from various studies such as string theory, loop quantum gravity, quantum geometry, or doubly special relativity theories that the uncertainty relation between positions and momenta acquire corrections due to gravity induced effects and should be modified accordingly [1–7]. These modifications implement, in one way or another, the minimal length scale and/or the maximum momentum. The ensuing modified uncertainty relations are known as the generalized uncertainty principles (GUPs). A paradigmatic form of GUP is the quadratic GUP, namely,

$$\delta x \delta p \geq \frac{\hbar}{2}\left(1+\beta\frac{\delta p^2}{m_p^2}\right), \tag{1}$$

where $c = 1$, $m_p = \sqrt{\hbar c/G} \approx 2.2 \times 10^{-8}$ kg is the Planck mass, and β is a dimensionless deformation parameter. The symbol δ denotes uncertainty of a given observable, and it does not need to be *a priori* related to the standard deviation. More like in the original Heisenberg uncertainty relation, δ can represent Heisenberg's "ungenauigkeiten" (i.e., error-disturbance uncertainties caused by the back-reaction in simultaneous measurement of x and p) or $\delta p = \langle\psi\|p\|\psi\rangle \equiv \langle|p|\rangle_\psi$; see, e.g., Ref. [8].

The quadratic GUP (1) has served as an incubator for a number of important studies in quantum mechanics [9–11], particle physics [12,13], finite-temperature quantum field

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theory [14], or cosmology [15]. In addition, the mass parameter in (1) does not need to be necessarily m_p, but it might be identified with a characteristic mass scale in the effective quantum description, e.g., in condensed matter and atomic physics or in nonlinear optics [16–18].

In cases when δ represents the standard deviation (henceforth denoted as Δ), the GUP inequality (1) can be deduced from the deformed (Jacobi identity satisfying) commutation relations (DCRs)

$$[\hat{x}, \hat{p}] = i\hbar\left(1 + \beta\frac{\hat{p}^2}{m_p^2}\right), \qquad [\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0, \quad (2)$$

by means of the Cauchy-Schwarz or covariance inequality [19–21], provided we focus on *mirror symmetric* states where $\langle\hat{x}\rangle_\psi = \langle\hat{p}\rangle_\psi = 0$, e.g., ψ are parity eigenstates [22].

Most of the recent discussions of GUP in quantum gravity have focused on heuristic applications in cosmology and astrophysics (for review, see, e.g., Ref. [17]). Comparably less attention has been devoted to the study of GUP in a quasiclassical (decoherence) domain. It is, however, the quasiclassical quantum regime that is pertinent in observational cosmology and astrophysics [23,24]. Important theoretical instruments used in quasiclassical quantum theory are coherent states (CSs). This is because CSs are least susceptible to the loss of quantum coherence [25]. In a sense, CSs are the privileged states in the transition to classical reality, as they are the only states that remain pure in the decoherence process [26,27].

Various classes of CSs have been studied. Here, we will discuss the Schrödinger-type minimum-uncertainty CSs [19,28] associated with GUP. We derive precise forms of such GUP CSs in both the momentum and position representations [29], though our focus will be on the momentum representation where CSs coincide with Tsallis probability amplitudes. For $\beta < 0$, we also reformulate the GUP in terms of one-parameter class of Tsallis entropy-power based uncertainty relations (EPUR), which are saturated by the GUP CSs. Since thermodynamics alongside with its various generalizations [30–35] crucially hinges on the maximum entropy principle (MEP) (i.e., thermodynamic entropy is the statistical entropy evaluated at the maximal entropy distribution), we are led to the conclusion that the combination of GUP CSs with Tsallis entropy provides a natural framework to discuss the quasiclassical regime of GUP in terms of nonextensive thermostatistics of Tsallis (NTT) [33]. In some quantum gravity theories, such as conformal gravity, aforementioned quasiclassical regime has relevant phenomenological consequences, some of which will be discussed.

II. COHERENT STATES FOR GUP

We first summarize the steps leading to (1) from the DCR (2). To this end, we quantify the uncertainty of an observable $\hat{\mathcal{O}}$ with respect to a density matrix ϱ via its standard deviation. In particular, for variance, i.e., square of the standard deviation, we have

$$(\Delta\hat{\mathcal{O}})^2_\varrho \equiv \mathrm{Tr}(\hat{\mathcal{O}}^2\varrho) - \mathrm{Tr}(\hat{\mathcal{O}}\varrho)^2 = \int_{\mathbb{R}} (\lambda - \langle\hat{\mathcal{O}}\rangle^2_\varrho)d\mathrm{Tr}(E^{(\hat{\mathcal{O}})}_\lambda\varrho). \quad (3)$$

Here, $E^{(\hat{\mathcal{O}})}_\lambda$ is the projection-valued measure of $\hat{\mathcal{O}}$ corresponding to spectral value λ. By confining our study to the observables $\hat{x}$ and $\hat{p}$, the passage from the DCR (2) to GUP (1) is as follows: we set $\hat{\mathcal{O}}_1 = \hat{x} - \langle\hat{x}\rangle_\varrho$ and $\hat{\mathcal{O}}_2 = \hat{p} - \langle\hat{p}\rangle_\varrho$ so that $(\Delta x)^2_\varrho = \langle\hat{\mathcal{O}}_1^2\rangle_\varrho$, $(\Delta p)^2_\varrho = \langle\hat{\mathcal{O}}_2^2\rangle_\varrho$ and $[\hat{x}, \hat{p}]\varrho = [\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2]\varrho$; then, for arbitrary vector $\psi \in \mathrm{Ran}\varrho$ and any $\gamma \in \mathbb{R}$, we have

$$0 \le \|(\hat{\mathcal{O}}_2 - i\gamma\hat{\mathcal{O}}_1)\psi\|^2 = \langle\psi|\hat{\mathcal{O}}_2^2|\psi\rangle + i\gamma\langle\psi|[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2]|\psi\rangle + \gamma^2\langle\psi|\hat{\mathcal{O}}_1^2|\psi\rangle, \quad (4)$$

and therefore

$$\mathrm{Tr}(\hat{\mathcal{O}}_2^2\varrho) + i\gamma\mathrm{Tr}([\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2]\varrho) + \gamma^2\mathrm{Tr}(\hat{\mathcal{O}}_1^2\varrho) \ge 0. \quad (5)$$

The lhs is smallest for $\gamma = i\mathrm{Tr}([\hat{\mathcal{O}}_2, \hat{\mathcal{O}}_1]\varrho)/(2\mathrm{Tr}(\hat{\mathcal{O}}_1^2\varrho))$, which turns (5) to

$$\mathrm{Tr}(\hat{\mathcal{O}}_1^2\varrho)\mathrm{Tr}(\hat{\mathcal{O}}_2^2\varrho) = (\Delta x)^2_\varrho(\Delta p)^2_\varrho \ge \frac{1}{4}\mathrm{Tr}(i[\hat{x}, \hat{p}]\varrho)^2. \quad (6)$$

This is nothing but quantum mechanical version of the *covariance inequality* [21]. Now, we can use (2) to obtain

$$(\Delta x)_\varrho(\Delta p)_\varrho \ge \frac{\hbar}{2}\left(1 + \beta\frac{(\Delta p)^2_\varrho + \langle\hat{p}\rangle^2_\varrho}{m_p^2}\right). \quad (7)$$

For mirror symmetric ϱ's satisfying $\langle\hat{p}\rangle_\varrho = 0$, inequality (7) clearly coincides with the GUP (1), with variances in place of generic δ's.

To find ϱ that saturates the GUP (7), we observe from (4) that the inequality is saturated if and only if for all $\psi \in \mathrm{Ran}\varrho$ the equation $(\hat{\mathcal{O}}_2 - i\gamma\hat{\mathcal{O}}_1)|\psi\rangle = 0$ holds. If this equation has for given γ, $\langle\hat{x}\rangle_\varrho$, and $\langle\hat{p}\rangle_\varrho$ more than one solution, the corresponding minimum-uncertainty ϱ is a mixture of CSs (i.e., pure minimum-uncertainty states). It is apparent, cf. Eq. (11), that on the class of mirror symmetric ϱ's the equation

$$(\hat{p} - i\gamma\hat{x})|\psi\rangle = 0 \quad (8)$$

has only one solution for $\psi \in L^2(\mathbb{R})$ so that the minimum-uncertainty ϱ is a pure state—CS. It is convenient to seek the solution to (8) in the momentum representation, i.e., $|\psi\rangle \mapsto \psi(p) = \langle p|\psi\rangle$. In the momentum space, $\hat{x}$ and $\hat{p}$ satisfying DCR can be represented as in Ref. [36]. However, by doing so, the nonsymmetric nature of $\hat{x}$ would provide inconsistent variance for the ensuing CS; cf. Eq. (19). For this reason, we resort to another representation of $\hat{x}$ and $\hat{p}$ complying with (2), namely,

$$\hat{p}\psi(p) = p\psi(p),$$
$$\hat{x}\psi(p) = i\hbar\left(\frac{d}{dp} + \frac{\beta}{2m_p^2}\left[p^2, \frac{d}{dp}\right]_+\right)\psi(p), \qquad (9)$$

with $[,]_+$ being an anticommutator. With this, we can cast (8) into an equivalent form,

$$\frac{d}{dp}\psi(p) = -\frac{(1+\frac{\beta\gamma\hbar}{m_p^2})}{\gamma\hbar(1+\beta\frac{p^2}{m_p^2})}p\psi(p), \qquad (10)$$

which admits the generic solution

$$\psi(p) = N[1+(\beta p^2)/m_p^2]^{-\frac{m_p^2}{2\beta\gamma\hbar}-\frac{1}{2}}. \qquad (11)$$

The coefficient N ensures that $\int|\psi(p)|^2 dp = 1$ and for $\beta > 0$

$$N_> = \sqrt{\sqrt{\frac{\beta}{m_p^2\pi}}\frac{\Gamma(\frac{m_p^2}{\beta\gamma\hbar}+1)}{\Gamma(\frac{m_p^2}{\beta\gamma\hbar}+\frac{1}{2})}}. \qquad (12)$$

Here, $\Gamma(x)$ is the gamma function [37].

The situation with $\beta < 0$ has been less explored in literature than the $\beta > 0$ case, though the related GUP has a number of important implications, e.g., in cosmology [18,38], astrophysics [39], or DSR [6,7]. Note that for $\beta < 0$ Eq. (11) involves noninteger powers of negative reals, which lead to multivalued CS. Because wave functions must be single valued, CS has to have bounded support, which in turn means that $\hat{p}$ must be bounded with spectrum $|\sigma(\hat{p})| \leq m_p/\sqrt{|\beta|}$. The ensuing operator $\hat{x}$ corresponding to the formal differential expression (9) is self-adjoint [29]. The resulting CS reads

$$\psi(p) = N_<[1-(|\beta|p^2)/m_p^2]_+^{\frac{m_p^2}{2|\beta|\gamma\hbar}-\frac{1}{2}}, \qquad (13)$$

where $[z]_+ = \max\{z, 0\}$ with

$$N_< = \sqrt{\sqrt{\frac{|\beta|}{m_p^2\pi}}\frac{\Gamma(\frac{1}{2}+\frac{m_p^2}{|\beta|\gamma\hbar})}{\Gamma(\frac{m_p^2}{|\beta|\gamma\hbar})}}. \qquad (14)$$

In passing, we observe that as $\beta \to 0$ both (11) and (13) reduce to the usual minimum uncertainty Gaussian wave packet (Glauber coherent state) associated with the conventional Heisenberg uncertainty relation.

To find a physical meaning for γ, we note [see the sentence after (5)] that for CS ψ

$$\gamma = -i\langle[\hat{x},\hat{p}]\rangle_\psi/[2(\Delta x)_\psi^2] = -2(\Delta p)_\psi^2/i\langle[\hat{x},\hat{p}]\rangle_\psi$$
$$= \frac{(\Delta p)_\psi}{(\Delta x)_\psi} = \frac{2(\Delta p)_\psi^2}{\hbar[1+\beta(\Delta p)_\psi^2/m_p^2]}, \qquad (15)$$

where in the second and third identities we utilized the fact that ψ saturates (7). Note also that CSs (11) satisfy $\langle\hat{p}\rangle_\psi = \langle\hat{x}\rangle_\psi = 0$.

III. TSALLIS DISTRIBUTION

Let us now consider the following substitutions (valid for $\beta \lessgtr 0$) in (11) and (13):

$$q = \frac{\beta\gamma\hbar}{m_p^2+\beta\gamma\hbar}+1, \qquad b = \frac{2m_p}{\gamma\hbar}+\frac{2\beta}{m_p}. \qquad (16)$$

With this, we can rewrite (11) and (13) as

$$\psi(p) = N_\lessgtr\left[1-b(1-q)\frac{p^2}{2m_p}\right]_+^{\frac{1}{2(1-q)}}. \qquad (17)$$

This is nothing but the probability amplitude for the Tsallis distribution of a free nonrelativistic particle,

$$q_T(p|q,b) = |\psi(p)|^2 = \frac{1}{Z}\left[1-b(1-q)\frac{p^2}{2m_p}\right]_+^{\frac{1}{1-q}}, \qquad (18)$$

with $Z = N_\lessgtr^{-2}$ being the "partition function."

A few remarks concerning (18) are now in order. Tsallis distribution of this type is also known as q-Gaussian distribution and denoted as $\exp_q(-bp^2/2m_p)$. In the limit $q \to 1$, $\exp_q(-bp^2/2m_p) \to \exp(-bp^2/2m_p)$. Note that because of (16) $q \to 1$ is equivalent to $\beta \to 0$. In addition, since for $\beta > 0$ the $\hat{p}$ operator is unbounded, CS (17) is normalizable only for values of $1 \leq q < 3$. For values $q < 1$ (i.e., $\beta < 0$), the distribution (18) has a finite support with $|p| < \sqrt{2m_p/b(1-q)}$. Moreover, for $q \geq 5/3$, the variance of (18) is undefined (infinite), and thus the GUP cannot even be formulated. When $q < 5/3$, then (see, e.g., Ref. [40])

$$(\Delta p)^2 = \frac{2m_p}{b(5-3q)} \Leftrightarrow \gamma = \frac{2(\Delta p)^2}{\hbar[1+\beta(\Delta p)^2/m_p^2]}, \qquad (19)$$

which coincides with (15) (this, in turn, justifies our choice of the representation of $\hat{x}$ and $\hat{p}$ operators). Furthermore, the mean value does not exist for $q > 2$, so such CS cannot be mirror symmetric. Thus, the only physically relevant domain of q in CS is $q < 5/3$, which ensures that β is a monotonically increasing function of q and that $\beta > -m_p^2/[3(\Delta p)_\psi^2]$.

IV. CONNECTION WITH ENTROPIC UNCERTAINTY RELATIONS

The probability distribution (18) decays asymptotically following the power law. If the variance and mean are the only observables, power-law type distributions are incompatible with the conventional MEP based on the Shannon-Gibbs entropy (SGE). Nonetheless, distribution (18) is a maximizer of Tsallis (differential) entropy (TE) $\mathcal{S}_{2-q}^T$, where

$$\mathcal{S}_q^T(\mathcal{F}) = \frac{k_B}{(1-q)}\left(\int_{\mathbb{R}} dp \mathcal{F}^q(p) - 1\right) \tag{20}$$

($\mathcal{F}$ is a probability density function) subject to a constraint $\langle \hat{p}^2 \rangle_\psi = 2m_p/[b(5-3q)]$; cf. Refs. [33,41–43]. k_B is the Boltzmann constant. In the limit $q \to 1$, the TE tends to SGE by l'Hôpital's rule.

When dealing with GUP that is saturated by Tsallis CS, it is convenient to employ the concept of Tsallis entropy power (TEP) [44]. TEP M_q^T of a random vector $\mathcal{X}$ is the unique number that solves the equation

$$\mathcal{S}_q^T(\mathcal{X}) = \mathcal{S}_q^T\left(\sqrt{M_q^T(\mathcal{X})} \cdot \mathcal{Z}^T\right). \tag{21}$$

Here, $\mathcal{Z}^T$ represents a Tsallis random vector with zero mean and unit covariance matrix. Such a vector is distributed with respect to the Tsallis distribution that extremizes $\mathcal{S}_q^T$. In the Supplemental Material [29], we use the Beckner-Babenko theorem [44] to prove that for $\beta < 0$ the DCR (2) implies the following one-parameter class of EPURs:

$$M_{q/2}^T(|\psi|^2) M_{1/(2-2/q)}^T(|\tilde{\psi}|^2) \geq \hbar^2/4, \qquad q \in [1,2). \tag{22}$$

Here, $\tilde{\psi}$ is the position-space wave function associated with ψ. The clear advantage of EPUR (22) over GUP (1) is in that the rhs has an irreducible and state-independent lower bound. Moreover, Eq. (22) is also saturated by the GUP CSs [29].

Numerical simulations based the Markovian master equations for the reduced density matrix coupled with the predictability sieve method [25,45–47] indicate that CSs belong among the so-called *pointer states*, i.e., states that are least affected by the interaction with the environment (external degrees of freedom). Such states belong to the quasiclassical domain of quantum theory as they are maximally predictable in spite of decoherence [47,48]. Among all pointer states in the would-be GUP driven universe, CSs (17) have the highest TE. Moreover, EPUR (22) indicates that TE is at the same time a pertinent entropy functional in the GUP context. So, when we want to discuss a statistical physics of an ensemble of noninteracting GUP particles that are monitored by quantum gravitational environment (bath of gravitons), we might invoke, similarly as in conventional statistical physics, MEP, but this time with TE in place of SGE. The ensuing NTT [33] can be then used to probe the quasiclassical domain of the GUP. We now illustrate potential implications of this observation with few examples.

V. PHYSICAL IMPLICATIONS

First, we consider modifications to Newton's law that should be expected in the quasiclassical epoch of a GUP-based universe with $\beta < 0$. To that end, we employ Verlinde's idea that gravity is an entropy-driven phenomenon—entropic gravity (EG) [49], along with the NTT [33,34].

Following Verlinde's EG, we suppose that true (unknown) microscopic degrees of freedom in any given part of space are stored in discrete bits on the holographic screen that surrounds them. A holographic screen can be considered to be a spherically symmetric of area $A = 4\pi R^2$. Outside of the screen is the *emergent world*, so the screen acts as an interface between known and unknown physics. When a test particle moves away from the screen, it feels an effective force F satisfying $F\delta x = T\delta S$, where T and δS are the temperature and the entropy change on the holographic surface, respectively, and δx is the distance of the particle from the screen. It should be stressed that Verlinde's thermodynamic relation is not directly related to the interior of the screen—it operates in the emergent world. In NTT, the heat one-form $T\delta S$ must be replaced with [34,35] $T\delta \mathcal{S}_q^T/[1 + ((1-q)/k_B)\mathcal{S}_q^T]$ (in our context $\mathcal{S}_q \mapsto \mathcal{S}_{2-q}$). If L is a (dimensionless) characteristic length scale (e.g., radius R/ℓ_p), then the Bekenstein-Hawking entropy $S_{\text{BH}} = \ln W(L) \propto L^2$, which implies that the total number of internal configurations W behaves for $L \gg 1$ as $W(L) = \phi(L)\nu^{L^2}$, where ϕ is any positive function satisfying $\lim_{L\to\infty} \ln \phi/L^2 = 0$ and $\nu > 1$ is some constant [50]. So, from the outside, the holographic screen has entropy

$$\mathcal{S}_{2-q}^T = k_B \ln_{2-q} W(L) = \frac{k_B}{q-1}[(\phi(L)\nu^{L^2})^{q-1} - 1]. \tag{23}$$

Consequently, the entropic force follows from

$$F\delta x = \frac{T\delta S_{2-q}^T}{1 + (q-1)(\omega_3 L^3 + \omega_2 L^2 + \cdots) + \cdots}, \tag{24}$$

where ω_2, $\omega_3 > 0$ are intensive coefficients known from Hills's entropy expansion in (conventional) thermodynamics of small and mesoscopic systems [51]. To comply with Hills's expansion, we have formally included term $\omega_3 L^3$ even if it is not supported by EG prescription. It will be seen that such a term is cosmologically unfeasible in the quasiclassical regime, so that $\omega_3 \approx 0$.

By holographic scaling, the energy residing inside the holographic screen is related with the on-screen degrees of freedom via the equipartition theorem $E = Nk_B T/2$, with $E = M$ being the total mass enclosed by the surface and $N = A/(G\hbar)$ being the number of bits connected with the area by the holographic principle [49].

The EG paradigm posits that the minimum possible increase in the screen entropy (equivalent to one bit of Shannon's information) happens if a particle of radius of Compton wavelength λ_C is added to a holographic sphere [49,52]. This happens when a pointlike quantum particle appears at the distance λ_C from the screen [53]. By setting $\delta x = \lambda_C = \hbar/m$ and using the nonextensive version of the Landauer principle [54,55], which states that the erasure of information leads to an entropy increase $\delta S_q^T = 2\pi k_B/(3-2q)$ per erased bit, we derive the modified Newton's law

$$F(R) = \frac{GMm}{wR^2}\frac{1}{1 - \kappa_3 \varepsilon_q R^3 - \kappa_2 \varepsilon_q R^2}, \tag{25}$$

with $\varepsilon_q = 1 - q$, $w = 1 + 2\varepsilon_q$, and $\kappa_n = \omega_n/\ell_p^n$, $n = 2, 3$. Since $2\varepsilon_q$ is small (see below), we can set $w = 1$. The ensuing gravitational potential up to the first order in ε_q is

$$V(R) = \frac{r_s}{2}\left[-\frac{1}{R} + \varepsilon_q\kappa_2 R + \frac{\varepsilon_q\kappa_3}{2}R^2\right], \qquad (26)$$

where $r_s = 2GM$ is the Schwarzschild radius. Equation (26) formally coincides with the Mannheim-Kazanas *external* gravitational potential of a static, spherically symmetric source of mass M in conformal gravity (CG) [56–59]. Strictly speaking, in CG, a given local gravitational source generates only a gravitational potential,

$$V_{MK}(R) = -\frac{r_s}{2R} + \frac{\chi}{2}R. \qquad (27)$$

The would-be term $\propto R^2$ corresponds to a trivial vacuum solution of CG and hence does not couple to matter sources [56–59]. Fitting with CG thus implies that $\omega_3 \approx 0$. The magnitude of the constant χ can be associated with the inverse Hubble radius [60], i.e., $\chi \simeq 1/R_H$. One should point out that by means of V_{MK} it has been successfully fitted more than two hundred galactic rotation curves (with no need for dark matter or other exotic modification of gravity) [56–59]. Besides CG, the spherically symmetric gravitational potential with a linear potential also occurs, e.g., in the dilaton-reduced action of gravity [61,62] or $f(R)$ gravity [63].

To be more quantitative, let us assume that the GUP particle in question is inflaton. In such a case, a quasi-classical (decoherence) description is valid at the late-inflation epoch (after the first Hubble radius crossing) and perhaps even after its end during reheating [64,65]. In this period, the NTT should be a pertinent framework for the description of the "inflaton gas." For example, by viewing the inflaton gas as the ideal gas, NTT predicts that the inflaton pressure should satisfy for $0 < q < 1$ a polytrope relation $p \propto \rho^{5/3}$ (ρ is energy density) [66,67]. The relation of this type frequently appears in phenomenological studies on late inflation [68,69]. We can fix β by matching the linear terms in Eqs. (26) and (27). By using $r_s \simeq R_H$, $\kappa_2 = \pi/\ell_p^2 \mathrm{cm}^{-2}$ (the Bekenstein-Hawking value), we get $\varepsilon_q = \ell_p^2/(\pi R_H^2)$. Note that in this setting the effect of the linear potential is comparable to that of the Newtonian potential on length scales $1/R^2 \simeq \varepsilon_q\kappa_2$, i.e., $R \simeq \ell_p\sqrt{1/(\pi\varepsilon_q)} = R_H$. By solving (16) with respect to β and employing (15), we obtain $|\beta| \simeq m_p^2\ell_p^2/(2\pi(\Delta p)_\psi^2 R_H^2)$. To estimate β, we express the Hubble radius as $R_H(t) = H^{-1}(t) = a(t)/\dot{a}(t)$, where H is the Hubble parameter and the scale factor $a(t)$ can be evaluated from the Vilenkin-Ford model [70]: $a(t) = A\sqrt{\sinh(Bt)}$, with $B = 2\sqrt{\Lambda/3}$ (Λ is the cosmological constant). On the other hand, from the relativistic equipartition theorem, we have $(\Delta p)_\psi^2 \simeq 12(k_B T)^2$; cf. Ref. [29]. A straightforward computation gives

$$|\beta| \equiv |\beta|(t) = \frac{m_p^2\ell_p^2\Lambda}{72\pi(k_B T)^2\tanh^2(2t\sqrt{\frac{\Lambda}{3}})}. \qquad (28)$$

For the sake of concreteness, let us consider the late-inflation/reheating epoch, i.e., timescale $t \simeq 10^{-33}$ s. By assuming T of the order of the reheating temperature $T_R \simeq 10^7 \div 10^8$ GeV, we obtain $|\beta| \sim 10^{-2} \div 1$, which is in agreement with the values predicted by string theory; cf. e.g., Refs. [1,2]. In passing, we stress that the above connection with the CG potential works only for $\beta < 0$, or else in (26), we would have a wrong sign in front of the linear potential.

VI. CONCLUSIONS

To conclude, we have derived the explicit form of coherent states for the generalized uncertainty principle and showed that in the momentum representation they coincide with Tsallis probability amplitudes. Furthermore, for $\beta < 0$, we have reformulated GUP in terms of Tsallis entropy–based entropic uncertainty relations, and by invoking the maximal entropy principle, we showed that in the semiclassical (decoherence) limit one can establish equivalence between the GUP quantum systems and nonextensive thermostatistics of Tsallis. This provides a novel framework to discuss transition between the GUP quantum substrate and classical reality and opens a viable route for tabletop experiments to explore possible GUP-based quantum gravitational phenomena via analog gravity models.

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