

Covariant formulation of the generalized uncertainty principle

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(Received 22 November 2021; accepted 5 May 2022; published 18 May 2022)

We present a formulation of the generalized uncertainty principle based on a commutator $[\hat{x}^i, \hat{p}_j]$ between position and momentum operators defined in a covariant manner using normal coordinates. We show how any such commutator can acquire corrections if the momentum space is curved. The correction is completely determined by the extrinsic curvature of the surface $p^2 = \text{constant}$ in the momentum space, and results in noncommutativity of normal position coordinates $[\hat{x}^i, \hat{x}^j] \neq 0$. We then provide a construction for the momentum space geometry as a suitable four dimensional extension of a geometry conformal to the three dimensional relativistic velocity space—the Lobachevsky space—whose curvature is determined by the dispersion relation $F(p^2) = -m^2$, with $F(x) = x$ yielding the standard Heisenberg algebra.

DOI: [10.1103/PhysRevD.105.L101501](https://doi.org/10.1103/PhysRevD.105.L101501)

I. INTRODUCTION

The Heisenberg algebra satisfied by position and momentum operators, \hat{x}^i and \hat{p}_j , forms the cornerstone of quantum mechanics, and in standard treatments, this is characterized by the commutators:

$$[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}^i, \hat{x}^j] = 0. \quad (1)$$

However, the generalization and interpretation of the above commutators in a fully relativistic theory which is also invariant under general coordinate transformations is not straightforward. The key issue is already evident from the explicit appearance of \hat{x}^i in the commutator relationships, which spoils manifest general covariance. This issue needs to be addressed since general covariance is also the first step toward generalization of a theory to a curved background space, or spacetime. Therefore, a curved space(-time) generalization of the Heisenberg algebra requires a covariant definition of the quantities appearing in the above relations. This, in itself, is not difficult if one identifies the coordinates appearing in the commutators via exponential map—the so called normal coordinates. These coordinates, by definition, depend on the geodesic spray from a given point \mathcal{P}_0 , and hence their characterization depends directly on the curvature of the manifold. We will discuss this in Sec. II.

A second generalization of the Heisenberg algebra, that has gained considerable interest over the past two decades [1] is tied to what we expect from measurements at very high energies or at very small length scales. Very general arguments rooted in basic principles of quantum mechanics and general relativity suggest that Planck scale might provide a fundamental operational bound on the measurement of observables such as spacetime intervals [2]. One therefore expects the above commutators to be modified at large momentum, with Planck scale as the characteristic scale at which the modification occurs. This has motivated a large amount of research exploring a modification of the form

$$[\hat{x}^i, \hat{p}_j] = i\hbar\Theta^i_j(\hat{p}_k), \quad (2)$$

with the most popular form of the function Θ being a quadratic in \hat{p}_k that reduces to δ^i_j at low momentum [3]. However, although there exists several arguments based on certain well-motivated requirements that have been used to constrain Θ^i_j at least in flat spacetime (using, for instance, translation and rotation invariance, see [4]), no such generic constraints are available for a curved spacetime.

Our first aim in this paper will be to provide a geometrically well motivated argument that gives a curved spacetime generalization of the Heisenberg algebra, in terms of quantities defined covariantly in a specific region of spacetime, provided only that it is a geodesically convex neighborhood of some spacetime point. Our analysis will

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also give us a hint toward a geometrical interpretation of $\Theta^i_j(\hat{p}_k)$, constructing which will be our next aim.

As a first step toward explicitly constructing $\Theta^i_j(\hat{p}_k)$, we will begin by first analyzing the geometry of the momentum space. To do this, we will start with the well-known expression for center-of-mass energy of a system of two point particles with rest masses m_1 and m_2 , and use it to obtain a metric on the three dimensional momentum space which turns out to be conformal to the well-known Lobachevsky metric on the relativistic velocity space. We then extend this metric to four dimensions by demanding that, for particles with zero relative velocities, it gives the (squared) difference of rest masses: $(\Delta m)^2 = (m_2 - m_1)^2$. The momentum space so constructed is flat (**Riemann** = 0) as long as the relation $p^2 = g^{ab} p_a p_b = -m^2$ holds for a given (p_a, m) . This also yields $\Theta^i_j(\hat{p}_k) = \delta^i_j$. For a general dispersion relation $F(p^2) = -m^2$, **Riemann** $\neq 0$ and we show that $\Theta^i_j(\hat{p}_k)$ is now determined by the extrinsic geometry of $p^2 = \text{constant}$ surface in momentum space.

To summarize, we present in this paper a novel formulation of the generalized uncertainty principle that incorporates and interconnects three key ideas: (i) covariant characterization of \hat{x}^i appearing in the uncertainty principle, (ii) momentum space geometry based on the relativistic velocity space, and (iii) modified dispersion relation $F(p^2) = -m^2$. For some earlier work that comments on similar issues, see [5]. Note, however, that our formalism involves only Lorentz invariant modifications of the dispersion relation, incorporated via a very specific construction of the four dimensional momentum space geometry. Our geometrical approach is also somewhat closer in spirit to Ref. [6]. However, the conceptual setup and mathematical implementation here are completely different.

Conventions: Throughout the paper, Latin indices (serif/nonserif) run from 0 to 3, and the sans-serif font represent frame indices. Boldface quantities represent four vectors.

II. COVARIANT CHARACTERIZATIONS OF COMMUTATORS

A. Normal coordinates

By definition, the normal coordinates are defined by $\Phi^a = -\eta^{ab} e_b^i \nabla_i \Omega$ where e_b^i is an orthonormal tetrad at \mathcal{P}_0 : $e_a \cdot e_b = \eta_{ab}$ (lorentzian) or δ_{ab} (euclidean), and $\Omega(\mathcal{P}_0, \mathcal{P}) = \sigma^2(\mathcal{P}_0, \mathcal{P})/2$ is the Synge world function [7] with $\sigma^2(\mathcal{P}_0, \mathcal{P}) = \pm \lambda^2$; see Fig. 1. In terms of frame components of the tangent vector, this is equivalent to assigning to \mathcal{P} the coordinates $\Phi^a(\mathcal{P}) = \lambda t^a(\mathcal{P}_0)$. The variations in these coordinates are therefore given by [8]

$$\delta\Phi^a(\mathcal{P}) = t^a(\mathcal{P}_0)[\delta\lambda]_{\lambda \text{ fixed}} + \lambda[\delta t^a(\mathcal{P}_0)]_{\lambda \text{ fixed}}. \quad (3)$$

We now wish to compute the variation which arises due to a shift in the origin \mathcal{P}_0 , which we characterize by a vector ϵ^k

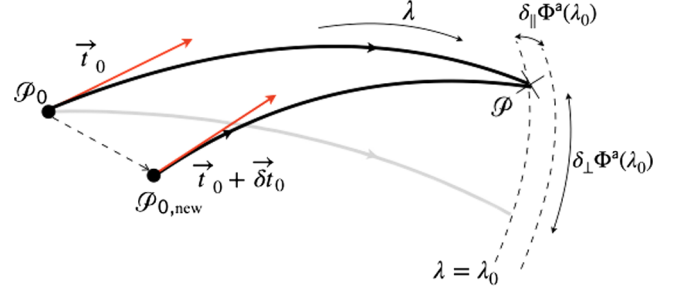


FIG. 1. Normal coordinates defined in the tangent space $T_{\mathcal{P}_0}(M)$. Their variations are completely determined by the geometry of the equigeodesic surfaces anchored at \mathcal{P}_0 (dashed lines), with λ the length of the geodesic connecting \mathcal{P} to \mathcal{P}_0 . The passive version of these variations correspond to shifting the base point \mathcal{P}_0 , in which case the change in normal coordinates is determined by extrinsic geometry anchored at \mathcal{P} .

in $T_{\mathcal{P}_0}(M)$. The above expression can then be written in a neat form by observing that the variation appearing in the second term on the right-hand side (rhs) lies purely within the equigeodesic surface $\lambda = \text{constant}$, and is therefore given by

$$[\delta t^a(\mathcal{P}_0)]_{\lambda \text{ fixed}} = K^a_b \epsilon^b, \quad (4)$$

where K^a_b represents the frame components of the extrinsic curvature tensor associated with the equigeodesic surfaces, and we have used $K^a_b t^b = 0$. We can therefore write

$$\delta\Phi^a(\mathcal{P}) = (\lambda K^a_b - t^a t_b) \epsilon^b. \quad (5)$$

The extrinsic curvature K^a_b of the equigeodesic surfaces is completely determined by curvature tensor of M , and its expansion in normal coordinates can be found in Appendix of the second reference in [9].

B. The commutator

The discussion so far has been general and applies to any manifold. For instance, it can be applied to the case when M represents curved spacetime, and moving to quantum mechanics by identifying the variation of $\Phi^a \equiv x^a$ with the commutator of x^a with the generator of this variation, which can then be identified with the momentum operator [10]. Alternatively, when M represents the momentum space, the variation of $\Phi_a \equiv p_a$ can again be identified, quantum mechanically, with the commutator p_a with the generator of this variation, which we can then identify with the position operator.

When M represents the spacetime manifold, the normal coordinates $\Phi^a(\mathcal{P}) \equiv x^a$, and the rhs of Eq. (5) suggests the following for the commutator between x^a and the operator \hat{p}_b generating the shift of origin by ϵ^b :

$$[\mathbf{x}^a, \tilde{\mathbf{p}}_b] \stackrel{\text{def}}{=} i\hbar(\lambda K^a_b - u^a u_b), \quad (6)$$

where $t^a = u^a$ is the four velocity. In flat spacetime, it follows from the geometry of the equigeodesic surfaces, that $\lambda K^a_b - u^a u_b = \delta^a_b$ [9,11] and one recovers the standard Heisenberg commutators. On curved spacetimes, the above arguments would imply that the Heisenberg commutators are modified to $[\mathbf{x}^a, \tilde{\mathbf{p}}_b] = i\hbar\Theta^a_b(\mathbf{x})$. However, demanding that the candidate for momentum operator be symmetric adds certain constraints, which we briefly discuss; a very detailed analysis can be found in the seminal papers by De Witt [12]. To begin with, it is easy to show that the symmetric momentum operator in position representation with respect to the integration measure, i.e., $d^n x \sqrt{-g}$, is given by

$$\tilde{\mathbf{p}}_a^{\text{sym}} = -i\hbar \left(\frac{1}{2} \Theta^m_a \Gamma^i_{mi} + \frac{1}{2} \Theta^m_{a,m} + \Theta^m_a \partial_m \right). \quad (7)$$

The standard commutator relations

$$[\mathbf{x}^a, \mathbf{p}_b] := i\hbar \delta^a_b, \quad (8)$$

can be obtained even in curved spacetime if we define $\mathbf{p}_a = (\Theta^{-1})^b_a \tilde{\mathbf{p}}_b$. But the new momentum operator is no more symmetric with respect to the integration measure. As soon as one tries to make it symmetric, it takes the following form

$$\mathbf{p}_a^{\text{sym}} = -i\hbar \left(\partial_a + \frac{1}{2} \Gamma^i_{ai} \right). \quad (9)$$

The point we want to emphasize is that, one does not need to modify the commutator relations even in curved spacetime if ‘ Θ ’ is a function of ‘ \mathbf{x} ’ only. As stated above, this is consistent with known results about quantization in curved spaces; see, for instance, [12].

Since one expects quantum gravitational effects to play some role in modifying the conventional Heisenberg algebra and hence the uncertainty principle, it is reasonable to expect the deformations to occur at large momenta (suitably identified). We will therefore now turn to the case when M represents the momentum space, and look for the kind of deformations that the geometry of the momentum space can produce in the commutators. To do so, we will now identify the normal coordinates ($-\mathbf{x}^a$) as generating shifts in \mathbf{p}_a [13]. To compute the commutators, we need information about the equigeodesic surface and its extrinsic curvature in the momentum space, and for this, we need to first characterize the geometry of the momentum space. We will do so in such a manner that the standard dispersion relation $p^2 = -m^2$ will represent.

III. GEOMETRY OF MOMENTUM SPACE

We will now characterize the geometry of the momentum space by starting with the following spacetime picture: two particles of rest energies (masses) m_1 and m_2 and four momenta p_1^i and p_2^i respectively, intersect at an spacetime event. The energy of this system in its center-of-momentum frame has the invariant form

$$\begin{aligned} E_{\text{com}}^2 &= -(p_1 + p_2)^2 \\ &= \mu_1^2 + \mu_2^2 - 2\mu_1\mu_2 \mathbf{u}_1 \cdot \mathbf{u}_2 \\ &= (\mu_1 + \mu_2)^2 + 2\mu_1\mu_2(\gamma_{\text{rel}} - 1), \end{aligned} \quad (10)$$

where we have defined $\mathbf{p}_{1,2} = \mu_{1,2} \mathbf{u}_{1,2}$ (with $\mathbf{u}_{1,2}^2 = -1$), so that $\mathbf{p}^2 = -\mu^2$ and $\gamma_{\text{rel}} = -\mathbf{u}_1 \cdot \mathbf{u}_2$ is the relative gamma factor corresponding to the relative velocity v_{rel} .

Note that, conventionally, μ would be identified with the rest mass m of the particle. However, we postpone this replacement at this stage since we will very soon be interested in generalization $\mu = f(m)$ which will correspond to the modified dispersion relation $F(p^2) = -m^2$. We will therefore keep the discussion general, but will refer to the case $f(m) = m$ to provide motivation for certain definitions and interpretation.

Consider now the quantity $\ell^2 > 0$ defined by:

$$\begin{aligned} \ell^2 &= 2\mu_1\mu_2(\gamma_{\text{rel}} - 1) \\ &= 2\mu_1\mu_2(\cosh\chi_1 \cosh\chi_2 - \sinh\chi_1 \sinh\chi_2 \cos\Omega - 1), \end{aligned} \quad (11)$$

where the second equality obtains by parameterizing u^a in terms of standard Lorentz transformations: $u^a(\chi, \Omega^A) = (\cosh\chi)T^a + (\sinh\chi)N^a$, where T^a, N^a are arbitrary unit timelike, spacelike vectors in the tangent space $T_{\mathcal{P}_0}(M)$, with $T^a N_a = 0$, χ is rapidity and $\Omega^A = (\theta, \phi)$. Note that, in the nonrelativistic limit $v_{\text{rel}} \ll c$, and with $\mu_{1,2} = m_{1,2}$, the above expression yields

$$\frac{\ell^2}{2M} \approx \frac{1}{2} \mu_{\text{red}} v_{\text{rel}}^2,$$

with $M = m_1 + m_2$ and $\mu_{\text{red}} = m_1 m_2 / M$ (reduced mass). This is therefore the relative kinetic energy of the system.

We will now take ℓ^2 as the measure of (squared) ‘‘three momentum distance’’ between the two particles.¹ Defining $\xi^a = (\mu, \chi, \Omega^A)$ The corresponding metric $g_{ab}^{3\text{-mom}}$ may be obtained by

¹Note that one can write $\ell^2 = E_{\text{com}}^2 - [E_{\text{com}}^2]_{v_{\text{rel}}=0}$, which has a nice physical interpretation. For a composite system, the 2nd term on the rhs represents the ‘‘rest energy’’ in the center-of-momentum frame since $v_{\text{rel}} = 0$. The above relation therefore has the structure ‘‘ $|\tilde{\mathbf{p}}|^2 = E^2 - m^2$ ’’ valid for a point particle.

$$\begin{aligned}
 g_{ab}^{3\text{-mom}} &= \lim_{\xi_2^a \rightarrow \xi_1^a} \frac{\partial^2}{\partial \xi_1^a \partial \xi_2^b} \left(\frac{\ell^2}{2} \right) \\
 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu^2 & 0 \\ 0 & 0 & \mu^2 \sinh^2(\chi) \sigma_{AB} \end{pmatrix} \quad (12)
 \end{aligned}$$

where σ_{AB} is the canonical metric on the 2-sphere. We note that, apart from a zero eigenvalue, the above metric is conformal to the Lobachevsky metric of the relativistic velocity space, $dl_{\text{rel}}^2 = d\chi^2 + \sinh^2\chi d\Omega^2$ [14]: $\mu^2 dl_{\text{rel}}^2 = -p^2 dl_{\text{rel}}^2$. This therefore gives a rigorous justification for our definition of distance measure: it correctly gives a locally Lorentz invariant measure of relative momentum on the space of three momenta. To extend it to a four dimensional metric $d\ell^2$, we demand that (i) the four momentum geometry is Lorentzian, and (ii) for points in momentum space that have zero relative velocity, that is, $dl_{\text{rel}} = 0$, the metric gives the difference in rest masses (or rest energies) associated with the corresponding momenta. That is, we require that $d\ell^2 = -dm^2$ when $dl_{\text{rel}}^2 = 0$. This then gives the full metric as:

$$d\ell^2 = -dm^2 - p^2 dl_{\text{rel}}^2 = -\frac{p^2 F'^2}{F} d\mu^2 + \mu^2 dl_{\text{rel}}^2, \quad (13)$$

where $F' = dF/dp^2$.

When $F(p^2) = p^2$, we have the standard dispersion relation $p^2 = -m^2$ and the above metric is easily recognized as flat Minkowski metric in hyperbolic coordinates, and hence **Riemann** = 0.

Modified dispersion relations which are Lorentz invariant, conventionally given in the form $F(p^2) = -m^2$, correspond to $f(x) = \sqrt{-F^{-1}(-x^2)}$. These would generically yield **Riemann** $\propto f''/f \neq 0$. In terms of the dispersion relation $F(p^2) = -m^2$, the commutators are given by,

$$[\mathbf{x}^a, \mathbf{p}_b] := i\hbar \left\{ \delta_b^a + \left(\frac{F(p^2)}{p^2 F'(p^2)} - 1 \right) h_b^a \right\}, \quad (14)$$

where h_b^a is the projector orthogonal to $\partial/\partial\mu$. One can write the above expression in a form in which it is easily compared with the existing works on generalized uncertainty principle (for instance, the now familiar version that was introduced in [1,4]). To do this, one uses the basic definition of normal coordinates, given in Sec. II, applied to the momentum space geometry, and recognize that m represents the geodesic length (from the origin) in the momentum space metric. The above expression then becomes:

$$[\mathbf{x}^a, \mathbf{p}_b] = i\hbar \left(\frac{F}{p^2 F'} \delta_b^a - \frac{1}{F} \left(\frac{F}{p^2 F'} - 1 \right) \mathbf{p}^a \mathbf{p}_b \right). \quad (15)$$

The remaining commutators then follow from a straightforward application of the Jacobi identity. The final form for the modified Heisenberg algebra is then given by:

$$[\mathbf{x}^a, \mathbf{p}_b] = i\hbar (G_1(p^2) \delta_b^a + G_2(p^2) \mathbf{p}^a \mathbf{p}_b), \quad (16)$$

$$[\mathbf{p}_a, \mathbf{p}_b] = 0, \quad (17)$$

$$[\mathbf{x}^a, \mathbf{x}^b] = 2i\hbar \left(G_2 - 2G_1' - \frac{2G_1' G_2 p^2}{G_1} \right) \mathbf{x}^a \mathbf{p}^b, \quad (18)$$

where $G_1 = F/(p^2 F')$ and $G_2 = -(G_1 - 1)/F$. (Note, however, that the last commutator above holds for arbitrary G_1, G_2 .) Evidently, the modified dispersion relation introduces a noncommutativity in normal coordinates. It will be interesting to study the implications of this noncommutativity vis-a-vis the its connection with modified dispersion [15]. The behavior of $[\mathbf{x}^a, \mathbf{p}_b]$ for some model dispersion relations are shown in the Fig. 2.

Some useful limits: To obtain some useful limits, let us write $F(p^2) = p^2(1 + q(p^2))$ which defines $q(p^2)$. If we

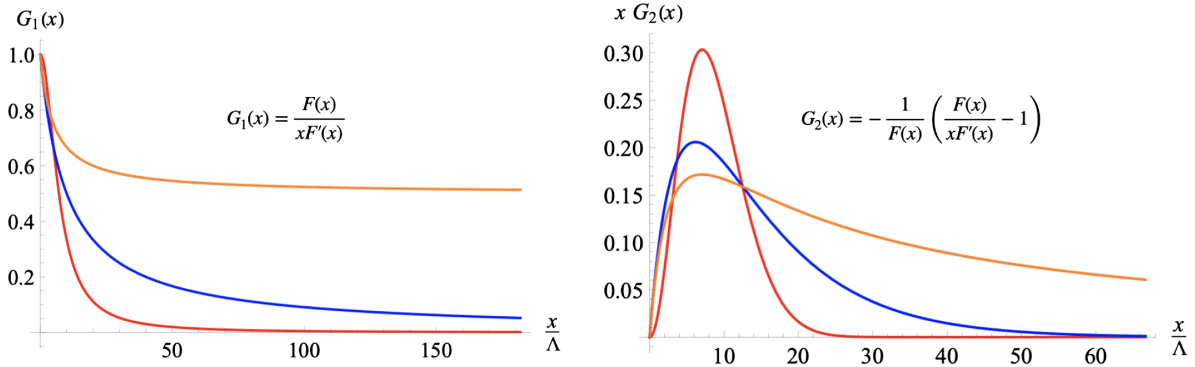


FIG. 2. The behavior of functions $G_1(p^2)$ and $G_2(p^2)$ for three different dispersion relations. These functions are defined by: $[\mathbf{x}^a, \mathbf{p}_b] = i\hbar (G_1(p^2) \delta_b^a + G_2(p^2) \mathbf{p}^a \mathbf{p}_b)$. The three curves correspond to $F(x)$: (i) $x + x[\exp(x/\Lambda)^2 - 1]$ (Red), (ii) $x \exp(x/\Lambda)$ (Blue), (iii) $x[1 + (x/\Lambda)^2]$ (Orange). Here, Λ is a momentum scale and $\Lambda \rightarrow \infty$ corresponds to $F(x) = x$.

demand $q(p^2)$ to be analytic at $p^2 = 0$, and $q(0) = 0$, then $F \sim p^2$ as $p^2 \rightarrow 0$ and $F'(0) = 1$. In this case, we have

$$\begin{aligned} G_1(p^2) &= 1 - q'(0)p^2 + O(p^4) \\ G_2(p^2) &= q'(0) + O(p^2) \end{aligned} \quad (19)$$

which induces a noncommutativity $[\mathbf{x}^a, \mathbf{x}^b] \sim 6i\hbar q'(0)x^{[a}\mathbf{p}^{b]}$ to the lowest order.

The opposite limit, $p^2 \rightarrow \infty$, depends very much on the fall-off behavior of G_1 , G_2 , which in turn would be determined by the theory that yields the modified dispersion. For instance, an exponential fall-off in, say, G_2 , of the form $\exp[-\alpha(p^2)^n]$ (α a real positive constant, $n \in$ positive integers) would give a divergent contribution for timelike momenta $p^2 < 0$ provided n is odd. As a concrete example that does not involve exponentials, consider a dispersion relation of the form $F(p^2) = p^2(1 + \alpha p^{2n})^{-1}$ (α a real constant, $n \in$ positive integers). For such cases, in the high energy limit, $G_1 \sim 1/(1 - n)$, while $G_2 \sim \alpha(n/(n - 1))p^{2(n-1)}$. We therefore see that G_2 would blow up in the high energy limit for $n > 1$. One can do a similar analysis for any dispersion relation; such analysis would be particularly interesting since it might connect analytic properties of $F(p^2)$ with the nature of commutators.

IV. DISCUSSION

We have presented a geometric formalism for the generalized uncertainty principle which is covariant and connects features of the underlying geometry with the deformation of canonical commutator relations. This deformation is tied to extrinsic geometry of the equigeodesic surfaces of the manifold. When the manifold in question is the momentum space, we characterized its geometry in terms of a four dimensional extension of the relative velocity (Lobachevsky) space, whose Riemann curvature is determined by the modified dispersion relation $F(p^2) = -m^2$. Our work therefore interconnects generalized uncertainty principle, momentum space geometry, and modified dispersion relations in a covariant setting, whose only free function is the one that yields the dispersion relation.

Let us highlight some key features inherent in the setup we have described:

As already stated in the Introduction and described in Sec. II, our choice of variables $(\mathbf{x}^a, \mathbf{p}_b)$ are intrinsically determined by geodesic flows in spacetime and momentum space. These flows are nicely described in terms of Sygne's World function and the van Vleck matrix. Therefore, the construction is completely covariant and in terms of physically well-defined variables.

It is also nonlocal, since one must choose a base point of the geodesic flow, in both position and momentum space. The nonlocality in position space is evident from the corresponding commutators in Eq. (18).

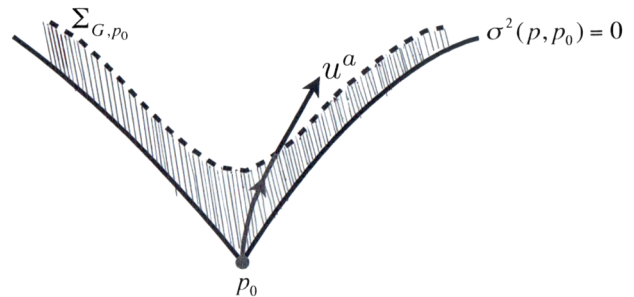


FIG. 3. The choice of reference point in the momentum space can be obtained from the vector u^a giving normal coordinates in position space.

In our description of the momentum space, we have chosen a coordinatization which is based on manifestly covariantly variables, viz the rest mass m and the relative velocity space metric. It will therefore not make much physical sense to talk about general coordinate transformations in the momentum space, although nothing in the formalism forbids it. The equations remain covariant under arbitrary coordinate reparametrizations of the momentum space.

More interesting is the issue of choice of origin in the momentum space. The astute reader would have noticed that we have taken $m = 0$ as the origin. However, we can keep the reference point arbitrary, in which case, the expression for extrinsic curvature K^a_b will be messy. Mathematically, one can still compute it, but conceptually, the interpretation is subtle since when choice of origin corresponds to $m \neq 0$, the coordinates in the relative velocity space (χ, θ, ϕ) must refer to some specific timelike vector. However, note that one does not have to introduce such a vector as an additional object. If $\mathbf{x} \in I^+(\mathbf{x}_0)$, then the tangent to the geodesic connecting \mathbf{x}_0 to \mathbf{x} already provides such a vector; see Fig. 3. This nicely intertwines the issue of reference point in momentum space with the normal coordinates in the position space, but also makes the former “position” dependent. It will be worth exploring the physical implications of this. Finally, we wish to highlight that our construction of the momentum space geometry has an elegant interpretation for the standard dispersion relation $p^2 = -m^2$: it describes flat spacetime in the Milne coordinates, with rest mass m giving the measure of geodesic length from origin.

ACKNOWLEDGMENTS

D. K. acknowledges the support from the Institute of Eminence scheme of IIT Madras funded by the Ministry of Education of India.

APPENDIX: COMMUTATORS AND PHASE SPACE REPRESENTATIONS

Here, we briefly discuss the commutators between position and momentum operators, their phase-space

representations, and constraints arising from the Jacobi identity.

$$[\mathbf{x}^a, \mathbf{p}_b]_{\mathbf{p}_0} = i\hbar\Theta^a_b(\mathbf{p}_0, \mathbf{p}). \quad (\text{A1})$$

One can show that the Jacobi identities yields,

$$[\mathbf{p}_c, [\mathbf{x}^a, \mathbf{x}^b]] = 2i\hbar(\Theta^b_c \mathbf{x}^a - \mathbf{x}^a \Theta^b_c) = 2i\hbar[\Theta^b_c, \mathbf{x}^a]. \quad (\text{A2})$$

This clearly indicate that $[\mathbf{x}^a, \mathbf{x}^b]$ is not going to commute. Let us try to write the possible representation of the position operator in the momentum basis, then we will give the $[\mathbf{x}^a, \mathbf{x}^b]$ relation in terms of $\Theta(\mathbf{p})$.

$$\mathbf{x}^a = i\hbar\Theta^a_m \partial^m. \quad (\text{A3})$$

But for being the physical observable the operator must be symmetric with respect to integration measure of the

momentum space i.e., $d^4\mathbf{p}\sqrt{-g}$. Using the definition of the symmetric operator,

$$\int d^4\mathbf{p}\sqrt{-g}\phi^*\hat{\mathbf{x}}\psi = \int d^4\mathbf{p}\sqrt{-g}(\hat{\mathbf{x}}\phi)^*\psi. \quad (\text{A4})$$

We write momentum representation of the position operator as,

$$\mathbf{x}^a_{\text{sym}} = i\hbar\left(\frac{1}{2}\Theta^{ma}\Gamma^i_{mi} + \frac{1}{2}\Theta_m^{a,m} + \Theta_m^a\partial^m\right). \quad (\text{A5})$$

where $f^m = \partial f / \partial p_m$. By using the Eqs. (A2) and (A5), we write the commutation relation between $\hat{\mathbf{x}}_i$ as

$$[\mathbf{x}^a, \mathbf{x}^b] = i\hbar\{\mathbf{x}^l, (\Theta^{-1})^m_l \Theta_n^{[a} \Theta_m^{b],n}\}. \quad (\text{A6})$$

The $\{, \}$ represent anticommutators bracket.

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- [13] This follows simply from the classical picture of a phase space $(\mathbf{x}^a, \mathbf{p}_b)$. A function $\mathcal{G}(\mathbf{x}^a, \mathbf{p}_b)$ on this space defines the vector field
- $$X_{\mathcal{G}} = \frac{\partial \mathcal{G}}{\partial \mathbf{p}_k} \frac{\partial}{\partial \mathbf{x}^k} - \frac{\partial \mathcal{G}}{\partial \mathbf{x}^k} \frac{\partial}{\partial \mathbf{p}_k}$$
- which implies, for any phase space function Φ , $\delta_{\mathcal{G}}\Phi = X_{\mathcal{G}}[\Phi] = (\Phi, \mathcal{G})_{\text{PB}}$ where ‘‘PB’’ stands for Poisson brackets. Our identification of the commutators is then the conventional one.
- [14] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Volume 2 in Course of Theoretical Physics (Elsevier Ltd., New York, 1975).
- [15] Various examples of such dispersion relations are discussed, for example, in: D. Nesterov and S. N. Solodukhin, *Nucl. Phys. B* **842**, 141 (2011); Also see: T. Padmanabhan, *Phys. Lett. B* **811**, 135912 (2020).